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THE STRUCTURE OF BIVARIATE POISSON DISTRIBUTION

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0. Summary.

In this paper we consider the structure of two dimensional Poisson distribution. In section 1 the famous Poisson's theorem and an example are stated, in section 2 two dimensional Bernoulli distribution is defined and by the n independent convolution, two dimensional binomial distribution is defined as in one dimensional case and in section 3 the main result of this paper is stated that under some conditions the two dimensional binomial distribution approaches to two dimensional Poisson distribution and adding another condition it approaches to the distribution of independent type.

1. Poisson's theorem.

It is well known fact as Poisson's theorem that for given sequence of probabilities (p_n) such that $p_n \to 0$ $(n \to \infty)$ we have

$$P_n(m) - \frac{\lambda_n{}^m}{m!} e^{-\lambda_n} \to 0 \qquad \text{as} \quad n \to \infty$$

for all non-negative integer m where

$$\lambda_n = np_n, \qquad P_n(m) = \binom{n}{m} p_n{}^m (1-p_n)^{n-m}.$$

Furthermore if $np_n \to \lambda$ $(n \to \infty)$ then we have

$$P_n(m) \to \frac{\lambda^m}{m!} e^{-\lambda} \qquad (n \to \infty).$$

As an example of this theorem we consider a Bernoulli trial that event S occurs on a given unit space with probability p and S doesn't occur on this space with probability $1-p$. If we have n independent observations of the Bernoulli trial and we put the number of occurence of S in the n observations as X then the random variable X takes the value $0, 1, \cdots, n$ and the distribution is binomial:

$$P(X=k) = b(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (0 \leq k \leq n).$$

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We assume that the number n of trials increases to infinity while the mean value np converges to λ. The probability of occurence of the event S and the area of the space or the length of the time interval of the Bernoulli trial are proportional. This assumption is natural one: we devide the whole space to n subspaces which are of same quality, then the probability of occurence of S becomes λ/n. By the Poisson's theorem the distribution of the number X of occurence of S as $n\to\infty$ under the condition is given by

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda} \qquad (k=0,1,2,\cdots).$$

2. Definition of bivariate binomial distribution.

1. Bivariate Bernoulli distribution.

Consider a pair of random variables (X, Y) which has a joint distribution

$$P(X=0,\ Y=0)=p_{00}, \qquad P(X=1,\ Y=0)=p_{10},$$

$$P(X=0,\ Y=1)=p_{01} \quad \text{and} \quad P(X=1,\ Y=1)=p_{11}$$

where

$$p_{00}+p_{10}+p_{01}+p_{11}=1.$$

In such case we say that this distribution has bivariate Bernoulli law.

The marginal distribution of X is given by $P(X=0)=p_{00}+p_{01}$ and $P(X=1)=p_{10}+p_{11}$, that is, X is distributed by univariate Bernoulli law with parameter $p_{10}+p_{11}$ then the mean value of X is given by $p_{10}+p_{11}$ and similarly we have the marginal distribution of Y is given by $P(Y=0)=p_{00}+p_{10}$ and $P(Y=1)=p_{01}+p_{11}$, that is, Y is distributed by univariate Bernoulli law with parameter $p_{01}+p_{11}$ then the mean value of Y is given by $p_{01}+p_{11}$:

$$E(X)=p_{10}+p_{11}, \qquad E(Y)=p_{01}+p_{11}.$$

The covariance of the pair (X, Y) is given by

$$\begin{aligned}\mathrm{Cov}(X, Y)&=E[(X-E(X))(Y-E(Y))]\\&=E(X\cdot Y)-E(X)E(Y).\end{aligned}$$

The first term of this equation becomes

$$\begin{aligned}E(X\cdot Y)&=\sum_{\substack{i=0,1\\j=0,1}} i\cdot j\, p_{ij}=0\cdot 0\, p_{00}+1\cdot 0\, p_{10}+0\cdot 1\, p_{01}+1\cdot 1\, p_{11}\\&=p_{11}.\end{aligned}$$

Then we have

$$\mathrm{Cov}(X, Y)=p_{11}-(p_{10}+p_{11})(p_{01}+p_{11})$$

$$=p_{00}p_{11}-p_{10}p_{01}.$$

The coefficient of the correlation $R(X, Y)$ of the pair (X, Y) is defined by

$$R(X, Y)=\frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\ \sqrt{\mathrm{Var}(Y)}}.$$

We have

$$\mathrm{Var}(X)=(p_{10}+p_{11})(1-(p_{10}+p_{11}))=(p_{10}+p_{11})(p_{00}+p_{01}),$$

$$\mathrm{Var}(Y)=(p_{01}+p_{11})(1-(p_{01}+p_{11}))=(p_{01}+p_{11})(p_{00}+p_{10}).$$

and

$$R(X, Y)=\frac{p_{00}p_{11}-p_{10}p_{01}}{\sqrt{(p_{10}+p_{11})(p_{00}+p_{01})}\ \sqrt{(p_{01}+p_{11})(p_{00}+p_{10})}}.$$

LEMMA 1. *If a pair of random variable* (X, Y) *has a bivariate Bernoulli law with parameters* p_{00}, p_{10}, p_{01} *and* p_{11} *summing up to unity and the covariance* $\mathrm{Cov}(X, Y)$ *equals to zero: the two random variables* X *and* Y *are uncorrelated, then the two random variables* X, Y *are independent.*

2. Bivariate binomial distribution.

We shall derive the distribution of the sum of n mutually independent random vectors $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ which have the same bivariate Bernoulli distribution law. We shall calculate the probabilities $P(\sum_{i=1}^n X_i=k,\ \sum_{i=1}^n Y_i=l)$ for all k and l satisfying $0\leqq k\leqq n$, $0\leqq l\leqq n$. If we assume the events $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ occur respectively α, β, γ and δ times with $\alpha+\beta+\gamma+\delta=n$ then the sum of pairs $(\sum_{i=1}^n X_i,\ \sum_{i=1}^n Y_i)$ equals to $(\beta+\delta,\ \gamma+\delta)$.

The probability of the event described above equals to

$$P_{\alpha\beta\gamma\delta}=\frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!}p_{00}{}^{\alpha}p_{10}{}^{\beta}p_{01}{}^{\gamma}p_{11}{}^{\delta}$$

by the notion of multinomial distribution. Then the probability

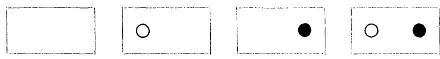
$$P\left(\sum_{i=1}^n X_i=k,\ \sum_{i=1}^n Y_i=l\right)$$

is given by the sum of the probabilities $P_{\alpha\beta\gamma\delta}$ where α, β, γ and δ take all over the values of non-negative integral values satisfying the conditions $\beta+\delta=k$, $\gamma+\delta=l$ and $\alpha+\beta+\gamma+\delta=n$:

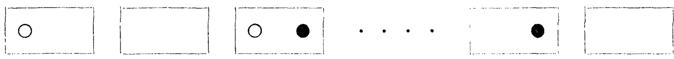
$$P\left(\sum_{i=1}^n X_i=k,\ \sum_{i=1}^n Y_i=l\right)=\sum_{\substack{\beta+\delta=k\\ \gamma+\delta=l\\ \alpha+\beta+\gamma+\delta=n}}\frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!}p_{00}{}^{\alpha}p_{10}{}^{\beta}p_{01}{}^{\gamma}p_{11}{}^{\delta}$$

where k and l are non-negative integers satisfying $0\leqq k,\ l\leqq n$.

For an example of this bivariate binomial distribution we consider the experiment that for given n spaces of same quality we distribute the following four events independently, a) neither white nor black ball exists, b) one white and no black ball exists, c) one black and no white ball exists and d) both white and black ball exist. The given four events have the pattern



with probabilities p_{00}, p_{10}, p_{01} and p_{11}. The probabilities of the n independent samples, for example,



is given by the product of n probabilities $p_{10}, p_{00}, p_{11}, \cdots, p_{01}, p_{00}$. The probability that the sum of the while ball and the sum of the black ball in first n independent samples equal k and l is given by the bivariate binomial distribution.

We shall derive the marginal distributions of the bivariate binomial distribution as follows:

$$P\left(\sum_{i=1}^{n} X_i=k\right)=\sum_{\substack{\beta+\delta=k \\ \alpha+\beta+\gamma+\delta=n}} \frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!} p_{00}{}^{\alpha} p_{10}{}^{\beta} p_{01}{}^{\gamma} p_{11}{}^{\delta}$$

$$=\sum_{\alpha+\gamma=n-k}\left(\sum_{\beta+\delta=k}\frac{k!}{\beta!\,\delta!}p_{10}{}^{\beta}p_{11}{}^{\delta}\right)\frac{n!}{\alpha!\,\gamma!\,k!}p_{00}{}^{\alpha}p_{01}{}^{\gamma}$$

$$=\frac{n!}{k!(n-k)!}(p_{10}+p_{11})^k\sum_{\alpha+\gamma=n-k}\frac{(n-k)!}{\alpha!\,\gamma!}p_{00}{}^{\alpha}p_{01}{}^{\gamma}$$

$$=\binom{n}{k}(p_{10}+p_{11})^k(p_{00}+p_{01})^{n-k} \qquad (0\leqq k\leqq n).$$

The marginal distribution of $\sum_{i=1}^n X_i$ is binomial distribution $b(k;\, n, p_{10}+p_{11})$ with parameter $p_{10}+p_{11}$. We can immediately understand this fact from the white and black ball model described above. Similary we have the fact that the marginal distribution of $\sum_{i=1}^n Y_i$ is binomial distribution $b(l;\, n, p_{01}+p_{11})$ with parameter $p_{01}+p_{11}$:

$$P\left(\sum_{i=1}^{n} Y_i=l\right)=\binom{n}{l}(p_{01}+p_{11})^l(p_{00}+p_{10})^{n-l} \qquad (0\leqq l\leqq n).$$

The expected value of $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ is given by

$$E(\textstyle\sum_{i=1}^n X_i)=nE(X)=n(p_{10}+p_{11}) \quad \text{and} \quad E(\sum_{i=1}^n Y_i)=nE(Y)=n(p_{01}+p_{11})$$

respectively. The covariance of $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n Y_i$ is given by

$$\mathrm{Cov}(\sum X_i, \sum Y_i)=E[(\sum X_i-E[\sum X_i])(\sum Y_i-E[\sum Y_i])]$$

$$=E[(\sum X_i)(\sum Y_i)]-E[\sum X_i]E[\sum Y_i]$$

$$=E\left[\sum_{i=1}^{n} X_i Y_i+\sum_{i\neq j} X_i Y_j\right]-(\sum E(X_i))(\sum E(Y_i))$$

$$=\sum_{i=1}^{n} EX_i Y_i+\sum_{i\neq j} EX_i Y_j-\sum_{i=1} EX_i EY_i-\sum_{i\neq j} EX_i EY_j$$

$$=\sum_{i=1}^{n} [EX_i Y_i-EX_i EY_i]+\sum_{i\neq j} [EX_i Y_j-EX_i EY_j].$$

If $i\neq j$ then (X_i, Y_i) and (X_j, Y_j) are mutually independent random vectors, that is, if $i\neq j$ then X_i and Y_j are mutually independent random variables. Therefore if $i\neq j$ the expected value $E(X_i Y_j)$ becomes $E(X_i)E(Y_j)$: if $i\neq j$ then $E(X_i Y_j)-E(X_i)E(Y_j)=0$. Then we have

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)=\sum_{i=1}^{n} [EX_i Y_i-EX_i EY_i].$$

The inside of the bracket [ ] on the right side of the equality above is the covariance of X_i and Y_i. X_i and Y_i $(i=1, 2, \cdots, n)$ have the same covariance $\mathrm{Cov}(X, Y)$. Then we have

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)=\sum_{i=1}^{n} \mathrm{Cov}(X, Y).$$

In section 2—1 we have derived the result

$$\mathrm{Cov}(X, Y)=p_{00}p_{11}-p_{10}p_{01}$$

then we have

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right)=n(p_{00}p_{11}-p_{10}p_{01}).$$

We shall show the modification of the joint distribution of $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$:

$$P\left(\sum_{i=1}^{n} X_i=k, \sum_{i=1}^{n} Y_i=l\right)=\sum_{\delta=\max(k+l-n,0)}^{\min(k,l)}$$

$$\frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!}p_{00}{}^{n-(k+l)+\delta}p_{10}{}^{k-\delta}p_{01}{}^{l-\delta}p_{11}{}^{\delta}.$$

If we assume p_{00}, p_{10}, p_{01} and p_{11} are positive and $\mathrm{Cov}(X, Y)=0$ then we have X and Y are independent by lemma 1. The sum $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)$ of n independent vectors $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ has

$$\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \operatorname{Cov}(X_i, Y_i) = n \operatorname{Cov}(X, Y).$$

Therefore if we assume $\operatorname{Cov}(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)=0$ then we have $\operatorname{Cov}(X, Y)=0$. If p_{00}, p_{10}, p_{01} and p_{11} are positive then X_i and Y_i are independent for all $i=1, 2, \cdots, n$. We have concluded that $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$ are mutually independent under the assumption described above.

LEMMA 2. *If the covariance* $\operatorname{Cov}(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)$ *of the sum of n independent random pairs* $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ *of same bivariate Bernoulli distribution with positive parameters* p_{00}, p_{10}, p_{01} *and* p_{11} *equals to zero then* $\sum_{i=1}^{n} X_i$ *and* $\sum_{i=1}^{n} Y_i$ *are mutually independent random variables. In another words if* (X, Y) *has bivariate binomial law and if the covariance* $\operatorname{Cov}(X, Y)$ *equals to zero then* X *and* Y *are mutually independent under the assumption* p_{00}, p_{10}, p_{01} *and* $p_{11}>0$.

Next we shall derive the bivariate generating functions of the bivariate Bernoulli distribution and the bivariate binomial distribution. Let us define the generating function of (X, Y) of bivariate Bernoulli law as

$$\begin{aligned} g(s_1, s_2) &= \sum_{\alpha=0,1,\beta=0,1} p_{\alpha\beta} {s_1}^{\alpha} {s_2}^{\beta} \\ &= p_{00} {s_1}^0 {s_2}^0 + p_{10} {s_1}^1 {s_2}^0 + p_{01} {s_1}^0 {s_2}^1 + p_{11} {s_1}^1 {s_2}^1 \\ &= p_{00} + p_{10} s_1 + p_{01} s_2 + p_{11} s_1 s_2. \end{aligned}$$

If we assume that p_{00}, p_{10}, p_{01} and p_{11} are positive and $\operatorname{Cov}(X, Y)=p_{00}p_{11}-p_{10}p_{01}=0$ then we have $p_{00}p_{11}=p_{10}p_{01}$ and

$$\begin{aligned} g(s_1, s_2) &= p_{00} + p_{10} s_1 + p_{01} s_2 + p_{11} s_1 s_2 \\ &= [P(X=0)+P(X=1)s_1][P(Y=0)+P(Y=1)s_2] \\ &= [(p_{00}+p_{01})+(p_{10}+p_{11})s_1][(p_{00}+p_{10})+(p_{01}+p_{11})s_2]. \end{aligned}$$

The generating function of n independent sum $(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)$ is given by

$$[g(s_1, s_2)]^n = (p_{00} + p_{10} s_1 + p_{01} s_2 + p_{11} s_1 s_2)^n.$$

If $\operatorname{Cov}(X, Y)=0$ and p_{00}, p_{10}, p_{01} and p_{11} are positive then we have

$$[g(s_1, s_2)]^n = [(p_{00}+p_{01})+(p_{10}+p_{11})s_1]^n[(p_{00}+p_{10})+(p_{01}+p_{11})s_2]^n.$$

3. Bivariate Poisson distribution.

1. In this section we consider the limiting distribution of bivariate binomial distribution as $n\to\infty$ when the probabilities are expressed as $p_{10}=\lambda_{10}/n$, $p_{01}=\lambda_{01}/n$ and $p_{11}=\lambda_{11}/n$. In the section 1 we have observed the famous Poisson's theorem. In this section we shall discuss the consideration of introduction to multivariate Poisson distribution. First we shall construct the bivariate binomial distribution having the condition that for any n the probabilities p_{10}, p_{01} and p_{11} are expressed by $p_{10}=\lambda_{10}/n$, $p_{01}=\lambda_{01}/n$ and $p_{11}=\lambda_{11}/n$ then the joint distribution of the sum vector of n independent vectors $(X_1, Y_1), \cdots, (X_n, Y_n)$ of bivariate Bernoulli law is given by the definition of the bivariate binomial distribution

$$P\left(\sum_{i=1}^{n} X_i=k,\ \sum_{i=1}^{n} Y_i=l\right)=\sum_{\delta=\max(k+l-n,0)}^{\min(k,l)}$$

$$\frac{n!}{(n-(k+l)+\delta)!\,(k-\delta)!\,(l-\delta)!\,\delta!}\left(1-\frac{\lambda_{10}+\lambda_{01}+\lambda_{11}}{n}\right)^{n-(k+l)+\delta}\left(\frac{\lambda_{10}}{n}\right)^{k-\delta}\left(\frac{\lambda_{01}}{n}\right)^{l-\delta}\left(\frac{\lambda_{11}}{n}\right)^{\delta}.$$

The term of the right side converges to

$$\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

as $n\to\infty$. See Kendall and Stuart [3]. And the sum of the right side becomes to δ varying $0, 1, 2, \cdots, \min(k, l)$ as n increases to infinity. Then we have the main theorem.

THEOREM 3.1. *The sum of n independent bivariate Bernoulli vectors (X_1, Y_1), $\cdots, (X_n, Y_n)$ of the same distribution p_{00}, p_{10}, p_{01} and p_{11} where $np_{01}=\lambda_{01}$, $np_{10}=\lambda_{10}$ and $np_{11}=\lambda_{11}$ are fixed values then the limiting distribution of the sum vector (X, Y) of the n vectors is given by the form*

$$P(X=k,\ Y=l)=\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}.$$

The marginal distribution of the bivariate Poisson distribution is given by the following lemma.

LEMMA 3.1. *We have*

$$P(X=k)=\frac{(\lambda_{10}+\lambda_{11})^k}{k!}e^{-(\lambda_{10}+\lambda_{11})} \qquad k=0, 1, 2, \cdots,$$

$$P(Y=l)=\frac{(\lambda_{01}+\lambda_{11})^l}{l!}e^{-(\lambda_{01}+\lambda_{11})} \qquad l=0, 1, 2, \cdots.$$

And the expected values and the variances of X and Y are given by

$$E(X)=\lambda_{10}+\lambda_{11}, \qquad \mathrm{Var}(X)=\lambda_{10}+\lambda_{11},$$

$$E(Y)=\lambda_{01}+\lambda_{11}, \qquad \mathrm{Var}(Y)=\lambda_{01}+\lambda_{11}.$$

Proof. We have

$$P(X=k)=\sum_{l=0}^{\infty}P(X=k,\ Y=l)$$

$$=\sum_{l=0}^{\infty}\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}.$$

The sum is expressed such that if $l=0$ then $\delta=0$ only and if $l=1$ then $\delta=0,1$ and if $\cdots$ and if $l=k$ then $\delta=0,1,\cdots,k$ and if $l=k+1$ then $\delta=0,1,\cdots,k$ and if $\cdots$ then the sum becomes to if $\delta=0$ then $l=0,1,2,\cdots,k,\ k+1,\cdots$ and if $\delta=1$ then $l=1,2,\cdots,k,\ k+1,\cdots$ and if $\cdots$ and if $\delta=k$ then $l=k,\ k+1,\cdots$ and if $\cdots$

$$P(X=k)=\left[\sum_{l=0}^{\infty}\frac{\lambda_{10}{}^{k}\lambda_{01}{}^{l}\lambda_{11}{}^{0}}{k!\,l!\,0!}+\sum_{l=1}^{\infty}\frac{\lambda_{10}{}^{k-1}\lambda_{01}{}^{l-1}\lambda_{11}{}^{1}}{(k-1)!\,(l-1)!\,1!}+\cdots+\sum_{l=k}^{\infty}\frac{\lambda_{10}{}^{0}\lambda_{01}{}^{l-k}\lambda_{11}{}^{k}}{0!\,(l-k)!\,k!}\right]e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

$$=\left[\frac{\lambda_{10}{}^{k}\lambda_{11}{}^{0}}{k!\,0!}e^{\lambda_{01}}+\frac{\lambda_{10}{}^{k-1}\lambda_{11}{}^{1}}{(k-1)!\,1!}e^{\lambda_{01}}+\cdots+\frac{\lambda_{10}{}^{0}\lambda_{11}{}^{k}}{0!\,k!}e^{\lambda_{01}}\right]e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

$$=\frac{(\lambda_{10}+\lambda_{11})^{k}}{k!}e^{-(\lambda_{10}+\lambda_{11})},\qquad k=0,1,2,\cdots.$$

And similarly we have

$$P(Y=l)=\sum_{k=0}^{\infty}P(X=k,\ Y=l)$$

$$=\sum_{k=0}^{\infty}\sum_{\delta=0}^{\min(k,l)}\frac{\lambda_{10}{}^{k-\delta}\lambda_{01}{}^{l-\delta}\lambda_{11}{}^{\delta}}{(k-\delta)!\,(l-\delta)!\,\delta!}e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}.$$

The sum is expressed as if $k=0$ then $\delta=0$ only and if $k=1$ then $\delta=0,1$ and if $\cdots$, and if $k=l$ then $\delta=0,1,\cdots,l$ and if $k=l+1$ then $\delta=0,1,\cdots,l$ and if $\cdots$ then the sum becomes to if $\delta=0$ then $k=0,1,\cdots,l,\ l+1,\cdots$ and if $\delta=1$ then $k=1,\cdots,l,\ l+1,\cdots$ and if $\cdots$ and if $\delta=l$ then $k=l,\ l+1,\cdots$ and if $\cdots$

$$P(Y=l)=\left[\sum_{k=0}^{\infty}\frac{\lambda_{10}{}^{k}\lambda_{01}{}^{l}\lambda_{11}{}^{0}}{k!\,l!\,0!}+\sum_{k=1}^{\infty}\frac{\lambda_{10}{}^{k-1}\lambda_{01}{}^{l-1}\lambda_{11}{}^{1}}{(k-1)!\,(l-1)!\,1!}+\cdots+\sum_{k=l}^{\infty}\frac{\lambda_{10}{}^{k-l}\lambda_{01}{}^{0}\lambda_{11}{}^{l}}{(k-l)!\,0!\,l!}\right]e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

$$=\left[\frac{\lambda_{01}{}^{l}\lambda_{11}{}^{0}}{l!\,0!}e^{\lambda_{10}}+\frac{\lambda_{01}{}^{l-1}\lambda_{11}{}^{1}}{(l-1)!\,1!}e^{\lambda_{10}}+\cdots+\frac{\lambda_{01}{}^{0}\lambda_{11}{}^{l}}{0!\,l!}e^{\lambda_{10}}\right]e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}$$

$$=\frac{(\lambda_{01}+\lambda_{11})^{l}}{l!}e^{-(\lambda_{01}+\lambda_{11})},\qquad l=0,1,2,\cdots.$$

Hence the marginal distribution of X is Poisson with parameter $\lambda_{10}+\lambda_{11}$ and the marginal distribution of Y is Poisson with parameter $\lambda_{01}+\lambda_{11}$. Then the mean values and the variances of X and Y equales to the parameters respectively. The covariance of bivariate Poisson distribution is given by the following lemma.

LEMMA 3.2. *If the random variables X and Y have the joint probability distribution given in the theorem* 3.1, *then we have the fact that the covariance of X and Y equals to λ_{11}.*

Proof. The generating function $h(s_1, s_2)$ of (X, Y) is given by

$$
\begin{aligned}
h(s_1, s_2) &= \lim_{n\to\infty} [g(s_1, s_2)]^n \\
&= \lim_{n\to\infty} [p_{00}+p_{10}s_1+p_{01}s_2+p_{11}s_1s_2]^n \\
&= \lim_{n\to\infty} \left[1-\frac{\lambda_{10}+\lambda_{01}+\lambda_{11}}{n}+\frac{\lambda_{10}}{n}s_1+\frac{\lambda_{01}}{n}s_2+\frac{\lambda_{11}}{n}s_1s_2\right]^n \\
&= \lim_{n\to\infty} \left[1-\frac{\lambda_{10}+\lambda_{01}+\lambda_{11}-\lambda_{10}s_1-\lambda_{01}s_2-\lambda_{11}s_1s_2}{n}\right]^n \\
&= e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})+\lambda_{10}s_1+\lambda_{01}s_2+\lambda_{11}s_1s_2};
\end{aligned}
$$

see Feller [1]. Then we have generally

$$h(s_1, s_2)=\sum_{k,l} P(X=k,\ Y=l)s_1{}^k s_2{}^l$$

and

$$\frac{\partial^2 h}{\partial s_1 \partial s_2}=\sum_{k,l} k\cdot l\, P(X=k,\ Y=l)s_1{}^{k-1}s_2{}^{l-1}.$$

We put $s_1=s_2=1$ then

$$\left[\frac{\partial^2 h}{\partial s_1 \partial s_2}\right]_{s_1=s_2=1}=\sum_{k,l} k\cdot l\, P(X=k,\ Y=l)=E(X\cdot Y)$$

Therefore we have

$$\frac{\partial^2 h}{\partial s_1 \partial s_2}=[(\lambda_{10}+\lambda_{11}s_2)(\lambda_{01}+\lambda_{11}s_1)+\lambda_{11}]e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})+\lambda_{01}s_2+\lambda_{10}s_1+\lambda_{11}s_1s_2}.$$

If we put $s_1=s_2=1$ then

$$E(XY)=(\lambda_{10}+\lambda_{11})(\lambda_{01}+\lambda_{11})+\lambda_{11}$$

and by the lemma 3.1 $E(X)=\lambda_{10}+\lambda_{11}$, $E(Y)=\lambda_{01}+\lambda_{11}$ we have

$$\mathrm{Cov}(X, Y)=E(XY)-E(X)E(Y)=\lambda_{11}.$$

And we have easily obtain the value of correlation coefficient of the bivariate Poisson distribution.

LEMMA 3.3. *If the random vector (X, Y) has the joint probability distribution given in the theorem 3.1, then we have that the coefficient of correlation of the vector equals to $\lambda_{11}/\sqrt{(\lambda_{10}+\lambda_{11})(\lambda_{01}+\lambda_{11})}$.*

Proof. The coefficient of correlation of the vector (X, Y), $R(X, Y)$ is defined by

$$R(X, Y)=\frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

Then by the fact $\mathrm{Var}(X)=\lambda_{10}+\lambda_{11}$, $\mathrm{Var}(Y)=\lambda_{01}+\lambda_{11}$ and $\mathrm{Cov}(X, Y)=\lambda_{11}$ we have

$$R(X, Y)=\frac{\lambda_{11}}{\sqrt{(\lambda_{10}+\lambda_{11})(\lambda_{01}+\lambda_{11})}}$$

as to be proved.

Then we have the following theorem.

THEOREM 3.2. *If the random vector* (X, Y) *has the joint probability distribution given in the theorem* 3.1 *and if we assume* X, Y *are uncorrelated then we have the joint distribution*

$$P(X=k,\ Y=l)=\frac{\lambda_{10}{}^{k}\lambda_{01}{}^{l}}{k!\ l!}e^{-(\lambda_{10}+\lambda_{01})}$$

for any integer $k=0, 1, 2, \cdots$, $l=0, 1, 2, \cdots$ *that is* X *and* Y *are independent random variables of Poisson laws.*

Proof. In the definition of the joint distribution we put in the sum $\delta=0, 1, 2, \cdots, \min(k, l)$ then the sum remain $\delta=0$ only and we generally consider $0^0=1$.

2. In the preceding section §3.1 we have discussed the limit distribution of the bivariate binomial distribution consists of n independent bivariate Bernoulli distribution

$$P(X=0,\ Y=0)=p_{00},\qquad P(X=1,\ Y=0)=p_{10},$$

$$P(X=0,\ Y=1)=p_{01}\quad \text{and}\quad P(X=1,\ Y=1)=p_{11}$$

as $n\to\infty$ where we assume $np_{10}=\lambda_{10}$, $np_{01}=\lambda_{01}$ and $np_{11}=\lambda_{11}$ are fixed numbers. We assume the two random variables X, Y of the bivariate Bernoulli random vector (X, Y) are independent and if we put $np_{10}=\lambda_{10}$ and $np_{01}=\lambda_{01}$ are fixed variables and np_{11} is bounded then if we put $np_{11}=\lambda_{11}$ then we have $\lambda_{11}\to 0$ as $n\to\infty$. Then we have $p_{10}=O(1/n)$, $p_{01}=O(1/n)$ and $p_{11}\to 0$ as $n\to\infty$. Since $p_{00}=1-p_{10}-p_{01}-p_{11}$ we have $p_{00}\to 1$ as $n\to\infty$.

By the independence of X, Y we have seen $p_{00}p_{11}=p_{10}p_{01}$ that is $p_{11}=p_{10}p_{01}/p_{00}=O(1/n^2)$, if we put $np_{11}=\lambda_{11}$ then we have $\lambda_{11}=O(1/n)$: $\lambda_{11}\to 0$ as $n\to\infty$. Therefore we have the fact that the limit distribution of the bivariate binimial distribution approaches to the bivariate Poisson distribution of $\lambda_{11}=0$ as to be proved.

Then we have the next theorem.

THEOREM 3.3. *The limit distribution as* $n\to\infty$ *of bivariate binomial distribu-*

tion of the sum of n independent bivariate Bernoulli vectors $(X_1, Y_1), \cdots, (X_n, Y_n)$ which have the same distribution p_{00}, p_{10}, p_{01} and p_{11} is given by the following form

$$P(X=k,\ Y=l)=\frac{\lambda_{10}{}^{k}\lambda_{01}{}^{l}}{k!\,l!}e^{-(\lambda_{10}+\lambda_{01})}$$

for any $k, l=0,1,2,\cdots$ where we assumed that $np_{10}=\lambda_{10}$, $np_{01}=\lambda_{01}$ are fixed values and np_{11} is bounded.

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