

THE STRUCTURE OF A RIEMANNIAN MANIFOLD ADMITTING A PARALLEL FIELD OF TANGENT VECTOR SUBSPACES

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1. Introduction. Let M be an n -dimensional connected complete Riemannian manifold of class C^2 admitting a parallel field of r -dimensional tangent vector subspaces. Then, M admits the parallel field of s -dimensional tangent vector subspaces, where $s = n - r$, orthogonal to the given field. M is also regarded as a Riemannian manifold whose homogeneous holonomy group fixes an r - (or s -) dimensional tangent vector subspace. The purpose of this note is to treat of the global structure of M . In a case where $r = n - 1$, i.e. $s = 1$, the author [3] already attempted to clarify geometrically the global structure. Here let us discuss the structure in the case where $1 \leq r$, $s \leq n - 1$, from the view-point of fibre bundle. For the main results, see Theorems 1-7. Especially Theorem 3 shows a general structure of M and from the other theorems we may know structures in respective cases. Notice that these theorems all hold good even if R and S in these theorems are exchanged for each other (see Remark 1).

From now on the word " k -dimensional" is abbreviated as " k -", say like k -space (but, such a prefix does not necessarily mean dimension). Let us suppose that indices run as follows: $a, b = 1, 2, \dots, r$; $i, j = r + 1, r + 2, \dots, n$; $\alpha = 1, 2, \dots, n$. The following conventions in a Riemannian manifold X are also applied to all of Riemannian manifolds: The *parallelism* in X means the one of Levi-Civita. A *neighborhood* in X is always an open set homeomorphic to Euclidean space. Take any $x, y \in X$. Let $[x, y]$ denote a geodesic arc joining x to y . And further, take a unit tangent vector v at x . Given a real number c , $g(x, v, c)$ is defined to be the geodesic arc issuing from x , whose length is $|c|$ and whose initial vector is v or $-v$ according as $c > 0$ or < 0 . Let (x, v, c) denote its terminal point. Note that a geodesic arc is not necessarily simple and sometimes may be closed. Let a curve $\alpha: x(t)$ (say, $0 \leq t \leq 1$) be given in X . At $x_0 = x(0)$ we take a unit vector v_0 tangent to X . Corresponding to each t , let $v(t)$ denote the unit vector at $x(t)$ parallel to v_0 along α . Moreover, if a geodesic arc $g(x_0, v_0, c)$ is given, each geodesic arc $g(x(t), v(t), c)$ is said to be *parallel* to $g(x_0, v_0, c)$ along α . And to displace the latter arc parallelly along α is to obtain the former arcs. A *covering manifold* $C(X)$ of X is defined to be a connected covering manifold of X .

with the Riemannian metric naturally induced from X by the covering map p . Especially, if $p^{-1}(x)$ ($x \in X$) consists of just k points, $C(X)$ is called a *k-covering manifold* of X . The notation “ \times ” always means the operation of metric product.

For models of *RS*-manifolds in Remarks 2–6, cf. [3].

2. Preliminaries. As already defined, let M be a connected complete Riemannian n -manifold ($n > 1$) of class C^2 admitting a parallel field of tangent vector r -subspaces ($1 \leq r \leq n - 1$). More precisely, to each point of M a tangent vector r -subspace is assigned so that all of them form a parallel field. We call it the *R-field* over M . Let us take the field of tangent vector s -subspaces, $s = n - r$, which is orthogonal to the *R-field* at each point of M . It is obvious that the field forms a parallel field over M . We call it the *S-field* over M . Throughout the whole discussion, M is such a manifold which will be called an *RS-manifold* of dimension n . In M the following fact is very well-known :

At any $x_0 \in M$ there is a coordinate neighborhood U with coordinate system (x^a) which satisfies the following properties :

- 1) *The transformation from the system (x^a) to an admissible coordinate system of M at x_0 is of class C^2 ;*
- 2) *The Riemannian metric in U is expressed by the form completely decomposed as follows :*

$$ds^2 = g_{ab}(x^1, \dots, x^r) dx^a dx^b + g_{ij}(x^{r+1}, \dots, x^n) dx^i dx^j$$

where g_{ab} and g_{ij} are functions of class C^1 independent of x^i and x^a respectively;

- 3) *A system of equations $x^i = \text{const.}$ expresses an integral manifold of the *R-field* and a system of equations $x^a = \text{const.}$ expresses one of the *S-field*.*

For the proof, see [1], say.

A coordinate neighborhood of $x_0 \in M$ with the same property as U above is called a *reduced coordinate neighborhood* of x_0 , if its coordinate system (x^a) consists of all of (x^a) 's such that $a^a < x^a < b^a$ (a^a, b^a are constants).

Let U, U' be two reduced coordinate neighborhoods of x_0 . Let (x^a) and (x'^a) be their coordinate systems respectively. Let W be the connected components of $U \cap U'$ containing x_0 . In W the coordinate systems (x^a) and (x'^a) are combined by the relations completely decomposed as follows :

$$x'^a = f'^a(x^1, \dots, x^r), \quad x'^i = f'^i(x^{r+1}, \dots, x^n)$$

where f'^a and f'^i are functions of class C^2 independent of x^i and x^a respectively. Moreover we can see that through $x_0 \in M$ there passes a pair

of the maximal connected integral manifolds of the R - and S -fields. Let $R(x_0)$ and $S(x_0)$ denote the ones respectively. We give them the Riemannian metric which is naturally induced from M and call them R - and S -submanifolds of M respectively. They form Riemannian manifolds of class C^1 . The following fact is well-known: *All of the R - and S -submanifolds are totally geodesic, and complete as Riemannian manifolds.* Let $I(x_0)$ denote the set $R(x_0) \cap S(x_0)$.

In M , suppose that there exists a connected open submanifold M^0 which satisfies the following conditions 1) and 2), or 1) and 3):

1) M^0 is a union set of R -submanifolds and the closure of M^0 is M ;

2) M^0 is the maximal subset in which each point x is a limit point of $I(x)$ relative to $S(x)$, or

3) M^0 is a maximal subspace which becomes a fibre bundle where each fibre is an R -submanifold. (By the word "maximal" it is meant that there are no subspaces, $\supset M^0, \neq M^0$, which have the same property.)

When M^0 satisfies 1) and 2), M is said to be of *almost R -clustered type* with *kernel* M^0 . In this case, if $M = M^0$, M is simply said to be of *R -clustered type*.

When M^0 satisfies 1) and 3), M is said to be of *almost R -fibred type* with *kernel* M^0 . In this case, if $M = M^0$, M is simply said to be of *R -fibred type*.

REMARK 1. Throughout this note, R -field and S -field, so R -submanifold and S -submanifold, can not be intrinsically distinguished. Accordingly, the statements all hold good even if we exchange the roles of them. If for example a definition is given, the new definition is obtained by exchanging R and S in it for each other. Of course it holds good. Let us suppose that the new definition is given there, although it is not explicitly stated. This is also applied to notations, lemmas, theorems, and so on. Besides, it is a matter of course that definitions, notations, and so on, in M are used for any RS -manifolds under the same senses.

3. Fundamental lemmas. Take any $x_0 \in M$. An R -neighborhood of x_0 is a neighborhood in $R(x_0)$. An R -normal vector at x_0 is a unit tangent vector at x_0 orthogonal to $R(x_0)$. Take a connected open subset O_R of $R(x_0)$ ($x_0 \in O_R$) and an R -normal vector n_0 at x_0 . At each $x \in O_R$ we plant an R -normal vector $n(x)$ where $n(x_0) = n_0$. If for any $x_1, x_2 \in O_R$, $n(x_1)$ is parallel to $n(x_2)$ along any of curves of class D^1 in O_R joining x_1 to x_2 , the set $\{n(x) | x \in O_R\}$ is called the R -normal vector field over O_R parallel to n_0 ($= n(x_0)$).

Again take $x_0 \in M$ and an R -normal vector n_0 at x_0 . For a constant c , put $y_0 = (x_0, n_0, c)$. Then we have

LEMMA 3.1. *There is an R -neighborhood U_R at x_0 which satisfies the following conditions:*

1) *Over U_R the R -normal vector field $\{n(x)|x \in U_R\}$ parallel to n_0 does exist;*

2) $(x, n(x), c) \in R(y_0)$ for all $x \in U_R$;

3) *The map*

$$f: U_R \rightarrow R(y_0) \text{ defined by } f(x) = (x, n(x), c)$$

is an isometric into-homeomorphism of class C^2 .

PROOF. First let us consider the case where the geodesic $g(x_0, n_0, c)$ is contained in a reduced coordinate neighborhood U . Let U_R be the connected component of $U \cap R(x_0)$ containing x_0 . In U , let (x_0^a) , (y_0^a) , and (n_0^a) denote x_0 , y_0 , and n_0 respectively. Here, $n_0^a = 0$. It is verified that all of the vectors $n(x)$, $x \in U_R$, which have in U the same components as n_0 , form the R -normal vector field parallel to n_0 . So, 1) holds good. In U let (x^a) denote any $x \in U_R$. Here, $x^i = x_0^i$. Moreover we can see that a point $(x, n(x), c)$ is denoted by (x^a, y_0^a) . As $x_0^a = y_0^a$, $(x, n(x), c) \in R(y_0)$, i. e., 2) holds true. 3) is now obvious by § 2.

Next let us consider the case where the geodesic $g(x_0, n_0, c)$ is not contained in a reduced coordinate neighborhood. Take a finite system of reduced coordinate neighborhoods $U_\lambda (\lambda = 1, 2, \dots, h)$ such that each U_λ contains a geodesic arc $[x_{\lambda-1}, x_\lambda]$ where the product curve $[x_0, x_1] \cdot [x_1, x_2] \cdot \dots \cdot [x_{h-1}, x_h]$ becomes $g(x_0, n_0, c)$. For each pair U_λ and $[x_{\lambda-1}, x_\lambda]$, there is an R -neighborhood of $x_{\lambda-1}$ which satisfies the conditions 1)–3), as already proved. Hence, it is easy to find an R -neighborhood U_λ which satisfies our conditions 1)–3).

Under the same notations as Lemma 3.1, let $x(t)$ ($a \leq t \leq b$), $x(a) = x_0$, be a curve of class D^1 in $R(x_0)$. For each t , let $n(t)$ be the R -normal vector at $x(t)$ parallel to n_0 along the curve. We put $y(t) = (x(t), n(t), c)$. Let $n'(t)$ be the vector at $y(t)$ parallel to $n(t)$ along the geodesic $g(x(t), n(t), c)$. Since $g(x(t), n(t), c) \subset S(x(t))$, $n'(t)$ is an R -normal vector. Then we have

LEMMA 3.2. 1) *The curve $y(t)$ ($a \leq t \leq b$) is a curve of class D^1 in $R(y_0)$;*

2) $\{n'(t)|a \leq t \leq b\}$ *consists of R -normal vectors parallel to one another along the curve $y(t)$.*

PROOF. For any t_0 ($a \leq t_0 \leq b$) if we cover the geodesic $g(x(t_0), n(t_0), c)$ by a finite number of reduced coordinate neighborhoods, it is seen that in a suitable interval of t containing t_0 , 1) and 2) hold good (cf. Proof of Lemma 3.1). Accordingly, 1), 2) are proved.

In M , let R, S be any R -, S -submanifolds respectively. Then we have

LEMMA 3.3. *The set $R \cap S$ is at most countable and non-empty.*

PROOF First note that the second countability axiom holds in M, R, S respectively. Now take a countable (or finite) system of reduced coordinate neighborhoods U_λ which cover M . Then $U_\lambda \cap R$ consists of a system of non-intersecting R -neighborhoods, which is at most countable. (Of course the system may be empty). For $U_\lambda \cap S$, too, it holds good. These properties are obvious by the second countability axiom.

Accordingly, $U_\lambda \cap R \cap S$ is at most countable. Hence $R \cap S$ is at most countable. For the assertion that $R \cap S$ is non-empty, see [2], p. 23.

4. A general structure. For any two points x_1, x_2 of the same R -submanifold in M , let $d_R(x_1, x_2)$ denote the length of a minimizing geodesic $[x_1, x_2]$ in the R -submanifold. Take any $x_0 \in M$ and a constant $a > 0$. Let $C_R(x_0; a)$ denote the part of $R(x_0)$ defined by the subset $\{x | x \in R(x_0), d_R(x_0, x) \leq a\}$. If a set $\{x | x \in R(x_0), d_R(x_0, x) < a\}$ forms an R -neighborhood of x_0 which can be covered by a normal coordinate system in $R(x_0)$ with center x_0 , this neighborhood is called a *normal R -neighborhood* of x_0 and denoted by $N_R(x_0; a)$. Moreover such a constant a is called a *normal R -radius* at x_0 . Let $T_R(x_0)$ denote the Euclidean vector r -space tangent to $R(x_0)$ at x_0 . The map \exp_R at $x_0 \in M$ is defined to be the map $T_R(x_0) \rightarrow R(x_0)$ such that $\exp_R v = x_0$ for the zero vector $v \in T_R(x_0)$ and $\exp_R v = (x_0, v/|v|, |v|)$ for any non-zero vector $v \in T_R(x_0)$, where $|v|$ denotes the length of v .

Again at $x_0 \in M$ let $e_R(x_0)$ denote the greatest lower bound of $\{d_R(x_0, x) | x \in I(x_0) - x_0\}$ if $I(x_0) - x_0$ is non-empty. If $I(x_0) - x_0$ is empty, we put $e_R(x_0) = \infty$. Accordingly, $0 \leq e_R(x) \leq \infty$ for any $x \in M$.

LEMMA 4.1. 1) *If $e_R(x_0) = 0$, $e_R(x) = 0$ for all $x \in S(x_0)$ (so, if $e_R(x_0) > 0$, $e_R(x) > 0$ for all $x \in S(x_0)$).*

2) *If $e_R(x_0) > 0$, there is a constant $a > 0$ such that the parts $C_R(x; a)$ for all $x \in S(x_0)$ do not intersect one another.*

3) *A necessary and sufficient condition for $e_R(x_0) > 0$ is that the topology of $S(x_0)$ coincides with the relative one induced from M .*

PROOF. 1) is evident by Lemma 3.1. To prove 2), at x_0 take a normal R -radius $c < e_R(x_0)$. Then for any $x \in S_0 = S(x_0)$, $C_R(x; c) \cap S_0$ consists of x only. For, otherwise, there is $x \in S_0$ such that $C_R(x; c) \cap S_0$ contains a point $x' (\neq x)$. Let $[x, x']$ be a minimizing geodesic in $R(x)$. And let $[x_0, x'']$ be the geodesic parallel to $[x, x']$ along a curve in S_0 . Hence, $x'' \in S_0$ by Lemma 3.2, $[x_0, x''] \subset C_R(x_0; c)$, and $x_0 \neq x''$. Here, $d_R(x_0, x'') \leq c$

$< e_R(x_0)$. These results contradict with the definition of $e_R(x_0)$. So, $C_R(x; c) \cap S_0$ for each $x \in S_0$ consists of x only. Now, put $a = c/2$. Then it is obvious that the constant a is a constant a in 2). 3) is easily proved by using 2).

Let x_0 be a point of M . Take a closed curve β of class D^1 in $R_0 = R(x_0)$ starting from x_0 . For any $v \in T_S(x_0)$, we obtain the vector v' at the terminal point x_0 by displacing v parallelly along β . Of course, $v' \in T_S(x_0)$. Then the map f_β of $T_S(x_0)$ onto itself, defined by $f_\beta(v) = v'$, is a congruent transformation in $T_S(x_0)$. This is said to be the congruent transformation *induced* from β . All of such transformations form a group. We denote it by $G(R_0, x_0)$ or $G(R_0)$ (it being independent of x_0 as abstract group).

LEMMA 4.2. $G(R_0, x_0)$ is isomorphic with a factor group of the fundamental group $\pi_1(R_0, x_0)$. Hence the order of $G(R_0, x_0)$ is at most countable.

PROOF. Let β_0 be a closed curve of class D^1 in R_0 , starting from x_0 and in R_0 homotopic to x_0 . Then the congruent transformation f_{β_0} in $T_S(x_0)$ induced from β_0 is the identity. For, otherwise, we can find a unit vector $v \in T_S(x_0)$ such that $f_{\beta_0}(v) \neq v$. Let c be a normal S -radius at x_0 . So, for a constant δ ($0 < \delta < c$), $g(x_0, v, \delta)$ is parallel to $g(x_0, f_{\beta_0}(v), \delta)$ along β_0 . Here if we deform β_0 to x_0 , we obtain a curve in $N_S(x_0; c)$ joining $y_{\beta_0} = (x_0, f_{\beta_0}(v), \delta)$ to $y = (x_0, v, \delta)$ as the locus of y_{β_0} . This curve is contained in $I(y)$. As $y_{\beta_0} \neq y$, this is contrary to Lemma 3.3. The fact above gives rise to the homomorphic map of $\pi_1(R_0, x_0)$ onto $G(R_0, x_0)$ naturally. So the former part is proved. The latter part is clear because $\pi_1(R_0, x_0)$ is at most countable.

In M , let x_0, y_0 be two points of an S -submanifold. Let $[x_0, y_0]$ be a geodesic arc in $S(x_0)$. Put $[x_0, y_0] = g(x_0, n_0, c)$. If $R(x_0)$ admits the R -normal vector field $\{n(x) | x \in R(x_0)\}$ parallel to n_0 , we can consider the map

$$f: R(x_0) \rightarrow R(y_0) \text{ defined by } f(x) = (x, n(x), c)$$

by Lemma 3.2. f is said to be the map *induced* from $[x_0, y_0]$.

LEMMA 4.3. f is locally an isometric homeomorphism of class C^2 and f is also a covering map.

PROOF. From Lemmas 3.1 and 3.2, the map f is onto and locally an isometric homomorphism of class C^2 . For any $y \in R(y_0)$ the subset $f^{-1}(y)$ of $R(x_0)$ is at most countable by Lemma 3.3. Let x_λ ($\lambda = 1, 2, \dots$) denote all of the points of $f^{-1}(y)$. Here, $f^{-1}(y)$ is contained in a compact subset $C_S(y; |c|)$. By covering $C_S(y; |c|)$ by a finite number of reduced coordinate neighborhoods, we can find an R -neighborhood $W_R(y)$ of y and the R -neighborhoods $W_R(x_\lambda)$ of x_λ for all λ , such that all $W_R(x_\lambda)$ are isometrically homeomorphic to $W_R(y)$

under f . Then, all of $W_R(x_\lambda)$ do not intersect one another. For, suppose that

$$W_R(x_\mu) \cap W_R(x_\nu) \neq 0 \quad \text{for } x_\mu, x_\nu \in f^{-1}(y) \ (x_\mu \neq x_\nu).$$

If we take a curve $\alpha \subset W_R(x_\mu) \cup W_R(x_\nu)$ joining x_μ to x_ν , then we have $f(\alpha) \subset W_R(y)$. This gives rise to a contradiction. Accordingly, our lemma is proved.

In M suppose that R_0 is an R -submanifold such that $G(R_0)$ consists of the identity only. Then R_0 is said to be R -maximal in M . Here, note the following property: Take $x_0 \in R_0$, $y_0 \in S(x_0)$. Let $[x_0, y_0]$ be a geodesic in $S(x_0)$. Put $R_1 = R(y_0)$. Then, there exists the map $f: R_0 \rightarrow R_1$ induced from $[x_0, y_0]$. By Lemma 4.3, R_0 is regarded as a covering manifold of R_1 under f . Moreover, if R_1 is R -maximal, it is easy to see

LEMMA 4.4. *The map f is an isometric homeomorphism of class C^2 , of R_0 onto R_1 .*

Let V be a Euclidean vector d -space which is topologized by regarding as Euclidean space. Let G be an effective group of congruent transformations in V , which is at most countable. We denote all of the elements of G by $g_\lambda (\lambda = 0, 1, 2, \dots)$, where g_0 is the identity. For each $g_\lambda (\lambda \neq 0)$, put $V_\lambda = \{v \mid g_\lambda v = v\}$. V_λ forms a subspace of dimension $< d$ in V . Let V^0 denote $V - \bigcup_{\lambda=1}^{\infty} V_\lambda$. Then, V^0 is non-empty. For any $v \in V^0$, the vectors

$$v, g_1 v, g_2 v, \dots$$

are all distinct from one another. This is easily verified. Such a vector v is said to be *completely variant* under G . It follows that V^0 consists of all of the vectors completely variant under G .

First, suppose that G is finite. Then we have

LEMMA 4.5. 1) V^0 is an open set of V and the closure of V^0 is V .
2) For any unit vectors $u, u' \in V^0$, there is a sequence of unit vectors:

$$u_1(=u), u_2, \dots, u_k(=u'), \subset V^0$$

such that $u_\mu, u_{\mu+1}$ belong to the same connected component of V^0 or are G -connected ($\mu = 1, 2, \dots, k-1$).

By the word " G -connected" it is meant that $u_{\mu+1} = g(\mu)u_\mu$ for a suitable $g(\mu) \in G$. k suffices to be an integer > 1 .

PROOF. As 1) is obvious, we prove 2). If V^0 is connected, the sequence: $u_1(=u), u_2(=u')$, satisfies our condition. Accordingly we consider the case where V^0 is not connected. Then, among $V_\lambda (\lambda = 1, 2, \dots, h-1; h = \text{the order}$

of G), there is at least one of dimension $d - 1$. Let us suppose V_1 to be such one. Take $v \in V^0$. The vector is represented by $u_1 + v_1$ where $u_1 (\neq 0)$ is perpendicular to V_1 and $v_1 \in V_1$. Then we have $g_1(u_1 + v_1) = -u_1 + v_1$. I. e., the vector v is G -connected with $v' = -u_1 + v_1$. The vector v' belongs to the side distinct from v with respect to V_1 . Here, if $v \in V^0$ is suitably chosen, v' belongs to V^0 . From this fact, 2) is proved.

Next, suppose that G is infinite (i. e., countable). Let V' be the set of all of $v \in V$ such that the vectors $g_\lambda v (\lambda = 0, 1, 2, \dots)$ indeed consist of a finite number of vectors distinct from one another. Any vector of $V - V'$ is said to be *infinitely variant* under G .

- LEMMA 4.6. 1) If $v \in V'$, $g_\lambda v \in V'$;
 2) V' forms a vector subspace of V ;
 3) For the dimension d of V' , $0 \leq d' < d - 1$;
 4) $V - V'$ is a connected open subset of V .

PROOF. 1) and 2) are obvious. To prove 3), suppose that $d' = d - 1$. By 1), $g_\lambda \cdot V' = V'$ for any $g_\lambda \in G$. Hence, for a vector e normal to V' , $g_\lambda e = e$ or $-e$. So, $e \in V'$. I. e., $V' = V$. This is contrary to the existence of vectors completely variant under G . Accordingly, 3) holds good. From 3), 4) follows immediately. This completes the proof.

THEOREM 1. In M suppose that the topology of every R -submanifold coincides with the relative one induced from M . Then there are R -maximal R -submanifolds. In all of them let M^0 be the subspace of M which is their union set. Then M^0 is a connected open submanifold of M whose closure is M and a maximal subspace which becomes a fibre bundle where each fibre is a R -submanifold. In other words, M is of almost R -fibred type with kernel M^0 .

PROOF. 1) For any R -submanifold R , $G(R)$ is finite. In fact, suppose that it is infinite (i. e., countable by Lemma 4.2). Denote all of the elements of $G(R, x)$, $x \in R$, by $g_\lambda (\lambda = 0, 1, 2, \dots)$ where g_0 is the identity. Then we can find a unit vector $v \in T_S(x)$ completely variant under $G(R, x)$. Let a be a normal S -radius at x . For a constant b ($0 < b < a$) put $x_\lambda = (x, g_\lambda v, b)$. Then $x_\lambda (\lambda = 0, 1, 2, \dots)$ are distinct from one another. And, $x_\lambda \in R(x_0) \cap C_S(x; b)$. $C_S(x; b)$ being compact, we have $e_S(x_0) = 0$. This contradicts with the assumption of our theorem by Lemma 4.1. So, $G(R)$ is finite.

2), Take an R -submanifold $R_0 = R(x_0)$. Let k_0 be the order of $G(R_0, x_0)$. By 1), k_0 is finite. Denote all of the elements of $G(R_0, x_0)$ by $g (\lambda = 0, 1, \dots, k_0 - 1)$ where g_0 is the identity. We have $e_S(x_0) > 0$. Let a be a normal

S -radius at x_0 such that $0 < a < e_s(x_0)/2$. Let v_0 be a unit vector of $T_s(x_0)$ completely variant under $G(R_0, x_0)$. For a constant δ ($0 < \delta < a$) put $y_\lambda = (x_0, g_\lambda v_0, \delta)$. Let $g(y_0, u_0, \delta)$ denote the geodesic $[y_0, x_0]$ in $N_s(x_0; a)$. Put $R_1 = R(y_0)$. Then, $R_1 \cap N_s(x_0; a) = \{y_\lambda | \lambda = 0, 1, \dots, k_0 - 1\}$. For, take $y \in R_1 \cap N_s(x_0; a)$ and displace $[y_0, x_0]$ parallelly along a curve of class D^1 in R_1 joining y_0 to y . At y , we obtain the geodesic $[y, x'_0]$ in $S(x_0)$. Here

$$d_s(x_0, x'_0) \leq d_s(x_0, y) + d_s(y, x'_0) < a + \delta < e_s(x_0).$$

So, $x_0 = x'_0$. From this manner, we can see that $y \in \{y_\lambda | \lambda = 0, 1, \dots, k_0 - 1\}$ by Lemma 3. 2. Accordingly, $R_1 \cap N_s(x_0; a)$ consists of y_λ ($\lambda = 0, 1, \dots, k_0 - 1$) only.

Now over R_1 there is the R -normal vector field parallel to u_0 . For, otherwise, by displacing u_0 parallelly along a suitable closed curve in R_1 we can obtain a vector u'_0 at y_0 distinct from u_0 . Of course, $u'_0 \in T_s(y_0)$. By Lemma 3. 2, $x'_0 = (y_0, u'_0, \delta) \in R(x_0)$. So $x'_0 \in I(x_0)$. Here, $x_0 \neq x'_0$ and we have

$$d_s(x_0, x'_0) \leq d_s(x_0, y_0) + d_s(y_0, x'_0) \leq 2\delta < e_s(x_0).$$

This is contrary to the definition of $e_s(x_0)$. So our assertion is true. Hence there is the map $f: R_1 \rightarrow R_0$ induced from the geodesic $[y_0, x_0]$. By Lemma 4. 3, R_1 is a k_0 -covering manifold of R_0 under f .

2)₂ We prove that R_1 is R -maximal. Denote all of the elements of $G(R_1, y_0)$ by h_μ ($\mu = 0, 1, \dots, k_1 - 1$) where h_0 is the identity and k_1 is the order. By 1), $1 \leq k_1 < \infty$. Now suppose $k_1 > 1$. We take a constant $\varepsilon < e_s(y_0)/2$, which becomes at each y_λ a normal S -radius, such that all $N_s(y_\lambda; \varepsilon)$ are contained in $N_s(x_0; a)$ and do not intersect one another. Here we can find a unit vector $w_0 \in T_s(y_0)$ completely variant under $G_s(R_1, y_0)$ and perpendicular to u_0 . All of the vector $h_\mu w_0$ are perpendicular to u_0 . Put $z_0 = (y_0, w_0, \varepsilon)$ for a constant ε , $0 < \varepsilon < \varepsilon$. Then there is a map $f': R(z_0) \rightarrow R_1$ induced from the geodesic $[z_0, y_0]$ in $N_s(y_0; \varepsilon)$. Under f' , $R(z_0)$ is the k_1 -covering manifold of R_1 . This is verified by the same way as 2)₁. Let $[z_0, x_0]$ be the geodesic in $N_s(x_0; a)$. Then, there is a map $f'': R(z_0) \rightarrow R_0$ induced from $[z_0, x_0]$. Under the map f'' , $R(z_0)$ is a $k_0 k_1$ -covering manifold of R_0 . This is easily verified, too. These results implies that $G(R_0, x_0)$ has order $\geq k_0 k_1$, so $> k_0$. This being a contradiction, k_1 must be one. I.e., R_1 is R -maximal.

2)₃ In the case where R_0 is R -maximal, we can see by 2)₂ that the R -submanifolds $R(y)$ for all $y \in N_s(x_0; a)$ are R -maximal.

Let us consider the case where R_0 is not R -maximal and where there is a unit vector $v \in T_s(x_0)$ which is not completely variant under $G(R_0, x_0)$. Put $y = (x_0, v, \delta)$ where $0 < \delta < a$. Then, $R(y)$ is not R -maximal. To prove this, at y take a normal S -radius $b < e_s(y)/2$. On the other hand, there is $g (\neq g_0) \in$

$G(R_0, x_0)$ such that $gv = v$. Hence, we can find a unit vector $v^* \in T_s(x_0)$ such that $y^* \neq y'_g$ and $y^*, y'_g \in N_s(y; b)$ for $y^* = (x_0, v^*, \delta), y'_g = (x_0, gv^*, \delta)$. Let $[y^*, y]$ be the geodesic in $N_s(y; b)$. Displace $[y^*, y]$ parallelly along a curve of class D^1 in $R(y^*)$ joining y^* to y'_g . At y'_g we obtain the geodesic $[y'_g, y']$. Of course $[y'_g, y'] \subset S(x_0)$. As $d_s(y, y') < e_s(y)$, we have $y = y'$. This means that the order of $G(R(y), y)$ is not one. So, $R(y)$ is not R -maximal.

3) Take an S -submanifold S of M . In S let S^0 be the subspace consisting of all $x \in S$ such that $R(x)$ is R -maximal. By 2)₃, S^0 is open in S . We prove that in S the closure of S^0 is S . It suffices to consider the case only where $S - S^0$ is non-empty. Take $x_0 \in S - S^0$ and at x_0 a normal S -radius $c < e_s(x_0)/2$. We put $R_0 = R(x_0)$. The order of $G(R_0, x_0)$ is greater than one. Let V^0 be the set of all vectors of $T_s(x_0)$, each of which has length $< c$ and is completely variant under $G(R_0, x_0)$. Then, $\exp_s V^0 = S^0 \cap N_s(x_0; c)$ by 2)₂, 2)₃. From Lemma 4.5, we can see that in $N_s(x_0; c)$ the closure of $\exp_s V^0$ is $N_s(x_0; c)$. So, x_0 is contained in the closure in S of S^0 . Accordingly our assertion is proved.

Now, by Lemma 3.3 M^0 is regarded as the union set of $\{R(x) | x \in S^0\}$. From the above facts and Lemma 3.1, M^0 is an open submanifold of M whose closure is M .

Next, we prove that M^0 is connected. For this, it suffices to show that any two points $x_1, x_2 \in S^0$ are joined by a curve in M^0 . Let α be a curve in S joining x_1 to x_2 . Cover α by a finite number of normal S -neighborhoods $N_s(y_\lambda; a_\lambda)$ ($\lambda = 1, 2, \dots, h$) where $y_\lambda \in \alpha, a_\lambda < e_s(y_\lambda)/2$. For some λ ; if $y_\lambda \in S^0, N_s(y_\lambda; a_\lambda) \subset S^0$ by 2)₃. If $y_\lambda \notin S^0$, we denote by W_λ the subspace of M which is the union set of $\{R(x) | x \in S^0 \cap N_s(y_\lambda; a_\lambda)\}$. Moreover let V_λ^0 be the set of all vectors of $T_s(y_\lambda)$, each of which has length $< a_\lambda$ and is completely variant under $G(R(y_\lambda), y_\lambda)$. Here, $\exp_s V_\lambda^0 = S^0 \cap N_s(y_\lambda; a_\lambda)$. If we give $T_s(y_\lambda)$ the topology by regarding as Euclidean space, V_λ^0 has the same property as the part of V^0 in Lemma 4.5. Hence, we can see that W_λ is open in M and connected. These facts, together with the property that in S the closure of S^0 is S , show that x_1, x_2 are joined by a curve in M^0 . So, M^0 is connected.

4) Take $x_0 \in S^0$ and at x_0 a normal S -radius $a < e_s(x_0)/2$. Then, $N_s(x_0; a) \subset S^0$ by 2)₃. For $z \in N_s(x_0; a)$ let $[x_0, z]$ be the geodesic contained in $N_s(x_0; a)$. Then there is the map $f_z: R(x_0) \rightarrow R(z)$ induced from $[x_0, z]$. This map f_z is an isometric homeomorphism by Lemma 4.4. Denote $R(x_0) \times N_s(x_0; a)$ by $V(x_0)$. Hence, any $x \in V(x_0)$ is represented by a pair (y, z) where $y \in R(x_0), z \in N_s(x_0; a)$. Define a map

$$f: V(x_0) \rightarrow M^0 \text{ by } f(x) = f_z(y).$$

The map f is one-to-one. For, otherwise, there are $x_1, x_2 \in V(x_0), x_1 \neq x_2,$

such that $f(x_1) = f(x_2)$. Represent x_1 by (y_1, z_1) and x_2 by (y_2, z_2) . Hence $f_{z_1}(y_1) = f_{z_2}(y_2)$, so $R(z_1) = R(z_2)$. As $z_1, z_2 \in N_S(x_0; a)$ and $R(z_1) \cap N_S(x_0; a)$ consists of z_1 only, we have $z_1 = z_2$. It follows that $y_1 = y_2$. I. e., $x_1 = x_2$. This is a contradiction. So, f is one-to-one. It is verified that f is an isometric into-homeomorphism such that $f(R(x_0), z) = R(z)$ for all $z \in N_S(x_0; a)$, $f(y, N_S(x_0; a)) \subset S(y)$ for all $y \in R(x_0)$.

In S^0 , if $x, y \in S^0$ belong to the same R -submanifold, we say that they are equivalent to each other. By this equivalence relation, we construct the quotient space of S^0 and denote it by B . Then, B becomes a manifold and over B a Riemannian metric is naturally induced from S^0 . Thus B is regarded as a connected Riemannian s -manifold of class C^1 . Next, for any $x \in M^0$, let $[x]$ denote the point of B representing $R(x) \cap S^0$. Then the map

$$\pi: M^0 \rightarrow B \text{ defined by } \pi(x) = [x]$$

is an onto-map. Thus we can prove that M^0 becomes a fibre bundle where each fibre is an R -submanifold, the base space is B , and the projection is π . The proof is omitted, as it is too long to give here (cf. [5]).

5) If $M = M^0$, our theorem holds good, M being of R -fibred type. So it remains to consider the case where $M \neq M^0$. For $x \in M - M^0$, the order of $G(R(x), x)$ is not one. Hence by 2)₂ it follows that any S -neighborhood of x contains at least two points of an R -submanifold which is contained in M^0 . This shows that there is no subspace, $\supset M^0$, $\neq M^0$, which is a union set of R -submanifolds and a fibre bundle where each fibre becomes an R -submanifold. Accordingly, M is of almost R -fibred type with kernel M^0 . This completes the proof of our theorem.

THEOREM 2. *In M suppose that the topology of at least one R -submanifold does not coincide with the relative one induced from M . In all of such R -submanifolds let M^0 be the subspace of M which is their union set. Then M^0 is a connected open submanifold of M whose closure is M , and the maximal subset of M in which each point x is a limit point of $I(x)$ relative to $S(x)$. In other words, M is of almost R -clustered type with kernel M^0 .*

PROOF. In the case where the topology of every R -submanifold does not coincide with the relative one induced from M , $e_S(x) = 0$ for all $x \in M$ by Lemma 4.1. So, M is of R -clustered type. Our theorem holds good. Accordingly consider the other case. Then, there is at least one R -submanifold R_0 whose topology coincides with the relative one. Let R_1 denote an R -submanifold whose topology does not coincide with the relative one.

1) For $x_0 \in R_0$, we have $e_S(x_0) > 0$ by Lemma 4.1. Let us prove that

$G(R_0)$ is infinite, i. e., countable. Take $y_0 \in R_1 \cap S(x_0)$, so $e_S(y_0) = 0$. Let $[y_0, x_0]$ be a minimizing geodesic in $S(x_0)$. Put $L = d_S(y_0, x_0)$. Let a be a normal S -radius at y_0 . We denote $N_S(y_0; a) \cap R_1$ by $\{y_\lambda | \lambda = 0, 1, 2, \dots\}$, it being countable. For each λ let β_λ be a curve of class D^1 in R_1 joining y_0 to y_λ . Displace $[y_0, x_0]$ parallelly along β_λ . As the locus of the terminal point x_0 we obtain a curve α_λ and at y_λ the geodesic $[y_\lambda, x_\lambda]$. x_λ is the terminal point of α_λ and by Lemma 3.2, $\alpha_\lambda \subset R_0$. Moreover, $\{x_\lambda | \lambda = 0, 1, 2, \dots\} \subset I(x_0)$, and $\subset C_S(y_0, a + L)$. From the compactness of $C_S(y_0, a + L)$ and Lemma 4.1, the set $\{x_\lambda | \lambda = 0, 1, 2, \dots\}$ must be finite. Hence, there is an infinite subset $\{\lambda_1, \lambda_2, \dots, \lambda_k, \dots\}$ of $\{0, 1, 2, \dots\}$ such that $x_{\lambda_1} = x_{\lambda_2} = \dots = x_{\lambda_k} = \dots$. Displace the geodesic $[y_{\lambda_1}, x_{\lambda_1}]^{-1} = g(x_{\lambda_1}, v'_0, c)$ ($c > 0$) parallelly along each product curve $\alpha_{\lambda_1}^{-1} \cdot \alpha_{\lambda_k}$. At $x_{\lambda_k} = x_{\lambda_1}$, we obtain the geodesic $[y_{\lambda_k}, x_{\lambda_k}]^{-1}$. $\{y_{\lambda_k}\}$ being however infinite, it follows that the vector v'_0 is infinitely variant under $G(R_0, x_{\lambda_1})$. So, $G(R_0, x_{\lambda_1})$, i. e., $G(R_0)$ is infinite.

From this proof, it is seen that if we put $[y_0, x_0]^{-1} = g(x_0, v_0, c)$, the vector v_0 is infinitely variant under $G(R_0, x_0)$.

2) Take any S -submanifold S . Let S^0 be the maximal subset of S such that each point x satisfies $e_S(x) = 0$. In our case, $S^0 \neq S$. For any $x_0 \in S - S^0$, $e_S(x_0) > 0$ and $G(R'_0, x_0)$, $R'_0 = R(x_0)$, is infinite by 1). Take a normal S -radius a at x_0 . Let V^0 be the set of all vectors in $T_S(x_0)$ with lengths $< a$, infinitely variant under $G(R'_0, x_0)$. Then, $\exp_S V^0 = N_S(x_0; a) \cap S^0$. For, it is obvious that $\exp_S V^0 \subset N_S(x_0; a) \cap S^0$. Take any $y_0 \in N_S(x_0; a) \cap S^0$. If $g(x_0, v_0, \delta)$ is the geodesic $[x_0, y_0]$ in $N_S(y_0; a)$, the vector v_0 is infinitely variant under $G(R'_0, x_0)$ by 1). I. e., $y_0 \in \exp_S V^0$. So, our assertion is true. Here, by using Lemma 4.6, it is shown that $\exp_S V^0$ is a connected open subset of $N_S(x_0; a)$ and its closure in $N_S(x_0; a)$ contains x_0 . Accordingly, in S the closure of S^0 is S .

Moreover, S^0 is open in S . For, if $y_0 \in S^0$ is not an inner point of S^0 relative to S , we can find $x_0 \in S - S^0$ and a normal S -radius a at x_0 such that $y_0 \in N_S(x_0; a)$. However, $N_S(x_0; a) \cap S^0$ is a connected open subset of $N_S(x_0; a)$ containing y_0 . This is a contradiction. So, S^0 is open in S .

Next, we prove that S^0 is a connected subset of S . In fact take two points $x_1, x_2 \in S^0$. Let α be a curve in S joining x_1 to x_2 . Cover α by a finite number of normal S -neighborhoods $N_S(y_\lambda; a_\lambda)$, $y_\lambda \in \alpha$ ($\lambda = 1, 2, \dots, h$), such that if $y_\lambda \in S^0$ for some λ , $N_S(y_\lambda; a_\lambda) \subset S^0$. Then, by the properties above, we can verify that x_1, x_2 are joined by a curve in S^0 . So, S^0 is connected.

3) By Lemmas 3.3 and 4.1, M^0 is regarded as the union set of $\{R(x) | x \in S^0\}$. In other words, M^0 is the maximal subset of M in which each point x is a limit point of $I(x)$ relative to $S(x)$. From 2), it follows that M^0 is a connected open submanifold of M whose closure is M . Therefore M is of almost R -

clustered type with kernel M^0 .

Summing up Theorems 1, 2, we have

THEOREM 3. *M is of almost R -fibred type or almost R -clustered type.*

REMARK 2. There exist RS -manifolds of the following respective type: R -fibred type; almost R -fibred type (not R -fibred type); R -clustered types; almost R -clustered type (not R -clustered type).

5. Fundamental groups and structures. Take any $x_0 \in M$ and put $R_0 = R(x_0)$, $S_0 = S(x_0)$. Let $i_R: R_0 \rightarrow M$ be the inclusion map. Let $i_R^*: \pi_1(R_0, x_0) \rightarrow \pi_1(M, x_0)$ be the homomorphism induced by the map i_R ([4], p. 75). It is already known that the map i_R^* is into-isomorphic ([2], p. 22). We denote the image $i_R^*\pi_1(R_0, x_0)$ by $i\pi_1(R_0, x_0)$. This is the subgroup of $\pi_1(M, x_0)$. Let $U(M)$ denote the universal covering manifold of M . Let p denote the covering map. So $U(M)$ becomes naturally an RS -manifold of dimension n . Take a point $\tilde{x}_0 \in p^{-1}(x_0)$. For the R -, S -submanifolds $R(\tilde{x}_0)$, $S(\tilde{x}_0)$ of $U(M)$, $R(\tilde{x}_0) \times S(\tilde{x}_0)$ is a Riemannian manifold of class C^1 . Then the following theorem is well-known: *There is the isometric homeomorphism*

$$j: R(\tilde{x}_0) \times S(\tilde{x}_0) \rightarrow U(M)$$

of class C^2 such that $j(\tilde{x}, \tilde{x}_0) = \tilde{x}$ for all $\tilde{x} \in R(\tilde{x}_0)$ and $j(\tilde{x}_0, \tilde{x}) = \tilde{x}$ for all $\tilde{x} \in S(\tilde{x}_0)$ [1]. Hence, $j(R(\tilde{x}_0), \tilde{x}) = R(\tilde{x})$ for $\tilde{x} \in S(\tilde{x}_0)$ and $j(\tilde{x}, S(\tilde{x}_0)) = S(\tilde{x})$ for each $\tilde{x} \in R(\tilde{x}_0)$. Such a map is always denoted by j . The fact above shows that $U(M)$ is completely decomposed with respect to the R -, S -submanifolds. Now, using these notations, let us prove the following lemma.

LEMMA 5.1. 1) *The R -submanifold $R(\tilde{x}_0)$ of $U(M)$ is a universal covering manifold of R_0 , where $p_R = p|_{R(\tilde{x}_0)}$ is the covering map.*

2) *The subgroups $i\pi_1(R_0, x_0)$, $i\pi_1(S_0, x_0)$ have no common element except the identity of $\pi_1(M, x_0)$.*

3) *If $I(x_0)$ is infinite, $\pi_1(M)$ is infinite.*

PROOF. To prove 1) it suffices to show $p(R(\tilde{x}_0)) = R_0$. $R(\tilde{x}_0)$ being simply-connected. For $\tilde{x} \in R(\tilde{x}_0)$, take a curve $\tilde{\alpha}$ in $R(\tilde{x}_0)$ joining \tilde{x}_0 to \tilde{x} . Then we can see $p(\tilde{\alpha}) \subset R_0$. Hence $p(R(\tilde{x}_0)) \subset R_0$. Conversely for $x \in R_0$ if we take a curve β in R_0 joining x_0 to x , we can find the curve $\tilde{\beta}$ in $R(\tilde{x}_0)$ with the initial point \tilde{x}_0 such that $p(\tilde{\beta}) = \beta$. Hence, $p(R(\tilde{x}_0)) \supset R_0$. So, $p(R(\tilde{x}_0)) = R_0$. I. e., 1) is proved.

To prove 2) suppose that $i\pi_1(R_0, x_0)$, $i\pi_1(S_0, x_0)$ have a common element

$A \in \pi_1(M, x_0)$ which is not the identity. Let α_R, α_S be two closed curves in R_0, S_0 respectively, starting from x_0 and representing A . Then the curves $\tilde{\alpha}_R, \tilde{\alpha}_S$ starting from \tilde{x}_0 such that $p(\tilde{\alpha}_R) = \alpha_R, p(\tilde{\alpha}_S) = \alpha_S$, must have the same terminal point. Moreover, this point is not \tilde{x}_0 , and $\tilde{\alpha}_R \subset R(\tilde{x}_0), \tilde{\alpha}_S \subset S(\tilde{x}_0)$. This contradicts with the fact that $U(M)$ is completely decomposed. So, 2) is true.

To prove 3) let us denote $I(x_0)$ by $\{x_\lambda | \lambda = 0, 1, 2, \dots\}$, $I(x_0)$ being countable by Lemma 3.3. For each λ , take $\tilde{x}_\lambda \in p^{-1}(x_\lambda) \cap R(\tilde{x}_0)$. As $x_0 \in S(x_\lambda)$, $p^{-1}(x_0) \cap S(\tilde{x}_\lambda)$ is non-empty. However, all of \tilde{x}_λ are distinct from one another. Hence all of $S(\tilde{x}_\lambda)$ are distinct from one another by the fact that $U(M)$ is completely decomposed. Accordingly $p^{-1}(x_0)$ is infinite and so $\pi_1(M)$ is infinite.

THEOREM 4. *In M suppose that $\pi_1(M)$ is finite. Then M is of almost R -fibred type and further almost S -fibred type.*

PROOF. For any $x_0 \in M$, $I(x_0)$ is finite by Lemma 5.1. Hence, $e_R(x_0) > 0$ and $e_S(x_0) > 0$. By Lemma 4.1 and Theorem 1, our theorem is evident.

REMARK 3. There exist RS -manifolds, whose fundamental groups are finite, of the following respective types: R -fibred type and further S -fibred type; almost R -fibred type (not R -fibred type) and further S -fibred type; R -fibred type and further almost S -fibred type (not S -fibred type).

In M suppose that all the R -submanifolds are simply-connected. Moreover if M is of almost R -fibred type, we have

LEMMA 5.2. *M is of R -fibred type.*

PROOF For any R -submanifold R , $G(R)$ consists of the identity only by Lemma 4.2. Hence, all the R -submanifolds are R -maximal. As M satisfies the assumption of Theorem 1, M is of R -fibred type.

THEOREM 5. *In M suppose that the order of $\pi_1(M)$ is finite and prime. Then M is of one of the following three structures:*

- 1) *R -fibred type, where all the R -submanifolds are simply-connected and $\pi_1(S_0)$ for at least one S -submanifold S_0 is isomorphic to $\pi_1(M)$.*
- 2) *S -fibred type, where all the S -submanifolds are simply-connected and $\pi_1(R_0)$ for at least one R -submanifold R_0 is isomorphic to $\pi_1(M)$.*
- 3) *R -fibred type and further S -fibred type, where all the R -, S -submanifolds are simply-connected.*

PROOF. For an S -submanifold S_0 , suppose that S_0 is not simply-connected.

Then it follows that $i\pi_1(S_0, x_0) = \pi_1(M, x_0)$ for any $x_0 \in S_0$. So, $R(x_0)$ is simply-connected by Lemma 5.1. As x_0 is any point of S_0 , all the R -submanifolds are simply-connected by Lemma 3.3. By Theorem 4 and Lemma 5.2, M is of R -fibred type. So, M is of the structure 1). Similarly, if we suppose that an R -submanifold R_0 is not simply-connected, we have the structure 2).

Finally, suppose that all the R -, S -submanifolds are simply-connected. Then, by Theorem 4 and Lemma 5.2, M is of the structure 3). This completes the proof of our theorem.

REMARK 4. There exist RS -manifolds, in which the orders of the fundamental groups are finite and prime, such that the conditions 1), 2), 3) of Theorem 5 hold good respectively. (Especially, for a model in the case 3) see § 4, [2].)

THEOREM 6. *In M suppose that $\pi_1(M)$ is infinite cyclic. Then M is of one of the following structures :*

- 1) *R -fibred type, where all the R -submanifolds are simply-connected and $\pi_1(S_0)$ for at least one S -submanifold S_0 is infinite cyclic.*
- 2) *S -fibred type, where all the S -submanifolds are simply-connected and $\pi_1(R_0)$ for at least one R -submanifold R_0 is infinite cyclic.*
- 3) *All the R -, S -submanifolds are simply-connected.*

PROOF. For any R -submanifold R , $\pi_1(R)$ is the group of identity only or an infinite cyclic group, being isomorphic into $\pi_1(M)$. This holds good for any S -submanifold, too. Moreover, there is not a pair of R -, S -submanifolds whose fundamental groups both are infinite cyclic. For, if such a pair (R_0, S_0) does exist, we can find $A \in \pi_1(M, x_0)$, $x_0 \in R_0 \cap S_0$, which is not the identity and belongs to both of $i\pi_1(R_0, x_0)$ and $i\pi_1(S_0, x_0)$. This is contrary to Lemma 5.1. So, by Lemma 3.3 the following three cases are considered :

- a) All the R -submanifolds are simply-connected and $\pi_1(S_0)$ for at least one S -submanifold S_0 is infinite cyclic.
- b) All the S -submanifolds are simply-connected and $\pi_1(R_0)$ for at least one R -submanifold R_0 is infinite cyclic.
- c) All the R -, S -submanifolds are simply-connected.

The case c) being the same as 3), it suffices to prove that M in the case a) is of R -fibred type. To prove this, take any $x_0 \in S_0$. Let α be a closed curve with endpoint x_0 which is a geodesic arc representing a generator of $\pi_1(M, x_0)$. As $i\pi_1(S_0, x_0)$ is infinite cyclic, we can find an integer $m > 0$ such that the product curve α^m represents a generator of $i\pi_1(S_0; x_0)$. Let $\tilde{x}_0 \in U(M)$ be a point of $p^{-1}(x_0)$. Let $\tilde{\beta}_1$ be the curve starting from \tilde{x}_0 such that $p(\tilde{\beta}_1) = \alpha^m$. Here, the terminal point \tilde{x}_m of $\tilde{\beta}_1$ is contained in the S -sub-

manifold $S(\tilde{x}_0)$ of $U(M)$ and $p^{-1}(x_0) \cap \tilde{\beta}_1$ consists of $m + 1$ points. Accordingly, we can find a part $C_R(\tilde{x}_0; c) \subset R(\tilde{x}_0)$ such that $j(C_R(\tilde{x}_0; c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0) \cap \tilde{\beta}_1$. Next $\tilde{\beta}_2$ be the curve starting from \tilde{x}_m such that $p(\tilde{\beta}_2) = \alpha^m$. The terminal point of $\tilde{\beta}_2$ is also contained in $S(\tilde{x}_0)$. Hence, $j(C_R(\tilde{x}_0; c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0) \cap \tilde{\beta}_2$. Thus we can verify that $j(C_R(\tilde{x}_0; c) \times S(\tilde{x}_0)) \supset p^{-1}(x_0)$.

Now suppose that $e_S(x_0) = 0$. For any constant $d > 0$, we can find a countable subset $\{x_\lambda | \lambda = 0, 1, 2, \dots\}$ of $I(x_0)$ such that $d_S(x_0, x_\lambda) < d$. For each x_λ , there is $\tilde{x}_\lambda \in S(\tilde{x}_0) \cap p^{-1}(x_\lambda)$ where $d_S(\tilde{x}_0, \tilde{x}_\lambda) < d$. By Lemma 5.1, $R(\tilde{x}_\lambda)$ contains a point of $p^{-1}(x_0)$. Here, all of \tilde{x}_λ are distinct from one another. Hence, all of $R(\tilde{x}_\lambda)$ are distinct from one another by the fact that $U(M)$ are completely decomposed. Accordingly, a part $j(C_R(\tilde{x}_0; c) \times C_S(\tilde{x}_0; d))$ of $U(M)$ contains an infinite subset of $p^{-1}(x_0)$. It being however compact, this contradicts with the property of covering. So, $e_R(x_0) > 0$. Since x_0 is any point of S_0 and the R -submanifolds of M are all simply-connected, M is of R -fibred type by Lemmas 4.1, 5.2, and Theorem 1. This completes the proof of our theorem.

REMARK 5. There exist RS -manifolds, whose fundamental groups are infinite cyclic, such that the conditions 1), 2), 3) of Theorem 6 hold good respectively

In Euclidean d -space E^d suppose that there are given a point set $Z = \{P_\lambda | \lambda = \text{integer}\}$ and a congruent transformation T leaving P_0 fixed, such that the vector $\vec{P_\lambda P_{\lambda+1}}$ is equal to the vector $T^\lambda \cdot \vec{P_0 P_1}$ for each λ . (P_λ 's are not necessarily distinct from one another.) Then we have

LEMMA 5.3. *There are two cases where Z is bounded or unbounded. In the latter case, P_0 is not limit point of Z .*

PROOF. We take an orthogonal coordinate system in E^d with origin P_0 , where T is represented by the following matrix :

$$\begin{pmatrix} E_1 & & & & & \\ & -E_2 & & & & 0 \\ & & (\theta_1) & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & & (\theta_k) \end{pmatrix}$$

where E_1, E_2 denote the unit matrices of degrees r_1, r_2 respectively and

$$(\theta_\nu) = \begin{pmatrix} \cos \theta_\nu & -\sin \theta_\nu \\ \sin \theta_\nu & \cos \theta_\nu \end{pmatrix}$$

for $0 < \theta_\nu < \pi$ ($\nu = 1, 2, \dots, k$; $r_1 + r_2 + 2k = d$). So, the matrix representation of T^λ is immediately obtained. Put $v_\lambda = \overrightarrow{P_\lambda P_{\lambda+1}}$. Let (v_λ^m) denote the vector v_λ and let (P_λ^m) denote the point P_λ , where $m = 1, 2, \dots, d$. Here, if $\lambda > 0$, $P_\lambda^m = v_0^m + \dots + v_{\lambda-1}^m$; if $\lambda < 0$, $P_\lambda^m = -(v_{-1}^m + \dots + v_\lambda^m)$. Then for all λ , we can verify the following facts: a) $P_\lambda^\beta = \lambda v_0^\beta$ ($\beta = 1, \dots, r_1$), b) $|P_\lambda^\gamma| < N$ ($\gamma = r_1 + 1, \dots, d$) where N is a constant independent of γ, λ . Hence, in the case $r_1 \neq 0$; if $v_0^\beta = 0$ for all β , Z is bounded, and if $v_0^\beta \neq 0$ for some β , Z is unbounded and P_0 is not limit point of Z . Next, in the case $r_1 = 0$, Z is bounded. So our lemma holds good.

THEOREM 7. *In M suppose that $\pi_1(M)$ is infinite cyclic and that all the R -submanifolds are Euclidean space forms. Then M is of R -fibred type or S -fibred type, both having simply-connected fibres.*

PROOF. It suffices to prove our theorem in the case 3) of Theorem 6. Accordingly, suppose that all the R -, S -submanifolds are simply-connected. For $x_0 \in M$ we take a closed curve α issuing from x_0 which is a geodesic arc representing a generator of $\pi_1(M, x_0)$. Let $\tilde{\beta}$ be the curve in $U(M)$ such that $p(\tilde{\beta})$ is the product curve

$$\dots \cdot \alpha \cdot \dots \cdot \alpha \cdot \alpha \cdot \alpha \cdot \dots \cdot \alpha \cdot \dots$$

Then $p^{-1}(x_0) \subset \tilde{\beta}$. We denote all of the points of $p^{-1}(x_0)$ by \tilde{x}_λ ($\lambda = \text{integer}$), where the subarc of $\tilde{\beta}$ from \tilde{x}_λ to $\tilde{x}_{\lambda+1}$ is mapped to the arc α by p . Any $\tilde{x} \in U(M)$ is represented by $j(P, Q)$ where $P \in R(\tilde{x}_0)$, $Q \in S(\tilde{x}_0)$. Define a map

$$f: U(M) \rightarrow R(\tilde{x}_0) \text{ by } f(\tilde{x}) = P.$$

We put $P_\lambda = f(\tilde{x}_\lambda)$. The curve $f(\tilde{\beta})$ contains P_λ and is a broken line in the Euclidean r -space $R(\tilde{x}_0)$. (Note that in our case all the R -submanifolds are Euclidean r -spaces.) Moreover we can see that the point set $Z = \{P_\lambda | \lambda = \text{integer}\}$ satisfies the condition of Lemma 5.3. Here T is the same as the congruent transformation in $T_R(x_0)$ which is induced from the element associated with α (or α^{-1}) of the homogeneous holonomy group of M at x_0 .

1) The case where Z is bounded. Take a part $C_R(\tilde{x}_0; c)$ of $R(\tilde{x}_0)$ which contains Z . Hence, a part $j(C_R(\tilde{x}_0; c) \times S(\tilde{x}_0))$ of $U(M)$ contains $p^{-1}(x_0)$. Then $e_S(y_0) > 0$, for any $y_0 \in S(x_0)$. For, otherwise, we can find a countable subset $\{y_\lambda | \lambda = 0, 1, 2, \dots\}$ of $I(y_0)$ such that $d_S(y_0, y_\lambda) < d$ for a constant $d > 0$. For each y_λ there is $\tilde{y}_\lambda \in S(\tilde{x}_0) \cap p^{-1}(y_\lambda)$ such that $d_S(\tilde{y}_0, \tilde{y}_\lambda) < d$. By Lemma 5.1, $R(\tilde{y}_\lambda)$ contains a point of $p^{-1}(y_0)$. Here all of \tilde{y}_λ are distinct from one another. Hence all of $R(\tilde{y}_\lambda)$ are distinct from one another by the fact that

$U(M)$ is completely decomposed. Accordingly a part $j(C_R(\bar{x}_0; c) \times C_S(\bar{y}_0; d))$ of $U(M)$ contains an infinite subset of $p^{-1}(y_0)$. It being however compact, this contradicts with the property of covering. So, $e_S(y_0) > 0$ for any $y_0 \in S(x_0)$. By Lemmas 4. 1, 5. 2, and Theorem 1, M is of R -fibred type.

2) The case where Z is unbounded. Then by Lemma 5. 3, P_0 is not limit point of Z . So there is a part $C_R(\bar{x}_0; c)$ of $R(\bar{x}_0)$ such that $R(\bar{x}_0) - C_R(\bar{x}_0; c) \supset Z$. Take a positive constant $d < c/3$. By using the property of covering, we can see that $e_R(x) > 0$ for any $x \in C_R(x_0; d)$. From Lemma 4. 1, Theorems 1 and 2, and Lemma 5. 2, M is of S -fibred type. This completes the proof.

REMARK 6. There exist RS -manifolds, whose fundamental groups are infinite cyclic and whose R -submanifolds are Euclidean space forms, of the following respective types: R -fibred type (not S -fibred type); S -fibred type (not R -fibred type); R -fibred type and further S -fibred type.

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BIBLIOGRAPHY

- [1] G. DE RHAM, Sur la réductibilité d'un espace de Riemann, Comment. Math. Helv., 26 (1952), 328-344.
- [2] S. KASHIWABARA, On the reducibility of an affinely connected manifold, Tôhoku Math. J., 8(1956), 13-28.
- [3] S. KASHIWABARA, The structure of a Riemannian manifold admitting a parallel field of one-dimensional tangent vector subspaces, Tôhoku Math. J., 11(1959), 327-350.
- [4] N. STEENROD, The topology of fibre bundles, Princeton Univ. Press., (1951).
- [5] A. G. WALKER, The fibering of Riemannian manifolds, Proc. London Math. Soc., (3) 3(1953), 1-19.

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