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KODAI MATH. J.
2 (1979), 337—345

THE STRUCTURE OF MULTIVARIATE POISSON DISTRIBUTION

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Summary

In this paper we shall derive a multivariate Poisson distribution and we shall discuss its structure.

Notations and Definitions

n	positive integer
$\boldsymbol{X}=(X_1, X_2, \cdots, X_n)$	n dimensional random vector
$\boldsymbol{i}=(i_1, i_2, \cdots, i_n)$	n dimensional vector with 0, 1 components
$\boldsymbol{k}=(k_1, k_2, \cdots, k_n)$	n dimensional vector with nonnegative integer components
$\boldsymbol{x}=(x_1, x_2, \cdots, x_n)$	observation of $\boldsymbol{X}$
$p(x, \lambda)$	Poisson density with parameter λ

Main Results

1. Multivariate Bernoulli distribution $B(1, p_i)$

Multivariate Bernoulli distribution is defined by

$$P(\boldsymbol{X}=\boldsymbol{i})=p_i$$

where $p_i \geqq 0$ and $\sum_i p_i=1$.

The moment generating function (m. g. f.) is given by

$$g(\boldsymbol{s})=\sum_i p_i s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} .$$

The marginal distribution of this multivariate Bernoulli distribution is also degenerated Bernoulli.

The covariance matrix of $B(1, p_i)$ is given by

$$\mathrm{Cov}(X_j, X_k)=\sum_{i_j=i_k=0} p_i \sum_{i_j=i_k=1} p_i - \sum_{i_j=1, i_k=0} p_i \sum_{i_j=0, i_k=1} p_i ,$$

$$\mathrm{Var}(X_j)=\sum_{i_j=0} p_i \sum_{i_j=1} p_i .$$

Received May 18, 1978

Proof.

$$\begin{aligned}\mathrm{Cov}(X_j, X_k) &= \sum_{i_j=i_k=1} x_j x_k p_i - \sum_{i_j=1} x_j p_i \sum_{i_k=1} x_k p_i \\ &= \sum_{i_j=i_k=1} p_i - \sum_{i_j=1} p_i \sum_{i_k=1} p_i \\ &= \sum_{i_j=i_k=1} p_i \Big(\sum_{i_j=i_k=0} p_i + \sum_{i_j=1, i_k=0} p_i + \sum_{i_j=0, i_k=1} p_i + \sum_{i_j=i_k=1} p_i \Big) \\ &\quad - \Big(\sum_{i_j=1, i_k=0} p_i + \sum_{i_j=i_k=1} p_i \Big) \Big(\sum_{i_j=0, i_k=1} p_i + \sum_{i_j=i_k=1} p_i \Big) \\ &= \sum_{i_j=i_k=0} p_i \sum_{i_j=i_k=1} p_i - \sum_{i_j=1, i_k=0} p_i \sum_{i_j=0, i_k=1} p_i \,. \end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X_j) &= \sum_{i_j=1} x_j^2 p_i - \Big(\sum_{i_j=1} x_j p_i \Big)^2 \\ &= \sum_{i_j=1} p_i - \Big(\sum_{i_j=1} p_i \Big)^2 .\end{aligned}$$

2. Multivariate binomial distribution $B(N, p_i)$

Multivariate binomial distribution is defined by

$$P(\boldsymbol{X}=\boldsymbol{k}) = \sum_{\Sigma \alpha_i i_j = k_j} \frac{N!}{\prod_\iota \alpha_\iota !} \prod_\iota p_i^{\alpha_\iota} \,.$$

This distribution is derived by N time convolution of $B(1, p_i)$. The m.g.f. of the distribution is given by

$$g_N(\boldsymbol{s}) = (\textstyle\sum_\iota p_i s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n})^N \,.$$

The marginal distribution of this multivariate binomial distribution is also degenerated binomial.

Covariance matrix of $B(N, p_i)$ is given by

$$\mathrm{Cov}(X_j, X_k) = N\Big(\sum_{i_j=i_k=0} p_i \sum_{i_j=i_k=1} p_i - \sum_{i_j=1, i_k=0} p_i \sum_{i_j=0, i_k=1} p_i \Big) ,$$

$$\mathrm{Var}(X_j) = N\Big(\sum_{i_j=0} p_i \sum_{i_j=1} p_i \Big) .$$

3. Multivariate Poisson distribution

Multivariate Poisson distribution is a limiting distribution of $B(N, p_i)$ as $N \to \infty$ under the condition of $N p_i = \lambda_\iota$ where λ_ι is a non-negative fixed parameter. If a random vector $\boldsymbol{X}$ has a binomial distribution $B(N, p_i)$ and if we assume $N p_i = \lambda_\iota$ then we have

$$\lim_{N \to \infty} P(\boldsymbol{X}=\boldsymbol{k}) = \sum_{\Sigma \alpha_i i_j = k_j} \prod_{\iota \neq 0} p(\alpha_\iota, \lambda_\iota)$$

where $p(\alpha_\iota, \lambda_\iota)$ is an univariate Poisson density.

THEOREM 1. *If a random vector $\boldsymbol{X}$ has a distribution $B(N, p_i)$ then we have*

$$\lim_{Np_i=\lambda_i,\, N\to\infty} P(\boldsymbol{X}=\boldsymbol{k})=\sum_{\Sigma\alpha_i i_j=k_j} \prod_{i\neq 0} p(\alpha_i, \lambda_i),$$

Proof. By the condition we get

$$P(\boldsymbol{X}=\boldsymbol{k})=\sum_{\Sigma\alpha_i i_j=k_j} \frac{N!}{\prod_i \alpha_i!} \prod_i p_i^{\alpha_i},$$

therefore the limit value of each term is

$$\lim_{Np_i=\lambda_i,\, N\to\infty} \frac{N!}{\prod_i \alpha_i!} \prod_i p_i^{\alpha_i}$$

$$=\lim \frac{N!}{\alpha_0!\prod_{i\neq 0}\alpha_i!}\left(1-\frac{\sum_{i\neq 0}\lambda_i}{N}\right)^{\alpha_0} \prod_{i\neq 0}\frac{\lambda_i^{\alpha_i}}{N^{\alpha_i}}$$

$$=\lim \frac{N!}{\alpha_0!\, N^{\sum_{i\neq 0}\alpha_i}} \lim\left(1-\frac{\sum_{i\neq 0}\lambda_i}{N}\right)^{N-\sum_{i\neq 0}\alpha_i} \prod_{i\neq 0}\lambda_i^{\alpha_i}/\prod_{i\neq 0}\alpha_i!$$

$$=\exp\{-\sum_{i\neq 0}\lambda_i\}\prod_{i\neq 0}\lambda_i^{\alpha_i}/\prod_{i\neq 0}\alpha_i!$$

$$=\prod_{i\neq 0} p(\alpha_i, \lambda_i).$$

Therefore we have

$$\lim_{Np_i=\lambda_i,\, N\to\infty} P(\boldsymbol{X}=\boldsymbol{k})=\lim_{Np_i=\lambda_i,\, N\to\infty}\sum_{\Sigma\alpha_i i_j=k_j} N!\prod_i\frac{p_i^{\alpha_i}}{\alpha_i!}$$

$$=\sum_{\Sigma\alpha_i i_j=k_j}\exp\{-\sum_{i\neq 0}\lambda_i\}\prod_{i\neq 0}\frac{\lambda_i^{\alpha_i}}{\alpha_i!}$$

$$=\sum_{\Sigma\alpha_i i_j=k_j}\prod_{i\neq 0} p(\alpha_i, \lambda_i).$$

THEOREM 2. *The moment generating function of the limiting distribution is given by*

$$h(\boldsymbol{s})=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i \boldsymbol{s}^i\}$$

$$=\prod_{i\neq 0}\exp\{-\lambda_i+\lambda_i\boldsymbol{s}^i\}.$$

Proof.

$$h(\boldsymbol{s})=\lim_{Np_i=\lambda_i,\, N\to\infty} g(\boldsymbol{s})^N=\lim_{Np_i=\lambda_i,\, N\to\infty}(\sum_i p_i s_1^{i_1}s_2^{i_2}\cdots s_n^{i_n})^N$$

$$=\lim_{N\to\infty}(1-\sum_{i\neq 0}p_i+\sum_{i\neq 0}p_i\boldsymbol{s}^i)^N$$

$$=\lim_{N\to\infty}\left(1-\sum_{i\neq 0}p_i+\sum_{i\neq 0}\frac{\lambda_i}{N}\boldsymbol{s}^i\right)^N$$

$$=\lim_{N\to\infty}\left(1-\sum_{i\neq 0}\frac{\lambda_i}{N}+\sum_{i\neq 0}\frac{\lambda_i}{N}\boldsymbol{s}^i\right)^N$$

$$=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i \boldsymbol{s}^{i}\} .$$

where $\boldsymbol{s}^{i}=s_1^{i_1}s_2^{i_2}\cdots s_n^{i_n}$.

THEOREM 3. *If a random vector* $\boldsymbol{X}$ *has the Poisson law, then we have an unique decomposition of the random vector by* $X_j=\sum_{i_j=1}X_i$ *where* $X_i(\boldsymbol{i}\neq\boldsymbol{0})$ *are mutually independent, univariate Poisson variables with parameter* λ_i.

Proof. This is a direct conclusion from the definition of the distribution of convolution type. Mathematical proof is given as followings. In the bivariate case $n=2$, if $\boldsymbol{X}=(X_1, X_2)$ has the Poisson law, our m. g. f. $h(\boldsymbol{s})$ becomes

$$h(\boldsymbol{s})=\exp\{-(\lambda_{10}+\lambda_{01}+\lambda_{11})+\lambda_{10}s_1+\lambda_{01}s_2+\lambda_{11}s_1s_2\}$$

$$=\exp(-\lambda_{10}+\lambda_{10}s_1)\exp(-\lambda_{01}+\lambda_{01}s_2)\exp(-\lambda_{11}+\lambda_{11}s_1s_2).$$

If we put $s_2=1$, then we get the m. g. f. of X_1

$$\exp(-\lambda_{10}+\lambda_{10}s_1)\exp(-\lambda_{11}+\lambda_{11}s_1) .$$

This is a m. g. f. of convolution type, and if we put $s_1=1$, we get

$$\exp(-\lambda_{01}+\lambda_{01}s_2)\exp(-\lambda_{11}+\lambda_{11}s_2)$$

the m. g. f. of X_2 of convolution type. Then we have a decomposition

$$X_1=X_{10}+X_{11}', \quad X_2=X_{01}+X_{11}'' .$$

If we put $X_{11}'\neq X_{11}''$ with positive probability then this is contradictory to the fact that X_{11}', X_{11}'' has a bivariate m. g. f.

$$\exp(-\lambda_{11}+\lambda_{11}s_1s_2) .$$

Therefore we have $X_{11}'=X_{11}''=X_{11}$ with probability one. And we can express the decomposition of given $\boldsymbol{X}=(X_1, X_2)$ as

$$X_1=X_{10}+X_{11}, \quad X_2=X_{01}+X_{11} .$$

In this equality X_{10}, X_{01} and X_{11} are mutually independent Poisson distribution with parameter λ_{10}, λ_{01} and λ_{11} respectively, as to be proved.

THEOREM 4. *The covariance matrix of the multivariate Poisson distribution is given by*

$$\mathrm{Var}(X_j)=\sum_{i_j=1}\lambda_i, \quad \mathrm{Cov}(X_j, X_k)=\sum_{i_j=i_k=1}\lambda_i .$$

Proof. This is a direct conclusion using the m. g. f. of the distribution (Theorem 2) or the preceding decomposition Theorem. Generally a m. g. f. is given by

$$h(\boldsymbol{s})=\sum_{\boldsymbol{k}} \boldsymbol{s}^{\boldsymbol{k}} P(\boldsymbol{X}=\boldsymbol{k}) .$$

And we have

$$\frac{\partial^2 h(\boldsymbol{s})}{\partial s_i \partial s_j}=\sum_{\boldsymbol{k}} k_i k_j s_1{}^{k_1}\cdots s_i{}^{k_i-1}\cdots s_j{}^{k_j-1}\cdots s_n{}^{k_n} P(\boldsymbol{X}=\boldsymbol{k}) ,$$

$$\left[\frac{\partial^2 h(s)}{\partial s_i \partial s_j}\right]_{s_1=s_2=\cdots=s_n=1}=\sum_{\boldsymbol{k}} k_i k_j P(\boldsymbol{X}=\boldsymbol{k})=E(X_i X_j) .$$

We use the result of Theorem 2, $h(\boldsymbol{s})$ becomes

$$h(\boldsymbol{s})=\exp\{-\sum_{\boldsymbol{i}\neq 0} \lambda_{\boldsymbol{i}}+\sum_{\boldsymbol{i}\neq 0} \lambda_{\boldsymbol{i}} \boldsymbol{s}^{\boldsymbol{i}}\} ,$$

then

$$\frac{\partial h(s)}{\partial s_k}=h(\boldsymbol{s})\{\sum_{\boldsymbol{i}\neq 0} i_k \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_k{}^{i_k-1}\cdots s_n{}^{i_n}\}$$

$$\frac{\partial^2 h(\boldsymbol{s})}{\partial s_j \partial s_k}=\frac{\partial h(\boldsymbol{s})}{\partial s_j}\{\sum_{\boldsymbol{i}\neq 0} i_k \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_k{}^{i_k-1}\cdots s_n{}^{i_n}\}$$

$$+h(\boldsymbol{s})\frac{\partial}{\partial s_j}\{\sum_{\boldsymbol{i}\neq 0} i_k \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_k{}^{i_k-1}\cdots s_n{}^{i_n}\}$$

$$=h(\boldsymbol{s})\{\sum_{\boldsymbol{i}\neq 0} i_j \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_j{}^{i_j-1}\cdots s_n{}^{i_n}\}\{\sum_{\boldsymbol{i}\neq 0} i_k \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_k{}^{i_k-1}\cdots s_n{}^{i_n}\}$$

$$+h(\boldsymbol{s})\{\sum_{\boldsymbol{i}\neq 0} i_j i_k \lambda_{\boldsymbol{i}} s_1{}^{i_1}\cdots s_j{}^{i_j-1}\cdots s_k{}^{i_k-1}\cdots s_n{}^{i_n}\}$$

$$\left[\frac{\partial^2 h(s)}{\partial s_j \partial s_k}\right]_{s_1=s_2=\cdots=s_n=1}$$

$$=h(\boldsymbol{s})|_{s_1=s_2=\cdots=s_n=1}[(\sum_{\boldsymbol{i}\neq 0} i_j \lambda_{\boldsymbol{i}})(\sum_{\boldsymbol{i}\neq 0} i_k \lambda_{\boldsymbol{i}})+\sum_{\boldsymbol{i}\neq 0} i_j i_k \lambda_{\boldsymbol{i}}]$$

$$E(X_j X_k)=\sum_{i_j=i_k=1} \lambda_{\boldsymbol{i}}+(\sum_{i_j=1} \lambda_{\boldsymbol{i}})(\sum_{i_k=1} \lambda_{\boldsymbol{i}}) .$$

And $E(X_j)$, $E(X_k)$ is given by $E(X_j)=\mathrm{Var}(X_j)=\sum_{i_j=1} \lambda_{\boldsymbol{i}}$, $E(X_k)=\mathrm{Var}(X_k)=\sum_{i_k=1} \lambda_{\boldsymbol{i}}$.

Finally we have

$$\mathrm{Cov}(X_j, X_k)=E(X_j X_k)-E(X_j)E(X_k)=\sum_{i_j=i_k=1} \lambda_{\boldsymbol{i}} ,$$

$$\mathrm{Var}(X_j)=E(X_j)=\sum_{i_j=1} \lambda_{\boldsymbol{i}} .$$

THEOREM 5. *If a random vector $\boldsymbol{X}$ has the Poisson law, then the marginal distribution is also a degenerated Poisson.*

Proof. Since $\boldsymbol{X}$ has a m. g. f. $h(\boldsymbol{s})$, it follows that $\boldsymbol{X}^{(j)}=(X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_n)$ $(j=1, 2, \cdots, n)$ has a m. g. f. $h(\boldsymbol{s})|_{s_j=1}$.

$$h(\boldsymbol{s})|_{s_j=1}=\exp\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i \boldsymbol{s}^{i}\}_{s_j=1}$$

$$=\exp\{-\sum_{i(j)\neq 0}(\sum_{i_j=0,1}\lambda_i)+\sum_{i(j)\neq 0}(\sum_{i_j=0,1}\lambda_i)s_1^{i_1}\cdots s_{j-1}^{i_{j-1}}s_{j+1}^{i_{j+1}}\cdots s_n^{i_n}\} .$$

This means that if $\boldsymbol{X}$ has a Poisson distribution, it follows that $\boldsymbol{X}^{(j)}$ has also a generated Poisson distribution. And, similarly, if we put

$$\boldsymbol{X}^{(j_1,j_2,\cdots,j_k)}=(X_1,\cdots,X_{j_1-1},X_{j_1+1},\cdots,X_{j_2-1},X_{j_2+1},\cdots,X_{j_k-1},X_{j_k+1},\cdots,X_n)$$

then the m. g. f. of the vector is given by

$$h(\boldsymbol{s})|_{s_{j_1}=s_{j_2}=\cdots=s_{j_k}=1}$$

$$=\exp\{-\sum_{i(j_1,j_2,\cdots,j_k)\neq 0}(\sum_{i_{j_1},i_{j_2},\cdots,i_{j_k}}\lambda_i)+\sum_{i(j_1,j_2,\cdots,j_k)\neq 0}$$

$$\times(\sum_{i_{j_1},i_{j_2},\cdots,i_{j_k}}\lambda_i)s_1^{i_1}\cdots s_{j_1-1}^{i_{j_1}-1}s_{j_1+1}^{i_{j_1}+1}\cdots s_{j_2-1}^{i_{j_2}-1}s_{j_2+1}^{i_{j_2}+1}\cdots$$

$$s_{j_k-1}^{i_{j_k}-1}s_{j_k+1}^{i_{j_k}+1}\cdots s_n^{i_n}\} .$$

Therefore, the random vector $\boldsymbol{X}^{(j_1,j_2,\cdots,j_k)}$ has a degenerated Poisson distribution as to be proved.

COROLLARY 1. *The marginal distribution X_j of $\boldsymbol{X}$ is Poisson with a parameter* $\sum_{i_j=1}\lambda_i$.

COROLLARY 2. *If* $\mathrm{Cov}(X_j, X_k)=0$ $(j\neq k)$, *then X_j and X_k are mutually independent random variables.*

THEOREM 6. *If $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N$ are mutually independent random vectors of the multivariate Poisson distribution, then the sum $\sum_{j=1}^{N}\boldsymbol{X}_j$ has a multivariate Poisson distribution.*

Proof. The m. g. f. of the sum vector is given by

$$h(\boldsymbol{s})^N=\exp N\{-\sum_{i\neq 0}\lambda_i+\sum_{i\neq 0}\lambda_i \boldsymbol{s}^{i}\}$$

$$=\exp\{-\sum_{i\neq 0}N\lambda_i+\sum_{i\neq 0}N\lambda_i \boldsymbol{s}^{i}\} .$$

This means that the sum vector is also a multivariate Poisson distribution with parameter $N\lambda_i$ $(\boldsymbol{i}\neq\boldsymbol{0})$.

Estimation of covariance matrix

We assume that $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_N$ are mutually independent multivariate Poisson random vectors with unknown parameter λ_i. Given a sequence of the random vectors, we shall estimate the mean vector and the covariance matrix, in this section.

A sequence of multivariate Poisson random vectors

$$\boldsymbol{X}_k=(X_{1k}, \cdots, X_{nk}) \qquad (k=1, 2, \cdots, N)$$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	sum
X_{1k}	1	0	0	0	1	0	2	1	1	0	0	0	0	1	0	0	0	0	1	0	8
X_{2k}	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	1	0	0	1	6
X_{3k}	1	3	0	0	1	0	1	1	1	1	0	2	0	0	0	0	1	2	1	1	16
X_{4k}	3	2	2	0	1	0	0	2	0	1	0	1	0	0	0	0	1	2	3	1	19
X_{5k}	5	4	5	15	10	7	5	11	6	4	9	11	11	3	10	7	14	8	5	7	157
X_{6k}	8	8	8	14	12	8	6	9	2	4	3	11	8	5	7	11	10	6	4	2	146
X_{7k}	1	3	2	0	0	0	0	1	0	1	1	2	2	1	0	0	1	1	1	0	17
X_{8k}	2	3	1	1	0	1	0	1	0	2	4	1	2	2	0	0	2	1	0	1	24
sum	21	23	18	30	25	16	14	27	10	14	17	29	23	13	17	18	30	20	15	13	

This sequence of random vectors is of $n=8$ dimensional Poisson distribution and the sample size $N=20$. In this paper, we use

$$\bar{X}_i=\frac{1}{N}\sum_{k=1}^{N} X_{ik}, \quad S_{ij}=\frac{1}{N}\sum_{k=1}^{N}(X_{ik}-\bar{X}_i)(X_{jk}-\bar{X}_j)$$

$$=\frac{1}{N}\sum_{k=1}^{N} X_{ik}X_{jk}-\bar{X}_i\bar{X}_j \qquad (1\leqq i, j\leqq n),$$

where we get easily $S_{ij}=S_{ji}$.

Estimated mean values and standard deviations

$$\bar{X}_i, \quad S_i^2=\frac{1}{N}\sum_{k=1}^{N}(X_{ik}-\bar{X}_i)^2 \qquad (i=1, 2, \cdots, n)$$

$\bar{X}_1=0.4$	$\bar{X}_2=0.3$	$\bar{X}_3=0.8$	$\bar{X}_4=0.95$
$S_1=0.5831$	$S_2=0.4583$	$S_3=0.8124$	$S_4=1.0235$
$\bar{X}_5=7.85$	$\bar{X}_6=7.3$	$\bar{X}_7=0.85$	$\bar{X}_8=1.2$
$S_5=3.3208$	$S_6=3.2573$	$S_7=0.8529$	$S_8=1.0770$

Sample mean of the sum	19.65
Standard deviation of the sum	5.9521

Estimated covariance matrix

The estimated covariance matrix is given by

$S=[S_{ij}]$

$$=\begin{pmatrix} 0.34 & -0.02 & 0.03 & 0.07 & -0.64 & -0.32 & -0.14 & -0.23 \\ -0.02 & 0.21 & 0.06 & 0.015 & 0.145 & -0.14 & 0.045 & 0.09 \\ 0.03 & 0.06 & 0.66 & 0.44 & -0.43 & -0.09 & 0.32 & 0.09 \\ 0.07 & 0.015 & 0.44 & 1.05 & -0.86 & -0.085 & 0.49 & 0.06 \\ -0.64 & 0.145 & -0.43 & -0.86 & 11.03 & 6.445 & -0.42 & -0.12 \\ -0.32 & -0.14 & -0.09 & -0.085 & 6.445 & 10.61 & 0.145 & -0.56 \\ -0.14 & 0.045 & 0.32 & 0.49 & -0.42 & 0.145 & 0.73 & 0.53 \\ -0.23 & 0.09 & 0.09 & 0.06 & -0.12 & -0.56 & 0.53 & 1.16 \end{pmatrix}$$

The main components

$$S_{ii}=S_i^2=\frac{1}{N}\sum_{k=1}^{N}(X_{ik}-\bar{X}_i)^2 \qquad (i=1, 2, \cdots, n)$$

will be refined by using estimated mean vector

$$\bar{X}_i=\frac{1}{N}\sum_{k=1}^{N}X_{ik} \qquad (i=1, 2, \cdots, n).$$

And the estimated sample covariances in S with negative values are not natural, because all parameters $\sum_{i_i=i_j=1}\lambda_i$ estimated by S_{ij} must be nonnegative. Therefore we shall refine the estimeter S as $S^+=[S_{ij}^+]$ where S_{ij}^+ equals to S_{ij} iff $S_{ij}\geqq 0$ and 0 iff $S_{ij}<0$. And a more refined estimater will be given by $\tilde{S}=[\tilde{S}_{ij}]$ where $\tilde{S}_{ij}$ is defined by

$$\tilde{S}_{ij}=\begin{cases} \bar{X}_i & \text{iff } i=j, \\ S_{ij}^+ & \text{iff } i\neq j. \end{cases}$$

Because the parameter estimated by S_{ii} is the variance value of X_i and equals to the mean value of X_i. And we get easily $S_{ii}^+=S_{ii}\geqq 0$, $S_{ij}^+=S_{ji}^+$ and $\tilde{S}_{ij}=\tilde{S}_{ji}$. Then we have

$$\tilde{S}=\begin{pmatrix} 0.4 & 0 & 0.03 & 0.07 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.06 & 0.015 & 0.145 & 0 & 0.045 & 0.09 \\ 0.03 & 0.06 & 0.8 & 0.44 & 0 & 0 & 0.32 & 0.09 \\ 0.07 & 0.015 & 0.44 & 0.95 & 0 & 0 & 0.49 & 0.06 \\ 0 & 0.145 & 0 & 0 & 7.85 & 6.445 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.445 & 7.3 & 0.145 & 0 \\ 0 & 0.045 & 0.32 & 0.49 & 0 & 0.145 & 0.85 & 0.53 \\ 0 & 0.09 & 0.09 & 0.06 & 0 & 0 & 0.53 & 1.2 \end{pmatrix}$$

Conclusion of this section.

1. The unknown mean values of $(X_1, \cdots, X_n)$

$$EX_1 = \sum_{\imath_1=1} \lambda_\imath, \cdots, EX_n = \sum_{\imath_n=1} \lambda_\imath$$

are to be estimated by

$$\bar{X}_1, \cdots, \bar{X}_n,$$

where

$$\bar{X}_\imath = \frac{1}{N} \sum_{k=1}^{N} X_{ik} \qquad (\imath = 1, 2, \cdots, n).$$

2. The unknown covariance matrix of $(X_1, \cdots, X_n)$

$$[\mathrm{Cov}(X_\imath, X_j)]; \qquad \mathrm{Cov}(X_\imath, X_j) = \sum_{\imath_i = \imath_j = 1} \lambda_\imath$$

will be estimated by the sample covariance matrix $\boldsymbol{S}$ or $\boldsymbol{S}^+$.

3. And a more refined covariance matrix will be given by $\tilde{\boldsymbol{S}}$.

References

[1] Feller, W., An introduction to probability theory and its applications, Vol. 1, sec. ed. 6-th print (1961).

[2] Kawamura, K., The structure of trivariate Poisson distribution, Vol. 28, No. 1, 1-8 (1976). Kōdai Math. Sem. Rep.

[3] Kendall, M.G. and A. Stuart, The advanced theory of statistics, Vol. 1. Distribution theory, second ed. Criffin (1963).

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