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The structure of piecewise monotonic transformations

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Abstract. Transformations on [0, 1] which are piecewise monotonic and piecewise continuous are considered. Using symbolic dynamics, the structure of their non-wandering set is determined. This is then used to prove results about maximal and absolutely continuous invariant measures.

0. *Introduction*

We consider dynamical systems $([0, 1], f)$, where $[0, 1] = \bigcup_{i=1}^{n} J_i$, the J_i are disjoint intervals and $f|J_i$ is continuous and increasing. The f-expansion gives rise to a shift space Σ_f^+ (cf. § 1). Our goal is to determine the structure of the non-wandering set Ω of Σ_f^+. In [**4**] it is shown how these results can be extended to the case where $f|J_i$ is either increasing or decreasing.

§ 1 gives a summary of results proved in [**3**] and needed in this paper. In § 2 it is shown that the non-wandering set Ω of Σ_f^+ can be written as $\bigcup_{i \geq 1} \Omega_i \cup Y \cup Z$. There are finitely or countably many Ω_i. The Ω_i and Y are closed, σ-invariant subsets of Σ_f^+, and Ω_i is topologically transitive. $\Omega_i \cap \Omega_j$, for $i \neq j$, and $\Omega_i \cap Y$ are empty or finite; Z is finite and wandering in Ω. The topological entropy of Y is zero. Ω_i is a finite union of intervals, a Cantor set or a periodic orbit. Furthermore, $\Omega_i = X_1 \cup X_2 \cup \cdots \cup X_q (q \geq 1)$, the X_i are closed and again pairwise disjoint up to finitely many points; $\sigma(X_i) \subset X_{i+1}$ for $1 \leq i \leq q-1$, $\sigma(X_q) \subset X_1$ and σ^q/X_i is topologically mixing.

The rest of the paper deals with invariant measures. It is shown in [**3**] that Ω_i has a unique measure with maximal entropy if $h_{\text{top}}(\Omega_i) > 0$. In § 3 this measure is characterized as the measure with respect to which the periodic points are uniformly distributed. § 4 considers invariant measures of $([0, 1], f)$ which are absolutely continuous with respect to Lebesgue measure and gives an example in connection with this.

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1. *Preliminaries*

We give a description of results proved in [3]. Let f be as in the introduction. Define the f-expansion $\phi: [0, 1] \to \Sigma_n^+ = \{1, \ldots, n\}^{\mathbf{N}}$ by $\phi(x) = i_0 i_1 i_2 \ldots$, where i_j is the number i of the interval J_i with $f^j(x) \in J_i$. If $J_k = (r, s)$, define $\mathbf{a}^k = \lim_{t \downarrow r} \phi(t)$ and $\mathbf{b}^k = \lim_{t \uparrow s} \phi(t)$. Set

$$\Sigma_f^+ = \{\mathbf{x} = x_0 x_1 \cdots \in \Sigma_n^+ : \mathbf{a}^{x_m} \leq x_m x_{m+1} \cdots = \sigma^m \mathbf{x} \leq \mathbf{b}^{x_m} \ \forall m \geq 0\}, \tag{1.1}$$

where $\leq$ denotes the lexicographic ordering and σ the shift transformation. We have $\overline{\phi([0, 1])} = \Sigma_f^+$, $\Sigma_f^+ \backslash \phi([0, 1])$ is countable, ϕ is order preserving and $\sigma \circ \phi = \phi \circ f$. ϕ is injective if and only if $(J_1, J_2, \ldots, J_n)$ is a generator for $([0, 1], f)$.

Σ_f^+ can be characterized by all blocks $x_0 x_1 \ldots x_{m-1}$ which are admissible in Σ_f^+, i.e. ${}_0[x_0 x_1 \ldots x_{m-1}] = \{\mathbf{z} \in \Sigma_f^+ : z_i = x_i \text{ for } 0 \leq i \leq m-1\}$ is not empty. This is equivalent to $\sigma^{m-1}({}_0[x_0 \ldots x_{m-1}]) \neq \varnothing$. We have ${}_0[x_0] = [\mathbf{a}^{x_0}, \mathbf{b}^{x_0}]$, which denotes a closed interval in Σ_f^+. We show by induction that there are $i, j, k \geq 1$ and $l \geq 1$ such that

$$\sigma^{m-1}({}_0[x_0 \cdots x_{m-1}]) = [\sigma^{k-1}\mathbf{a}^i, \sigma^{l-1}\mathbf{b}^j] \quad \text{with} \quad a^i_{k-r} = b^j_{l-r} \quad \text{for } 1 \leq r \leq \min(k, l). \tag{1.2}$$

The induction step is

$$\begin{aligned}
\sigma^m({}_0[x_0 \cdots x_m]) &= {}_0[x_m] \cap \sigma^m({}_0[x_0 \cdots x_{m-1}]) \\
&= [\mathbf{a}^{x_m}, \mathbf{b}^{x_m}] \cap [\sigma^k \mathbf{a}^i, \sigma^l \mathbf{b}^j] \\
&= \begin{cases}
\varnothing, & \text{if } x_m < a^i_k \quad \text{or} \quad x_m > b^j_l, \\
[\sigma^k \mathbf{a}^i, \mathbf{b}^{x_m}], & \text{if } x_m = a^i_k \quad \text{and} \quad x_m < b^j_l, \\
[\mathbf{a}^{x_m}, \sigma^l \mathbf{b}^j], & \text{if } x_m > a^i_k \quad \text{and} \quad x_m = b^j_l \\
[\sigma^k \mathbf{a}^i, \sigma^l \mathbf{b}^j], & \text{if } x_m = a^i_k = b^j_l \\
[\mathbf{a}^{x_m}, \mathbf{b}^{x_m}], & \text{if } a^i_k < x_m < b^j_l.
\end{cases}
\end{aligned} \tag{1.3}$$

It is easy to see that $\sigma^m({}_0[x_0 \cdots x_m])$ is either empty or satisfies (1.2). Hence we have shown (1.2) by induction. In particular, $\sigma^{m-1}({}_0[x_0 \cdots x_{m-1}])$ is a closed sub-interval of some ${}_0[i] \subset \Sigma_f^+ (1 \leq i \leq n)$. If it is not empty, it is either $\sigma^{k-1}({}_0[a^i_0 \cdots a^i_{k-1}])$ or $\sigma^{k-1}({}_0[b^i_0 \cdots b^i_{k-1}])$ for some i and k, because all intervals one obtains by repeated use of (1.3) are such sets:

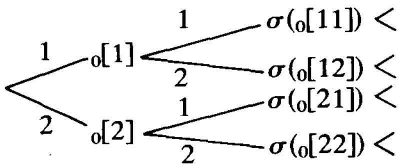
$$[\sigma^{k-1}\mathbf{a}^i, \sigma^{l-1}\mathbf{b}^j] = \sigma^{k-1}({}_0[a^i_0 \cdots a^i_{k-1}]) \quad \text{if } k \geq l$$

and

$$[\sigma^{k-1}\mathbf{a}^i, \sigma^{l-1}\mathbf{b}^j] = \sigma^{l-1}({}_0[b^j_0 \cdots b^j_{l-1}]) \quad \text{if } k \leq l.$$

This follows from lemma 11 of [3]. Hence many of the sets $\sigma^{m-1}({}_0[x_0 \cdots x_{m-1}])$ coincide.

We form a diagram with the sets $\sigma^{m-1}({}_0[x_0 \cdots x_{m-1}])$ (we take $n = 2$ for convenience). It will be called M.



(1.4)

There is a 1–1 correspondence between paths in M which begin with one of the arrows ending at some $_0[k](1 \le k \le n)$ and which do not lead to an empty set and the points $\mathbf{x} \in \Sigma_f^+(x_0, x_1 \cdots$ are the numbers of the arrows on this path). We shall call such paths special paths in order to distinguish them from paths beginning with any other arrow of (1.4) (they represent also points of Σ_f^+, but an $\mathbf{x} \in \Sigma_f^+$ may have many such representations).

M serves also for another purpose. Set

$$D = \{\sigma^{m-1}(_0[x_0 \cdots x_{m-1}]) : \sigma^{m-1}(_0[x_0 \cdots x_{m-1}]) \ne \varnothing\}$$
$$= \{\sigma^{m-1}(_0[a_0^i \cdots a_{m-1}^i]), \sigma^{m-1}(_0[b_0^i \cdots b_{m-1}^i]) : 1 \le i \le n, m \ge 1\}.$$

Together with the arrows $\sigma^{m-1}(_0[x_0 \cdots x_{m-1}]) \xrightarrow{x_m} \sigma^m(_0[x_0 \cdots x_m])$, D becomes the diagram M of (1.4). In [**3**] we have used $\sigma^m(_0[x_0 \cdots x_{m-1}])$ instead of $\sigma^{m-1}(_0[x_0 \cdots x_{m-1}])$. This makes no difference for the results and the proofs of [**3**], but the new definition is more convenient. For example, D need not be a set of pairs $(x_{m-1}, \sigma^m(_0[x_0 \cdots x_{m-1}]))$ as in [**3**] because x_{m-1} is determined by $\sigma^{m-1}(_0[x_0 \cdots x_{m-1}]) \subset {_0[x_{m-1}]}$.

Define $\Sigma_M = \{\mathbf{y} \in D^{\mathbf{Z}}$: there is an arrow from y_i to y_{i+1} in M $\forall i \in \mathbb{Z}\}$. Now $\Sigma_f = \{\mathbf{x} \in \{1, \ldots, n\}^{\mathbf{Z}} : x_m x_{m+1} \cdots \in \Sigma_f^+ \forall m \in \mathbb{Z}\}$, the natural extension of Σ_f^+, can be written as disjoint union of sets N and X which are σ-invariant and measurable. N contains no periodic points and is a null set for every measure with maximal entropy. (X, σ) and (Σ_M, σ) are isomorphic, the isomorphism ψ is given by representing $\mathbf{y} \in \Sigma_M$, which is a two-sided path of vertices in the diagram M, by the numbers of the arrows on this path giving an $\mathbf{x} \in X \subset \Sigma_f$ (cf. [**3**]). Two-sided paths exist in M because many of the sets $\sigma^{m-1}(_0[x_0 \cdots x_{m-1}])$ coincide. Examples can be found in [**3**]. The map $\chi: (\Sigma_M, \sigma) \xrightarrow{\psi} (\Sigma_f, \sigma) \xrightarrow{\pi} (\Sigma_f^+, \sigma)$ is the composition of this isomorphism ψ and the projection π to positive coordinates. χ is continuous.

We conclude § 1 with two remarks. This first one explains how the results of this paper can be extended to maps f, for which $f|J_i$ is continuous and increasing for some is and decreasing for the other is. The only difference to the piecewise increasing case is that we have another order relation in the shift space such that ϕ is order preserving (cf. [**4**]). Σ_f^+ is defined as in (1.1), but with this different order relation. σ is then not order preserving, hence the intervals occurring in (1.3) may have also $\sigma^k \mathbf{b}^i$ for some i and k as initial point or $\sigma^l \mathbf{a}^j$ for some j and l as endpoint or both. One can define the diagram M as in (1.4) and also the map χ. In [**4**], a piecewise increasing transformation g is constructed such that (Σ_f^+, σ) is a two-to-one factor of (Σ_g^+, σ). The only proofs in this paper which will use the explicit form of the intervals in (1.3) are those of lemmas 1,4 and (ii) of lemma 7. These proofs can be extended to the piecewise increasing–decreasing case in the same way as one obtains the diagram M for f from that of g (cf. [4]). The definitions and all other proofs work unchanged. Hence all results of this paper are also valid for piecewise increasing–decreasing transformations.

The second remark shows how one can determine the structure of the non-wandering set of $([0,1], f)$ from that of (Σ_f^+, σ). If $(J_1, \ldots, J_n)$ is a generator for $([0,1], f)$, then ϕ is injective and ϕ^{-1} can be easily extended on all of Σ_f^+ to a map ρ which is continuous and preserves the ordering. An $x \in [0,1]$ is wandering under f if and only if $\phi(x) \notin \Omega$, unless x is an inverse image under some iterate of f of an endpoint of some J_i, not equal to 0 or 1, which can be non-wandering, and $\phi(x) \notin \Omega$. These are exactly those $x \in [0,1]$ such that $\rho^{-1}(x)$ is not a single point but two points $\mathbf{x}$ and $\mathbf{x}'$. If x is non-wandering and $\mathbf{x} \notin \Omega$, $\mathbf{x}' \notin \Omega$, then there is an $\varepsilon > 0$ such that $f^k(x-\varepsilon, x) \cap (x-\varepsilon) = \emptyset$ and $f^k(x, x+\varepsilon) \cap (x, x+\varepsilon) = \emptyset$ for all $k \geq 1$ (the intervals $(x-\varepsilon, x)$ and $(x, x+\varepsilon)$ correspond to neighbourhoods of $\mathbf{x}$ and $\mathbf{x}'$ respectively). Hence x is isolated in the non-wandering set of $([0,1], f)$ and non-periodic (otherwise $\mathbf{x}$ or $\mathbf{x}'$ is periodic). If we transfer the structure of Ω to $([0,1], f)$ via ρ, then we can add these points to $\rho(Z)$. Hence the non-wandering set of $([0,1], f)$ has the same structure as that of (Σ_f^+, σ) described in § 0: one has only to allow that the set corresponding to Z is countable.

If $(J_1, \ldots, J_n)$ is not a generator, then ϕ maps certain intervals to single points (cf. [4]).

2. *The non-wandering set of* (Σ_f^+, σ)

We show that (Σ_f^+, σ) has the structure described in § 0. To this end we consider M as a 0–1-matrix with index set D. $M_{de} = 1$ if and only if there is an arrow from d to e in M. We divide M into irreducible submatrices M_i with index set $D_i (i \geq 1)$, i.e. D_i is a maximal subset of D such that, if $d, e \in D_i$, then there is a path from d to e in M, and $M_i = M/D_i$. $\Sigma_{M_i} \subset \Sigma_M$ denotes the shift space corresponding to M_i. If $i \neq j$, then $D_i \cap D_j = \emptyset$ and $\bigcup_{i \geq 1} D_i \subset D$. It may happen that $\bigcup_{i \geq 1} D_i \neq D$. As

$$D = \{\sigma^m({}_0[a_0^i \cdots a_m^i]), \sigma^m({}_0[b_0^i \cdots b_m^i]) : 1 \leq i \leq n, m \geq 0\}$$

and because of the arrows

$$\sigma^m({}_0[a_0^i \cdots a_m^i]) \to \sigma^{m+1}({}_0[a_0^i \cdots a_{m+1}^i]),$$
$$\sigma^m({}_0[b_0^i \cdots b_m^i]) \to \sigma^{m+1}({}_0[b_0^i \cdots b_{m+1}^i]),$$

it is easy to see that, for every D_i, there are p_j, q_j, u_j, v_j $(1 \leq j \leq n)$ with $0 \leq p_j \leq q_j \leq \infty$ and $0 \leq u_j \leq v_j \leq \infty$ such that

$$D_i = \{\sigma^l({}_0[a_0^j \cdots a_l^j]), \sigma^m({}_0[b_0^k \cdots b_m^k]) : 1 \leq j, k \leq n, \quad p_j \leq l < q_j, u_k \leq m < v_k\}. \tag{2.1}$$

We introduce an order relation among the D_is as follows:

$$D_i \leq D_j, \text{ if there is a path from } D_i \text{ to } D_j \text{ in } M. \tag{2.2}$$

As the M_is are the irreducible submatrices of M, this is an order relation. Let the indices $i \in \mathbb{N}$ of the D_i be such that $D_i \leq D_j$ implies $i \leqslant j$. Set

$$\bar{D}_i = \left\{ d \in D\text{: there is a path from some } e \in \bigcup_{j \geq i} D_j \text{ to } d \right\},$$
$$D_i' = \bar{D}_i \backslash D_i.$$

We then have $\bar{D}_i \supset D_i' \supset \bar{D}_{i+1}$. It is easy to see that $\bar{D}_i$ and D_i' are subsets C of D, which have the following property.

$$\text{If } d \in C \text{ and there is a path from } d \text{ to } e \text{ in } M\text{, then } e \in C. \tag{2.3}$$

Remark. Suppose $\sigma^k({}_0[a_0^i \cdots a_k^i]) \in D$ (or $\sigma^k({}_0[b_0^i \cdots b_k^i])$) consists only of the single point $\mathbf{y} = \sigma^k \mathbf{a}^i$ ($\sigma^k \mathbf{b}^i$) such that $\sigma^m \mathbf{y}$ is not periodic for $m \geq k$. Then $\sigma^m({}_0[a_0^i \cdots a_m^i]) = \{\sigma^m \mathbf{a}^i\} \in D$ has only one successor in M for all $m \geq k$. We then cancel the vertices $\sigma^m({}_0[a_0^i \cdots a_m^i])$ for $m \geq k$ in M and the corresponding (via χ^{-1}) set $\bigcup_{j=0}^{\infty} \sigma^{-j}(\sigma^k \mathbf{a}^i)$ in Σ_f^+ which is open, countable, σ^{-1}-invariant and consists only of wandering points. We denote the remainder of Σ_f^+ again by Σ_f^+, which is closed, σ-invariant and contains all non-wandering points of the original Σ_f^+. After this modification, every element of D is either a non-trivial interval or a single point $\mathbf{y}$, such that $\sigma^k \mathbf{y}$ is periodic for some k. As there are only finitely many $\mathbf{a}^i$ and $\mathbf{b}^i$, there are among the elements of D only finitely many single points, i.e. trivial intervals.

LEMMA 1. *Let $C \subset D$ have the property* (2.3). *Then $\bigcup \{d\colon d \in C\}$ is a finite union of intervals and is σ-invariant.*

Proof. Set $\mathfrak{A}_j = \{d \in C\colon d = [\mathbf{a}^j, \sigma^l \mathbf{b}^k] \text{ for some } l, k\}$ and $\mathfrak{B}_j = \{d \in C\colon d = [\sigma^l \mathbf{a}^k, \mathbf{b}^j] \text{ for some } l, k\}$. $A_j = \bigcup_{d \in \mathfrak{A}_j} d$ and $B_j = \bigcup_{d \in \mathfrak{B}_j} d$ are intervals in Σ_f^+ or are empty. Let $d \in D$ be non-trivial and γ_d the minimal number of steps to go on a path in M from $d \in \mathfrak{A}_j$ or $\mathfrak{B}_j$ to an element of some $\mathfrak{A}_m$ or $\mathfrak{B}_m$. Because d is not a single point, $\gamma_d < \infty$. This part of M looks as follows ($d = [\mathbf{a}^j, \sigma^l \mathbf{b}^k]$, cf. (1.3)):

$$\begin{array}{ccccc} [\mathbf{a}^j, \sigma^l \mathbf{b}^k] \to \cdots \to & [\sigma^{\gamma_d - 1} \mathbf{a}^j, & \sigma^{l+\gamma_d-1} \mathbf{b}^k] & \to [\mathbf{a}^t, \sigma^{l+\gamma_d} \mathbf{b}^k] & \\ & \downarrow & \downarrow & \searrow & \\ & [\sigma^{\gamma_d} \mathbf{a}^j, \mathbf{b}^s] & [\mathbf{a}^{s+1}, \mathbf{b}^{s+1}] & \cdots & [\mathbf{a}^{t-1}, \mathbf{b}^{t-1}]. \end{array}$$

Set $\alpha_j = \min\{\gamma_d\colon d \in \mathfrak{A}_j\}$ and $\beta_j = \min\{\gamma_d\colon d \in \mathfrak{B}_j\}$. We show that

$$\sigma^{\alpha_j}(A_j) \subset B_s \cup A_t \cup \bigcup_{s<k<t} A_k \quad \text{for some } s \text{ and } t \ (s < t).$$

Let $d = [\mathbf{a}^j, \sigma^l \mathbf{b}^k] \in \mathfrak{A}_j$ with $\gamma_d = \alpha_j$. Then

$$\sigma^{\alpha_j}(d) = [\sigma^{\alpha_j} \mathbf{a}^j, \mathbf{b}^s] \cup [\mathbf{a}^{s+1}, \mathbf{b}^{s+1}] \cup \cdots \cup [\mathbf{a}^{t-1}, \mathbf{b}^{t-1}]$$
$$\cup [\mathbf{a}^t, \sigma^{l+\alpha_j} \mathbf{b}^k] \subset B_s \cup A_t \cup \bigcup_{s<k<t} A_k.$$

If $\gamma_d > \alpha_j$, then

$$\sigma^{\alpha_j}(d) = [\sigma^{\alpha_j}\mathbf{a}^j, \sigma^{l+\alpha_j}\mathbf{b}^k].\ \sigma^{l+\alpha_j}\mathbf{b}^k \le \mathbf{b}^s,$$

because $b^k_{l+\alpha_j} = a^j_{\alpha_j} = s$ (cf. (1.2)) and $\mathbf{b}^k \in \Sigma_f^+$ (cf. (1.1)). Hence

$$\sigma^{\alpha_j}(d) \subset [\sigma^{\alpha_j}\mathbf{a}^j, \mathbf{b}^s] \subset B_s.$$

Similar results hold for $d \in \mathfrak{B}_j$. Hence

$$\bigcup_{d\in C} d = \bigcup_{j=1}^{n}\left(\bigcup_{k=0}^{\alpha_j-1}\sigma^k(A_j) \cup \bigcup_{k=0}^{\beta_j-1}\sigma^k(B_j)\right) \cup \bigcup_{d\in T} d,$$

where $T \subset C$ is a finite (or empty) set whose elements are trivial intervals (cf. the remark above). This is a finite union of intervals in Σ_f^+. It is σ-invariant, because C satisfies (2.3). □

By lemma 1, $F_i := \bigcup\{d: d \in \bar{D}\}$ and $G_i := \bigcup\{d: d \in D_i'\}$ are finite unions of intervals which are σ-invariant. We have $F_i \supset G_i \supset F_{i+1}$. We set $G_0 := \Sigma_f^+$. Hence we have split Σ_f^+ into a sequence of decreasing sets which are σ-invariant. For $i \ge 1$ we define

$$\Omega_i = \bigcap_{k=0}^{\infty} \overline{\sigma^{-k}(F_i \backslash G_i)},$$

and for $i \ge 0$ we define

$$\tilde{\Omega}_i = \bigcap_{k=0}^{\infty} \overline{\sigma^{-k}(G_i \backslash F_{i+1})}.$$

If D_{i+1} does not exist, we set $F_{i+1} = \emptyset$. $\tilde{\Omega}_i$ may be empty. As $\Omega_i \subset \overline{F_i \backslash G_i}$ and $\tilde{\Omega}_i \subset \overline{G_i \backslash F_{i+1}}$, which are finite unions of closed intervals, it follows that the sets $\Omega_i \cap \Omega_j$, $\tilde{\Omega}_i \cap \tilde{\Omega}_j$, for $i \ne j$, and $\Omega_i \cap \tilde{\Omega}_j$, for any i and j, are at most finite.

If there are infinitely many M_i, we have to consider also an $\tilde{\Omega}_\infty$ and an Ω_∞. Set $D_\infty' = \bigcap_{i=1}^{\infty} \bar{D}_i$. This set has the property (2.3), because every $\bar{D}_i$ has the property (2.3). Hence $G_\infty := \bigcup\{d: d \in D_\infty'\}$ and $\tilde{\Omega}_\infty := \bar{G}_\infty$ are finite unions of intervals which are σ-invariant. Because $\tilde{\Omega}_\infty \subset \bar{F}_{i+1}$ for all i, we have that $\Omega_i \cap \tilde{\Omega}_\infty$ and $\tilde{\Omega}_i \cap \tilde{\Omega}_\infty$ are at most finite. If there are only finitely many M_is, we set $\tilde{\Omega}_\infty = \emptyset$. Now set $H_i = \overline{F_i \backslash G_\infty}$. Then H_i is closed and $H_i \supset H_{i+1}$. Hence $H = \bigcap_{i\ge 1} H_i$ is closed and not empty. Set $\Omega_\infty = \bigcap_{k=0}^{\infty} \sigma^{-k}(H)$, which is closed and σ-invariant. If $D_\infty' = \emptyset$, then $\Omega_\infty = H$. As $\Omega_\infty \subset H_{i+1}$ for all i, the sets $\Omega_\infty \cap \Omega_i$, $\Omega_\infty \cap \tilde{\Omega}_i$ and $\Omega_\infty \cap \tilde{\Omega}_\infty$ are at most finite. If there are only finitely many M_i, we set $\Omega_\infty = \emptyset$.

We need one more definition. Let Z_i be the set of all $\mathbf{x} \in \operatorname{bd} F_i \backslash \operatorname{bd} G_i$ (bd means boundary) such that there is a k with $\sigma^m \mathbf{x} \in \operatorname{bd} G_i$ for all $m \ge k$ and of all $\mathbf{x} \in \operatorname{bd} G_i \backslash \operatorname{bd} F_{i+1}$ such that there is a k with $\sigma^m \mathbf{x} \in \operatorname{bd} F_{i+1}$ for all $m \ge k$. If $G_\infty \ne \emptyset$, let Z_∞ be the set of all $\mathbf{x} \in \bigcap_{j\ge i} \operatorname{bd} F_j \backslash \operatorname{bd} G_\infty$ for some i such that there is a k with $\sigma^m \mathbf{x} \in \operatorname{bd} G_\infty$ for all $m \ge k$.

Now we have the following result.

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LEMMA 2. *The set Ω of non-wandering points of Σ_f^+ is contained in*

$$\bigcup_{1\le i\le\infty}\Omega_i\cup\bigcup_{0\le i\le\infty}\tilde{\Omega}_i\cup\bigcup_{0\le i\le\infty}Z_i.$$

Proof. If $\mathbf{x}\in\bar{F}_i$ for all i and there are infinitely many M_is, then $\mathbf{x}\in H\cup G_\infty$. We consider this case below. As $G_0=\Sigma_f^+$, we find otherwise an $i<\infty$ with $\mathbf{x}\in\bar{F}_i\backslash\bar{G}_i$ or $\mathbf{x}\in\bar{G}_i\backslash\bar{F}_{i+1}$. We consider first the case $\mathbf{x}\in\bar{F}_i\backslash\bar{G}_i$. The following three possibilities can occur.

(i) $\mathbf{x}\in\Omega_i$.

(ii) $\mathbf{x}\notin\Omega_i$ and $\mathbf{x}\in\operatorname{int}(F_i\backslash G_i)$ (int means interior). As $\mathbf{x}\notin\Omega_i$, there is a k with $\mathbf{x}\notin\overline{\sigma^{-k}(F_i\backslash G_i)}$. Since this set is closed, there is a neighbourhood U of $\mathbf{x}$ contained in $\operatorname{int}(F_i\backslash G_i)\cap{}_0[x_0\cdots x_k]$ such that $U\cap\sigma^{-k}(F_i\backslash G_i)=\varnothing$. But then $\sigma^k(U)\cap F_i\backslash G_i=\varnothing$ and $\sigma^k(U)\subset G_i$, because $\sigma(F_i)\subset F_i$. Because $\sigma(G_i)\subset G_i$, we have $\sigma^m(U)\subset G_i$ for all $m\ge k$. This means that $\mathbf{x}$ is wandering, i.e. $\mathbf{x}\notin\Omega$.

(iii) $\mathbf{x}\notin\Omega_i$ and $\mathbf{x}\in\operatorname{bd}F_i\backslash\operatorname{bd}G_i$ As $\mathbf{x}\notin\Omega_i$, there is a k with $\mathbf{x}\notin\sigma^{-k}(\overline{F_i\backslash G_i})$, i.e. $\sigma^k\mathbf{x}\notin\overline{F_i\backslash G_i}$, hence $\sigma^k\mathbf{x}\in G_i$. If $\sigma^m\mathbf{x}\in\operatorname{int}G_i$ for some m, then there is a neighbourhood U of $\mathbf{x}$ with $U\cap G_i=\varnothing$ and $\sigma^m(U)\subset G_i$, because σ is continuous. Because $\sigma(G_i)\subset G_i$, we then have $\mathbf{x}\notin\Omega$. If $\sigma^m\mathbf{x}\in\operatorname{bd}G_i$ for all $m\ge k$, then $\mathbf{x}\in Z_i$.

Now we consider the case $\mathbf{x}\in\bar{G}_i\backslash\bar{F}_{i+1}$. We have the same three possibilities.

(i) $\mathbf{x}\in\tilde{\Omega}_i$.

(ii) $\mathbf{x}\notin\tilde{\Omega}_i$ and $\mathbf{x}\in\operatorname{int}(G_i\backslash F_{i+1})$. As above, it follows that $\mathbf{x}\notin\Omega$.

(iii) $\mathbf{x}\notin\tilde{\Omega}_i$ and $\mathbf{x}\in\operatorname{bd}G_i\backslash\operatorname{bd}F_{i+1}$. As above, we have either $\mathbf{x}\notin\Omega$ or $\mathbf{x}\in Z_i$.

If now $\mathbf{x}\in\bar{F}_i$ for all i and there are infinitely many M_i, then $\mathbf{x}\in H\cup G_\infty$. We consider again the three possibilities as above.

(i) $\mathbf{x}\in\Omega_\infty\cup\tilde{\Omega}_\infty$.

(ii) $\mathbf{x}\notin\Omega_\infty\cup\tilde{\Omega}_\infty$ and $\mathbf{x}\in\operatorname{int}(F_i\backslash G_\infty)$ for all i. As

$$\Omega_\infty=\bigcap_{k=0}^{\infty}\sigma^{-k}\left(\bigcap_{i=0}^{\infty}H_i\right)=\bigcap_{i=0}^{\infty}\bigcap_{k=0}^{\infty}\sigma^{-k}(H_i),$$

there is an i with $\mathbf{x}\notin\bigcap_{k=0}^{\infty}\sigma^{-k}(H_i)$. As $\mathbf{x}\in\operatorname{int}H_i$, it follows as above that $\mathbf{x}\notin\Omega$.

(iii) $\mathbf{x}\notin\Omega_\infty\cup\tilde{\Omega}_\infty$ and $\mathbf{x}\in\operatorname{bd}F_i$ for some i. Because $F_{j+1}\subset F_j$ and $\mathbf{x}\in\overline{F_j}$ for all j, one has from $\mathbf{x}\in\operatorname{bd}F_i$ that $\mathbf{x}\in\operatorname{bd}F_j$ for all $j\ge i$. Then it follows again as above that either $\sigma^m\mathbf{x}\in\operatorname{int}G_\infty$ for some m, and hence $\mathbf{x}\notin\Omega$ (note that $\mathbf{x}\notin\bar{G}_\infty=\tilde{\Omega}_\infty$), or $\mathbf{x}\in Z_\infty$.

In any case, we have shown that either $\mathbf{x}\notin\Omega$ or $\mathbf{x}\in\bigcup\Omega_i\cup\bigcup\tilde{\Omega}_i\cup\bigcup Z_i$. This proves the lemma. □

Examples which show how the Ω_i look like can be found in [**3**]. We give here an example where a Z_i occurs. The transformation f on $[0, 1]$ shown in figure 1 is a modification of an example given by L. Block and L. S. Young (cf. [**7**]). We have $0<x<d<c<p<e<1$, f is increasing on the intervals $[0, d)$, $[d, c)$, $[c, e)$ and $[e, 1]$, and it satisfies

$$f([0,d))=[c,1),\quad f([d,c))=[x,1),\quad f([c,e))=[x,1)\quad\text{and}\quad f([e,1])=[p,1].$$

Furthermore, $f(x)=f(p)=p$.

The diagram M is as follows (we take the elements of D here as sub-intervals of $[0, 1]$).

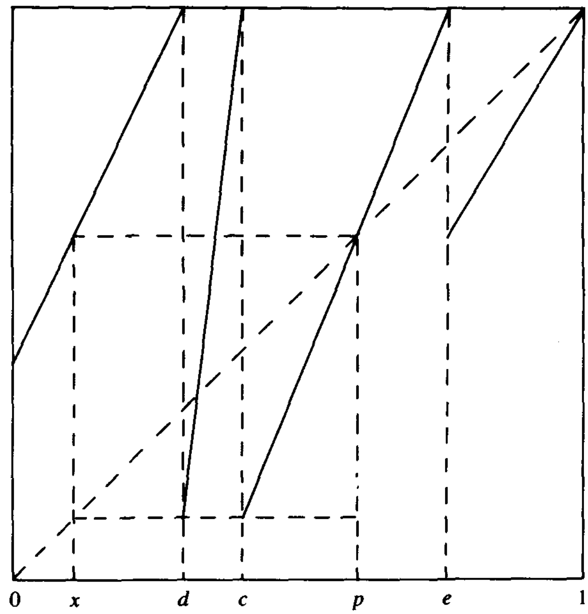
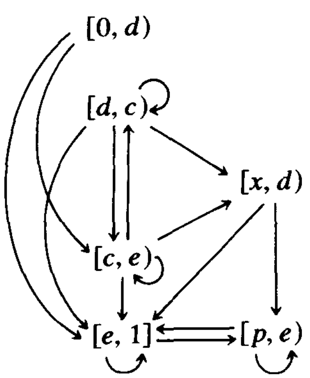


FIGURE 1



$D_1 = \{[d, c), [c, e)\}$

$D_2 = \{[e, 1], [p, e)\}$

$\bar{D}_1 = D_1 \cup D_2 \cup \{[x, d)\}$

$D_1' = D_2 \cup \{[x, d)\}$

$\bar{D}_2 = D_2$

$D_2' = \varnothing$

$G_0 = [0, 1]$, $F_1 = [x, 1]$, $G_1 = [x, d) \cup [p, 1]$, $F_2 = [p, 1]$, $G_2 = \varnothing$. $\tilde{\Omega}_0 = \varnothing$, $\Omega_1 = \bigcap_{k=0}^{\infty} f^{-k}([d, p])$, a Cantor set, $\tilde{\Omega}_1 = \varnothing$, $\Omega_2 = [p, 1]$. $Z_1 = \{x\}$ and $\Omega = \Omega_1 \cup \Omega_2 \cup \{x\}$.

One sees that $x \notin \Omega_1$ and $x \notin \Omega_2$. Hence the Z_i in lemma 2 are necessary.

Before we are able to investigate Ω_i and $\tilde{\Omega}_i$ further, we need two technical lemmas.

LEMMA 3. *Let $d \in D$. Then $\mathbf{x} \in d$, if and only if $\mathbf{x}$ can be represented as a path in M, which begins with an arrow ending at d.*

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Proof. Suppose $d=\sigma^k({}_0[a_0^i \cdots a_k^i])$. Let $\mathbf{x}\in d$. This means $\mathbf{y}=a_0^i \cdots a_{k-1}^i x_0 x_1 \cdots \in \Sigma_f^+$. By the 1–1 correspondence of points in Σ_f^+ and special paths in M (cf. § 1), we have that $\mathbf{y}$ corresponds to the special path

$$\xrightarrow{a_0^i} {}_0[a_0^i] \xrightarrow{a_1^i} \sigma({}_0[a_0^i a_1^i]) \xrightarrow{a_2^i} \cdots \xrightarrow{x_0} \sigma^k({}_0[a_0^i \cdots a_k^i]) \xrightarrow{x_1} \cdots$$

in M. This means $\mathbf{x}$ can be represented as a path in M beginning with an arrow which ends at d. Now let $\mathbf{x}$ correspond to a path which begins with an arrow ending at d. Then $a_0^i \cdots a_{k-1}^i x_0 x_1 \cdots \in \Sigma_f^+$ because it corresponds to a special path as above. Hence $\mathbf{x}\in\sigma^k({}_0[a_0^i \cdots a_k^i])=d$. □

LEMMA 4. *Let* $\mathbf{y}^1, \mathbf{y}^2 \in \Sigma_M$. *If* $\chi(\mathbf{y}^1)=\chi(\mathbf{y}^2)=\mathbf{x}\notin E$, *where* $E=\{\sigma^m\mathbf{a}^i, \sigma^m\mathbf{b}^i\colon 1\le i\le n, m\ge 0\}$, *then there is a* K *with* $y_t^1=y_t^2$ *for all* $t\ge K$.

Proof. Suppose y_0^1 is the interval $[\sigma^k\mathbf{a}^i, \sigma^{k'}\mathbf{b}^{i'}]$. Using (1.3) one can determine what y_1^1 is. If $x_1=a_{k+1}^i$, then y_1^1 has initial point $\sigma^{k+1}\mathbf{a}^i$. If $x_1\ne a_{k+1}^i$, then y_1^1 has initial point $\mathbf{a}^{i_2}$, where $i_2=x_1$. Proceeding in this way, we can determine the initial points of y_t^1 for $t\ge 0$. We obtain the following result. Determine $r_1, r_2, \ldots$ ($r_l\ge 1$) and $i_1=i, i_2, \ldots$ ($1\le i_l\le n$) inductively according to

$$a_{k+t}^{i_1}=x_t \quad \text{for } 0\le t<r_1, \qquad a_{k+r_1}^{i_1}\ne x_{r_1},$$
$$a_{t-R_l}^{i_{l+1}}=x_t \quad \text{for } R_l\le t<R_{l+1}, \qquad a_{r_{l+1}}^{i_{l+1}}\ne x_{R_{l+1}} \qquad (l\ge 1), \tag{2.4}$$

where we have written R_t for $r_1+\cdots+r_t$. Then it follows from (1.3) that the initial point of y_t^1 is $\sigma^{k+t}\mathbf{a}^{i_1}$ for $0\le t<r_1$ and $\sigma^{t-R_l}\mathbf{a}^{i_{l+1}}$ for $R_l\le t<R_{l+1}$ ($l\ge 1$).

We do the same for $\mathbf{y}^2$. Suppose $y_0^2=[\sigma^m\mathbf{a}^j, \sigma^{m'}\mathbf{b}^{j'}]$. We determine $s_1, s_2, \ldots$ ($s_l\ge 1$) and $j_1=j, j_2, \ldots$ ($1\le j_l\le n$) inductively according to

$$a_{m+t}^{j_1}=x_t \quad \text{for } 0\le t<s_1, \qquad a_{m+s_1}^{j_1}\ne x_{s_1},$$
$$a_{t-S_l}^{j_{l+1}}=x_t \quad \text{for } S_l\le t<S_{l+1}, \qquad a_{s_{l+1}}^{j_{l+1}}\ne x_{S_{l+1}} \qquad (l\ge 1), \tag{2.5}$$

where we have written S_t for $s_1+\cdots+s_t$. By (1.3) we have that the initial point of y_t^2 is $\sigma^{m+t}\mathbf{a}^{j_1}$ for $0\le t<s_1$ and $\sigma^{t-S_l}\mathbf{a}^{j_{l+1}}$ for $S_l\le t<S_{l+1}$ ($l\ge 1$).

Without loss of generality, we can assume that $r_1\le s_1$. Because $\mathbf{x}\notin E$, we have $s_1<\infty$. If $r_1<s_1$, it follows then from lemma 4 of [**3**] that there is a u with $r_1+\cdots+r_u=s_1$. If $r_1=s_1$, we set $u=1$. Now it follows from (2.4) and (2.5) that $r_{u+l}=s_{1+l}$ and $i_{u+l}=j_{1+l}$ for $l\ge 1$. Hence y_t^1 and y_t^2 have the same initial points for all $t\ge s_1:=K'$.

We can perform the same also for endpoints of y_t^1 and y_t^2 and find a K'' such that y_t^1 and y_t^2 have the same endpoints for all $t\ge K''$. If one now sets $K=\max(K', K'')$, one has $y_t^1=y_t^2$ for all $t\ge K$ and the lemma is proved. □

Remarks. (i) Lemma 4 asserts a kind of injectivity of χ. In particular, if $\mathbf{y}^1$ and $\mathbf{y}^2$ are distinct periodic points in $\Sigma_M\backslash\chi^{-1}(E)$, then $\chi(\mathbf{y}^1)\ne\chi(\mathbf{y}^2)$.

(ii) The proof of lemma 4 works also if one has instead of $\mathbf{y}^1$ and $\mathbf{y}^2$ only one-sided paths $y_0^1y_1^1y_2^1\cdots$ and $y_0^2y_1^2y_2^2$ in M, which correspond to the same $\mathbf{x}\in\Sigma_f^+\backslash E$. We shall sometimes apply lemma 4 in this form.

We return to the investigation of Ω_i and $\tilde{\Omega}_i$.

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LEMMA 5. *If Σ_{M_i} does not consist of only a periodic orbit, then $\chi(\Sigma_{M_i}) \subset \Omega_i$.*

Proof. Let $\mathbf{y} = \cdots d_0 d_1 d_2 \cdots \in \Sigma_{M_i}$ and suppose $\sigma^k \mathbf{x} \notin E$ for $k \geq 0$, where $\mathbf{x} = \chi(\mathbf{y})$. By definition of χ and lemma 3, we have $\mathbf{x} \in d_0 \in D_i$, hence $\mathbf{x} \in F_i$. We show $\mathbf{x} \notin G_i$. If $\mathbf{x} \in G_i$, then $\mathbf{x} \in d'_0$ for some $d'_0 \in D'_i$. By lemma 3, there is a path in M, beginning with an arrow ending at d'_0, which corresponds to $\mathbf{x}$. Let $d'_0 d'_1 d'_2 \cdots$ be the vertices on this path. By lemma 4 there is a $K > 0$ such that $d_k = d'_k$ for $k \geq K$, since $\mathbf{x} \notin E$. As $d_k \in D_i$, we have found a path from D'_i to D_i, a contradiction to the definition of D'_i. Hence $\mathbf{x} \in F_i \backslash G_i$. But also $\sigma^k(\mathbf{x}) = \chi(\sigma^k \mathbf{y}) \notin E$ and therefore, as for $\mathbf{x}$ above, it follows that $\sigma^k \mathbf{x} \in F_i \backslash G_i$ for all $k \geq 0$. This implies $\mathbf{x} \in \Omega_i$.

Now suppose that $\sigma^k \mathbf{x} \in E$ for some k. As Σ_{M_i} does not consist of only a periodic orbit, we can find $\mathbf{y}^n \in \Sigma_{M_i}$ with $\sigma^k(\chi(\mathbf{y}^n)) \notin E$, for all k, converging to $\mathbf{y}$ in Σ_{M_i} (M_i is irreducible). As χ is continuous and Ω_i is closed, it follows that $\chi(\mathbf{y}) = \lim \chi(\mathbf{y}^n) \in \Omega_i$. Hence we have shown that $\chi(\Sigma_{M_i}) \subset \Omega_i$. □

Now set $\tilde{D}_i = D'_i \backslash \bar{D}_{i+1}$, $\tilde{D}_0 = D \backslash \bar{D}_1$, $\tilde{D}_\infty = D'_\infty$ and $\tilde{M}_i = M/\tilde{D}_i$ for $0 \leq i \leq \infty$. Let $S_i \subset \Sigma_f^+$ be the set of all $\mathbf{x}$ which can be represented as one-sided paths in $\tilde{M}_i$. If $\tilde{D}_i$ is finite, then $S_i = \emptyset$, because $\tilde{M}_i$ contains no closed paths. All closed paths must be contained in the M_is by definition.

We have now:

LEMMA 6. $\Omega_i \subset \overline{\chi(\Sigma_{M_i})}$ *for* $i < \infty$, $\tilde{\Omega}_i \subset \bar{S}_i$ *for* $i \leq \infty$ *and the set* $\Omega_i \backslash \chi(\Sigma_{M_i})$ *is at most countable and contains only finitely many periodic points.*

Proof. We give the proof only for Ω_i. It is the same for $\tilde{\Omega}_i$. First we show that $\chi(\Sigma_{M_i})$ is dense in Ω_i. Suppose ${}_0[x_0 \cdots x_{k-1}] \cap \Omega_i \neq \emptyset$. Because ${}_0[x_0 \cdots x_{k-1}]$ is open, there is at least one $d \in D_i$ with ${}_0[x_0 \cdots x_{k-1}] \cap d \neq \emptyset$. For every such d, $x_0 \cdots x_{k-1}$ corresponds to a path of length k beginning with an arrow ending at d (lemma 3). If all these paths leave M_i, we have

$$ {}_0[x_0 \cdots x_{k-1}] \cap \Big(F_i \backslash \bigcup_{m=0}^{k-1} \sigma^{-m}(G_i) \Big) = \emptyset. $$

As ${}_0[x_0 \cdots x_{k-1}]$ is open, this implies ${}_0[x_0 \cdots x_{k-1}] \cap \Omega_i = \emptyset$, a contradiction. Hence we have a path of length k in M_i which corresponds to $x_0 \cdots x_{k-1}$. Let $d_0 \cdots d_{k-1}$ be the vertices on this path. As M_i is irreducible, we can extend $d_0 \cdots d_{k-1}$ to a two-sided path in M_i giving rise to a $\mathbf{y} \in \Sigma_{M_i}$. By definition of χ, $\chi(\mathbf{y}) \in {}_0[x_0 \cdots x_{k-1}]$. (In the case of $\tilde{\Omega}_i$ we extend $d_0 \cdots d_{k-1}$ to a one-sided path in $\tilde{D}_i$ giving rise to an element of S_i.) This proves that $\Omega_i \subset \overline{\chi(\Sigma_{M_i})}$.

Now let V be the set of all $\mathbf{x} \in \Omega_i$ with $\sigma^k \mathbf{x} \in E$ for some k or with $\mathbf{x} \in \operatorname{bd} F_i$. Then V is countable and contains at most finitely many periodic points. In order to show the second assertion it suffices to prove that $\Omega_i \backslash V \subset \chi(\Sigma_{M_i})$.

Let $\mathbf{x} \in \Omega_i \backslash V$. As $\mathbf{x} \notin \operatorname{bd} F_i$ there is a $d_0 \in D_i$ with $\mathbf{x} \in d_0$. By lemma 3 there is a path in M which begins with an arrow ending at d_0 and which corresponds to $\mathbf{x}$. Let $d_0 d_1 d_2 \cdots$ be the vertices on this path. We show that $d_k \in D_i$. Suppose $d_k \notin D_i$, then we have $d_k \in D'_i$. By lemma 3 we have $\sigma^k \mathbf{x} \in d_k$. As the endpoints of the interval d_k are in E and $\mathbf{x} \notin V$, $\sigma^k \mathbf{x}$ is in the interior of d_k. Hence $\sigma^k \mathbf{x} \in \operatorname{int} G_i$, i.e. $\sigma^k \mathbf{x} \notin \overline{F_i \backslash G_i}$,

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a contradiction to $\mathbf{x} \in \Omega_i$. Hence $d_k \in D_i$ for all k. Because M_i is irreducible, we can extend the path $d_0 d_1 d_2 \cdots$ to a two-sided one in M_i, which gives a $\mathbf{y} \in \Sigma_{M_i}$. By definition of χ, $\chi(\mathbf{y}) = \mathbf{x}$, hence $\mathbf{x} \in \chi(\Sigma_{M_i})$. This proves that $\Omega_i \setminus V \subset \chi(\Sigma_{M_i})$. □

Remarks. (i) If Σ_{M_i} is only a periodic orbit, then $\Omega_i = \chi(\Sigma_{M_i})$ or is empty. In this case we redefine Ω_i as $\chi(\Sigma_{M_i})$. Then we have $\Omega_i = \overline{\chi(\Sigma_{M_i})}$ for all i by lemmas 5 and 6.

(ii) If D_i is finite, then Σ_{M_i} is compact. As χ is continuous, this implies that $\chi(\Sigma_{M_i})$ is closed. Hence $\chi(\Sigma_{M_i}) = \Omega_i$. An open question is whether this also happens if D_i is infinite.

(iii) If $\tilde{D}_i$ is finite, then $S_i = \emptyset$. By lemma 6 we then have $\tilde{\Omega}_i = \emptyset$.

The next two lemmas give properties of Ω_i for $i \neq \infty$.

LEMMA 7. (i) *$\sigma|\Omega_i$ is topologically transitive.*

(ii) *If $\mathbf{x} \in \Sigma_f^+$ is periodic, then $\mathbf{x} \in \Omega_i$ for some i.*

Proof. (i) Because M_i is irreducible, we can find a $\mathbf{y} \in \Sigma_{M_i}$ such that $\{\sigma^k \mathbf{y}: k \geq 0\}$ is dense in Σ_{M_i}. As χ is continuous and commutes with σ, the set $\{\sigma^k \mathbf{x}: k \geq 0\}$, where $\mathbf{x} = \chi(\mathbf{y})$, is dense in Ω_i by lemma 6.

(ii) Let $\mathbf{x} \in \Sigma_f^+$ satisfy $\sigma^p \mathbf{x} = \mathbf{x}$ for some p. Choose a path $y_0 y_1 \cdots$ in $M(y_i \in D)$ which corresponds to $\mathbf{x}$. Suppose first that $\mathbf{x} \notin E$. Because $\sigma^p \mathbf{x} = \mathbf{x}$, the path $y_p y_{p+1} \cdots$ also corresponds to $\mathbf{x}$. Hence it follows from lemma 4 that there is a K with $y_{m+p} = y_m$ for all $m \geq K$. Now suppose that $\mathbf{x} \in E$, say $\mathbf{x} = \sigma^k \mathbf{b}^j$. Set $y_m = \sigma^{k+m}({}_0[b_0^j \cdots b_{k+m}^j])$. Then $y_0 y_1 \cdots$ is a path in M which corresponds to $\mathbf{x}$. As in the proof of lemma 4 define $r_1, r_2, \ldots (r_l \geq 1)$ and $i_1, i_2, \ldots$ inductively such that, for $l \geq 0$,

$$a_{t-R_l}^{i_{l+1}} = b_t^j \quad \text{for } R_l \leq t < R_{l+1}, \qquad a_{r_{l+1}}^{i_{l+1}} \neq b_{R_{l+1}}^j, \tag{2.6}$$

where we have written R_m for $r_1 + \cdots + r_m (R_0 = 0)$. It follows again from (1.3) that $y_m = [\sigma^{k+m-R_l} \mathbf{a}^{i_{l+1}}, \sigma^{k+m} \mathbf{b}^j]$, where l is such that $R_l \leq k+m < R_{l+1}$.

If $r_t = \infty$ for some t, then $y_m = \{\sigma^{k+m} \mathbf{b}^j\}$ for $m \geq R_{t-1} - k =: K$ and hence $y_{m+p} = y_m$ for all $m \geq K$. If $r_l < \infty$ for all l, let u and v be such that $R_{u-1} < k \leq R_u$ and $R_{v-1} < k+p \leq R_v$. If $R_u + p < R_v$ it follows from $\sigma^{k+p} \mathbf{b}^j = \sigma^k \mathbf{b}^j$ and lemma 4 of [3] that there is a $w > u$ with $R_w + p = R_v$. If $R_u + p > R_v$, we can increase v to obtain equality. Hence it follows from (2.6) that $r_{w+l} = r_{v+l}$ and $i_{w+l} = i_{v+l}$ for all $l \geq 1$. This implies $y_{m+p} = y_m$ for all $m \geq R_w - k =: K$.

Hence in any case we have found a closed path $y_K \to y_{K+1} \to \cdots \to y_{K+p} = y_K$ in M, which must then belong to some M_i by definition of the M_js. As $\mathbf{x}$ is the point which corresponds to this closed path, we have by lemma 5 that $\mathbf{x} \in \Omega_i$ (cf. also the remark after lemma 6). □

LEMMA 8. (i) *If D_i is maximal with respect to* (2.2) *and $\tilde{D}_i = \emptyset$, then Ω_i is a finite union of intervals.*

(ii) *If Σ_{M_i} is a periodic orbit, Ω_i is also a periodic orbit.*

(iii) *Otherwise, Ω_i is a Cantor set.*

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Proof. (i) Let $\mathbf{x}\in F_i\backslash G_i$. Then $\mathbf{x}\in d$ for some $d\in D_i$. By lemma 3, $\mathbf{x}$ corresponds to a path which begins with an arrow ending at d. Because D_i is maximal and $\tilde{D}_i=\emptyset$, this path cannot leave M_i. Extending it to a two-sided path gives a $\mathbf{y}\in\Sigma_{M_i}$ with $\chi(\mathbf{y})=\mathbf{x}$. Hence $\Omega_i=\overline{\chi(\Sigma_{M_i})}=\overline{F_i\backslash G_i}$, a finite union of intervals.

(ii) We have $\Omega_i=\chi(\Sigma_{M_i})$ by definition (cf. the remark after lemma 6).

(iii) The set $\bigcap_{k=0}^{m}\overline{\sigma^{-k}(F_i\backslash G_i)}$ is a finite union of closed intervals. The intersection of these sets for $m\geq 1$ gives Ω_i. Furthermore, Ω_i cannot contain an interval. Suppose ${}_0[x_0\cdots x_{k-1}]\subset\Omega_i$. Then one finds a path of length k in M_i which corresponds to $x_0\cdots x_{k-1}$. We can extend this path to an infinite one which leaves M_i and find an $\mathbf{x}\in{}_0[x_0\cdots x_{k-1}]$ with $\sigma^m\mathbf{x}\in\operatorname{int} G_i$ for some m. Hence $\mathbf{x}\notin\Omega_i$, a contradiction to ${}_0[x_0\cdots x_{k-1}]\subset\Omega_i$. As Σ_{M_i} contains uncountably many elements, the same is true for Ω_i by lemmas 5 and 4. Therefore Ω_i is a Cantor set. □

Next we investigate $\tilde{\Omega}_i$ for $0\leq i\leq\infty$.

LEMMA 9. (i) $h_{\text{top}}(\tilde{\Omega}_i)=0$.

(ii) *Among the* $\tilde{\Omega}_i$ *for* $0\leq i\leq\infty$ *there are only finitely many which are not empty.*

Proof. (i) We have shown in lemma 6 that $\tilde{\Omega}_i\subset\overline{S_i}$. Hence it suffices to show that $\lim_{k\to\infty}(1/k)\log N_k=0$, where N_k is the number of admissible blocks of length k in $\overline{S_i}$. But if the block $x_0\cdots x_{k-1}$ is admissible in $\overline{S_i}$, i.e. ${}_0[x_0\cdots x_{k-1}]\cap\overline{S_i}\neq\emptyset$, then we also have ${}_0[x_0\cdots x_{k-1}]\cap S_i\neq\emptyset$, because ${}_0[x_0\cdots x_{k-1}]$ is open. Hence N_k is also the number of admissible blocks of length k in S_i. Furthermore, we can suppose that G_i is all of Σ_f^+. Otherwise we restrict f to $\phi^{-1}(G_i)$, which again gives a piecewise monotonic transformation. Then N_k is the number of special paths of length k in $\tilde{M}_i$.

Fix some $\varepsilon>0$. By lemma 13 of [**3**] there is a finite subset A of $\tilde{D}_i$ such that the spectral radius $r(\tilde{M}_i/B)$ of $\tilde{M}_i$ restricted to $B=\tilde{D}_i\backslash A$ is less than ε. This implies that the number of paths of length l in $\tilde{M}_i/B$ which begin with one of the finitely many arrows leading from A to B, is less than $C\exp(2\varepsilon l)$ for some constant C (cf. § 3 of [**3**]). Making A larger if necessary, we can also suppose that the finitely many elements of $\tilde{D}_i$ at which special paths begin are contained in A.

Now let K be the cardinality of A. In a special path of length k in $\tilde{M}_i$ every element of $\tilde{D}_i$ can occur at most once, because $\tilde{M}_i$ contains no closed paths. Hence this special path contains at most K blocks consisting of elements of A, each of which has length at most K. One of these blocks is at the beginning of the special path. Hence there are not more than $\binom{k}{K}$ possibilities to choose the places of these blocks in the special path of length k. In between there are blocks consisting of elements of B which begin with an arrow leading from A to B. They have lengths $l_1, l_2, \ldots, l_j$ $(j\leq K)$ with $\sum_{i=1}^{j} l_i\leq k$. Hence we have

$$N_k\leq\binom{k}{K}n^{K^2}\cdot C\exp(2\varepsilon l_1)\cdots C\exp(2\varepsilon l_j)\leq\binom{k}{K}n^{K^2}\cdot C^j\exp(2\varepsilon k).$$

This implies that $\lim(1/k)\log N_k \le 2\varepsilon$. As ε was arbitrary, we obtain $\lim(1/k)\log N_k = 0$.

(ii) If $\tilde{D}_i$ is finite, then $S_i = \varnothing$, and hence $\tilde{\Omega}_i = \varnothing$ by lemma 6. If $\tilde{D}_i$ is infinite, then it contains a set of the form

$$\{\sigma^k({}_0[a_0^i \cdots a_k^i]): k \ge m\} \quad \text{or} \quad \{\sigma^k({}_0[b_0^i \cdots b_k^i]): k \ge m\} \quad \text{for some } i \text{ and } m,$$

because D_i' and $\bar{D}_{i+1}$ have the property (2.3), which implies that $\tilde{D}_i$ satisfies (2.1). Hence there can be at most $2n$ different $\tilde{D}_i$, which contain infinitely many elements, and hence at most $2n$ different non-empty $\tilde{\Omega}_i$. □

Remark. The transformation $x \to x+\alpha \pmod 1$ on $[0, 1]$, $\alpha \notin \mathbb{Q}$, is an example where $\tilde{D}_0 = D$ and $\tilde{\Omega}_0 = \Sigma_f^+$. It would be interesting either to find an example where $\tilde{\Omega}_i \ne \varnothing$ for some i such that there is a $j > i$ with $D_j \ge D_i$, or to show that this cannot happen.

LEMMA 10. *If there are infinitely many M_is, then $h_{\text{top}}\left(\bigcap_{i=1}^{\infty} \bar{F}_i\right) = 0$. In particular, $h_{\text{top}}(\Omega_\infty) = 0$.*

Proof. Fix some $\varepsilon > 0$. We have $\bar{D}_i = (\bar{D}_i \backslash D'_\infty) \cup D'_\infty$. As $\bigcap_{i=1}^{\infty} (\bar{D}_i \backslash D'_\infty) = \varnothing$, there is an i with $r(L_i) < \exp\varepsilon$, where $L_i = M/(\bar{D}_i \backslash D'_\infty)$, by lemma 13 of [**3**].

Let $x_0 x_1 \cdots x_{k-1}$ be an admissible block of length k in $\bar{F}_i$. It follows from the proof of lemma 1 that F_i is a finite union of intervals

$$\sigma^m A_j = \bigcup_{r,l} ([\sigma^m \mathbf{a}^j, \sigma^{m+l}\mathbf{b}^r]) \quad \text{and} \quad \sigma^m B_j = \bigcup_{r,l} ([\sigma^{m+l}\mathbf{a}^r, \sigma^m \mathbf{b}^j]).$$

If ${}_0[x_0 \cdots x_{k-1}] \cap \overline{\sigma^m A_j} \ne \varnothing$, then there are r and l such that

$$[\sigma^m \mathbf{a}^j, \sigma^{m+l}\mathbf{b}^r] \cap {}_0[x_0 \cdots x_{k-1}] \ne \varnothing.$$

By lemma 3, $x_0 \cdots x_{k-1}$ can then be represented as a path of length k in $M/\bar{D}_i$ which begins at $[\sigma^m \mathbf{a}^j, \sigma^{m+l}\mathbf{b}^r] \in \bar{D}_i$. Let z be the number of intervals of which F_i consists. Then for every k, we can find z elements $d_1, \ldots, d_z$ of $\bar{D}_i$ such that every admissible block in $\bar{F}_i$ of length k can be represented as path of length k in $M/\bar{D}_i$ beginning at one of these z elements.

By definition of D'_∞, no path leads from D'_∞ to $\bar{D}_i \backslash D'_\infty$. Hence for every admissible block $x_0 \cdots x_{k-1}$ there is an l $(0 \le l \le k)$ such that $x_0 \cdots x_{l-1}$ corresponds to a path in $\bar{D}_i \backslash D'_\infty$ and $x_l \cdots x_{k-1}$ to a path in D'_∞. Hence the number of admissible blocks of length k beginning at some d_j $(1 \le j \le z)$ is less than $\sum_{l=0}^{k} N_l' N_{k-l}''$, where N_l' is the number of admissible blocks of length l in $\bar{D}_i \backslash D'_\infty$ beginning at d_j and N_{k-l}'' is the maximal number of blocks of length $k-l$ in D'_∞ beginning at some element of D'_∞.

Let u be the vector, with index set $\bar{D}_i \backslash D'_\infty$, which has entry 1 at the d_jth coordinate and entry 0 otherwise. If $d_j \in D'_\infty$, then u has only zero entries (this corresponds to the case $l = 0$). Then $N_l' = \|L_i^l u\|_1 \le \|L_i^l\|_1 \le C \exp(2\varepsilon l)$, for some constant C, because $r(L_i) < \varepsilon$.

In the proof of lemma 9, we have shown that $h_{top}(S_\infty) = 0$. Hence $N''_{k-l} \le C' \exp(2\varepsilon(k-l))$ for some constant C'. If N_k is the number of admissible blocks of length k in $\tilde{\mathbf{D}}_i$, we therefore have $N_k \le zkCC' \exp(2\varepsilon k)$. This implies that $h_{top}(\bar{F}_i) < 2\varepsilon$ and hence

$$h_{top}\left(\bigcap_{i=1}^{\infty} \bar{F}_i\right) < 2\varepsilon.$$

As ε was arbitrary, this gives the desired result. □

Remark. An example, where one has infinitely many M_is and $\Omega_\infty \neq \emptyset$, is $x \to ax(1-x)$ on [0, 1] for certain values of $a \in [2, 4]$.

Let Z be the set of all non-wandering $\mathbf{x} \in \bigcup_{0 \le i < \infty} Z_i$ which are not contained in some Ω_i or $\tilde{\Omega}_i$.

LEMMA 11. (i) *If* $\mathbf{x} \in Z$, *then* $\mathbf{x}$ *is an isolated point of* Ω *which is not periodic. Hence* $\mathbf{x}$ *is wandering in* Ω.

(ii) Z *is finite.*

Proof. (i) Let $\mathbf{x} \in Z \cap Z_i$ $(i < \infty)$. Then

$$\mathbf{x} \notin \bar{F}_{i+1} \supset \bigcup_{j>i} (\Omega_j \cup \tilde{\Omega}_j \cup Z_j).$$

Also $\mathbf{x} \notin \bigcup_{j \le i} (\Omega_j \cup \tilde{\Omega}_j)$, which is a closed set because Ω_i and $\tilde{\Omega}_i$ are closed. By lemma 2, we have

$$\Omega \subset \bigcup_{j \le i} (\Omega_j \cup \tilde{\Omega}_j) \cup \bar{F}_{i+1} \cup \bigcup_{j \le i} Z_j.$$

As $\bigcup_{j \le i} Z_j$ is finite, we find a neighbourhood U of $\mathbf{x}$ such that $U \cap \Omega = \{\mathbf{x}\}$. The non-periodicity of $\mathbf{x}$ follows from the definition of Z_i.

(ii) It follows from (2.1) that there are only finitely many D_i, say $D_{i_1}, \ldots, D_{i_r}$, which have infinitely many elements. By the proof of (ii) of lemma 9, there are only finitely many $\tilde{D}_i$, say $\tilde{D}_{j_1}, \ldots, \tilde{D}_{j_s}$, which have infinitely many elements.

If D_i is finite, then $\bigcup \{d : d \in D_i\}$ is already closed, because every $d \in D$ is closed. Hence $\text{bd}\, F_i \backslash \text{bd}\, G_i \subset \bigcup \{d : d \in D_i\}$ and an $\mathbf{x} \in \text{bd}\, F_i \backslash \text{bd}\, G_i$ is at the boundary of some $d \in D_i$. But then $\mathbf{x} = \sigma^k \mathbf{a}^m$ or $\sigma^k \mathbf{b}^m$ for some k and m (cf. (1.2)). As $\sigma^l \mathbf{x}$ is periodic for some l, we then have $\sigma^{k+l} \mathbf{a}^m$ (or $\sigma^{k+l} \mathbf{b}^m$) is periodic. Similar arguments apply for a finite $\tilde{D}_i$. Hence

$$\begin{aligned} Z \subset \{\sigma^{k_i} \mathbf{a}^i, \sigma^{m_j} \mathbf{b}^j : 1 \le i, j \le n, 0 \le k_i \le K_i - 1, 0 \le m_j \le M_j - 1, \\ \sigma^{K_i} \mathbf{a}^i \text{ and } \sigma^{M_j} \mathbf{b}^j \text{ are periodic}\} \\ \cup \bigcup_{t=1}^{r} Z_{i_t} \cup \bigcup_{t=1}^{s} Z_{j_t}. \end{aligned}$$

This is a finite set. □

We collect the results in the following theorem.

THEOREM 1. $\Omega = \bigcup_{i\geq 1} \Omega_i \cup Y \cup Z$ *(finite or countable union) such that*:

(i) Ω_i *and* Y *are closed,* σ*-invariant sets.* Z *is finite and wandering in* Ω.

(ii) $\Omega_i \cap \Omega_j$ *for* $i \neq j$ *and* $\Omega_i \cap Y$ *are at most finite.* $\Omega_i \cap Z = \emptyset$, $Y \cap Z = \emptyset$.

(iii) *All periodic points are contained in* $\bigcup_{i\geq 1} \Omega_i$.

(iv) Ω_i *is topologically transitive. It is either a finite union of intervals, a Cantor set, or a periodic orbit.*

(v) $h_{\text{top}}(Y) = 0$.

Proof. We set $Y = \left(\bigcup_{0\leq i\leq\infty} \tilde{\Omega}_i \cup \bigcap_{i=1}^{\infty} \bar{F}_i \right) \cap \Omega$ if there are infinitely many M_is, and $Y = \bigcup_{0\leq i\leq\infty} \tilde{\Omega}_i \cap \Omega$ if there are finitely many M_is. Because $\Omega_\infty \cup Z_\infty \subset \bigcap_{i=1}^{\infty} \bar{F}_i$, it follows from lemma 2 that $\Omega = \bigcup_{i\geq 1} \Omega_i \cup Y \cup Z$. By (ii) of lemma 9, $\bigcup_{0\leq i\leq\infty} \tilde{\Omega}_i$ is a finite union of closed sets, hence Y is closed. Y is σ-invariant, because $\tilde{\Omega}_i$, $\bar{F}_i$ and Ω are σ-invariant. Together with lemma 11, this implies (i). (ii) follows from definitions, because $\bigcup_{0\leq i\leq\infty} \tilde{\Omega}_i$ is a finite union. (iii) and (iv) are lemmas 7 and 8, (v) follows from lemmas 9 and 10. □

Now we investigate Ω_i for some fixed $i \neq \infty$. One says that the 0–1-matrix M_i has period q if $D_i = C_1 \cup C_2 \cup \cdots \cup C_q$ (disjoint) and if $d \in C_k$ and $M_{dd'} = 1$ imply $d' \in C_{k+1}$ (we take the indices of the C_js modulo q). q is taken as large as possible. If $q = 1$, M_i is called aperiodic. Set $K_j = (F_i \backslash G_i) \cap \bigcup \{d : d \in C_j\}$.

LEMMA 12. K_j *is a finite union of intervals.* $K_j \cap K_m$ $(j \neq m)$ *is empty or finite.*

Proof. One shows that, if $d_1, d_2 \in D_i$ and the interval $d_1 \cap d_2$ contains more than one point, then d_1 and d_2 are in the same C_j. This proves both assertions. It proves the first assertion, because it implies that the A_is and B_is in the proof of lemma 1 are subsets of one of the K_js. If $d_1 \cap d_2$ is a non-trivial interval, one can choose $\mathbf{y}^1$, $\mathbf{y}^2 \in \Sigma_{M_i}$ with $y_0^1 = d_1$ and $y_0^2 = d_2$ such that $\chi(\mathbf{y}^1) = \chi(\mathbf{y}^2) \notin E$ (lemma 3 and E is countable). By lemma 4 there is a t with $y_t^1 = y_t^2 \in C_m$ for some m. Hence y_{t-i}^1 and y_{t-i}^2 are in C_{m-i}. In particular, $y_0^1 = d_1$ and $y_0^2 = d_2$ are both in C_{m-t}, proving the lemma. □

Set $X_j = \overline{K_j \cap \Omega_i}$. We have

THEOREM 2. $\Omega_i = \bigcup_{j=1}^{q} X_j$. X_j *is closed and* $X_j \cap X_k$ *is empty or finite* $(j \neq k)$. $\sigma(X_j) \subset X_{j+1}$. $\sigma^p : X_j \to X_j$ *is topologically mixing.*

Proof. It remains to prove the last assertion. (X_j, σ^q) is an Ω_m for the piecewise monotonic transformation $(\phi^{-1}(K_j), f^q)$. The matrix $\tilde{M}$ corresponding to (X_j, σ^q) can be derived from M_i as follows. Set

$$\tilde{D} = \{y_1 \cdots y_q : y_k \in C_{j+k-1}, M_{y_k y_{k+1}} = 1 \quad \text{for } 1 \leq k \leq q-1\}.$$

We have an arrow $y_1 \cdots y_q \to y_1' \cdots y_q'$ in $\tilde{M}$ iff there is an arrow $y_q \to y_1'$ in M_i. $\tilde{M}$ is irreducible because M_i is. $\tilde{M}$ is aperiodic. If $\tilde{M}$ has period $\tilde{q}$, then it follows that

M_i has period $q\tilde{q}$. In the next section we shall see that there is a σ-invariant, mixing measure on $\Sigma_{\tilde{M}}$ which is positive on open sets if $h_{top}(\Sigma_{M_i}) \neq 0$, i.e. Σ_{M_i} is not a periodic orbit. It follows from proposition (6.7) of [**2**] that $\Sigma_{\tilde{M}}$ and hence also (X_j, σ^q) are topologically mixing. If Σ_{M_i} is a periodic orbit, the desired result is trivial.

3. *The maximal measure*

In this section we consider a fixed Ω_i satisfying $h_{top}(\Omega_i) > 0$, i.e. Ω_i is not only a periodic orbit. Therefore we denote Ω_i, F_i, M_i, D_i simply by Ω, F, M, D respectively. $\Omega \subset \Sigma_f^+ \subset \Sigma_n^+$ is expansive, hence Ω has at least one measure with maximal entropy (cf. [**2**]). It is proved in [**3**] that (Ω, σ) and (Σ_M, σ) have isomorphic sets of maximal measures via $\mu \to \mu \circ \chi^{-1}$ and that there is a unique maximal measure μ on Σ_M given by $\mu(_0[y_0 \cdots y_{k-1}]) = \pi_{y_0} P_{y_0 y_1} \cdots P_{y_{k-2} y_{k-1}}$, where $\pi_d = u_d v_d$, $P_{de} = M_{de} v_e / \lambda v_d$ $(d, e \in D)$. u is the unique (up to constant factors) positive left and v the unique positive right eigenvector of M for the eigenvalue λ satisfying $\Sigma u_d v_d = 1$. $\lambda = r(M)$ is the spectral radius of the l^1-operator $w \to wM (w \in l^1)$. $\lambda > 1$, because $\log \lambda = h_{top}(\Omega) > 0$. § 3 of [**3**] shows that $h_{top}(\Omega) \leq \log \lambda$. The converse inequality follows from the variational principle (cf. [**2**]): $h_{top}(\Omega) \geq h(\mu) = \log \lambda$.

π is a vector and P is a matrix with index set D satisfying $\pi_d > 0$, $P_{de} \geq 0$ $(d, e \in D)$, $\pi P = \pi$, $\sum_{e \in D} P_{de} = 1$ $(d \in D)$ and $\sum_{d \in D} \pi_d = 1$. Hence π and P give rise to a Markov chain with countable state space. Assuming that M and hence also P are aperiodic, we can use the results proved in probability theory (cf. [**1**]).

$$(\pi, P) \text{ is recurrent.} \tag{3.1}$$

$$P_{de}^{(n)} \to \pi_e \ (n \to \infty) \quad \text{for } d, e \in D. \tag{3.2}$$

$P_{de}^{(n)}$ denotes an entry of the matrix P^n. It follows from (3.2) that

$\mu(_0[y_0 \cdots y_{i-1}] \cap {_k[z_0 \cdots z_{j-1}]})$ converges to $\mu(_0[y_0 \cdots y_{i-1}])\mu(_0[z_0 \cdots z_{j-1}])$ as $k \to \infty$.

Hence μ is mixing. This also completes the proof of theorem 2. $M_{de}^{(n)}$ is the number of admissible blocks of length $n+1$ in Σ_M beginning with d and ending with e. Therefore $p_n = \sum_{d \in D} M_{dd}^{(n)}$ is the number of periodic points of period n in Σ_M ($p_n < \infty$ by remark (i) after lemma 4). Set $\mu_n = (1/p_n) \sum_{\mathbf{y} \in Q_n} \delta_{\mathbf{y}}$, where Q_n is the set of all periodic points of period n in Σ_M and $\delta_{\mathbf{y}}$ the measure concentrated in $\mathbf{y} \in \Sigma_M$.

THEOREM 3. *If M is aperiodic, we have*:

(i) $\lim_{n \to \infty} \lambda^{-n} p_n = 1$.

(ii) *μ_n converges in the weak topology to the unique maximal measure μ of Σ_M.*

Proof. Choose a sequence n_i of integers such that $\lambda^{-n_i} p_{n_i}$ converges to C $(0 \leq C \leq \infty)$. The sequence μ_{n_i} has a limit point ν. We suppose that it converges (take again a subsequence, if necessary). We have

$$\begin{aligned}\mu_{n_i}({}_0[y_0\cdots y_{k-1}]) &= p_{n_i}^{-1}\operatorname{card}\{\mathbf{x}\in Q_{n_i}: x_m=y_m \quad \text{for } 0\le m\le k-1\}\\ &= p_{n_i}^{-1}M_{y_{k-1}y_0}^{(n_i-k+1)}\\ &= p_{n_i}^{-1}\lambda^{n_i}\lambda^{-k+1}P_{y_{k-1}y_0}^{(n_i-k+1)}v_{y_{k-1}}/v_{y_0}\\ &\to C^{-1}\lambda^{-k+1}\pi_{y_0}v_{y_{k-1}}/v_{y_0}\\ &= C^{-1}\pi_{y_0}P_{y_0y_1}\cdots P_{y_{k-2}y_{k-1}}\\ &= C^{-1}\mu({}_0[y_0\cdots y_{k-1}]).\end{aligned}$$

Hence $\nu({}_0[y_0\cdots y_{k-1}])=C^{-1}\mu({}_0[y_0\cdots y_{k-1}])$. As ν is a probability measure, we have $C=1$ proving (i). But then the computation above shows that every limit point of μ_n is μ. This is (ii). □

If M has period $q>1$, then $\Omega=X_1\cup\cdots\cup X_q$ and every X_j is an Ω_i for the piecewise monotonic transformation f^q. In the proof of theorem 2 we have computed $\tilde{M}$ corresponding to (X_j,σ^q), which has period 1. Applying theorem 3 to $\tilde{M}$ we have:

COROLLARY. *If M has period q, we have*:

(i) $p_n=0$, *if q is not a divisor of n*, $\lim_{n\to\infty} p_{nq}\lambda^{-nq}=1$.

(ii) μ_{nq} *converges in the weak topology to the unique maximal measure μ of Σ_M.*

By lemmas 4, 5 and the second assertion of lemma 6 these results are also valid for Ω.

Now we turn to a result about the number b_k of admissible blocks of length k in Ω. For this we need that the left eigenvector u of M is in l^1 (the proof for this result is not published).

THEOREM 4. *If M is aperiodic, we have that $\lim_{k\to\infty}\lambda^{-k}b_k$ exists and is greater than zero.*

Proof. Without loss of generality assume that $F=F_i=\Sigma_f^+$, considering $f|\phi^{-1}(F_i)$ instead of f. Set $D'=\{{}_0[k]: 1\le k\le n\}\subset D$. b_k is the number of admissible blocks of length k in Σ_M which begin with an element of D', because of the 1–1 correspondence of special paths in M and points in Σ_f^+. Hence

$$b_k=\sum_{d\in D'}\sum_{e\in D}M_{de}^{(k-1)}.$$

We have for $d\in D'$ and $e\in D$

$$\begin{aligned}\lambda^{-m}M_{de}^{(m)} &= P_{de}^{(m)}v_dv_e^{-1}=\pi_d^{-1}v_dv_e^{-1}\pi_dP_{de}^{(m)}\\ &\le (u_dv_e)^{-1}\sum_{g\in D}\pi_gP_{ge}^{(m)}=(u_dv_e)^{-1}\pi_e\\ &= u_e/u_d\le c^{-1}u_e,\quad \text{where } c=\min_{d\in D'}u_d>0.\end{aligned}$$

As $u\in l^1$, we have for every $\varepsilon>0$ a subset D_ε of D with $D\backslash D_\varepsilon$ finite, such that

$$\sum_{e\in D_\varepsilon}\lambda^{-m}M_{de}^{(m)}<\varepsilon\quad \text{for all } m\ge 1.$$

Hence we can interchange the limit and the sums.

$$\lim_{k\to\infty} \lambda^{-k} b_k = \lambda^{-1} \sum_{d\in D'} \sum_{e\in D} \lim P_{de}^{(k-1)} v_d / v_e$$

$$= \lambda^{-1} \sum_{d\in D'} \sum_{e\in D} \pi_e v_d / v_e = \lambda^{-1} \sum_{d\in D'} \sum_{e\in D} u_e v_d.$$

This is a positive constant, since $u \in l^1$. □

As for theorem 3 we can generalize this result to the periodic case.

4. *Absolutely continuous invariant measures*

We consider the problem of finding an invariant measure μ on $([0, 1], f)$, which is absolutely continuous with respect to the Lebesgue measure λ on $[0, 1]$. It suffices to consider this problem for $\phi^{-1}(\Omega_i)$ and $\phi^{-1}(\tilde{\Omega}_i)$ (we assume that ϕ is injective and denote $\phi^{-1}(\Omega_i)$ again by Ω_i), because every invariant measure is a linear combination of measures concentrated on these sets.

If f is piecewise C^2 and $|f'(x)| \ge d > 1$ for all $x \in [0, 1]$, the sets L_j considered in [**6**] are exactly those Ω_is, which are finite unions of intervals. There is exactly one ergodic invariant measure μ absolutely continuous with respect to λ on every L_j. All other Ω_is and all $\tilde{\Omega}_i$s have Lebesgue measure zero (theorem 2 of [**6**]).

In [**5**] one finds an example of an f on $[0, 1]$, piecewise C^2, $f'(x) > 1$ for $x \in (0, 1]$ and $f'(0) = 1$, which has no finite invariant measure absolutely continuous with respect to λ. Below, we give an example of an f, piecewise C^1, $f'(x) \ge 2$ for all x, which has a Cantor set Ω_1 with $\lambda(\Omega_1) > 0$ and λ/Ω_1 is f-invariant.

We consider $[0, 2]$ instead of $[0, 1]$. Define f on $(1, 2]$ by $f(x) = 2x - 1$ for $1 < x \le \frac{3}{2}$ and $2x - 2$ for $\frac{3}{2} < x \le 2$ (or in any other way such that $f((1, 2]) \subset (1, 2])$. $\Omega_2 = [1, 2]$.

Now consider $[0, 1]$. Set $a_i = c/i^2 2^{i-1}$, for some c with $0 < c < (\sum_{i=1}^{\infty} i^{-2})^{-1}$. We define open sub-intervals A_m^k $(1 \le k \le 2^{m-1},\ m \ge 1)$ of $[0, 1]$. A_1^1 has midpoint $\frac{1}{2}$ and length a_1. Let B_2^1 and B_2^2 be the two closed sub-intervals of which $[0, 1] \backslash A_1^1$ consists. The midpoint of A_2^k is the midpoint of B_2^k and the length of A_2^k is $a_2 (k = 1, 2)$. The mth step is as follows. Let $B_m^k (1 \le k \le 2^{m-1})$ be the closed intervals of equal length of which

$$[0, 1] \backslash \bigcup_{i=1}^{m-1} \bigcup_{j=1}^{2^{i-1}} A_i^j$$

consists. Then the midpoint of A_m^k is the midpoint of B_m^k and the length of A_m^k is $a_m (1 \le k \le 2^{m-1})$. The A_m^ks are pairwise disjoint, because the length of all the A_m^ks $(1 \le k \le 2^{m-1},\ m \ge 1)$ together is

$$\sum_{m=1}^{\infty} 2^{m-1} a_m = \sum_{m=1}^{\infty} c m^{-2} < 1. \tag{4.1}$$

Set

$$C = [0, 1] \backslash \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} A_i^j = \bigcap_{i=2}^{\infty} \bigcup_{j=1}^{2^{i-1}} B_i^j.$$

To define f on $[0, 1]$, we define first $f' = g \in C([0, 1])$. Set $g(x) = 2$ for $x \in C$. On A^k_m define g such that it is greater than or equal to 2 and continuous (i.e. $\lim g(x) = 2$, where $x \in A^k_m$ approaches one of the endpoints of A^k_m and

$$\sup\left\{|g(x)-2| : x \in \bigcup_k A^k_m\right\} \to 0 \quad \text{as } m \to \infty)$$

and that

$$\int_{A^k_m} g(x)\,dx = \begin{cases} a_{m-1} & \text{for } m \geq 2, \\ 1 & \text{for } m = 1. \end{cases} \tag{4.2}$$

This is possible, because $a_i/a_{i+1} = 2[(i+1)/i]^2 \downarrow 2$, if $i \to \infty$. Define

$$f(x) = \begin{cases} \displaystyle\int_0^x g(t)\,dt & \text{for } x \in B^2_1 \cup A^1_1 \\ \displaystyle\int_d^x g(t)\,dt & \text{for } x \in B^2_2, \end{cases}$$

where $d = \frac{1}{2}(1 + a_1)$ is the initial point of B^2_2. Then $f/B^2_1 \cup A^1_1$ and f/B^2_2 are increasing and C^1. $f'(x) = g(x) \geq 2$ for $x \in [0, 1] \setminus \{d\}$. We have

$$\lambda(B^k_m \cap C) = 2^{-m+1}\left(1 - \sum_{i=1}^{m-1} 2^{i-1} a_i\right) - \sum_{i=1}^{\infty} 2^{i-1} a_{i+m-1}.$$

From this it follows that

$$2\lambda(B^k_m \cap C) = \lambda(B^{k'}_{m-1} \cap C), \tag{4.3}$$

where $k' = k \pmod{2^{m-2}}$. We show that $f(A^k_m) = A^{k'}_{m-1}(=[1, 2]$, if $m = 1)$. To this end we prove that $\lambda(f(A^k_m)) = \lambda(A^{k'}_{m-1})(=\lambda([1, 2])$, if $m = 1)$ and $\lambda(f(B^k_m)) = \lambda(B^{k'}_{m-1})$ (set $B^1_1 = [0, 1]$). The first assertion follows because of (4.2). For the second assertion remark that

$$B^k_m = (C \cap B^k_m) \cup A^k_m \cup A^{2k-1}_{m+1} \cup A^{2k}_{m+1} \cdots.$$

This is a union of disjoint sets. Hence

$$\begin{aligned} \lambda(f(B^k_m)) &= \int_{B^k_m} g(x)\,dx = \int_{C \cap B^k_m} 2\,dx + \int_{A^k_m} g(x)\,dx + \cdots \\ &= 2\lambda(B^k_m \cap C) + a_{m-1} + 2a_m + 4a_{m+1} + \cdots \\ &= \lambda(B^{k'}_{m-1} \cap C) + \lambda(A^{k'}_{m-1}) + \lambda(A^{2k'-1}_m \cup A^{2k'}_m) + \cdots \\ &= \lambda(B^{k'}_{m-1}). \end{aligned}$$

Because $f|B^1_2 \cup A^1_1$ and $f|B^2_2$ are increasing and $[0, 1]$ is the disjoint union of the intervals A^j_i $(1 \leq j \leq 2^{i-1}$, $1 \leq i \leq m-1)$ and B^i_m $(1 \leq i \leq 2^{m-1})$, it follows that $f(A^k_{m-1}) = A^{k'}_{m-2}$ and $f(B^k_m) = B^{k'}_{m-1}$ $(m \geq 2)$.

Now it follows that

$$\Omega_1 = \bigcap_{k=1}^{\infty} f^{-k}([0, 1]) = [0, 1] \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} A^j_i = C.$$

By (4.3) we have that λ/Ω_1 is invariant and by (4.1) that $\lambda(\Omega_1) > 0$.

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