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The structure of nonseparable Banach spaces with uncountable unconditional bases

Carlos Finol and Marek Wójtowicz

Abstract. Let X be a Banach space with an uncountable unconditional Schauder basis, and let Y be an arbitrary nonseparable subspace of X. If X contains no isomorphic copy of $\ell_1(J)$ with J uncountable then (1) the density of Y and the weak*-density of Y^* are equal, and (2) the unit ball of X^* is weak* sequentially compact. Moreover, (1) implies that Y contains large subsets consisting of pairwise disjoint elements, and a similar property holds for uncountable unconditional basic sets in X.

La estructura de los espacios de Banach no separables que tienen bases incondicionales no numerables

Resumen. Sea X un espacio de Banach con una base incondicional de Schauder no numerable, y sea Y un subespacio arbitrario no separable de X. Si X no contiene una copia isomorfa de $\ell_1(J)$ con J no numerable entonces (1) la densidad de Y y la débil*-densidad de Y^* son iguales, y (2) la bola unidad de X^* es débil* sucesionalmente compacta. Además, (1) implica que Y contiene subconjuntos grandes formados por elementos disjuntos dos a dos, y una propiedad similar se verifica para las bases incondicionales no numerables de X.

1 Introduction

Throughout this paper X will denote a Banach space with an unconditional basis $(x_\gamma)_{\gamma\in\Gamma}$, where Γ is an uncountable set, and Y will be its nonseparable closed linear subspace. The best known examples of such spaces X are $\ell_p(\Gamma)$, $1 \leq p < \infty$, and $c_0(\Gamma)$ (another examples are addressed in [10, 15, 20, 24]). By $\ell_1(\aleph_1)$ we denote the space $\ell_1(\Gamma)$ with $\mathrm{card}(\Gamma) = \aleph_1$.

This paper deals with the structure of nonseparable subspaces of X whose study is motivated by the result below, included implicitly in the proofs of two results by Rodriguez-Salinas ([15, Proposition 2]) and Granero ([5, Proposition 1]):

(RSG) *Let Y be a nonseparable subspace of X with $\chi(Y) = \chi^*(Y^*)$. Then Y contains a set of the cardinality of $\chi(Y)$ consisting of elements of norm one with pairwise disjoint supports.*

(Here and in what follows $\chi(Y)$ and $\chi^*(Y^*)$, respectively, denote the density and weak*-density character of Y and Y^*, respectively, whose definition is given below.) Since the condition $\chi(Y) = \chi^*(Y^*)$ holds true for Y reflexive or weakly compactly generated, the above result gives almost immediately a description of complemented subspaces of $\ell_p(\Gamma)$, with $1 < p < \infty$, and $c_0(\Gamma)$ (see [15, 5]), generalizing Pełczyński's

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classical theorem asserting that *every infinite dimensional complemented subspace of ℓ_p, with $1 \leq p \leq \infty$, (respectively, c_0) is isomorphic to ℓ_p* (respectively, c_0) (see [13, Theorem 2.a.3]). One should mention here that in 1966 a similar result for $\ell_1(\Gamma)$ was obtained by Köthe [12].

In the next section we give a characterization of those spaces X where the condition $\chi(Y) = \chi^*(Y^*)$ holds for *all* subspaces Y of X (Theorem 1); this appears to be equivalent to the non-containment (by X) of an isomorphic copy of the space $\ell_1(\aleph_1)$ or, under the continuum hypothesis, to the weak* sequential compactness of the unit ball of X^* (Proposition 1). The latter equivalence relates to the problem posed in 1977 by Rosenthal [17]: *Suppose that the dual unit ball B_{W^*} of a Banach space W is not weak* sequentially compact; can we then conclude that W contains a subspace isomorphic to ℓ_1?*, which was answered in 1978 in the negative by Hagler and Odell [7] (cf. [6, 11]). Moreover, in 1977 Haydon constructed a Banach space Z (of the type $C(K)$) not containing isomorphic copies of $\ell_1(\aleph_1)$ such that B_{Z^*} is not weak* sequentially compact [9]. This shows that our equivalence does depend on the structure of the given Banach space (for other results concerning the embeddability of $\ell_1(\aleph_1)$ into Banach spaces see [19] and the references given therein). The characterization given in Theorem 1 allows us to generalize, by (RSG), the cited results of Rodriguez-Salinas and Granero (Theorem 2); it also shows that if X contais no copy of $\ell_1(\aleph_1)$ then Y, containing large (unconditional) basic set consisting of pairwise disjoint elements, has "big" unconditional structure (see the comment in ([4, p. 396]) on atomic Banach lattices). In Section 3, complementing the previous theorems, we show that every uncountable unconditional basic set $(y_j)_{j\in J}$ in X contains a subset of the same cardinality as J consisting of pairwise disjoint elements provided that $(y_j)_{j\in J}$ has no uncountable subsets of the ℓ_1-type (Theorem 3).

The restrictive role of $\ell_1(J)$, with J uncountable, in Theorems 1, 2, and 3 explains the following result obtained in 1975 by Troyanski [21]

(T) *Let the basis $(x_\gamma)_{\gamma\in\Gamma}$ of X be symmetric. If X has a subspace isomorphic to $\ell_1(J)$ [resp., $c_0(J)$] for some uncountable set J, then the basis is equivalemt to the natural basis of $\ell_1(\Gamma)$ [resp., $c_0(\Gamma)$],*

and generalized in 1988 by Drewnowski [3] who showed that if the basis in (T) is merely unconditional then it contains "large" subbases of the ℓ_1-[resp., c_0-]type. Therefore, in the context of Troyanski's result, the last section is devoted only to the structure of nonseparable subspaces of $X = \ell_1(\Gamma)$, and the basic tool we use in our studies is the notion of ε-disjoint systems. In Theorem 4 we prove the existence, for every $\varepsilon > 0$, of such systems in X, which allows one to strengthen the above-cited result of Köthe (Corollary 6) and to give its shorter proof (Corollary 7).

Our terminology and notation is that of [13] and [20]. All subspaces are assumed to be linear and closed. Recall that a family $(x_\gamma)_{\gamma\in\Gamma}$ in X is said to be an (unconditional) basis of X if, for every $x \in X$ there is a unique family of scalars $(t_\gamma)_{\gamma\in\Gamma}$ such that $x = \sum_{\gamma\in\Gamma} t_\gamma x_\gamma$ (unconditional convergence). By $(x^*_\gamma)_{\gamma\in\Gamma}$ we denote the dual family, biorthogonal to $(x_\gamma)_{\gamma\in\Gamma}$; then

$$x = \sum_{\gamma\in\Gamma} x^*_\gamma(x) x_\gamma \qquad \text{for every } x \in X, \tag{1}$$

and the support of $x \in X$ is defined as $\operatorname{supp}(x) := \{\gamma \in \Gamma : x^*_\gamma(x) \neq 0\}$. We say that two elements $u, v \in X$ are *disjoint* if their supports, $\operatorname{supp}(u)$ and $\operatorname{supp}(v)$, are disjoint subsets of Γ. From (1) it follows that every element $x^* \in X^*$ has the representation

$$x^* = \sum_{\gamma\in\Gamma} x^*(x_\gamma) x^*_\gamma \qquad \text{(weak*-convergence)}, \tag{2}$$

which allows one to define the support of x^* as $\operatorname{supp}(x^*) := \{\gamma \in \Gamma : x^*(x_\gamma) \neq 0\}$. The basis $(x_\gamma)_{\gamma\in\Gamma}$ is called *symmetric* if, for every sequence (γ_n) in Γ the basic sequence (x_{γ_n}) is symmetric in the usual sense ([13, p. 113]). We say that a family $(v_j)_{j\in J}$ is a basic set in X if it is a basis of the closed linear span of this family (denoted by $[v_j]_{j\in J}$). Two basic sets $(u_j)_{j\in J}$, $(v_j)_{j\in J}$ in a Banach space W are said to be equivalent if the linear operator $G : [u_j]_{j\in J} \to [v_j]_{j\in J}$ of the form $G(u_j) = v_j$ is an isomorphism.

An isomorphism T between two Banach spaces V and W is said to be an $(1+\varepsilon)$-isometry provided that $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$.

A subspace W_0 of a Banach space W is said to be complemented [k-complemented for some $k \ge 1$, resp.] in W if it is the range of a continuous projection P [with $\|P\| = k$, resp.]. If F is a nonempty subset of Γ, then X_F denotes the subspace of X consisting of the elements with supports included in F, and P_F denotes the continuous projection from X onto X_F of the form $P_F x = x \cdot \mathbf{1}_F$, where $\mathbf{1}_F$ is the characteristic function of F. Notice that if $X = \ell_p(\Gamma)$, then the spaces X_F and $\ell_p(F)$ are isometric. From (2) it easily follows that for every $x^* \in X^*$ the element $x_F^* := \sum_{\gamma\in F} x^*(x_\gamma)x_\gamma^*$ (weak*-convergence) is well defined, and hence the operator $\widehat{P}_F$ on X^* of the form $\widehat{P}_F(x^*) := x_F^*$ is a continuous projection (in fact, $\widehat{P}_F = P_F^*$).

If Y is a subspace of X then S_Y denotes its unit sphere, $\mathrm{supp}(Y)$ stands for the support of Y ($= \cup_{y\in Y}\mathrm{supp}(y)$), and $\chi(Y)$ [resp., $\chi^*(Y^*)$] denotes the density character of Y [resp., the weak*-density character of Y^*], i.e., the smallest cardinal α such that Y [resp., Y^*] contains a subset A with $\mathrm{card}(A) = \alpha$ and such that A is linearly norm-dense in Y [resp., weak*-dense in Y^*]. Recall that $\chi^*(Y^*)$ equals also $\min\{\mathrm{card}(\mathcal{F}) : \mathcal{F} \subset Y^* \text{ and } \mathcal{F} \text{ is total over } Y\}$, and that $\chi^*(Y^*) \le \chi(Y)$ ([20, p. 599]).

2 The weak* cardinality property and weak* sequential compactness

Following Vašak [22], we say that a Banach space W has *the weak* cardinality property* (W*CP, for short) if, for every subspace V of W we have $\chi(V) = \chi^*(V^*)$. In [22, Corollary 2] Vašak proved that every weakly countably determined (in particular, every weakly compactly generated (WCG)) Banach space possesses this property. In the theorem below we give a characterization of the class of those X's which have the W*CP.

Theorem 1 *Let X be a Banach space with an uncountable unconditional basis $(x_\gamma)_{\gamma\in\Gamma}$. Then X has the W*CP if and only if one of the following equivalent conditions is satisfied:*

(i) X contains no isomorphic copy of the space $\ell_1(\aleph_1)$.

(ii) No subbasis $(x_\gamma)_{\gamma\in J}$, with $\mathrm{card}(J) = \aleph_1$, is equivalent to the standard basis of $\ell_1(\aleph_1)$.

(iii) Every element of X^* has countable support.

PROOF. *(i)* ⇒ *(ii)*. Obvious.

non-*(iii)* ⇒ non-*(ii)*. Let $J := \mathrm{supp}(x^*)$ be uncountable. From (2) it follows there exist an uncountable subset J_0 of J and $\varepsilon_0 > 0$ such that $|x^*(x_\gamma)| > \varepsilon_0$ for all $\gamma \in J_0$, whence we obtain (since the basis $(x_\gamma)_{\gamma\in\Gamma}$ is unconditional) that the series $\sum_{\gamma\in J_0} x_\gamma^*(x)$ converges unconditionally for every $x \in X$ and defines an element x_0^* of X^*. Put $W := [x_\gamma]_{\gamma\in J_0}$. For every $w \in W$ of the form $w = \sum_{\gamma\in J_0} t_\gamma x_\gamma$ we thus have $x_0^*(w) = \sum_{\gamma\in J_0} t_\gamma$ (unconditional convergence), and so $\sum_{\gamma\in J_0} |t_\gamma| < \infty$. It follows that the basis of W and the standard basis of $\ell_1(J)$ are equivalent, and hence W is an isomorphic copy of $\ell_1(J_0)$, with J_0 uncountable.

*(iii) implies X has W*CP*. We may assume that a subspace Y of X is nonseparable. Let Λ denote the class $\{\mathcal{F} \subset X^* : \mathcal{F} \text{ is total over } Y\}$. By the Hahn-Banach theorem, $\chi^*(Y^*) = \min\{\mathrm{card}(\mathcal{F}) : \mathcal{F} \in \Lambda\} = \mathrm{card}(\mathcal{F}_0)$ for some $\mathcal{F}_0 \in \Lambda$. Now we define two subsets of Γ:

$$A = \bigcup_{x^*\in\mathcal{F}_0} \mathrm{supp}(x^*), \quad \Gamma_Y = \mathrm{supp}(Y).$$

From *(iii)* we obtain that

$$\mathrm{card}(A) \le \mathrm{card}(\mathcal{F}_0) = \chi^*(Y^*), \tag{3}$$

and from the defintion of the sets A and Γ_Y we get $x^*(P_{\Gamma_Y \setminus A} y) = 0$, and hence $x^*(P_A y) = x^*(y)$ for every $y \in Y$ and $x^* \in \mathcal{F}_0$. It follows that the set $\mathcal{F}_0$ is total over $P_A(Y)$; thus the operator P_A restricted to Y is injective which, together with (3), gives

$$\chi(Y) \leq \operatorname{card}(Y) = \operatorname{card}(P_A(Y)) = \operatorname{card}(\Gamma_Y \cap A) \leq \chi^*(Y^*).$$

Finally, $\chi(Y) = \chi^*(Y^*)$, as claimed.

*If X has W*CP then (i) holds.* Assume that X contains an isomorphic copy of $\ell_1(J)$ with $\operatorname{card}(J) = \aleph_1$. The remaining part of the proof depends on the observation that *if W is a separable Banach space then for every infinite dimensional subspace Y of W^* we have $\chi^*(Y^*) = \aleph_0$*, which we apply to the space $W = C[0,1]$ whose dual contains $Y := \ell_1([0,1])$. ∎

As a by-product of the equivalence of *(i)* and *(ii)* in Theorem 1 we obtain the Troyanski's result (T) (see Introduction) which immediately gives

Corollary 1 *Let X be a Banach space with an uncountable symmetric basis $(x_\gamma)_{\gamma\in\Gamma}$. Then X has the W*CP if and only if the basis is not equivalent to the standard basis of $\ell_1(\Gamma)$.*

The above Corollary applies to "big" Orlicz spaces $h_\varphi(\Gamma)$, where φ is an Orlicz function (for exact definition of $h_\varphi(\Gamma)$ see e.g. [10]), giving that $h_\varphi(\Gamma)$ *has the W*CP if and only if φ is not equivalent to the linear function $\psi(t) = t$ at* 0.

The next theorem is an immediate consequence of Theorem 1 and the result (RSG); it applies to the spaces $h_\varphi(\Gamma)$, in particular to $\ell_p(\Gamma)$, $1 < p < \infty$, and $c_0(\Gamma)$ (cf. [5, 15]). It also complements a similar result obtained in [15, Proposition 2] for Y reflexive.

Theorem 2 *If X contains no isomorphic copies of $\ell_1(\aleph_1)$, then every nonseparable subspace Y of X contains a set of the cardinality of $\chi(Y)$ consisting of pairwise disjoint elements of norm one.*

The proposition below deals with weak* sequential compactness of the dual unit ball of X^*. The proof of the first implication is a discrete version of the proof given in 1968 by Lozanovskii [14] for a class of Banach lattices (cf. [23, Theorem 4.4]), and is included here for the convenience of the reader who is not familiar with the theory of Banach lattices (one should also note that the original proof works for *real* Banach lattices).

Proposition 1 *Let X be a Banach space with an uncountable unconditional basis. Then statement (iii) in Theorem 1 implies that*

(iv) the dual unit ball B_{X^*} of X^* is weak sequentially compact.*

Moreover, under the continuum hypothesis (CH) statements (i) and (iv) are equivalent.

PROOF. *(iii)⇒ (iv)* Let (x_n^*) be a sequence in B_{X^*}, and put $V := \{x^* \in X^* : \operatorname{supp}(x^*) \subset A\}$, where $A = \bigcup_{n=1}^\infty \operatorname{supp}(x_n^*)$. We obviously have $V = \widehat{P}_A(X^*)$, and $x_n^* \in V$ for all n's. We set $Y := P_A(X)$. Since A is countable, the space Y is separable. It is easy to check that the annihilator $Y^\perp$ of Y in X^* equals $\widehat{P}_{\Gamma\setminus A}(X^*)$, and hence Y^* can be identified with $\widehat{P}_A(X^*)(= V)$. The separability of Y implies that the ball B_{Y^*} is $\sigma(Y^*, Y)$-sequentially compact, and using the above identification of Y^* and V, we can find a $\sigma(X^*, X)$-convergent subsequence $(x_{n_k}^*)$ of (x_n^*).

(iv)⇒(i) (under CH; cf [23, pp. 78–79]). It is known that condition *(iv)* implies X cannot contain isomorphic copies of $\ell_1(\mathbb{R})$, where $\mathbb{R}$ denotes the set of all real numbers (see e.g. [1, p. 226]), and hence, under CH, the space X cannot contain any copy of $\ell_1(\aleph_1)$. ∎

From Corollary 1 and Proposition 1 we immediately obtain

Corollary 2 *Let the basis $(x_\gamma)_{\gamma\in\Gamma}$ of X be symmetric. Under the continuum hypothesis, the dual unit ball of X is weak* sequentially compact if and only if the basis is not equivalent to the standard basis of $\ell_1(\Gamma)$.*

3 Uncountable unconditional basic sets in X

In the theorem below we show that large unconditional basic sets in X have "nice" structure (the conclusion *(i)* below was obtained in [15, Proposition 6] under more restrictive assumption); its application is given in three corollaries following it.

Theorem 3 *Let X be a Banach space with an uncountable unconditional basis, and let $(y_j)_{j\in J}$ be an uncountable unconditional normalized basic set in X. Then the following alternative holds:*

(i) There is a subset J_0 of J with $\mathrm{card}(J_0) = \mathrm{card}(J)$ such that the elements of $(y_j)_{j\in J_0}$ are pairwise disjoint.

(ii) For every infinite cardinal number $\alpha_0 < \mathrm{card}(J)$ there exists a subset J_0 of J with $\mathrm{card}(J_0) > \alpha_0$ such that $(y_j)_{j\in J_0}$ is equivalent to the unit vector basis of $\ell_1(J_0)$.

In particular, the conclusion of part (i) holds if $(y_j)_{j\in J}$ is equivalent to the unit vector basis of $c_0(J)$ or $\ell_p(J)$, with $1 < p < \infty$.

We would like to comment on the above property *(ii)* in Theorem 3. One should note that it is impossible, in general, to choose a pairwise disjoint subsequence even from a *sequence* (y_n) in X equivalent to the unit vector basis of ℓ_1: it is enough to take any $\gamma_0 \in \Gamma \setminus \bigcup_{n=1}^{\infty} \mathrm{supp}(y_n)$ and consider the sequence (y_n'), equivalent to (y_n), of the form $y_n' = y_n + x_{\gamma_0}$, $n = 1, 2, \ldots$. On the other hand, it is known that a Banach space with an unconditional Schauder basis contains a copy of ℓ_1 iff it contains a normalized block basic sequence of the basis equivalent to the unit vector basis of ℓ_1 (see e.g. [13, Theorem 1.c.9]).

The proof of Theorem 3 depends essentially on Lemma 1 below and it is a modification of the arguments used in the proof of [2, Lemma 3]. To shorten the text we say that a family $(y_j)_{j\in J}$ of non-null elements of a Banach space W is totally non-$\ell_1(\alpha)$, where α is an infinite cardinal number with $\alpha \le \mathrm{card}(J)$ (TN$\ell_1(\alpha)$, for short) if, for every subset C of J with $\mathrm{card}(C) = \alpha$ there is a family $(t_j)_{j\in C}$ of scalars such that the series $\sum_{j\in C} t_j y_j$ converges unconditionally, but $\sum_{j\in C} |t_j| = \infty$. (For $\alpha = \aleph_0$ this notion coincides with the notion of a totally non-ℓ_1 family considered by Drewnowski in [2].) If $(y_j)_{j\in J}$ is a basic set in X then it is totally non-$\ell_1(\alpha)$ whenever, for every subset C of J with $\mathrm{card}(C) = \alpha$, the basic set $(y_j)_{j\in C}$ is *not equivalent* to the standard basis of $\ell_1(C)$. We have that if $\alpha_1 < \alpha_2$, then TN$\ell_1(\alpha_1)$ implies TN$\ell_1(\alpha_2)$; thus, if $(y_j)_{j\in J}$ is totally non-ℓ_1 then it is TN$\ell_1(\alpha)$ for every infinite $\alpha \le \mathrm{card}(J)$.

Lemma 1 *Let X be a Banach space with an uncountable unconditional basis, let α_0 be an infinite cardinal number, and let J be a set with $\mathrm{card}(J) > \alpha_0$. If, for every cardinal α with $\alpha_0 < \alpha \le \mathrm{card}(J)$ a family $(y_j)_{j\in J}$ of non-null elements of X is $TN\ell_1(\alpha)$, then there exists a subset J_0 of J with $\mathrm{card}(J_0) = \mathrm{card}(J)$ such that the elements of the subfamily $(y_j)_{j\in J_0}$ are pairwise disjoint.*

PROOF. It is an immediate consequence of the following combinatorial fact, the proof of which is similar to the proof of [2, Lemma 2] and therefore omitted:

> *Let J be an uncountable set, and let $\mathbf{m}$ be an infinite cardinal number with $\mathbf{m} < \mathrm{card}(J)$. Let $(S_j)_{j\in J}$ be a family of subsets of a set Γ such that*:
>
> (a) *for every $j \in J$ we have $\mathrm{card}(S_j) \le \mathbf{m}$, and*
>
> (b) *for every $\gamma \in \Gamma$ we have $\mathrm{card}\{j \in J : \gamma \in S_j\} \le \mathbf{m}$.*
>
> *Then there exists a subset J_0 of J with $\mathrm{card}(J_0) = \mathrm{card}(J)$ such that the elements of the family $(S_j)_{j\in J_0}$ are pairwise disjoint.* ■

THE PROOF OF THEOREM 3. Assume condition *(ii)* is false. Then $(y_j)_{j\in J}$ is TN$\ell_1(\alpha)$ for all cardinal numbers α with $\alpha_0 < \alpha \le \mathrm{card}(J)$. Now we apply Lemma 1. ■

The two below corollaries of Theorem 3 show that uncountable unconditional basic sets in the spaces $\ell_p(\Gamma)$ and $c_0(\Gamma)$ contain long symmetric subsets. (One should note here that subspaces with symmetric uncountable bases in Orlicz spaces $\ell_\varphi(\Gamma)$ were described by Rodriguez-Salinas [16]; see also [10].)

The first corollary is now obvious (the case $X = c_0(\Gamma)$ and $X = \ell_p(\Gamma)$, with $1 < p < \infty$, was studied in [5] and [15], respectively).

Corollary 3 *Let X be a Banach space with an uncountable unconditional basis, and let $(y_j)_{j\in J}$ be an uncountable unconditional normalized basic set in X. If X contains no isomorphic copy of the space $\ell_1(\aleph_1)$, then there exists a subset J_0 of J with $\mathrm{card}(J_0) = \mathrm{card}(J)$ such that the elements of $(y_j)_{j\in J_0}$ are pairwise disjoint.*

Each of the either cases of Theorem 3 proves the next corollary.

Corollary 4 *Let $(y_j)_{j\in J}$ be an uncountable unconditional normalized basic set in $\ell_1(\Gamma)$. Then for every infinite cardinal number $\alpha_0 < \mathrm{card}(J)$ there exists a subset J_0 of J with $\alpha_0 < \mathrm{card}(J_0)$ and such that the basic subset $(y_j)_{j\in J_0}$ is equivalent to the natural symmetric basis of $\ell_1(J_0)$.*

It is known that every symmetric basic sequence in the sequence space ℓ_p (or c_0) is equivalent to the unit vector basis of the given space [13, Remark following Proposition 3.b.5]. From Corollaries 3 and 4 we immediately obtain a similar property for the spaces $\ell_p(\Gamma)$ and $c_0(\Gamma)$.

Corollary 5 *Let $X(\Gamma)$ denote the space $\ell_p(\Gamma)$, $1 \le p < \infty$, or $c_0(\Gamma)$. Every uncountable, normalized and symmetric basic set $(y_j)_{j\in J}$ in $X(\Gamma)$ is equivalent to the natural basis of $X(J)$.*

4 ε-disjoint systems in X

The main result of this section is motivated by the remark following Theorem 3 (see also the proof of Theorem 1 in [3]). Here we show that the structure of infinite dimensional subspaces of X can also be studied effectively by the use of "almost" disjoint elements.

Let $\varepsilon \in [0, 1)$, and let Y be a subspace of X. We say that two elements $y_1, y_2 \in X \setminus \{0\}$ are *ε-disjoint* if there exist disjoint elements $u_1, u_2 \in X \setminus \{0\}$ such that $\|x_i - u_i\| \le \varepsilon$, $i = 1, 2$. A system $(y_j)_{j\in J} \subset S_Y$ is said to be ε-disjoint provided that there exists a system $(u_j)_{j\in J}$ of pairwise disjoint elements of X with $\|y_j - u_j\| \le \varepsilon$ for all $j \in J$. A concrete ε-disjoint system $(y_j)_{j\in J}$ with the corresponding pairwise disjoint system $(u_j)_{j\in J}$ will be denoted by $(y_j, u_j)_{j\in J}$.

Remark 1 *It is obvious that every 0-disjoint system is pairwise disjoint.*

Remark 2 *From inequality $\min\{|a| + |b|, |c|\} \le \min\{|a|, |c|\} + \min\{|b|, |c|\}$, for all scalars a, b, c (see [18, Corollary, p. 53]), we easily obtain that $\min\{|a|, |b|\} \le 2|a - u| + |b - v| + \min\{|u|, |v|\}$ for all a, b, u, v. Hence, if the elements $y_i = \sum_{\gamma\in\Gamma} t_\gamma^{(i)} x_\gamma$, $i = 1, 2$, are ε-disjoint, with corresponding disjoint elements $u_i = \sum_{\gamma\in\Gamma} s_\gamma^{(i)} x_\gamma$, $i = 1, 2$, then $\|\sum_{\gamma\in\Gamma} \min\{|t_\gamma|, |s_\gamma|\} x_\gamma\| \le 2K\|y_1 - u_1\| + K\|y_2 - u_2\| \le 3K\varepsilon$, where K is the basis constant. It follows that the supports of two ε-disjoint elements intersect at "norm-min-small" subsets.*

Remark 3 *Let $\varepsilon \in (0, 1/2)$, and let a system $(y_j, u_j)_{j\in J}$ be ε-disjoint in $\ell_1(\Gamma)$. Then $(y_j)_{j\in J}$ is equivalent to the standard basis of $\ell_1(J)$, with $\|u_j\| \in (1 - \varepsilon, 1 + \varepsilon)$ for all j's, and similarly for $(y_j)_{j\in J}$:*

$$\sum_{j\in J} |t_j| \ge \|\sum_{j\in J} t_j y_j\| \ge (1 - 2\varepsilon) \sum_{j\in J} |t_j|,$$

for all $(t_j)_{j\in J} \in \ell_1(J)$. Proving as in [13, Proposition 1.a.9 and Theorem 2.a.3], we obtain that for $\varepsilon \in (0, \sqrt{2} - 1)$, the spaces $[y_j]_{j\in J}$ and $[u_j]_{j\in J}$ are $(1 + \varepsilon)$-isometric and δ-complemented in $\ell_1(\Gamma)$, where $\delta \le 1 + \frac{2\varepsilon}{1 - 2\varepsilon - \varepsilon^2}$.

The main result of this section reads as follows.

Theorem 4 *Let Y be an infinite dimensional subspace of X. Then for every $\varepsilon \in (0,1)$ the space Y contains an ε-disjoint system $(y_j, u_j)_{j\in J}$ with $\operatorname{card}(J) = \chi(Y)$ and such that $\operatorname{supp}(u_j) \subset \operatorname{supp}(y_j)$ for all $j \in J$.*

PROOF. Put $F = \operatorname{supp}(Y)$. We first consider the case $\chi(Y) = \aleph_0$. Then F is countable, and hence Y is a subspace of the space X_F with the countable unconditional basis $(x_\gamma)_{\gamma\in F}$. By [13, Proposition 1.a.11], Y contains an ε-disjoint countable infinite system $(y_n, u_n)_{n\geq 1}$ with $\operatorname{supp}(u_n) \subset \operatorname{supp}(y_n)$ for all n's.

Now assume $\chi(Y) > \aleph_0$, and let $\mathcal{E}$ be the class of all ε-disjoint systems $(y_j, v_j)_{j\in J}$ with $\operatorname{card}(J) \geq \aleph_0$ and $\operatorname{supp}(u_j) \subset \operatorname{supp}(y_j)$ for all $j \in J$. By the previous case, $\mathcal{E} \neq \emptyset$. We introduce the following partial ordering in $\mathcal{E}$: $(y'_j, u'_j)_{j\in J} \preceq (y''_l, u''_l)_{l\in L}$ iff $J \subset L$ and $y'_j = y''_j$ and $u'_j = u''_j$ for all $j \in J$, and let $(y^M_j, u^M_j)_{j\in J_M}$ be a maximal element in $\mathcal{E}$. We define the cardinal number $\lambda_M := \operatorname{card}(J_M)$, and we put $I_M := \bigcup_{j\in J_M} \operatorname{supp}(u^M_j)$. Then we have

$$\lambda_M = \operatorname{card}(I_M) \leq \chi(Y). \tag{4}$$

We claim we have two equalities in (4). Assume this is not so, i.e., $\lambda_M < \chi(Y)$. Then we must have:

$$\text{for every } \eta > 0 \text{ there is } y_\eta \in S_Y \text{ with } \|P_{I_M} y_\eta\| < \eta \tag{*}$$

(in the oposite case the number $\inf_{y\in S_Y} \|P_{I_M} y\|$ were positive, and hence the operator P_{I_M} restricted to Y would be injective; this and (4) would then imply that $\chi(Y) \leq \operatorname{card}(Y) = \operatorname{card}(P_{I_M}(Y)) = \operatorname{card}(F \cap I_M) \leq \lambda_M < \chi(Y)$, a contradiction). Now choose $y_\eta \in S_Y$ fulfilling (*) with $\eta = \varepsilon$, and put $w_\eta = P_{F\setminus I_M} y_\eta$; we see that $\operatorname{supp}(w_\eta) \subset \operatorname{supp}(y_\eta)$. Next, from (*) we obtain $\|y_\eta - w_\eta\| < \varepsilon$, and since $\operatorname{supp}(w_\eta) \cap I_M = \emptyset$, we also have that for every $j \in J_M$ the elements u_η and u_j are disjoint. It follows that for the set $J^\eta := J_M \cup \{\eta\}$ the system $(y_j, u_j)_{j\in J^\eta}$ is ε-disjoint, and it strictly dominates $(y_j, u_j)_{j\in J}$. This contradiction proves our claim and finishes the proof. ■

From Theorem 4 and Remark 3 we get

Corollary 6 *Let Y be a nonseparable subspace of $\ell_1(\Gamma)$. Then for every $\varepsilon \in (0, \sqrt{2} - 1)$ the space Y contains an $(1+\varepsilon)$-isometric and δ-complemented copy of $\ell_1(J)$, where $\operatorname{card}(J) = \chi(Y)$ and $\delta \leq 1 + \frac{2\varepsilon}{1-2\varepsilon-\varepsilon^2}$.*

Consequently, from Corollary 6 and Pełczyński's decomposition method we obtain Köthe's result ([12, Theorem (6), p. 187]):

Corollary 7 *Every complemented subspace of $\ell_1(\Gamma)$ is isomorphic to $\ell_1(J)$ for some $J \subset \Gamma$.*

References

[1] Diestel, J. (1984). *Sequences and Series in Banach Spaces*, Springer, Berlin.

[2] Drewnowski, L. (1987). *On symmetric bases in nonseparable Banach spaces*, Studia Math., **85**, 157-161.

[3] Drewnowski, L. (1988). *On uncountable unconditional bases in Banach spaces*, Studia Math., **90**, 191–196.

[4] Figiel, T., Johnson, W.B. and Tzafriri, L. (1975). *On Banach Lattices and Spaces Having Local Unconditional Structure, with Applications to Lorentz Function Spaces*, J. Approx. Theory, **13**, 395–412.

[5] Granero, A.S. (1998). *On the complemented subspaces of* $c_0(I)$, Atti Sem. Mat. Fis. Univ. Modena, **46**, 35–36.

[6] Hagler, J. and Johnson, W.B. (1977). *On Banach spaces whose dual balls are not weak* sequentially compact*, Israel J. Math., **28**, 325–330.

[7] Hagler, J. and Odell, E. (1978). *A Banach space not containing ℓ_1 whose dual ball is not weak* sequentially compact*, Illinois J. Math., **22**, 290–294.

[8] Hagler, J. and Sullivan, F. (1980). *Smoothness and weak* sequential compactness*, Proc. Amer. Math. Soc., **78**, 497–503.

[9] Haydon, R. (1977). *On Banach spaces which contain $\ell_1(\tau)$ and types of measures on compact spaces*, Israel J. Math., **28**, 313–324.

[10] Hernández, F.L. and Troyanski, S.T. (1993). *On the representation of uncountable symmetric basic sets and its applications*, Studia Math., **107**, 287–304.

[11] Josefson, B. (2001). *A Gelfand-Philips space not containing ℓ_1 whose dual ball is not weak* sequentially compact*, Glasg. Math. J., **43**, 125–128.

[12] Köthe, G. (1966). *Hebbare lokalkonvexe Raüme*, Math. Ann., **165**, 181–195.

[13] Lindenstrauss, J. and Tzafriri, L. (1977). *Classical Banach Spaces I.*, Springer, Berlin.

[14] Lozanovskii, G. Ya. (1968). *Certain topological properties of Banach lattices and conditions for their reflexivity* (in Russian), Dokl. Akad. Nauk SSSR, **183**, 521–523.

[15] Rodriguez-Salinas, B. (1994). *On the Complemented Subspaces of $c_0(I)$ and $\ell_p(I)$ for $1 < p < \infty$*, Atti Sem. Mat. Fis. Univ. Modena, **42**, 399–402.

[16] Rodriguez-Salinas, B. (1996). *Subspaces with symmetric basis in the Orlicz space $\ell_F(I)$*, Comment. Math. Prace Mat., **36**, 201–222.

[17] Rosenthal, H.P. (1977). *Point-wise compact subsets of the first Baire class*, Amer. J. Math., **99**, 362–378.

[18] Schaefer, H.H. (1974). *Banach Lattices and Positive Operators*, Springer, Berlin.

[19] Schlumprecht, T. (1989). *On dual spaces with bounded sequences without weak* convergent convex blocks*, Proc. Amer. Math. Soc., **107**, 395–408.

[20] Singer, I. (1981). *Bases in Banach Spaces II*. Springer, Berlin.

[21] Troyanski, S. (1975). *On non-separable Banach spaces with a symmetric basis*, Studia Math., **53**, 253–263.

[22] Vašak, L. (1981). *On one generalization of weakly compactly generated Banach spaces*, Studia Math., **70**, 11–19.

[23] Wnuk, W. (1999). *Banach Lattices with Order Continuous Norms*, Polish Scientific Publishers, Warszawa.

[24] Zizler, V. (2003). *Nonseparable Banach spaces*, in: W. B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, Vol. 2, North-Holland, Amsterdam, pp. 1743–1816.

Marek Wójtowicz
Instytut Matematyki
Akademia Bydgoska
Pl. Weyssenhoffa 11
85-072 Bydgoszcz, Poland
mwojt@ab.edu.pl

Carlos Finol
Escuela de Matemática
Universidad Central de Venezuela
Caracas 1020
Venezuela
cfinol@euler.ciens.ucv.ve