THE STRUCTURE OF APPLIED GENERAL EQUILIBRIUM MODELS

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Introduction

This book describes the structure of general equilibrium models. It is written for the researcher who intends to construct or study applied general equilibrium (AGE) models and has a special interest in their theoretical background. Both general equilibrium theory and AGE modeling continue to be active fields of research, but the styles of presentation differ greatly. Whereas the applied model builder often finds the style of theoretical papers inaccessible, the theoretician can hardly recognize the concepts he is used to in the list of equations of applied models. The main purpose of the book is to present the theoretical models in a unified way and to indicate how the main concepts can find their way into applications.

To make the models more accessible and their structure more transparent, we unify the presentation in four ways. First, we use standardized, though possibly not the weakest, assumptions on basic model components like utility functions and production sets (chapters 1 and 2), and we only deviate from the standard when the topic requires it. Second, in chapter 3 we define five basic formats to represent and to analyze the same model. For every specific model to be discussed in chapters 4 to 12, we apply the format that proves most convenient. Third, beside fixed point theorems (appendix A.4), we almost exclusively use theorems from the theory of convex programming as mathematical background (appendixes A.1 to A.3). Finally, every model in chapters 4 to 12 analyzes a topic within a common basic scheme, as follows:

1. The chapter starts introducing the topic (e.g., taxes or finite horizon dynamics) and then proceeds with the discussion of the issues related to the topic itself as well as its incorporation in a general equilibrium model. This may require nonstandard assumptions (e.g., relaxing convexity).

2. Existence proofs are given, using the most convenient format defined in chapter 3. The proofs are not meant as a theoretical contribution and mainly serve to highlight the roles of the various assumptions and to indicate how the fixed point mapping can be set up for computing a solution via fixed point algorithms. Issues in computation are briefly dealt with in appendixes A.7 to A.9.

3. Various properties of the equilibrium solution of the model are analyzed, with a focus on efficiency. We derive conditions under which the inefficiencies due to specific imperfections (taxes, price rigidities, external effects) can be reduced through Pareto-improving reforms. However, these reforms only represent an idealized situation, since they require losers to be compensated and all imperfections to be reduced simultaneously.

4. Policy reforms can rarely eliminate or even reduce all imperfections at the same time. Therefore their consequences cannot be predicted from theory alone, and numerical simulation is called for. This requires construct-

ing applied models that can be used to run alternative scenarios. Every chapter ends with a section that describes how applied models that incorporate the theoretical concepts can be built and briefly surveys existing applications.

The following limitations should be mentioned:

1. For clarity of exposition, the topics will often be treated in isolation. For example, we limit the presentation of finite and infinite horizon dynamics to the case of the closed economy. Occasionally some issues need to be treated together (e.g., taxes, tariffs, and foreign trade in chapter 5).

2. In chapter 2 we compare several analytical forms (CES, translog, etc.) for the functions that should be estimated econometrically, but in the sections on applications, we do not enter into the details of these specifications.

3. We sketch the framework for national and social accounting that is needed as a database (the social accounting matrix, SAM) to calibrate a model, but we hardly discuss the elaborate process that leads from statistical publications to such a database.

4. Computation of equilibria has received much attention in the early years of AGE modeling. Advances in computing capacity and speed as well as the availability of user friendly packages, such as GAMS, have almost eliminated computational concerns for most applications. Therefore computational issues are only treated briefly in appendixes.

5. The numerical solutions of AGE models should be made easily interpretable for the policy analyst who will be reluctant to decipher the standard printings from software packages. The presentation should thus be customized, and the results should be cast in the form of tables the analyst is familiar with. Deriving such tables for every model is hardly meaningful without a numerical illustration, and giving this in every chapter would cause excessive duplication.

We overcome some of these limitations and avoid repetition by describing, in appendix B, a complete numerical application in GAMS language, incorporating taxes, trade, price rigidities, buffer stocks, transportation costs, and simple dynamics. The application uses simple functional forms and yields social and national accounts in report quality form.[1]

To summarize, chapters 1 to 3 and appendix A set the framework for the topics covered in chapters 4 to 12. Chapter 1 provides an elementary but almost comprehensive treatment of the competitive model. It ends, in section 1.5, with an overview of the properties that could be relaxed so as to make the model more useful for policy analysis and introduces the subjects covered in chapters 4 to 12. Chapter 2 deals with relatively standard elements of the theory of producer and consumer behavior, most of which can be found in textbooks such as Varian's *Microeconomic Analysis*. The reader who is familiar with this material can proceed to section 2.3, which summarizes the microeconomic assumptions that are used in later chapters. Section 2.4 presents results on welfare analysis that are needed in proofs. Chapter 3 describes the basic formats and introduces the methods of proof used in the book. Its last section discusses the main steps in constructing a numerical application and gives a full GAMS application for the simplest possible model. The reader may choose freely the order in which he wants to study the topics covered in chapters 4 to 12. Occasionally he may have to consult material from earlier chapters and from appendix A, and this will be pointed out explicitly when needed.

A user's guide and library of GAMS-models (Keyzer 1997) accompany this volume. Both are freely accessible at the MIT Press Internet site (http://www.mitpress.mit.edu). The guide extends the material covered in appendix B. The library contains a set of computer programs with illustrative applications for most of the models covered in chapters 3 to 12 of this book, except for the infinite horizon models of chapter 8.

Notation

We denote the n-dimensional real space by R^n, the nonnegative orthant by R^n_+, and the positive orthant by R^n_{++}. Let x and y be two vectors in R^n. The following notations are equivalent:

Enumeration	all $h \Leftrightarrow h = 1, \ldots, n$
Summation	$\Sigma_h x_h \equiv \sum_{h=1}^{n} x_h$
If and only if	iff or $\Leftrightarrow$
Complementary slackness	$(x \geqslant 0 \perp y \geqslant 0) \Leftrightarrow (x \geqslant 0,\ xy = 0,\ y \geqslant 0)$
Inner product[1]	$xy \equiv x \cdot y \equiv \Sigma_h x_h y_h$
Inequality	$x \geqslant y \Leftrightarrow x_h \geqslant y_h$ for all h, and similarly for $\leqslant$; $x > y \Leftrightarrow x_h > y_h$ for all h, and similarly for $<$.
Partial derivative (Jacobian)	$F'(x) \equiv \partial F(x)/\partial x$
Fixed point or equilibrium	superscript *
Optimal value of a variable[2]	superscript °
Vector norm[3]	$\|x\|_p \equiv (\Sigma_{h=1}^{n} |x_h|^p)^{1/p}$ for given $1 \leqslant p < \infty$; $\|x\|_\infty \equiv \max_{1 \leqslant h \leqslant n} |x_h|$, for the limiting case; $\|x\| \equiv \|x\|_p$ for some p to be specified.
Matrix norm	$\|A\| \equiv \sup_{\|x\|=1} \|Ax\|$

Notation in the Mathematical Program

The notation will be introduced by means of an example. Define the function $u: X \subset R^n \rightarrow R$ and the sets $Y_j \subset R^n$, $j = 1, \ldots, J$, and consider the mathematical program:

$\max u(x)$,

$x \geqslant 0,\ y_j$, all j,

subject to

$x \leqslant \Sigma_j y_j \qquad (p)$,

$y_j \in Y_j$.

For simplicity we write "max" instead of "sup" because we only consider programs for which the maximum will be attained. The choice variables x and y_j are placed under the maximand. Here x is constrained to be

1. When no confusion is possible, we avoid using the transposition sign for inner products and for premultiplication of a matrix by a vector.

2. When no distinction with equilibrium values is needed, the superscript * is also used.

3. More general definitions can be used. Here we restrict ourselves to l_p-norms.

nonnegative while y_j is unconstrained. We write "all j" as equivalent for "$j = 1, \ldots, J$"; "all j" is assumed to apply for all constraints that use the subscript without summation sign (here it applies to the second but not to the first constraint). The brackets (p) denote a vector of Lagrange multipliers associated with the constraint $x \leqslant \Sigma_j y_j$ (the vector p is a member of the set of multipliers).

We also use a more compact notation; for example, for the utility maximization of the consumer, we write

$$V(p, h) = \max_{x \geqslant 0}\{u(x) \mid px \leqslant h\}$$

or

$$x(p, h) = \operatorname{argmax}_{x \geqslant 0}\{u(x) \mid px \leqslant h\},$$

where p and h are the given price and income, respectively, $V(p, h)$ is the value function, and $x(p, h)$ is the unique optimal choice. If the solution is not unique, we consider one optimal choice x° and write

$$x^\circ \in \operatorname{argmax}_{x \geqslant 0}\{u(x) \mid px \leqslant h\}$$

1 Competitive Equilibrium

In this chapter we describe an economy, say, of a village, a town, or a country, that does not entertain trading relations with an outside world. We call this a closed economy.

After introducing basic concepts in section 1.1, we prove existence of equilibrium in the simplest possible aggregate framework, without reference to individuals' behavior, and we discuss computation of equilibrium and properties like multiplicity and stability in section 1.2. In section 1.3 we impose a first set of assumptions on the behavior of producers and consumers and show that under these assumptions, a competitive equilibrium exists. Section 1.4 is devoted to the welfare properties of equilibrium allocations. Finally, section 1.5 points to the need for extensions that will yield an applied general equilibrium model with a more realistic structure.

1.1 Basic Concepts

1.1.1 Commodities and Agents; Demands and Supplies

Let us consider an economy with r commodities indexed by $k = 1, 2, \ldots, r$. The commodity space is thus an r-dimensional space, denoted by R^r, and all vectors belong to that space.

An orange available today in New York is more or less the same as an orange in Marseilles tomorrow, but each will be defined as a different commodity in a general equilibrium model, which distinguishes commodities by location and date of delivery. By definition, each commodity k will be traded at a single price p_k. Agents are characterized by preferences over commodities and by capabilities to satisfy these preferences through actions like production, purchases, sales, storage, and consumption, now or in the future. This situation may be formalized by representing each agent as maximizing his utility subject to technological and trading constraints. In the simplest case the budget constraint, which requires expenditure not to exceed revenue, is the only restriction on exchanges. Under these conditions and some specific assumptions to be discussed later on, we can decompose the decisions made by each agent into two subproblems:[1] profit maximization subject to technological constraints, and utility maximization subject to a budget constraint.

These subproblems enable us to consider two types of agents who make decisions: producers (or firms) and consumers. There will be n producers, indexed by $j = 1, 2, \ldots, n$ who will produce (and sell) commodities using (and buying) some other commodities, like labor, steel, and machines. Let $y_j(p)$ be the production plan of producer j, where p denotes the price vector; outputs will carry a positive sign and inputs a negative sign. There will be

1. See Koopmans (1957).

m consumers, indexed by $i = 1, 2, \ldots, m$. Every consumer offers for sale his commodity endowment ω_i and expresses the wish to buy a commodity bundle $x_i(p)$ at given prices p.

We define the excess demand vector $z(p)$ as

$$z(p) = \Sigma_i x_i(p) - \Sigma_j y_j(p) - \Sigma_i \omega_i.$$

The typical component $z_k(p)$ of this vector will represent the excess of demand over supply (which includes the initial endowment) of commodity k.

A natural solution concept is to require that no commodity be in excess demand; otherwise, some agents would not be able to carry out their demands. We accept, however, that there can be excess supply for some commodities, which can be disposed of freely (the free disposal assumption). An excess demand equilibrium is then defined as follows:

DEFINITION 1.1 (Excess demand equilibrium) The price $p^* \geqslant 0$, $p^* \neq 0$, and the excess demand $z(p^*)$ define an excess demand equilibrium if $z(p^*) \leqslant 0$.

1.1.2 The Behavior of Producers and Consumers

Our description of the agents' behavior starts with the assumption that prices exist for all commodities and that all agents take these prices as given, none of them being sufficiently "large" or "important" to think that he can influence a price, even less set a price. Prices are thus considered as signals on the basis of which agents compute their plans. Such an institutional setting is commonly referred to as perfect competition.

Production Plans

Each producer j is endowed with a technology, represented by a set Y_j, which belongs to R^r and is the set of feasible production plans. A producer formulates a production plan y_j that must be feasible: This is expressed as $y_j \in Y_j$. Obviously further assumptions that characterize the mathematical properties of the set Y_j are needed. The competitive model assumes that from the set of feasible plans y_j, the producer chooses those that maximize his profit, defined as $\Sigma_k p_k y_{jk}$ or py_j.

The problem of producer j can thus be stated as follows: Given the price vector p, and the technological set Y_j, producer j chooses y_j so as to maximize profits py_j subject to a feasibility constraint $y_j \in Y_j$, or

$$\Pi_j(p) = \max_{y_j}\{py_j \mid y_j \in Y_j\}, \tag{1.1}$$

where $\Pi_j(p)$ is the resulting maximal profit.

Consumption Plans

The choice made by consumer i is restricted in two ways. First, his consumption plans need to be feasible: He cannot consume negative quantities of any commodity: $x_i \in R^r_+$. Second, he is faced with a budget constraint: He cannot spend more than his income h_i. Since at given prices p, a consumption plan x_i costs px_i, his budget constraint can be written as

$$px_i \leqslant h_i.$$

The income h_i of the consumer consists of two parts: The proceeds $p\omega_i$ of selling the endowment ω_i and distributed profits. The latter are defined as follows: It is assumed that consumer i owns a nonnegative share θ_{ij} in firm j and that he receives dividends $\theta_{ij}\Pi_j(p)$ from this firm. All profits are distributed so that $\Sigma_i \theta_{ij} = 1$ for every j. Consumer i's income is now

$$h_i = p\omega_i + \Sigma_j \theta_{ij}\Pi_j(p).$$

Consumer i is also characterized by a utility function $u_i(x_i)$, which associates to every consumption plan x_i a utility level $u_i(x_i)$; under further assumptions on $u_i(x_i)$, this makes it possible to consistently rank alternative consumption bundles.

The problem of consumer i can now be stated as follows: Given the price vector p and the revenue h_i, consumer i chooses x_i so as to maximize his utility $u_i(x_i)$ subject to a feasibility constraint $x_i \geqslant 0$ and to his budget constraint $px_i \leqslant h_i$, or

$$\max_{x_i \geqslant 0}\{u_i(x_i) \mid px_i \leqslant h_i\}. \tag{1.2}$$

1.1.3 General Competitive Equilibrium

The excess demand equilibrium of definition 1.1 is a general equilibrium because it covers all agents and all commodities of the economy. We can now define a general competitive equilibrium as an excess demand equilibrium, in which producers and consumers behave according to (1.1) and (1.2), respectively.

DEFINITION 1.2 (General competitive equilibrium) The allocation y_j^*, all j, x_i^*, all i, supported by the price vector $p^* \geqslant 0$, $p^* \neq 0$ is a general competitive equilibrium if the following conditions are satisfied:

1. For every producer j, y_j^* solves $\max_{y_j}\{p^* y_j \mid y_j \in Y_j\}$.
2. For every consumer i, x_i^* solves $\max_{x_i \geqslant 0}\{u_i(x_i) \mid p^* x_i \leqslant h_i^*\}$, where $h_i^* = p^*\omega_i + \Sigma_j \theta_{ij} p^* y_j^*$.
3. All markets are in equilibrium, $\Sigma_i x_i^* - \Sigma_j y_j^* - \Sigma_i \omega_i \leqslant 0$.

In definition 1.2 there are four components. First, agents have behavioral rules that they follow to compute their optimal decisions. Second, in doing so, they take into account signals, without trying to affect these: Producers react on prices only, consumers take into account prices and their income. Third, there is a price for every commodity, so a competitive market exists for every commodity. Fourth, there are conditions on excess demands, which agents do not take into account when making their decisions and which are satisfied in equilibrium.

Characteristics of a General Competitive Equilibrium

So far, endowments ω_i and shares in profits θ_{ij} are the only explicit parameters. In applied models other parameters like tax rates or institutional rigidities will appear, and the model will be used to compute solutions when variations are imposed on some of these parameters. When analyzing the response of the model to such changes, several issues have to be addressed. To introduce these, we consider the response of the system $z(p, \omega) \leqslant 0$ to variations in ω only, where $\omega = (\omega_1, \omega_2, \dots, \omega_m)$ is the vector of endowments of all consumers.

Various questions now come up: (1) What are natural properties for $z(p, \omega)$ (assumptions)? (2) Does the system $z(p, \omega) = 0$ have a solution with nonnegative prices (existence)? (3) How many such solutions are there (multiplicity)? (4) Do we know anything on the direction of change of prices and allocations when ω changes (general properties of $z(p, \omega)$)? (5) How to compute solutions given a numerical specification of $z(p, \omega)$? (6) Are some policy changes "better" than others (welfare analysis)? Finally, when building an applied general equilibrium model, the main issue is (7) how to specify numerically the model (empirical implementation). We will discuss these questions by turns.

1.2 Excess Demand Equilibrium

1.2.1 Assumptions on the Excess Demands

We now formulate assumptions on the excess demand function $z(p)$, along the lines of Arrow and Hahn (1971, ch. 2). In section 1.3 we will specify assumptions on individual agents and the properties of the excess demand function will follow from there:

ASSUMPTION Z1 (Single-valuedness and continuity) $z(p)$ is a single-valued and continuous function, which is defined for $p \geqslant 0$, $p \neq 0$.[2]

2. We use the following notation: $p \geqslant 0$ means that $p_k \geqslant 0$ for all k, $p \geqslant 0$ and $p \neq 0$ mean that $p_k > 0$ for some k, $p > 0$ means that $p_k > 0$ for all k.

ASSUMPTION Z2 (Continuous differentiability) $z(p)$ is continuously differentiable for $p > 0$.

ASSUMPTION Z3 (Homogeneity) $z(p)$ is homogeneous of degree zero in p.

ASSUMPTION Z4 (Walras's law) $pz(p) = 0$.

ASSUMPTION Z5 (Desirability) $p_k = 0$ implies that $z_k(p) > 0$ for $k = 1, 2, \ldots, r$.

In assumption Z1 single-valuedness is fairly restrictive and will not be satisfied in many practical applications (e.g., when production takes place under constant returns to scale).[3] The assumption is made here in order to keep the first proof of existence of an equilibrium simple, and it will be relaxed later on. The continuity assumption is also restrictive, particularly when $p_k = 0$ for some k, because at that price we may expect excess demand to rise to infinity and hence to be discontinuous.

Assumption Z2 is restrictive, but it will only be used when we discuss multiplicity of equilibria and the response to changes in parameters. Assumption Z3 implies that for every scalar $\lambda > 0$, $z(\lambda p) = z(p)$. We can thus multiply each nonzero, nonnegative price vector by some positive number, without changing the value of the excess demand function $z(p)$: The absolute level of prices does not affect outcomes. Without loss of generality, and since we require some prices to be positive anyway, we can assume that $\Sigma_k p_k = 1$. Prices now belong to a set $S^r = \{p \mid \Sigma p_k = 1, p_k \geqslant 0\}$, called the price simplex. This scaling is known as price normalization. Other normalizations are possible. For example, we may choose commodity 1 as numéraire, that is, set $p_1 = 1$. But this can only be done if p_1 is positive in equilibrium.

Assumption Z4 (Walras's law) plays an important role in general equilibrium models. Replacing $z(p)$ by its components yields

$$p\Sigma_i x_i(p) = p\Sigma_j y_j(p) + p\Sigma_i \omega_i.$$

The assumption requires that for every nonzero, nonnegative price, the value of aggregate demand must be equal to the value of aggregate supply.

Finally, assumption Z5 states that demand for a commodity will be larger than supply whenever its price is zero. This assumption is not essential. It merely ensures that all prices are positive in equilibrium.

1.2.2 An Existence Proof

Existence proofs play an essential role in economic theory. This is because a model that does not possess any solution is inconsistent and therefore meaningless, but also because the existence proof itself highlights the role of

3. In that case $y_j \in Y_j$ implies that $\lambda y_j \in Y_j$ for every positive scalar λ. We will later see that this implies nonuniqueness of optimal y_j.

the assumptions made and, by that, facilitates the search for weaker assumptions. In doing so, it enlarges the field to which the theory applies.

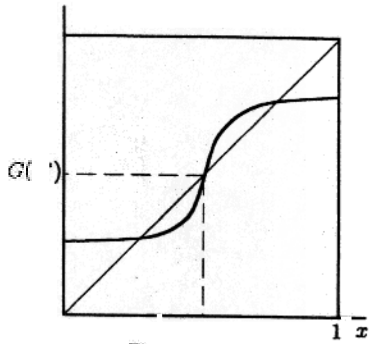
The applied modeler may think that he can dispense of existence proofs because a reasonable model can always be calibrated so as to possess a solution. However, such a calibration will not help when the modeler seeks to compute a new solution after having changed the value of some parameter. Then he needs to know the range of parametric variations for which a solution will exist at all. The theoretical assumptions limit the range of variations of these parameters, and the existence proof makes clear why these restrictions are needed. Finally, the existence proof is useful because it will in general construct a fixed point mapping that can serve as the basis of the fixed point algorithms that solve the model numerically.

We will now prove that an equilibrium exists in this economy, using the assumptions made in section 1.2.1. The proof of this basic result rests on a fixed point theorem, due to Brouwer, that states that a continuous function G that maps a compact convex set A into itself ($G: A \rightarrow A$) has a fixed point x^*, that is, a point such that $x^* = G(x^*)$.[4]

Hence Brouwer's theorem imposes three requirements: (1) the function $G(x)$ should be continuous, (2) it should map from a compact, convex domain A, (3) the mapping should be "into itself": The range A should be the same as the domain. We illustrate the role of each of these requirements. In figure 1.1a we draw a function that meets all the requirements with the unit interval as the set A. There are three fixed points (intersections with the 45° line). Note first that unless the function is tangent to the 45° line, the number of fixed points will always be finite, and there will be at least one intersection. Then, unless the function starts at $(0, 0)$ and ends at $(1, 1)$, the number of fixed points must be odd. In figure 1.1b the mapping is not single valued (it is not a function but a correspondence); nevertheless, it maps from A into A and is convex valued (it is obviously so when single valued, but it also is at x° where it is set valued, since the interval BB' is a convex set). In this case a fixed point will exist, but now by virtue of the Kakutani theorem[5] instead of the Brouwer theorem. In figure 1.1c the correspondence is compact and maps into A, but it is not convex valued (and not continuous) at x° and there is no fixed point. Figure 1.1d shows that the three requirements are sufficient but not necessary: A fixed point exists, though the function is not continuous, not defined everywhere, not compact valued (it goes to infinity), and does not map into itself (C does not lie in A). In figure 1.1e the domain of the function is compact, and the function is continuous, so its range is compact. But the range is not contained in the

4. See theorem A.4.1 in the mathematical appendix A.

5. See theorem A.4.2.



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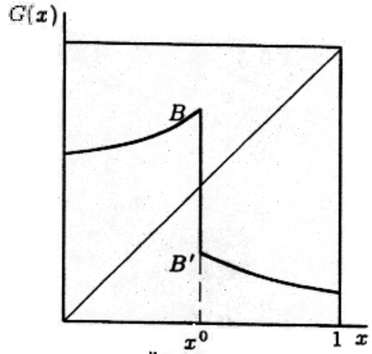

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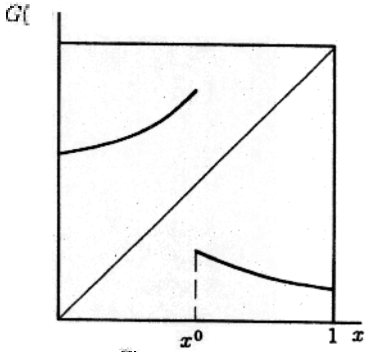

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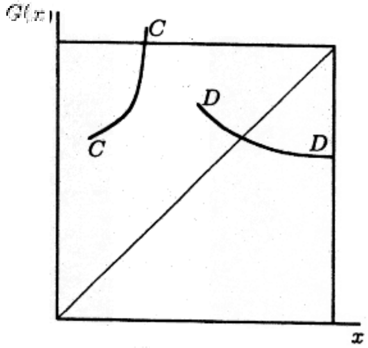

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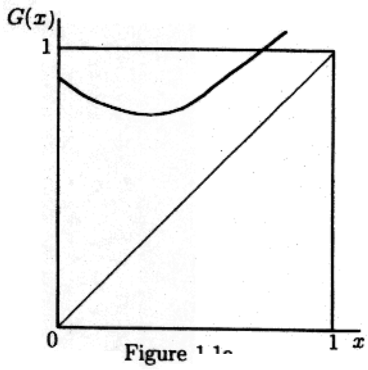

Figure 1.1e

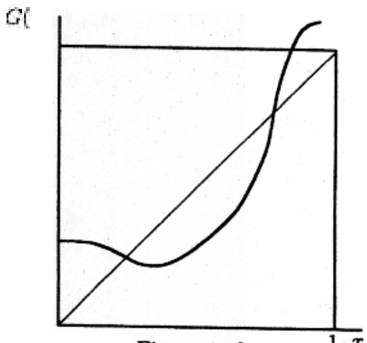

Figure 1.1f

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domain, and no fixed point exists. Finally, figure 1.1f illustrates that the requirement (3) of "mapping into itself" is not necessary.

Though the proof of existence of equilibrium is standard, we give it because it clarifies the role played by the various assumptions.

PROPOSITION 1.1 (Existence of an excess demand equilibrium) If assumptions Z1, Z3, and Z4 hold, then there exists a price vector $p^* \in S^r$ such that $z(p^*) \leqslant 0$.

Proof To apply the Brouwer theorem, we first define a continuous function G that maps the simplex S^r into itself. Second, we show that at such a fixed point, the equilibrium conditions are satisfied.

1. Definition of the function. Let

$$G_k(p) = \frac{p_k + \max[0, z_k(p)]}{\Sigma_j p_j + \Sigma_j \max[0, z_j(p)]}.$$

Then, since $p \in S^r$ and $z(p)$ is continuous, $G(p)$ maps the simplex into a compact set. Moreover $\max[0, z_k(p)] \geqslant 0$, $\Sigma_j \max[0, z_j(p)] \geqslant 0$, and $\Sigma_j p_j = 1$. Therefore the denominator is strictly positive; hence the function $G(p)$ is continuous and maps S^r into itself. At this point we can invoke Brouwer's theorem. There exists a p_k^* such that $p_k^* = G_k(p^*)$. By the definition of $G_k(p)$, in the fixed point, we have

$$p_k^* = \frac{p_k^* + \max[0, z_k(p^*)]}{1 + \Sigma_j \max[0, z_j(p^*)]}. \tag{1.3}$$

2. We have still to show that $z_k(p^*) \leqslant 0$ in the fixed point. Indeed, multiplying the two sides of (1.3) by the denominator leads to

$$p_k^* \Sigma_j \max[0, z_j(p^*)] = \max[0, z_k(p^*)]. \tag{1.4}$$

Multiplying each term of (1.4) by $z_k(p^*)$ and summing over all k yields

$$\Sigma_k p_k^* z_k(p^*)\, \Sigma_j \max[0, z_j(p^*)] = \Sigma_k z_k(p^*) \max[0, z_k(p^*)]. \tag{1.5}$$

On the left-hand side of (1.5), there is a term $\Sigma_k p_k^* z_k(p^*)$ that is equal to zero, by assumption Z4, and hence

$$\Sigma_k z_k(p^*) \max[0, z_k(p^*)] = 0.$$

Each term in this sum is equal to 0 if $z_k(p^*) \leqslant 0$ and to $[z_k(p^*)]^2$ if $z_k(p^*) > 0$. The zero terms do not contribute to this sum. All others are positive, but then the expression on the left-hand side cannot be equal to zero. Therefore none of the $z_k(p^*)$ can be positive. ∎

Proposition 1.1 guarantees that there exists at least one equilibrium. But, as suggested by figure 1.1a, there may exist many equilibria. We return to this in section 1.2.3.

We note that by the homogeneity assumption Z3 and because we disregard the case with $p = 0$, it was possible to normalize prices to the simplex S^r, the compact, convex domain of the mapping. Assumption Z4 (Walras's law) was used to prove that the fixed point is an equilibrium. Hence assumptions Z1, Z3, and Z4 are sufficient for existence of an equilibrium. However, models that fail to satisfy some of these may possess solutions, as was illustrated in figures 1.1d and f.

The intuition behind the artificial function $G(p)$ in the proof is to increase the price p_k of a commodity that is in excess demand ($z_k(p) > 0$). However, such a mapping should not be interpreted as a theory of the dynamics of price adjustment. Producers and consumers carry out their plans only in equilibrium; the model has no clear "out-of-equilibrium" interpretation; in particular, no exchange is specified to take place out of equilibrium. In section 1.2.5 we further discuss the issue when we consider computation of equilibrium.

Since $z_k(p^*) \leqslant 0$ in equilibrium, we do not exclude strict inequality and hence allow for excess supply of some commodity. But then, as stated in proposition 1.2, the associated price p_k^* will be zero.

PROPOSITION 1.2 (Free goods) Under assumptions Z1, Z3, and Z4, $z_k(p^*) < 0$ implies that $p_k^* = 0$.

Proof In equilibrium $z(p^*) \leqslant 0$ and $p^* \geqslant 0$, so $p_k^* z_k(p^*) \leqslant 0$ for every k. Now, if, for some k, $z_k(p^*) < 0$ and $p_k^* > 0$, then $p_k^* z_k(p^*) < 0$. However, for Walras's law to hold, it must be that $p_h^* z_h(p^*) > 0$ for at least one commodity. This cannot happen, since $p_k^* z_k(p^*) \leqslant 0$ for every k, a contradiction. ■

Proposition 1.2 shows that all goods in excess supply have a zero price. Of course in the real world economy, they may have a positive price. For example, on the labor market, unemployment may prevail at a positive wage rate. We will return to this in chapter 6.

In the existence proof we do not use assumption Z5. However, when excess demand is obtained from aggregation of consumers' and producers' plans, it is difficult to maintain continuity of $z(p)$ if the price of some commodity is zero. We will make assumptions on consumers' utility functions and endowments that make prices positive in equilibrium. We will be interested only in goods that carry positive prices and will discard from our model goods we know to be available freely.

PROPOSITION 1.3 (Positive prices) Under assumptions Z1 and Z3 to Z5, equilibrium prices p^* are positive.

Proof Evidently $p_k^* = 0$ implies that $z_k(p^*) > 0$ by assumption Z5 and this cannot be an equilibrium. ■

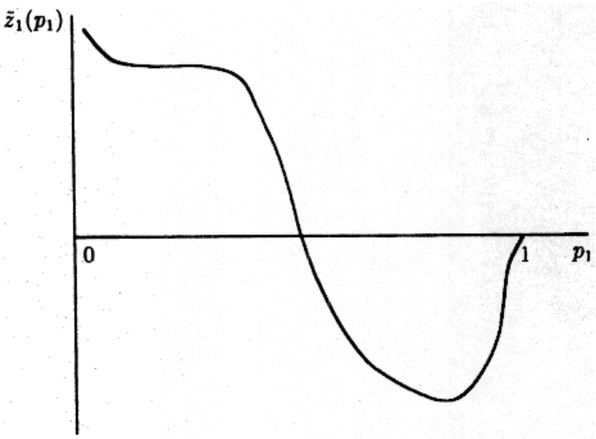

Figure 1.2
The excess demand function $\tilde{z}_1(p_1)$ in the two-commodity case.

When assumption Z5 holds, and if there are only two commodities, an easy geometrical argument for existence of equilibrium can be given. Under assumption Z5 no price can be zero in equilibrium. Since $p_2 = 1 - p_1$, we can draw the curve $\tilde{z}_1(p_1) = z_1(p_1, 1 - p_1)$, shown in figure 1.2. At $p_1 = 0$, we have $\tilde{z}_1(p_1) > 0$, by assumption Z5; by continuity, $\tilde{z}_1(p_1) > 0$ for p_1 close to zero. This makes it possible to start the curve above the p_1-axis. On the other hand, for $p_1 = 1$, Walras's law implies that $\tilde{z}_1(p_1) = 0$. However, for p_1 close to 1, we have p_2 close to zero, and therefore $\tilde{z}_2(p_1) > 0$, $\tilde{z}_1(p_1) < 0$, by Walras's law, where $\tilde{z}_2(p_1) = z_2(p_1, 1 - p_1)$. Hence $\tilde{z}_1(p_1)$ must lie under the p_1-axis for p_1 close to 1. Since $\tilde{z}_1(p_1)$ is a continuous function, it must cross the p_1-axis at least once, and there exists a value p_1 (and p_2), say, $p_1^\dagger \in (0, 1)$ such that $\tilde{z}_1(p_1^\dagger) = 0$ (and $\tilde{z}_2(p_1^\dagger) = 0$), again by Walras's law.

1.2.3 Multiplicity of Equilibrium

The multiplicity issue is important in applied modeling because we need to know whether a change in parameter will lead to a unique new solution; otherwise, the impact of the change is ambiguous. We also want this solution to vary continuously under the change. We discuss multiplicity under the additional assumption of continuous differentiability (assumption Z2) and of desirability (assumption Z5). Clearly, for positive prices, continuous differentiability implies single-valuedness and continuity.

Because of homogeneity (assumption Z3), we can normalize prices on the simplex and define the price of commodity r residually. We define the excess demand function for the $r-1$ commodities as

$$\tilde{z}_k(\tilde{p}) = z_k(p_1, p_2, \quad -\Sigma_{k \neq r} p_k),$$