

# The Structure of Groups of Motions Admitted by Einstein-Maxwell Space-Times

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**Abstract.** The known symmetry of the non-null electromagnetic field, which acts as the source of a four-dimensional space-time satisfying the Einstein-Maxwell equations, is used to show that when such a space-time admits a group of motions, generated by a Killing vector, the structure constants for the group must satisfy an additional relation to the known relations of group theory.

## 1. Introduction

The work of Rainich (1925), and subsequently Misner and Wheeler (1957), has shown that in the absence of sources the equations of electromagnetism and gravitation can be expressed in a purely geometric form. A consequence of this was shown by Misner and Wheeler to be that the non-null electromagnetic field tensor  $F_{\mu\nu}$  is determined up to a duality rotation by the metric tensor  $g_{\mu\nu}$ .

In the work which follows we shall see that this leads to the conclusion that when a four dimensional vacuum space-time, having a non-null electromagnetic field as its source, admits a group of motions generated by a Killing vector the infinitesimal transformations  $\mathcal{L}_v F_{\mu\nu}$  of the electromagnetic field tensor  $F_{\mu\nu}$  must be such that one of the equations

$$\mathcal{L}_v F_{\mu\nu} = 0$$

or

$$\mathcal{L}_v^2 F_{\mu\nu} = -F_{\mu\nu}$$

is satisfied. This has the consequence that the structure constants  $c_{\alpha\beta}^\gamma$  for the group of motions must satisfy an additional relation to the known relations of group theory.

## 2. The Infinitesimal Transformations

We will consider a four dimensional space-time which satisfies the Einstein-Maxwell equations. These may be written

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 4\pi\{F_{\mu\sigma}F_{\nu}^{\sigma} + *F_{\mu\sigma}*F_{\nu}^{\sigma}\} \quad (2.1)$$

$$\text{and} \quad \left. \begin{aligned} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} \{ \sqrt{-g} F^{\mu\nu} \} &= 0 \\ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} \{ \sqrt{-g} *F^{\mu\nu} \} &= 0 \end{aligned} \right\} \quad (2.2)$$

where  $F_{\mu\nu}$  is the electromagnetic field tensor and  $*F^{\mu\nu}$  is its dual. We will assume that  $F_{\mu\nu}$  is not null so that

$$\text{and} \quad \left. \begin{aligned} F_{\mu\nu}F^{\mu\nu} &\neq 0 \\ F_{\mu\nu}*F^{\mu\nu} &\neq 0 \end{aligned} \right\} \quad (2.3)$$

The mixed energy momentum tensor for the electromagnetic field has vanishing trace so that (2.1) may be written

$$R_{\mu\nu} = 4\pi\{F_{\mu\sigma}F_{\nu}^{\sigma} + *F_{\mu\sigma}*F_{\nu}^{\sigma}\}. \quad (2.4)$$

We will now consider that our space-time admits an  $r$ -parameter group of motions which is generated by a Killing vector. This requires that there exist  $r$  linearly independent vectors  $v_{\alpha}^{\sigma}$  which satisfy the equations of Killing. These may be written (Yano, 1955)

$$\mathcal{L}_v g_{\mu\nu} = v^{\sigma} \frac{\partial}{\partial x^{\sigma}} g_{\mu\nu} + g_{\mu\sigma} \frac{\partial v^{\sigma}}{\partial x^{\nu}} + g_{\sigma\nu} \frac{\partial v^{\sigma}}{\partial x^{\mu}} = 0 \quad (2.5)$$

and for each Killing vector  $v_{\alpha}^{\sigma}$  we have an infinitesimal operator

$$\mathcal{L}_v \equiv \mathcal{L}_{\alpha} \quad (2.6)$$

such that (2.5) is satisfied.

If we denote any of the independent vectors  $v_{\alpha}^{\sigma}$  by  $v^{\sigma}$  we find that the infinitesimal transformations  $\mathcal{L}_v F_{\mu\nu}$  of the electromagnetic field tensor  $F_{\mu\nu}$  must have the forms

$$\text{and} \quad \left. \begin{aligned} \mathcal{L}_v F_{\mu\nu} &= \frac{\partial A_{\mu}}{\partial x^{\nu}} - \frac{\partial A_{\nu}}{\partial x^{\mu}} \\ \mathcal{L}_v F^{\mu\nu} &= \frac{\varepsilon_{\mu\nu\sigma\tau}}{\sqrt{-g}} \frac{\partial B_{\tau}}{\partial x^{\sigma}} \end{aligned} \right\} \quad (2.7)$$

where the two vectors  $A_\sigma$  and  $B_\sigma$  are defined by

$$\text{and} \quad \left. \begin{aligned} A_\sigma &= v^\nu F_{\sigma\nu} \\ B_\sigma &= v^{\nu*} F_{\sigma\nu} \end{aligned} \right\} \quad (2.8)$$

respectively. From the relations (2.7) we find that  $\mathcal{L}_v F_{\mu\nu}$  satisfies Maxwell's Eqs. (2.2). We may use this fact in order to determine firstly the infinitesimal mode of transformation of the non-null field and subsequently the structure of the group of motions.

There are two distinct cases which have to be considered. Firstly it is possible for  $\mathcal{L}_v F_{\mu\nu}$  to vanish. This is certainly a solution to the vacuum Maxwell equations and moreover the first integrability condition

$$\mathcal{L}_v R_{\mu\nu\sigma\tau} = 0 \quad (2.9)$$

of Killing's equation ensures that  $\mathcal{L}_v R_{\mu\nu} = 0$ . It is therefore possible for a field satisfying

$$\mathcal{L}_v F_{\mu\nu} = 0 \quad (2.10)$$

to be a solution of the Einstein-Maxwell equations. The condition (2.10) expresses the invariance of the electromagnetic field under the action of the transformations generated by the Killing vector  $v^\sigma$ . We shall now consider the second case in which (2.10) is not satisfied.

When the electromagnetic field is not invariant its Lie derivative, with respect to the Killing vector  $v^\sigma$ , is a non-trivial solution of Maxwell's equations (2.2) with the metric tensor  $g_{\mu\nu}$ . Thus if

$$f_{\mu\nu} = \mathcal{L}_v F_{\mu\nu} \quad (2.11)$$

this electromagnetic field must have an energy-momentum tensor

$$T_{\mu\nu} = -\frac{1}{2}(f_{\mu\sigma}f_\nu^\sigma + *f_{\mu\sigma} *f_\nu^\sigma) \quad (2.12)$$

and Einstein's theory of gravitation requires that

$$G_{\mu\nu} = -8\pi T_{\mu\nu}. \quad (2.13)$$

But in a four-dimensional space-time the Einstein tensor  $G_{\mu\nu}$  is unique to within the cosmological term (e.g. Lovelock, 1971 and 1972) – which we are not considering here. Thus (2.13) requires that

$$R_{\mu\nu} = 4\pi(f_{\mu\sigma}f_\nu^\sigma + *f_{\mu\sigma} *f_\nu^\sigma) \quad (2.14)$$

and we may use this relation together with (2.4) in order to determine the precise relation between  $f_{\mu\nu}$  and  $F_{\mu\nu}$  in this case.

If we define the complex fields  $\Gamma_{\mu\nu}$  and  $\gamma_{\mu\nu}$  by

$$\text{and } \left. \begin{aligned} \Gamma_{\mu\nu} &= F_{\mu\nu} + i^*F_{\mu\nu} \\ \gamma_{\mu\nu} &= \mathcal{L}_v \Gamma_{\mu\nu} \end{aligned} \right\} \quad (2.15)$$

respectively then (2.4) and (2.14) may be written

$$\text{and } \left. \begin{aligned} R_{\mu\nu} &= 4\pi\Gamma_{\mu\sigma}\bar{F}_v^\sigma \\ R_{\mu\nu} &= 4\pi\gamma_{\mu\sigma}\bar{\gamma}_v^\sigma \end{aligned} \right\} \quad (2.16)$$

respectively where the bar denotes the complex conjugate. In addition, the theorem of Misner and Wheeler (1957) asserts that there exists an  $\varepsilon$  such that

$$f_{\mu\nu} = F_{\mu\nu}\text{Cos}\varepsilon + {}^*F_{\mu\nu}\text{Sin}\varepsilon \quad (2.17)$$

or

$$\gamma_{\mu\nu} = e^{-i\varepsilon}\Gamma_{\mu\nu}. \quad (2.18)$$

But

$$\mathcal{L}_v R_{\mu\nu} = 0 \quad (2.19)$$

implies that

$$\Gamma_{\mu\sigma}\bar{\gamma}_v^\sigma + \gamma_{\mu\sigma}\bar{F}_v^\sigma = 0 \quad (2.20)$$

and this yields

$$R_{\mu\nu}\text{Cos}\varepsilon = 0 \quad (2.21)$$

which, since  $R_{\mu\nu} \neq 0$ , requires that  $\varepsilon$  has the values  $\left(\frac{\pi}{2} + 2m\pi\right)$  or  $\left(\frac{3\pi}{2} + 2m\pi\right)$  only where  $m$  is any integer<sup>1</sup>. On substituting this in (2.18) and equating real and imaginary parts we conclude that

$$\text{and } \left. \begin{aligned} \mathcal{L}_v F_{\mu\nu} &= \pm {}^*F_{\mu\nu} \\ \mathcal{L}_v {}^*F_{\mu\nu} &= \mp F_{\mu\nu} \end{aligned} \right\} \quad (2.22)$$

represent the infinitesimal mode of transformation of the non-null non-invariant electromagnetic field.

The two relations (2.22) are equivalent to

$$\mathcal{L}_v^2 F_{\mu\nu} + F_{\mu\nu} = 0 \quad (2.23)$$

which, when the Killing vector is known, is a second order linear partial differential equation for the functional form of  $F_{\mu\nu}$  which satisfies the Einstein-Maxwell equations. As an example, if the Killing vector has

<sup>1</sup> I would like to thank Professor A. H. Taub for pointing out this short-cut in the calculation. M.L.W.

the form  $\delta_{(k)}^\sigma$ , (2.23) reduces to

$$\frac{\partial^2 F_{\mu\nu}}{\partial x^k{}^2} + F_{\mu\nu} = 0 \quad (2.24)$$

and its solution, which must satisfy (2.22), has the form

$$\left. \begin{aligned} F_{\mu\nu} &= a_{\mu\nu} \text{Cos } \theta + {}^*a_{\mu\nu} \text{Sin } \theta \\ \theta &= x^k + \varphi \end{aligned} \right\} \quad (2.25)$$

and both  $a_{\mu\nu}$  and  $\phi$  are independent of  $x^k$ .

A physical interpretation of (2.23) is obtained if we consider the vectors  $A_\nu$  and  $B_\nu$  which give the local electric and magnetic fields respectively of a test observer who follows a path everywhere tangent to the Killing vector  $v^\sigma$ . Then (2.23) shows that the observer will find these field vectors rotating as he moves along and, for example, when  $x^k$  has the role of a time co-ordinate this would amount to a rotation in time of the local field vectors.

We have now seen that if a source-free four dimensional space-time has integrable equations of Killing the non-null electromagnetic field must transform as either (2.10) or (2.22) and we are in a position to see what implications those relations have for the structure of the corresponding of motions.

### 3. The Structure Relations

In arriving at the relations (2.10) and (2.22) we considered an arbitrary member  $v^\sigma$  of the set of  $r$  generators for the group of motions. It follows that either (2.10) or (2.22) must be satisfied by each of the infinitesimal operators  $\mathcal{L}_\alpha$  for the group in question. We will now consider this fact in more detail.

The infinitesimal operators  $\mathcal{L}_\alpha$  can be shown to obey the commutation relations (Yano, 1955, p. 29)

$$\left[ \mathcal{L}_\alpha, \mathcal{L}_\beta \right] G_A = c_{\alpha\beta}^\gamma \mathcal{L}_\gamma G_A \quad (3.1)$$

where

$$\left[ \mathcal{L}_\alpha, \mathcal{L}_\beta \right] \equiv \mathcal{L}_\alpha \mathcal{L}_\beta - \mathcal{L}_\beta \mathcal{L}_\alpha \quad (3.2)$$

and the  $c_{\alpha\beta}^\gamma$  are the fundamental structure constants of the group. The quantity  $G_A$  in (3.1) represents any linear differential geometric object.

We will now replace  $G_A$  in (3.1) with  $F_{\mu\nu}$ , and consider these relations together with (2.10) and (2.22) – one of which must be satisfied by each  $\mathcal{L}_\alpha$  when it acts on  $F_{\mu\nu}$ .

Firstly we define the set  $S_I$  of independent vectors  $v_\alpha^\sigma$  which generate the invariance group of  $F_{\mu\nu}$ . Thus

$$S_I = \{v_\alpha^\sigma | \mathcal{L}_\alpha F_{\mu\nu} = 0\} \quad (3.3)$$

and  $S_I$  is clearly a subset of the set of generators for the group of motions.

A consideration of the right hand side of (3.1) shows that this becomes

$$c_{\alpha\beta}^\gamma \mathcal{L}_\gamma F_{\mu\nu} = \pm c_{\alpha\beta} {}^*F_{\mu\nu} \quad (3.4)$$

where

$$c_{\alpha\beta} = \sum_{\substack{\gamma \\ v \notin S_I}} c_{\alpha\beta}^\gamma. \quad (3.5)$$

In considering the left hand side of (3.1) there are three distinct cases which we must take into account separately. These correspond to whether or not both, one or none of  $v_\alpha$  and  $v_\beta$  are contained in  $S_I$ .

If both of  $v_\alpha$  and  $v_\beta$  are in  $S_I$  then the left hand side of (3.1) must vanish since  $\mathcal{L}_\alpha F_{\mu\nu}$  and  $\mathcal{L}_\beta F_{\mu\nu}$  both vanish. On the other hand, if one of  $v_\alpha$  and  $v_\beta$  is in  $S_I$  the other must be such that

$$\mathcal{L}_v F_{\mu\nu} = \pm {}^*F_{\mu\nu} \quad (3.6)$$

and, since the invariance of  $F_{\mu\nu}$ , under the transformations generated by a Killing vector, implies the invariance of its dual we see that the left hand side of (3.1) vanishes in this case too. Finally, if neither of  $v_\alpha$  and  $v_\beta$  are contained in  $S_I$  they must both be such that (3.6) is true. We must then have

$$\mathcal{L}_\alpha \mathcal{L}_\beta F_{\mu\nu} = \pm \mathcal{L}_\alpha {}^*F_{\mu\nu} \quad (3.7)$$

$$= -F_{\mu\nu} \quad (3.8)$$

and, since this result is independent of the order of operation of  $\mathcal{L}_\alpha$  and  $\mathcal{L}_\beta$  it follows that the left hand side of (3.1) must again vanish.

We have now seen that if  $F_{\mu\nu}$  satisfies (2.10) and (2.22) the left hand side of (3.1) must vanish. This means that the right hand side must vanish also, since (3.1) is an identity for a given group of motions. The only conclusion which may now be reached is that the quantity  $c_{\alpha\beta}$  defined by (3.5) must vanish, i.e. we must have

$$\sum_{\substack{\gamma \\ v \notin S_I}} c_{\alpha\beta}^\gamma = 0. \quad (3.9)$$

This relation is purely a consequence of the possible symmetry of the non-null Einstein-Maxwell electromagnetic field as expressed by the relations (2.10) and (2.22).

#### 4. Conclusions

It is well known that the structure constants  $c_{\alpha\beta}^\gamma$  for an  $r$ -parameter group of motions must satisfy the relations

$$c_{\alpha\beta}^\gamma + c_{\beta\alpha}^\gamma = 0$$

and

$$c_{\alpha\beta}^\gamma c_{\gamma\delta}^\sigma + c_{\delta\alpha}^\gamma c_{\gamma\beta}^\sigma + c_{\beta\delta}^\gamma c_{\gamma\alpha}^\sigma = 0.$$

We have seen in the work here that for a group of motions which leave unchanged the metric tensor  $g_{\mu\nu}$  of a four dimensional vacuum Einstein-Maxwell spacetime, having a non-null electromagnetic field  $F_{\mu\nu}$  as its source, we must consider in addition the relations (3.9). These, together with the further relations

$$\mathcal{L}_v F_{\mu\nu} = 0$$

$$\mathcal{L}_v^2 F_{\mu\nu} = -F_{\mu\nu}$$

and

$$\mathcal{L}_v g_{\mu\nu} = 0$$

should allow a complete group theoretical characterization, of this class of electromagnetic space-times, to be carried out.

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