Karl H. Hofmann · Sidney A. Morris

The Structure of Compact Groups

A Primer for the Student – A Handbook for the Expert

Walter de Gruyter
Berlin · New York 1998

Contents

Chapter 1. Basic Topics and Examples 1

Definitions and Elementary Examples 2
Actions, Subgroups, Quotient Spaces 5
Products of Compact Groups 10
Applications to Abelian Groups 11
Projective Limits 17
Totally Disconnected Compact Groups 22
Some Duality Theory 23
Postscript 29
References for This Chapter—Additional Reading 30

Chapter 2. The Basic Representation Theory of Compact Groups 31

Some Basic Representation Theory for Compact Groups 31
The Haar Integral 34
Consequences of Haar Measure 37
The Main Theorem on Hilbert Modules for Compact Groups 39
Postscript 51
References for This Chapter—Additional Reading 51

Chapter 3. The Ideas of Peter and Weyl 52

Part 1: The Classical Theorem of Peter and Weyl 52
An Excursion into Linear Algebra 57
The G-modules $E' \otimes E$, $\mathrm{Hom}(E, E)$ and $\mathrm{Hom}(E, E)'$ 59
The Fine Structure of $R(G, \mathbb{K})$ 61
Part 2: The General Theory of G-Modules 68
Vector Valued Integration 68
The First Application: The Averaging Operator 74
Compact Groups Acting on Convex Cones 78
More Module Actions, Convolutions 79
Complexification of Real Representations 84
Postscript 90
References for This Chapter—Additional Reading 90

Chapter 4. Characters 91

Part 1: Characters of Finite Dimensional Representations 91
Part 2: The Structure Theorem of E_{fin} 98
Cyclic Modules 108
Postscript 109
References for This Chapter—Additional Reading 110

Chapter 5. Linear Lie Groups 111
Preliminaries 111
The Exponential Function and the Logarithm 113
Differentiating the Exponential Function in a Banach Algebra 125
Local Groups for the Campbell–Hausdorff Multiplication 131
Subgroups of A^{-1} and Linear Lie Groups 134
Analytic Groups 138
The Intrinsic Exponential Function of a Linear Lie Group 139
The Adjoint Representation of a Linear Lie Group 146
Subalgebras, Ideals, Lie Subgroups, Normal Lie Subgroups 151
Normalizers, Centralizers, Centers 157
The Commutator Subgroup 162
Forced Continuity of Morphisms between Lie Groups 166
Quotients of Linear Lie Groups 169
The Topological Splitting Theorem for Normal Vector Subgroups . . 173
Postscript 184
References for This Chapter—Additional Reading 186

Chapter 6. Compact Lie Groups 187
Compact Lie Algebras 188
The Commutator Subgroup of a Compact Lie Group 199
The Structure Theorem for Compact Lie Groups 204
Maximal Tori 208
The Second Structure Theorem for Connected Compact Lie Groups . 217
Compact Abelian Lie Groups and Their Linear Actions 222
Action of a Maximal Torus on the Lie Algebra 226
The Weyl Group Revisited 238
The Commutator Subgroup of Connected Compact Lie Groups . . . 248
On the Automorphism Group of a Compact Lie Group 249
Auerbach's Generation Theorem 272
The Topology of Connected Compact Lie Groups 280
Postscript 290
References for This Chapter—Additional Reading 292

Chapter 7. Duality for Abelian Topological Groups 293
The Compact Open Topology and Hom-Groups 294
Local Compactness and Duality of Abelian Topological Groups . . . 298
Basic Functorial Aspects of Duality 305
The Annihilator Mechanism 308
Character Groups of Topological Vector Spaces 319
The Exponential Function 334
Weil's Lemma and Compactly Generated Abelian Groups 342
Reducing Locally Compact Groups to Compact Abelian Groups . . . 346
The Duality Theorem 350
The Identity Component 358

The Weight of Locally Compact Abelian Groups 360
Postscript 362
References for This Chapter—Additional Reading 365

Chapter 8. Compact Abelian Groups 366

Part 1: Aspects of the Algebraic Structure 367
Divisibility, Torsion, Connectivity 367
Compact Abelian Groups as Factor Groups 373
Part 2: Aspects of the Point Set Topological Structure 381
Topological Dimension of Compact Abelian Groups 381
Arc Connectivity 388
Local Connectivity 393
Compact Metric Abelian Groups 403
Part 3: Aspects of Algebraic Topology—Homotopy 407
Free Compact Abelian Groups 407
Homotopy of Compact Abelian Groups 412
Exponential Function and Homotopy 418
The Fine Structure of Free Compact Abelian Groups 419
Part 4: Aspects of Homological Algebra 425
Injective, Projective, and Free Compact Abelian Groups 425
Part 5: Aspects of Algebraic Topology—Cohomology 429
Cohomology of Compact Abelian Groups 429
Postscript 434
References for This Chapter—Additional Reading 437

Chapter 9. The Structure of Compact Groups 438

Part 1: The Fundamental Structure Theorems of Compact Groups . 439
Approximating Compact Groups by Compact Lie Groups 439
The Closedness of Commutator Subgroups 440
Semisimple Compact Connected Groups 441
The Levi–Mal'cev Structure Theorem for Compact Groups 455
Maximal Connected Abelian Subgroups 463
The Splitting Structure Theorem 468
Supplementing the Identity Component 469
Part 2: The Structure Theorems for the Exponential Function 474
The Exponential Function of Compact Groups 474
The Dimension of Compact Groups 481
Locally Euclidean Compact Groups Are Compact Lie Groups 485
Part 3: The Connectivity Structure of Compact Groups 487
Arc Connectivity 487
Local Connectivity 490
Compact Groups and Indecomposable Continua 493
Part 4: Some Homological Algebra for Compact Groups 495
The Projective Cover of Connected Compact Groups 495
Part 5: The Automorphism Group of Compact Groups 503

The Iwasawa Theory of Automorphism Groups 505
Postscript 513
References for This Chapter—Additional Reading 515

Chapter 10. Compact Group Actions 516
A Preparation Involving Compact Semigroups 517
Orbits, Orbit Space, and Isotropy 517
Equivariance and Cross Sections 521
The Triviality Theorem 526
Quotient Actions, Totally Disconnected G-Spaces 536
Compact Lie Group Actions on Locally Compact Spaces 537
The Existence of Global Cross Sections 539
Split Morphisms 543
Postscript 556
References for This Chapter—Additional Reading 557

Chapter 11. The Structure of Free Compact Groups 558
The Category Theoretical Background 558
Splitting the Identity Component 563
The Center of a Free Compact Group 564
The Commutator Subgroup of a Free Compact Group 573
Freeness Versus Projectivity 591
Postscript 593
References for This Chapter—Additional Reading 595

Chapter 12. Cardinal Invariants of Compact Groups 596
Suitable Sets 596
Generating Degree and Density 601
The Cardinal Invariants of Connected Compact Groups 604
Cardinal Invariants in the Absence of Connectivity 607
On the Location of Special Generating Sets 609
Postscript 614
References for This Chapter—Additional Reading 615

Appendix 1. Abelian Groups 616
Examples 616
Free Abelian Groups 620
Projective Groups 627
Torsion Subgroups 628
Pure Subgroups 629
Free Quotients 631
Divisibility 632
Some Homological Algebra 640
Exact Sequences 643
Whitehead's Problem 654

Postscript 667
References for This Appendix—Additional Reading 667

Appendix 2. Covering Spaces and Groups 668

Covering Spaces and Simple Connectivity 668
The Group of Covering Transformations 679
Universal Covering Groups 680
Groups Generated by Local Groups 682
Postscript 686
References for This Appendix—Additional Reading 686

Appendix 3. A Primer of Category Theory 687

Categories, Morphisms 687
Pointed Categories 695
Types of Morphisms 696
Functors 703
Natural Transformations 715
Equivalence of Categories 719
Limits 720
The Left Adjoint Existence Theorem 724
Commutative Monoidal Categories and Its Monoid Objects 729
Part 1: The Quintessential Diagram Chase 729
Part 2: Connected Graded Commutative Hopf Algebras 738
Part 3: Duality of Graded Hopf Algebras 755
Part 4: An Application to Compact Monoids 757
Postscript 759
References for This Appendix—Additional Reading 759

Appendix 4. Selected Results in Topology and Topological Groups 760

The Arc Component Topology 760
The Weight of a Topological Space 763
Metrizability of Topological Groups 768
Duality of Vector Spaces 775
Subgroups of Topological Groups 776
Postscript 778
References for This Appendix—Additional Reading 779

References 781
List of Symbols 795
Index 799