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THE STRUCTURE OF RADIAL SOLUTIONS FOR ELLIPTIC EQUATIONS ARISING FROM THE SPHERICAL ONSAGER VORTEX

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Abstract. In this paper, we consider a nonlinear elliptic equation on the plane away from the origin, which arises from the spherical Onsager vortex theory in physics or the problem of prescribing Gaussian curvature in geometry. Depending on various situations for the prescribed function in the nonlinear term, the complete structure of radial solutions in terms of initial data will be offered.

1. Introduction. This paper is concerned with the structure of radial solutions of the nonlinear elliptic equation

$$\Delta u(x) + K(|x|)e^{2u(x)} = 0 \quad \text{in} \quad \boldsymbol{R}^2 \setminus \{\boldsymbol{0}\}, \tag{1.1}$$

where $\Delta = \sum_{i=1}^{2} \partial^2/\partial x_i^2$ is the Laplacian operator of $\boldsymbol{R}^2$ and $K(|x|)$ is a given nonnegative function in $\boldsymbol{R}^2 \setminus \{\boldsymbol{0}\}$. One interesting motivation in studying (1.1) arises from the spherical Onsager vortex theory, which bridges the gap between statistical mechanics of classical vortices and the random surface problem. We give a brief description as follows. Let $S^2 = \{y = (y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 = 1\}$ be the unit sphere of $\boldsymbol{R}^3$ and consider the mean field equation

$$\Delta_{S^2}\phi(y) + \frac{e^{\beta\phi(y)-\gamma\langle n,y\rangle}}{\int_{S^2} e^{\beta\phi(y)-\gamma\langle n,y\rangle}\, d\mu} - \frac{1}{4\pi} = 0, \quad y \in S^2, \tag{1.2}$$

where Δ_{S^2} is the Beltrami-Laplace operator with respect to the standard metric on S^2, $\beta \geq 0$, γ is a constant in $\boldsymbol{R}$, n is a unit vector on S^2 and $d\mu$ is the uniform measure or the surface element on S^2. We note that equation (1.2) comes from the spherical Onsager vortex theory which was studied in [2], [8] and [11]. To rewrite (1.2) in the coordinates of the plane $\boldsymbol{R}^2$, we assume $n = (0, 0, 1)$ and let

$$y_1 = \frac{2x_1}{1+r^2}, \quad y_2 = \frac{2x_2}{1+r^2}, \quad y_3 = \frac{r^2-1}{1+r^2}, \tag{1.3}$$

where $x_1, x_2 \in \boldsymbol{R}$ and $r^2 = x_1^2 + x_2^2$. Then the above correspondence, denoted by $\pi((x_1, x_2)) = (y_1, y_2, y_3)$, is the inverse of the stereographic projection from n (the north pole of S^2) onto

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$\boldsymbol{R}^2$. Moreover, it is easy to see that the standard metric ds^2 on S^2 is

$$\begin{aligned} ds^2 &= dy_1^2 + dy_2^2 + dy_3^3 \\ &= \left(\frac{2}{1+r^2}\right)^2 (dx_1^2 + dx_2^2)\,, \end{aligned}$$

and hence, in $\boldsymbol{R}^2$, we obtain

$$\Delta_{S^2} = \left(\frac{1+r^2}{2}\right)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) = \left(\frac{1+r^2}{2}\right)^2 \Delta \tag{1.4}$$

and

$$d\mu = \left(\frac{2}{1+r^2}\right)^2 dx_1 dx_2 = \left(\frac{2}{1+r^2}\right)^2 dx\,. \tag{1.5}$$

Now, set $v(x) = \phi(\pi(x))$ for $x \in \boldsymbol{R}^2$. Then, by combining the facts (1.2), (1.4) and (1.5), we know $v(x)$ satisfies

$$\Delta v(x) + \frac{I^2(x)e^{\beta v(x) - \gamma\psi(x)}}{\int_{\boldsymbol{R}^2} I^2(x) e^{\beta v(x)-\gamma\psi(x)}\,dx} - \frac{I^2(x)}{4\pi} = 0\,, \quad x \in \boldsymbol{R}^2\,, \tag{1.6}$$

where

$$I(x) = \frac{2}{1+|x|^2} \quad \text{and} \quad \psi(x) = \frac{|x|^2-1}{1+|x|^2}\,.$$

Furthermore, if we define

$$u(x) = \left(\frac{\beta}{2}\right)\left[v(x) - \frac{1}{4\pi}\ln(1+|x|^2)\right] + J$$

with

$$J = \frac{1}{2}\left\{\gamma + \ln\left(\frac{2}{\beta}\int_{\boldsymbol{R}^2} I^2(x)e^{\beta v - \gamma\psi}\,dx\right)\right\},$$

then $u(x)$ satisfies

$$\Delta u(x) + K(x)e^{2u(x)} = 0\,, \quad x \in \boldsymbol{R}^2\,, \tag{1.7}$$

where

$$K(x) = \left(1+|x|^2\right)^{(-8\pi+\beta)/(4\pi)} e^{\gamma I(x)}\,,$$

and (1.7) is exactly the form we are dealing with in (1.1).

Another reason for studying (1.1) comes from the problem of prescribing Gaussian curvature. Let (M, g_0) be a Riemannian manifold of two dimension. For a given function K on M, one would like to ask whether there exists a metric g, which is conformal to g_0, such that $K/2$ is the Gaussian curvature of g. Let $g = e^{2u}g_0$ for some function u on M. Then the problem above is equivalent to solving the equation

$$\Delta_{g_0} u - k_0 + Ke^{2u} = 0 \quad \text{in } M\,,$$

where Δ_{g_0} is the Beltrami-Laplace operator and k_0 is the scalar curvature of g_0. If (M, g_0) is the standard flat plane $\boldsymbol{R}^2$, then we have that $k_0 \equiv 0$ and the above equation is reduced to (1.1). We note that any radial solution $u = u(r)$, $r = |x|$, reduces (1.1) to the following ordinary differential equation

$$u''(r) + \frac{1}{r}u'(r) + K(r)e^{2u(r)} = 0\,, \quad r > 0\,.$$

Before getting into the main theme of this paper, we first consider the specific case of $K(|x|) \equiv 1/2$ in (1.1) which is known as the Liouville equation. By virtue of [3], it is well-known that any solution $u(x)$ of (1.1) which is defined in the whole space $\boldsymbol{R}^2$ is radially symmetric if

$$\int_{\boldsymbol{R}^2} e^{2u(x)}dx < +\infty\,,$$

and can be explicitly expressed as the form

$$u(x) = u(|x|) = \frac{1}{2}\ln\frac{32A^2}{(4 + A^2|x|^2)^2}\,, \quad A > 0\,,\ x \in \boldsymbol{R}^2\,.$$

Therefore, we see that

$$\lim_{r\to\infty}\frac{u(r)}{2\ln r} = -1\,.$$

Moreover, using the change of variables, we can derive that each radial solution $u(r)$ of (1.1) in such case is of the form

$$u(r) = \frac{1}{2}\ln\frac{8B^2}{r^2((Cr)^B + (Cr)^{-B})^2}\,, \quad B, C > 0\,,\ r > 0\,. \tag{1.8}$$

Indeed, by letting $t = -\ln r^2$ and $w(t) = 2u(r) - t - \ln 4$, the original equation can be transformed into the equation $w'' + e^w = 0$, which possesses the general solution which is displayed as

$$w(t) = \ln\frac{B^2}{1 + \cosh(Bt + b)}\,,$$

where $B > 0$ and $b \in \boldsymbol{R}$. According to (1.8), we remark that the structure of solutions for (1.1) in the case of $K(|x|) \equiv 1/2$ is exactly illustrated as in Theorem 1.4 and Figure 4 which will be offered later.

On the other hand, Cheng-Lin [5] shows that if $K(|x|)$ is nonconstant and non-increasing in $|x|$, then the solution of (1.1) defined on the whole plane is radially symmetric under certain conditions on $K(|x|)$.

In this article, we are interested in studying the following initial value problem

$$\begin{cases} u''(r) + \dfrac{1}{r}u'(r) + K(r)e^{2u(r)} = 0\,, & r > 0\,,\\ u(1) = \theta\,, \quad u'(1) = \eta\,, \end{cases} \tag{1.9}$$

where $\theta, \eta \in \boldsymbol{R}$ are given initial data, $K(r)$ is a non-negative C^1 function on $(0, \infty)$ satisfying

$$\begin{cases} K(r) = K_0 r^{2p} + O(r^{2p+k}) & \text{near } r = 0\,,\\ K(r) = K_\infty r^{2q} + O(r^{2q-l}) & \text{near } r = \infty\,,\\ K'(r) = 2qK_\infty r^{2q-1} + O(r^{2q-1-m}) & \text{near } r = \infty \end{cases} \tag{1.10}$$

for some positive constants K_0 and K_∞, where $p > -1$, $q < 1$ and $k, l, m > 0$. We note that (1.9) possesses a unique solution on $(0, \infty)$ which will be denoted by $u(r; \theta, \eta)$. It will be shown that solutions of (1.9) can be categorized into various types introduced in Definitions 1.1 and 1.2.

DEFINITION 1.1. Any solution $u(r)$ of (1.9) is classified as follows according to its behavior as $r \to 0$.

Type R-$*$: $u(r)$ is regular at 0, i.e., $u(r)$ converges to a constant as $r \to 0$.
Type P-$*$: $u(r)$ is positively singular at 0, i.e., $u(r) \to +\infty$ as $r \to 0$.
Type N-$*$: $u(r)$ is negatively singular at 0, i.e., $u(r) \to -\infty$ as $r \to 0$.

REMARK 1.1. It will be shown, as in Lemma 2.1, that any solution of (1.9) behaves like $C \ln r + O(1)$ at infinity for some $C \in \boldsymbol{R}$.

DEFINITION 1.2. Any solution $u(r)$ of (1.9) is classified as follows according to its behavior as $r \to \infty$.

Type $*$-R$^\pm$: $\pm u(r)/(2 \ln r)$ converges to 1 as $r \to \infty$.
Type $*$-F$^\pm$: $\pm u(r)/(2 \ln r)$ converges to a positive constant which is greater than 1 as $r \to \infty$.
Type $*$-S$^\pm$: $\pm u(r)/(2 \ln r)$ converges to a positive constant which is less than 1 as $r \to \infty$.
Type $*$-C: $u(r)$ converges to a constant as $r \to \infty$.

In this paper, we offer the structures of solutions for (1.9) under various conditions involving (1.10). We will apply the shooting arguments (see, e.g., [1], [7], [12] and references therein) to deal with our problem. To achieve our goal, we introduce the following initial value problems:

$$\begin{cases} \{rU_R'(r)\}' + rK(r)e^{2U_R(r)} = 0\,, & r > 0\,, \\ U_R(0) = a\,, \end{cases} \tag{1.11}$$

$$\begin{cases} \{rU_{R^-}'(r)\}' + rK(r)e^{2U_{R^-}(r)} = 0\,, & r > 0\,, \\ \lim_{r\to\infty} (U_{R^-}(r) + 2\ln r) = b\,, \end{cases} \tag{1.12}$$

where $a, b \in \boldsymbol{R}$. By $p > -1$ and $q < 1$ (ref. [6], [10]), we denote the unique solutions of (1.11) and (1.12) by $U_R(r; a)$ and $U_{R^-}(r; b)$, respectively. We note that from (1.10) and (1.11), $rU_R'(r) \to 0$ as $r \to 0$ and

$$rU_R'(r) = -\int_0^r sK(s)e^{2U_R(s)}ds\,, \quad r > 0$$

since $p > -1$. Define

$$\gamma_1(a) = (U_R(1; a), U_R'(1; a))$$

and

$$\gamma_2^-(b) = (U_{R^-}(1; b), U'_{R^-}(1; b))$$

for $a, b \in \boldsymbol{R}$, and let Γ_1 and Γ_2^- be the ranges of γ_1 and γ_2^- over $\boldsymbol{R}$, respectively. We note that both γ_1 and γ_2^- are smooth by the assumptions $p > -1$ and $q < 1$ again. In fact, Γ_1 and Γ_2^- are the collections of initial data corresponding to solutions of Type R-$*$ and $*$-R$^-$ for (1.9), respectively.

REMARK 1.2. In [9] and [12], the authors introduce the idea of the canonical forms to convert results for one problem to that of others. Based on this idea, the structure of radial solutions including solutions with singularity of several equations, such as the Matukuma's equation or equations with power nonlinearities, can be investigated more precisely. Briefly speaking, suppose that an equation for radial solutions can be rewritten in the form

$$(g(r)u')' + h(r, u) = 0 \quad \text{in } (r_1, r_2)$$

with a boundary condition, where $g(r)$ is positive and $-\infty \le r_1 < r_2 \le +\infty$. Then, by changing both dependent and independent variables, the original equation can be reduced to a canonical form

$$v''(t) + k(t, v) = 0 \quad \text{in } (0, 1)$$

with a suitable boundary condition if $g(r)$ satisfies certain assumptions. Nevertheless, in our problem, it is not easy to find a way to transform (1.9) into a desirable canonical form. In fact, we couldn't obtain a canonical form based on the transformation process given in [9] (Theorem D) because the corresponding function $g(r)$ is r and $1/g(r)$ is not integrable near the origin and infinity.

In this article, we provide another approach to studying the structure of radial solutions in terms of initial data prescribed at $r = 1$ and clarify the types of solutions completely, as in [9], but more complicated.

REMARK 1.3. According to [4], the asymptotic behaviors of γ_1 and γ_2^- at $\pm\infty$ have been determined. More precisely,

$$\lim_{a \to -\infty} \gamma_1(a) = (-\infty, 0), \qquad \lim_{a \to +\infty} \gamma_1(a) = (-\infty, -2(1+p))$$

and

$$\lim_{b \to -\infty} \gamma_2^-(b) = (-\infty, -2), \qquad \lim_{b \to +\infty} \gamma_2^-(b) = (-\infty, -2q).$$

A brief description will be presented in Section 3. We note that each of Γ_1 and Γ_2^- divides the (θ, η)-plane into two regions, one is bounded and the other is unbounded in the direction of the η-axis.

In addition, if $K'(r)$ satisfies

$$K'(r) = 2pK_0 r^{2p-1} + O(r^{2p-1+\lambda}) \quad \text{near } r = 0 \tag{1.13}$$

for some $\lambda > 0$, we set

$$G(r) = \int_0^r s^2 K'(s) ds, \quad r > 0. \tag{1.14}$$

Note that $G(r)$ is well-defined by (1.13) and $p > -1$.

Now, we present our main results. To clarify the whole structures of solutions for readers, the figure is illustrated following the description of each main theorem.

THEOREM 1.1. *Suppose $p \geq 0$ and either $q = 0$ with $0 < l < 2$ or $0 < q < 1$. If $K'(r)$ satisfies* (1.13) *and $G(r)$, defined in* (1.14), *is nonnegative and is not equal to zero identically, then the following assertions concerning the solution $u(r; \theta, \eta)$ of* (1.9) *in (θ, η)-plane are true.*

(a) *The curves Γ_1 and Γ_2^- do not intersect.*

(b) *If (θ, η) belongs to Γ_1, then $u(r; \theta, \eta)$ is of Type R-F^-; if (θ, η) belongs to Γ_2^-, then $u(r; \theta, \eta)$ is of Type P-R^-.*

(c) *If (θ, η) lies on the region bounded by Γ_1 which is bounded* (*resp.*, *unbounded*) *in the direction of the η-axis, then $u(r; \theta, \eta)$ is of Type P-$*$* (*resp.*, *N-$*$*).

(d) *If (θ, η) lies on the region bounded by Γ_2^- which is bounded* (*resp.*, *unbounded*) *in the direction of the η-axis, then $u(r; \theta, \eta)$ is of Type $*$-S^-* (*resp.*, *$*$-F^-*).

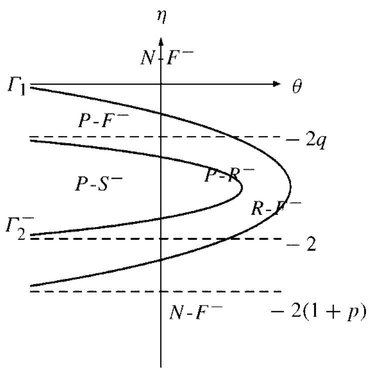

FIGURE 1.

THEOREM 1.2. *Suppose either $-1 < p < 0$ with $-1 \leq q \leq 0$ or $p = 0$ with $0 < k < 2$ and $-1 \leq q < 0$. If $K'(r)$ satisfies* (1.13) *and $G(r)$, defined in* (1.14), *is non-positive and not equal to zero identically, then the following assertions concerning the solution $u(r; \theta, \eta)$ of* (1.9) *in (θ, η)-plane are true.*

(a) *Γ_1 does not intersect Γ_2^-.*

(b) *If (θ, η) belongs to Γ_1, then $u(r; \theta, \eta)$ is of Type R-S^-; if (θ, η) belongs to Γ_2^-, then $u(r; \theta, \eta)$ is of Type N-R^-.*

(c) *Both assertions of* (c) *and* (d) *in Theorem* 1.1 *hold.*

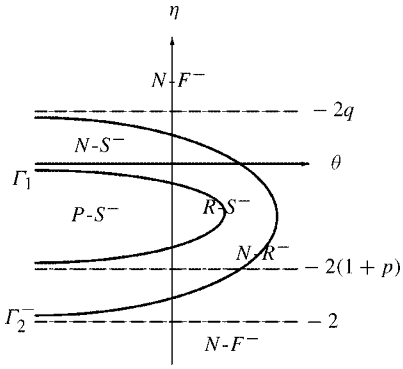

FIGURE 2.

THEOREM 1.3. *Suppose either $p > 0$ with $-1 \le q < 0$ or $-1 < p < 0 < q < 1+p$. Then the following assertions concerning the solution $u(r; \theta, \eta)$ of* (1.9) *in (θ, η)-plane are true.*

(a) *Γ_1 and Γ_2^- intersect.*

(b) *Both assertions of* (c) *and* (d) *in Theorem* 1.1 *hold.*

Based on Theorem 1.3 above, the following figure illustrates the structure of solutions in the case that Γ_1 and Γ_2^- intersect exactly once.

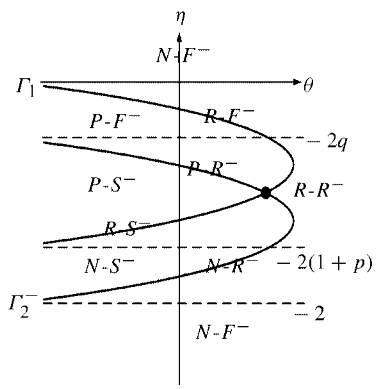

FIGURE 3.

THEOREM 1.4. *If $K(r)$ is a constant function, then the following assertions concerning the solution $u(r; \theta, \eta)$ of* (1.9) *in (θ, η)-plane are true.*

(a) *Γ_1 and Γ_2^- are identical.*

(b) *If (θ, η) lies on the region bounded by Γ_1 which is bounded* (*resp., unbounded*) *in the direction of the η-axis, then $u(r; \theta, \eta)$ is of Type P-S^- (resp., N-F^-).*

(c) *$u(r; \gamma_1(a))$ is of Type R-R^- for all $a \in \mathbf{R}$.*

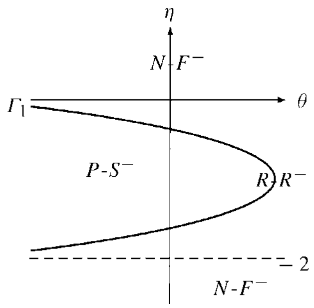

FIGURE 4.

To describe the final two consequences, we need to introduce additional two initial value problems:

$$\begin{cases} \{rU'_{R^+}(r)\}' + rK(r)e^{2U_{R^+}(r)} = 0\,, \quad r > 0\,, \\ \lim_{r\to\infty}(U_{R^+}(r) - 2\ln r) = c\,, \end{cases} \tag{1.15}$$

$$\begin{cases} \{rU'_{C}(r)\}' + rK(r)e^{2U_{C}(r)} = 0\,, \quad r > 0\,, \\ \lim_{r\to\infty} U_{C}(r) = d\,, \end{cases} \tag{1.16}$$

where $c, d \in \boldsymbol{R}$. As usual, we denote the unique solutions of (1.15) and (1.16) by $U_{R^+}(r; c)$ and $U_C(r; d)$, respectively. Define

$$\gamma_2^+(c) = (U_{R^+}(1; c), U'_{R^+}(1; c))$$

and

$$\gamma_3(d) = (U_C(1; d), U'_C(1; d))$$

for $c, d \in \boldsymbol{R}$, and let Γ_2^+ and Γ_3 be the ranges of γ_2^+ and γ_3 over $\boldsymbol{R}$, respectively. It is easy to see that both curves γ_2^+ and γ_3 are also smooth by $p > -1$ and $q < 1$ again. Also, Γ_2^+ and Γ_3 are the collections of initial data corresponding to solutions of Type $*$-R$^+$ and $*$-C for (1.9), respectively.

REMARK 1.4. We also have the following asymptotic behaviors of γ_2^+ and γ_3 at $\pm\infty$:

$$\lim_{c\to-\infty} \gamma_2^+(c) = (-\infty, 2)\,, \quad \lim_{c\to+\infty} \gamma_2^+(c) = (-\infty, -2(q+2))$$

and

$$\lim_{d\to-\infty} \gamma_3(d) = (-\infty, 0)\,, \quad \lim_{d\to+\infty} \gamma_3(d) = (-\infty, -2(q+1))$$

by [4].

Moreover, in Theorems 1.5 and 1.6 below, the following extra hypothesis is assumed:

$$\lim_{r\to\infty} \frac{rK'(r)}{K(r)} < -2\,. \tag{1.17}$$

THEOREM 1.5. *Suppose either* $-1 < p < 0$ *or* $p = 0$ *with* $0 < k < 2$. *If* $K(r)$ *satisfies* (1.13), (1.17) *and* $G(r)$, *defined in* (1.14), *is non-positive and not equal to zero identically, then the following assertions concerning the solution* $u(r; \theta, \eta)$ *of* (1.9) *in* (θ, η)-*plane are true.*

Case 1: $q < -3$. *Then* Γ_2^+ *exists* (*nonempty*).

(a) *Any two curves of* Γ_1, $\Gamma_2^\pm$ *and* Γ_3 *do not intersect.*

(b) *The assertion of* (b) *in Theorem* 1.2 *holds.*

(c) *The assertion of* (c) *in Theorem* 1.1 *holds.*

(d) *If* (θ, η) *lies on the region bounded by* Γ_3 *and* $\Gamma_2^\pm$, *the* $u(r; \theta, \eta)$ *is of Type* $*$-$S^\pm$. *In addition,* $u(r; \theta, \eta)$ *is of Type* $*$-F^+ (*resp.,* $*$-F^-) *if* (θ, η) *lies on the region bounded by* Γ_2^+ (*resp.,* Γ_2^-) *which is bounded* (*resp., unbounded*) *in the direction of the* η-*axis.*

Case 2: $-3 \le q < -1$. *Then* Γ_2^+ *does not exist* (*empty*).

All conclusions of (a) to (d) *in Case* 1 *are true if the statements related to* Γ_2^+ *are removed.*

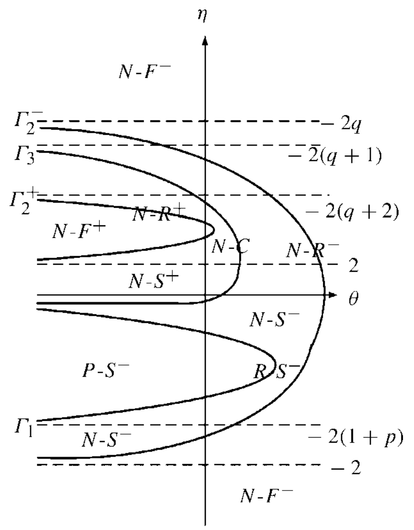

FIGURE 5.

THEOREM 1.6. *Suppose* (1.17) *holds and* $p > 0$. *Then the following assertions concerning the solution* $u(r; \theta, \eta)$ *of* (1.9) *in* (θ, η)-*plane are true.*

(a) *Among* Γ_1, $\Gamma_2^\pm$ *and* Γ_3, *only* Γ_1 *and* Γ_2^- *intersect each other.*

(b) *All assertions in Theorem* 1.5 *hold except* (a) *and* (b) *in Case* 1.

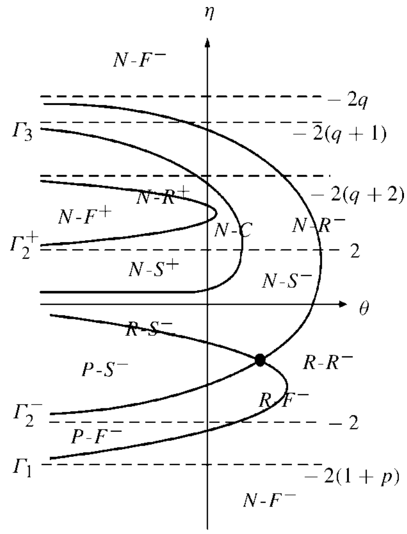

FIGURE 6.

This article is organized as follows. First, we make preparations for demonstrating our main results in Section 2. Finally, Section 3 is devoted to presenting complete verifications of Theorems 1.1 to 1.6.

2. Preliminaries. In this section, we are going to derive some preparatory works which are essential elements for us to study the structure of solutions for (1.9).

LEMMA 2.1. *Suppose* $u(r) \equiv u(r;\theta,\eta)$ *is a solution of* (1.9), *then the following are true.*

(a)

$$\int_0^\infty rK(r)e^{2u(r)}dr < \infty .$$

(b) $u(r) = c_0 \ln r + O(1)$ *at* $r = 0$ *and* $u(r) = c_\infty \ln r + O(1)$ *at* $r = \infty$ *for some* $c_0, c_\infty \in \boldsymbol{R}$ *satisfying* $1 + p + c_0 > 0$ *and* $1 + q + c_\infty < 0$.

PROOF. (a) We divide the proof into two steps.

Step 1. First, we prove $\int_1^\infty rK(r)e^{2u}dr < \infty$. Suppose $\int_1^\infty rK(r)e^{2u}dr = \infty$. Then from (1.9), we see that

$$ru'(r) = \eta - \int_1^r sK(s)e^{2u}ds$$

for $r \geq 1$. Hence,

$$\lim_{r\to\infty} ru'(r) = -\infty . \tag{2.1}$$

We choose $c < 0$ with $1 + c + q < 0$. Then (2.1) implies $u(r) \leq c \ln r$ for $r \geq R$ for a real number $R \geq 1$. Therefore, we obtain that

$$\begin{aligned} \infty &= \int_1^\infty sK(s)e^{2u} ds \\ &\leq \int_1^R sK(s)e^{2u} ds + \int_R^\infty s^{1+2c} K(s) ds \\ &< \infty\,, \end{aligned}$$

which leads to a contradiction. This step is established.

Step 2. Now, we prove $\int_0^1 rK(r)e^{2u} dr < \infty$. Similarly as in Step 1, suppose $\int_0^1 rK(r)e^{2u} dr = \infty$. Then by (1.9) again, we have

$$ru'(r) = \eta + \int_r^1 sK(s)e^{2u} ds$$

for $0 < r \leq 1$. Hence,

$$\lim_{r \to 0} ru'(r) = \infty\,, \tag{2.2}$$

which implies $\lim_{r \to 0} u(r) = -\infty$. Indeed, for any $M > 0$, there exists $0 < \delta < 1$ such that $ru'(r) \geq M$ for all $r \in (0, \delta]$ and hence $u(r) \leq C(\delta) + M \ln r$ for all $r \in (0, \delta]$ for some $C(\delta) \in \boldsymbol{R}$. Therefore, $u(r)$ is bounded from above on $(0, 1]$. Since $p > -1$, we obtain inequalities

$$\begin{aligned} \infty &= \int_0^1 sK(s)e^{2u} ds \\ &\leq C \int_0^1 sK(s) ds \\ &< \infty \end{aligned}$$

for some $C > 0$. This is a contradiction.

Hence (a) is proved.

(b) Note that (1.9) and (a) imply

$$\lim_{r \to 0} ru'(r) = c_0 \quad \text{and} \quad \lim_{r \to \infty} ru'(r) = c_\infty$$

for some $c_0, c_\infty \in \boldsymbol{R}$, and

$$c_\infty \leq ru'(r) \leq c_0$$

for $r > 0$. Hence by combining (a), (1.10) and the above fact, we obtain $1 + p + c_0 > 0$ and $1 + q + c_\infty < 0$. To show the remaining assertions, we split the proof into two steps.

Step 1. First, we prove $u(r) = c_\infty \ln r + O(1)$ at $r = \infty$. Let $v(r) = u(r) - c_\infty \ln r$, then $v(r)$ satisfies

$$v''(r) + \frac{1}{r} v'(r) = -K(r)e^{2u}\,, \quad r > 0\,. \tag{2.3}$$

Let $\varepsilon_1 > 0$ with $1 + q + c_\infty + \varepsilon_1 < 0$. Then there exists $R > 0$ such that

$$u(r) \leq c_1 + (c_\infty + \varepsilon_1)\ln r \quad \text{and} \quad K(r) \leq 2K_\infty r^{2q}$$

for $r \geq R$ for some $c_1 \in \boldsymbol{R}$. Hence, we have

$$rK(r)e^{2u} \leq c_2 r^{1+2q+2c_\infty+2\varepsilon_1} \quad \text{for } r \geq R$$

for some $c_2 > 0$. Now, choose $\varepsilon > 0$ such that $2(1 + q + c_\infty + \varepsilon_1) + \varepsilon < 0$. Then

$$\frac{rK(r)e^{2u}}{\varepsilon r^{-\varepsilon-1}} \leq \left(\frac{c_2}{\varepsilon}\right) r^{2(1+q+c_\infty+\varepsilon_1)+\varepsilon} \quad \text{for } r \geq R\,,$$

which implies

$$\lim_{r\to\infty} \frac{rK(r)e^{2u}}{\varepsilon r^{-\varepsilon-1}} = 0\,.$$

By means of (2.3), the above result and the fact $\lim_{r\to\infty} rv'(r) = 0$, we get $v'(r) > 0$ for $r > 0$ and

$$\lim_{r\to\infty} \frac{rv'(r)}{r^{-\varepsilon}} = \lim_{r\to\infty} \frac{rK(r)e^{2u}}{\varepsilon r^{-\varepsilon-1}} = 0\,.$$

Therefore, there exist $c > 0$ and $r_0 > 0$ such that

$$v'(r) < cr^{-1-\varepsilon}\,, \quad r \geq r_0\,,$$

which implies

$$v(r) < v(r_0) + \left(\frac{c}{\varepsilon}\right)(r_0^{-\varepsilon} - r^{-\varepsilon}) < C$$

for $r \geq r_0$ for some $C > 0$. This step is finished.

Step 2. Now, we prove $u(r) = c_0 \ln r + O(1)$ at $r = 0$. By following the similar arguments as in Step 1, let $w(r) = u(r) - c_0 \ln r$. Then $w(r)$ satisfies

$$w''(r) + \frac{1}{r}w'(r) = -K(r)e^{2u}\,, \quad r > 0\,.$$

Let $\delta_1 > 0$ with $1 + p + c_0 - \delta_1 > 0$. Then there exists $r_0 > 0$ such that

$$u(r) \leq k_1 + (c_0 - \delta_1)\ln r \quad \text{and} \quad K(r) \leq 2K_0 r^{2p}$$

for $0 < r \leq r_0$ for some $k_1 \in \boldsymbol{R}$. Hence, we get

$$rK(r)e^{2u} \leq k_2 r^{1+2p+2(c_0-\delta_1)} \quad \text{for } 0 < r \leq r_0$$

for some $k_2 > 0$. Now, take $\delta > 0$ such that $2(1 + p + c_0 - \delta_1) - \delta > 0$. Then

$$\frac{rK(r)e^{2u}}{\delta r^{\delta-1}} \leq \left(\frac{k_2}{\delta}\right) r^{2(1+p+c_0-\delta_1)-\delta} \quad \text{for } 0 < r \leq r_0\,,$$

which implies

$$\lim_{r\to 0} \frac{rK(r)e^{2u}}{\delta r^{\delta-1}} = 0\,.$$

Since $\lim_{r\to 0} rw'(r) = 0$, we conclude that $w'(r)$ is negative for $r > 0$ and

$$\lim_{r\to 0} \frac{rw'(r)}{r^\delta} = -\lim_{r\to 0} \frac{rK(r)e^{2u}}{\delta r^{\delta-1}} = 0\,.$$

Therefore, there exist $c > 0$ and $r_1 > 0$ such that

$$w'(r) > -cr^{-1+\delta}\,, \quad 0 < r \le r_1\,,$$

and Step 2 is completed.

Hence (b) is also proved. □

REMARK 2.1. (i) We note that both Γ_1 and Γ_2^- are nonempty by $2p > -2$, $-2q > -2$, (1.10) and using the Kelvin transformation. In addition, Γ_2^+ is empty if $-3 \le q < 1$ and nonempty if $q < -3$. Furthermore, if $q \ge -1$, then (1.9) does not possess solutions of Type $*$-S$^+$ and Type $*$-F$^+$.

(ii) By Lemma 2.1(b), the existence of solutions of (1.9) of Type $*$-F$^+$ and Type $*$-S$^+$ implies $q < -3$ and $q < -1$, respectively.

To realize the structure of solutions, we introduce an auxiliary function associated with solutions. Let $u(r)$ be a solution of (1.9). We define

$$P(r; u; L, M) = (ru' + L)(ru' + 2 - M) + r^2K(r)e^{2u} \tag{2.4}$$

for $r > 0$, where $L, M \ge 0$. By straightforward computations, we obtain

$$\frac{d}{dr}P(r; u; L, M) = \{(M - L)K(r) + rK'(r)\}re^{2u} \tag{2.5}$$

for $r > 0$. To simplify the notations, we denote $P(r; u; 0, 0)$ by $P(r; u)$ and $P(r; u(r; \theta, \eta); L, M)$ by $P(r; \theta, \eta; L, M)$.

We now present some facts, stated in Lemma 2.2 below, which are involving the characterization of solutions of various types in terms of $P(r; u)$.

LEMMA 2.2. *Suppose $u(r)$ is a solution of* (1.9), *then the following assertions are true.*

(a) *If $u(r)$ is of Type $*$-R$^-$, then $P(r; u) \to 0$ as $r \to \infty$.*

(b) *If $u(r)$ is of Type $*$-F$^-$, then $P(r; u) \to C$ for some $C > 0$ as $r \to \infty$.*

(c) *If $u(r)$ is of Type $*$-S$^-$, then $P(r; u) \to C$ for some $C < 0$ as $r \to \infty$. Furthermore, $C > (q - 1)(q + 3)$ if $q \ge -1$.*

(d) *If $q < -3$ and $u(r)$ is of Type $*$-F$^+$, then $P(r; u) \to C$ for some $8 < C < q^2 - 1$ as $r \to \infty$.*

(e) *If $q < -1$ and $u(r)$ is of Type $*$-S$^+$, then $P(r; u) \to C$ for some $0 < C < \min\{8, q^2 - 1\}$ as $r \to \infty$.*

(f) *If $\lim_{r\to\infty} ru'(r) > 0$ and $\lim_{r\to\infty} P(r; u) = \alpha$, then $u(r)$ is of Type $*$-F$^+$ if $\alpha > 8$; Type $*$-S$^+$ if $0 < \alpha < 8$.*

(g) *If $\lim_{r\to\infty} ru'(r) < 0$ and $\lim_{r\to\infty} P(r; u) > 0$, then $u(r)$ is of Type $*$-F$^-$.*

PROOF. First, by Lemma 2.1(b), we get

$$\lim_{r\to\infty} r^2 K(r)e^{2u(r)} = 0$$

for any solution $u(r)$ of (1.9).

(a) If $u(r)$ is of Type $*$-R$^-$, then $ru'(r) \to -2$ as $r \to \infty$ and hence $P(r; u) \to 0$ as $r \to \infty$.

(b) If $u(r)$ is of Type $*$-F$^-$, then $ru'(r) \to c_1$ for some $c_1 < -2$ as $r \to \infty$. Hence $P(r; u) \to c_1(c_1 + 2) \equiv C > 0$ as $r \to \infty$.

(c) If $u(r)$ is of Type $*$-S$^-$, then $ru'(r) \to c_2$ for some $-2 < c_2 < 0$ as $r \to \infty$. Hence $P(r; u) \to c_2(c_2 + 2) \equiv C < 0$ as $r \to \infty$. Moreover, if $q \ge -1$, then $-c_2 > 1 + q \ge 0$ by Lemma 2.1(b) which implies

$$C = c_2^2 + 2c_2 > (1 + q)^2 - 4 = (q - 1)(q + 3)\,.$$

(d) If $u(r)$ is of Type $*$-F$^+$, then $ru'(r) \to c_3$ for some $c_3 > 2$ as $r \to \infty$ which implies $P(r; u) \to c_3(c_3 + 2) \equiv C > 8$ as $r \to \infty$. In addition, we also have $c_3 < -1 - q$ by Lemma 2.1(b) and hence

$$C < (-1 - q)(1 - q) = q^2 - 1\,.$$

(e) If $u(r)$ is of Type $*$-S$^+$, then $ru'(r) \to c_4$ for some $0 < c_4 < 2$ as $r \to \infty$. Hence $P(r; u) \to c_4(c_4 + 2) \equiv C < 8$ as $r \to \infty$. Also, $c_4 < -1 - q$ implies $C < q^2 - 1$.

(f) Let $\lim_{r\to\infty} ru'(r) = c > 0$. Then $\lim_{r\to\infty} P(r; u) = c(c + 2) = \alpha$. Hence

$$c = -1 + \sqrt{1 + \alpha}$$

and the assertions in (f) are easily obtained.

(g) By the similar arguments in the proof of (f), we conclude $\lim_{r\to\infty} ru'(r) < -2$. □

In order to clarify the regions of initial data corresponding to certain types of solutions, the properties of openness for such regions play significant roles. The following propositions provide us with this substantial concept.

PROPOSITION 2.1. *The following assertions on the solution $u(r; \theta, \eta)$ of* (1.9) *are true.*

(i) *If $u(r; \theta_0, \eta_0)$ is of Type $*$-F$^-$, then there exists $\delta > 0$ such that $u(r; \theta, \eta)$ is of Type $*$-F$^-$ for $(\theta, \eta) \in B_\delta((\theta_0, \eta_0))$.*

(ii) *If $u(r; \theta_0, \eta_0)$ is of Type $*$-S$^-$, then there exists $\delta > 0$ such that $u(r; \theta, \eta)$ is of Type $*$-S$^-$ for $(\theta, \eta) \in B_\delta((\theta_0, \eta_0))$.*

PROOF. (i) Suppose there existed a sequence $\{(\theta_j, \eta_j)\}_{j=1}^{\infty}$ with $(\theta_j, \eta_j) \to (\theta_0, \eta_0)$ as $j \to \infty$ and $u(r; \theta_j, \eta_j)$ is of Type $*$-S$^\pm$ or $*$-F$^+$ for all j. Then $ru'(r; \theta_j, \eta_j) > -2$ for $r > 0$ by the fact that $ru'(r; \theta_j, \eta_j)$ is decreasing on $(0, \infty)$ for all j. Moreover, since $u(r; \theta_0, \eta_0)$ is of Type $*$-F$^-$, there exists $r_1 > 0$ such that $r_1 u'(r_1; \theta_0, \eta_0) < -2$. Hence, we obtain

$$\begin{aligned} -2 &\le \lim_{j\to\infty} r_1 u'(r_1;\theta_j,\eta_j) \\ &= r_1 u'(r_1;\theta_0,\eta_0) \\ &< -2\,, \end{aligned}$$

which is a contradiction. The proof of (i) is completed.

(ii) We divide this proof into two steps.

Step 1. First, by using the same arguments described in the proof of (i), we have the following assertion: If $u(r;\theta,\eta)$ is of Type $*$-S$^-$ or Type $*$-F$^-$, then there doesn't exist a sequence $\{(\theta_j,\eta_j)\}$ such that $\{(\theta_j,\eta_j)\}\to(\theta,\eta)$ as $j\to\infty$ and $u(r;\theta_j,\eta_j)$ is of Type $*$-S$^+$ or Type $*$-F$^+$ for all j.

Step 2. Next, we show that if $u(r;\theta,\eta)$ is of Type $*$-F$^+$ or Type $*$-S$^\pm$, then there doesn't exist a sequence $\{(\theta_j,\eta_j)\}$ such that $\{(\theta_j,\eta_j)\}\to(\theta,\eta)$ as $j\to\infty$ and $u(r;\theta_j,\eta_j)$ is of Type $*$-F$^-$ for all j. To prove this, we first note that by (1.10) and $q<1$, $\lim_{r\to\infty} rK'(r)/K(r)<2$. Then from (2.5), there exist $0<L<2$ and $R_0>1$ such that

$$\frac{d}{dr}P(r;v;L,0)\le 0\,,\quad r\ge R_0 \tag{2.6}$$

for any solution $v(r)$ of (1.9). Let $u(r;\theta,\eta)$ be of Type $*$-S$^-$. Suppose that there exists a sequence $\{(\theta_j,\eta_j)\}$ with $(\theta_j,\eta_j)\to(\theta,\eta)$ as $j\to\infty$ such that $u(r;\theta_j,\eta_j)$ is of Type $*$-F$^-$ for all j. To continue this proof, we need the following assertion.

Claim. For any $0<\varepsilon<\min\{2-L,2+c_\infty\}$*, where* c_∞ *is selected in Lemma* 2.1 (b) *with respect to* $u(r;\theta,\eta)$*, there exists* $R_\varepsilon>R_0$ *such that*

$$ru'(r;\theta_j,\eta_j)<-2+\frac{\varepsilon}{2}\,,\quad r\ge R_\varepsilon$$

for all j.

Proof of Claim. Suppose there existed $0<\varepsilon_0<\min\{2-L,2+c_\infty\}$ and a sequence $\{r_j=r_j(\varepsilon_0)\}$ such that $\lim_{j\to\infty} r_j=\infty$ and

$$r_j u'(r_j;\theta_j,\eta_j)=-2+\frac{\varepsilon_0}{2}\quad\text{for all } j\,.$$

Since

$$\lim_{r\to\infty} ru'(r;\theta,\eta)=\xi>-2\quad\text{and}\quad \lim_{r\to\infty} ru'(r;\theta_j,\eta_j)=\xi_j<-2\quad\text{for all } j\,,$$

we have

$$\xi_j<-2<\xi<-(1+q)\quad\text{for all } j$$

by Lemma 2.1(b). Then there exist constants $R_1>R_0$, $\delta>0$ and $N>0$ such that $R_1u'(R_1;\theta_j,\eta_j)<-(1+q+\delta)$ for $j\ge N$, and hence $ru'(r;\theta_j,\eta_j)<-(1+q+\delta)$ for $r\ge R_1$ and $j\ge N$. Therefore, by (1.10) and the above result, we get that $r^2K(r)e^{2u(r;\theta_j,\eta_j)}$ is bounded by a constant times $r^{-2\delta}$ from above for $r\ge R_1$ and $j\ge N$, which implies

$$r^2K(r)e^{2u(r;\theta_j,\eta_j)}<-\left(-2+\frac{\varepsilon_0}{2}+L\right)\left(\frac{\varepsilon_0}{2}\right),\quad r\ge R_2\,,\ j\ge N \tag{2.7}$$

for some $R_2 \geq R_1$ since the right-hand side of (2.7) is positive. In addition, by (2.6) and (2.7), we have

$$\begin{aligned} P(r;\theta_j,\eta_j;L,0) &\leq P(r_j;\theta_j,\eta_j;L,0) \\ &= \left(-2+\frac{\varepsilon_0}{2}+L\right)\left(\frac{\varepsilon_0}{2}\right)+r^2K(r)e^{2u(r;\theta_j,\eta_j)} \\ &< 0 \end{aligned}$$

for $r \geq r_j$ and $j \geq J$, where J is chosen such that $r_J \geq R_2$. However, this contradicts the fact $P(r;\theta_j,\eta_j;L,0) \geq 0$ for $r \geq R_0$ by $\lim_{r\to\infty} P(r;\theta_j,\eta_j;L,0) \geq 0$ and (2.6). We complete the proof of this claim.

Finally, by Claim, we obtain

$$\begin{aligned} -2+\frac{\varepsilon}{2} &\geq \lim_{j\to\infty} R_\varepsilon u'(R_\varepsilon;\theta_j,\eta_j) \\ &= R_\varepsilon u'(R_\varepsilon;\theta,\eta) \\ &\geq c_\infty \\ &\geq -2+\varepsilon \end{aligned}$$

for any $0 < \varepsilon < \min\{2-L, 2+c_\infty\}$. This yields a contradiction.

For the cases of $u(r;\theta,\eta)$ being of Type $*$-S$^+$ or Type $*$-F$^+$, we also have $ru'(r;\theta_j,\eta_j) < -(1+q+\delta_1)$ for $r \geq R_3$ and $j \geq N_1$ for some constants $R_3 > R_0$, $\delta_1 > 0$ and $N_1 > 0$. The rest of proofs of the two cases are the same as above and we omit them. This step is finished.

By Steps 1 and 2, (ii) is obtained. □

REMARK 2.2. We note that the existence of the solution of (1.9) which goes to infinity as $r \to \infty$ implies $q < -1$, and hence (1.17) holds

PROPOSITION 2.2. *The following assertions on the solution $u(r;\theta,\eta)$ of* (1.9) *are true.*

(i) *If $u(r;\theta_0,\eta_0)$ is of Type $*$-S$^+$, then there exists $\delta > 0$ such that $u(r;\theta,\eta)$ is of Type $*$-S$^+$ for $(\theta,\eta) \in B_\delta((\theta_0,\eta_0))$.*

(ii) *If $u(r;\theta_0,\eta_0)$ is of Type $*$-F$^+$, then there exists $\delta > 0$ such that $u(r;\theta,\eta)$ is of Type $*$-F$^+$ for $(\theta,\eta) \in B_\delta((\theta_0,\eta_0))$.*

PROOF. Due to Remark 2.2, we only consider $q < -1$. The proof of this proposition is split into the following two steps.

Step 1. First, by following the similar arguments as in the proof of Proposition 2.1(i), we conclude that

(a) if $u(r;\theta,\eta)$ is of Type $*$-S$^+$, then there doesn't exist a sequence $\{(\theta_j,\eta_j)\}$ such that $\{(\theta_j,\eta_j)\} \to (\theta,\eta)$ as $j \to \infty$ and $u(r;\theta_j,\eta_j)$ is of Type $*$-F$^+$ for all j;

(b) if $u(r;\theta,\eta)$ is of Type $*$-F$^\pm$, then there doesn't exist a sequence $\{(\theta_j,\eta_j)\}$ such that $\{(\theta_j,\eta_j)\} \to (\theta,\eta)$ as $j \to \infty$ and $u(r;\theta_j,\eta_j)$ is of Type $*$-S$^+$ or Type $*$-S$^-$ for all j.

Step 2. In the following, we prove that if $u(r;\theta,\eta)$ is of Type $*$-S$^+$ or Type $*$-F$^+$, then there doesn't exist a sequence $\{(\theta_j,\eta_j)\}$ such that $\{(\theta_j,\eta_j)\} \to (\theta,\eta)$ as $j \to \infty$ and $u(r,\theta_j,\eta_j)$ is of Type $*$-S$^-$ or Type $*$-F$^-$ for all j. To see this, note that $\lim_{r\to\infty} rK'(r)/K(r) < -2$ since $q < -1$. Then by (2.5), there exist $M > 2$ and $R_0 > 1$ such that

$$\frac{d}{dr}P(r;w;0,M) \le 0\,, \quad r \ge R_0 \tag{2.8}$$

for any solution $w(r)$ of (1.9). Let $u(r;\theta,\eta)$ be of Type $*$-S$^+$. Suppose that there exists a sequence $\{(\theta_j,\eta_j)\}$ with $(\theta_j,\eta_j) \to (\theta,\eta)$ as $j \to \infty$ such that $u(r;\theta_j,\eta_j)$ is of Type $*$-F$^-$ or Type $*$-S$^-$ for all j. We first give the following assertion.

Claim. For any $0 < \varepsilon < \min\{M-2, c_\infty\}$*, where* c_∞ *is selected in Lemma* 2.1 (b) *with respect to* $u(r;\theta,\eta)$*, there exists* $R_\varepsilon > R_0$ *such that*

$$ru'(r;\theta_j,\eta_j) < \frac{\varepsilon}{2}\,, \quad r \ge R_\varepsilon$$

for all j.

Proof of Claim. If there exist $0 < \varepsilon_0 < \min\{M-2, c_\infty\}$ and a sequence $\{r_j = r_j(\varepsilon_0)\}$ such that $\lim_{j\to\infty} r_j = \infty$ and

$$r_j u'(r_j;\theta_j,\eta_j) = \frac{\varepsilon_0}{2} \quad \text{for all } \ j.$$

Let

$$\lim_{r\to\infty} ru'(r;\theta,\eta) = \xi \quad \text{and} \quad \lim_{r\to\infty} ru'(r;\theta_j,\eta_j) = \xi_j \quad \text{for all } \ j\,.$$

From (b) of Lemma 2.1, we have

$$\text{either} \quad \xi_j < -2 < \xi < -(1+q) \quad \text{or} \quad -2 < \xi_j < 0 < \xi < -(1+q)$$

for each j. Therefore, there exist constants $R_1 > R_0$, $\delta > 0$ and $N > 0$ such that $ru'(r;\theta_j,\eta_j) < -(1+q+\delta)$ for $r \ge R_1$ and $j \ge N$. Similarly as in the proof of (a), we obtain

$$r^2K(r)e^{2u(r;\theta_j,\eta_j)} < -\left(\frac{\varepsilon_0}{2}\right)\left(\frac{\varepsilon_0}{2}+2-M\right), \quad r \ge R_2\,,\ j \ge N \tag{2.9}$$

for some $R_2 \ge R_1$, and hence by (2.8) and (2.9),

$$P(r;\theta_j,\eta_j;0,M) < 0 \quad \text{for } \ r \ge r_j\,.$$

On the other hand, since $\lim_{r\to\infty} P(r;\theta_j,\eta_j;0,M) \ge 0$, we have

$$P(r;\theta_j,\eta_j;0,M) \ge 0 \quad \text{for } \ r \ge R_0$$

by (2.8). This is impossible and the proof of this claim is completed.

For $0 < \varepsilon < \min\{M-2, c_\infty\}$, the above claim implies

$$\begin{aligned}\frac{\varepsilon}{2} &\ge \lim_{j\to\infty} R_\varepsilon u'(R_\varepsilon;\theta_j,\eta_j)\\ &= R_\varepsilon u'(R_\varepsilon;\theta,\eta)\\ &\ge c_\infty\\ &> \varepsilon\,,\end{aligned}$$

which is a contradiction.

By the similar arguments as above, we omit the detailed proof of the other case of $u(r;\theta,\eta)$, and Step 2 is established.

Now, by combining the above steps, the assertions of (i) and (ii) are obtained. We complete the proof of Proposition 2.2. □

Moreover, to determine the existences for certain types of solutions, we need the following transformation. Let $u(r) = u(r;\theta,\eta)$ be a solution of (1.9). For any $c \in \boldsymbol{R}$, we set $z(s;c) = u(r) + 2c\ln r$, where $r = 1/s$. Then $z(s;c)$ satisfies

$$
(2.10)\qquad \begin{cases} z''(s;c) + \dfrac{1}{s}z'(s;c) + \tilde{K}(s)e^{2z} = 0\,, & s \in (0,\infty)\,, \\ z(1;c) = \theta\,, \quad z'(1;c) = -(\eta + 2c)\,, \end{cases}
$$

where $\tilde{K}(s) = s^{-4-4c}K(1/s)$.

REMARK 2.3. From (2.10) and Lemma 2.1(b), the solution for (1.9) with behavior being like $-2c\ln r$ at the origin or infinity exists if $p > -3 - 2c$ and $q < -1 - 2c$ for any $c \in \boldsymbol{R}$. Therefore, the existences of solutions of Type $*$-S$^-$ and P-$*$ are derived if $p > -1$ and $q < 1$. Moreover, it is easy to see that the solution $u(r;\theta;\eta)$ of (1.9) is of Type $*$-F$^-$ if $\eta < -2$; Type N-$*$ if $\eta > 0$.

PROPOSITION 2.3. *The following assertions on the solution* $u(r;\theta,\eta)$ *of* (1.9) *are true.*

(i) *If* $u(r;\theta_0,\eta_0)$ *is of Type P-$*$, then there exists* $\delta > 0$ *such that* $u(r;\theta,\eta)$ *is of Type P-$*$ for* $(\theta,\eta) \in B_\delta((\theta_0,\eta_0))$.

(ii) *If* $u(r;\theta_0,\eta_0)$ *is of Type N-$*$, then there exists* $\delta > 0$ *such that* $u(r;\theta,\eta)$ *is of Type N-$*$ for* $(\theta,\eta) \in B_\delta((\theta_0,\eta_0))$.

PROOF. (i) Let $z(s;1)$ be defined in (2.10) with $u(r) = u(r;\theta_0,\eta_0)$. Then we have

$$
\frac{d}{ds}\tilde{P}(s;z(s;1);L,0) \le 0
$$

for $s > 0$ since $p > -1$, where $\tilde{P}$ is defined as in (2.4) with respect to solutions of (2.10). Hence, the proof is completed by Step 2 in the proof of Proposition 2.1(ii).

(ii) Since $u(r;\theta_0,\eta_0)$ is of Type N-$*$, there exists $r_0 > 0$ such that $u'(r_0;\theta_0,\eta_0) > 0$. Then, (ii) is proved by the fact $(ru')' \le 0$ on $(0,\infty)$ and the continuity of solutions with respect to initial data. □

3. Proofs of Main Results. In this section, we present complete verifications for our main results mentioned in Section 1. First, we derive the asymptotic behaviors of the curves

γ_1, $\gamma_2^{\pm}$ and γ_3. To attain this end, we introduce the following two initial value problems:

$$\begin{cases} v''(r) + \frac{1}{r} v'(r) + K(r) e^{2v(r)} = 0\,, & r \in (0, 1]\,, \\ v(r) = A \ln r + a + o(1) \quad \text{as } r \to 0\,, \end{cases}$$

$$\begin{cases} w''(r) + \frac{1}{r} w'(r) + K(r) e^{2w(r)} = 0\,, & r \in [1, \infty)\,, \\ w(r) = -(2 + B) \ln r + b + o(1) \quad \text{as } r \to \infty\,, \end{cases}$$

where $A > -(1 + p)$, $B > q - 1$ and a, b are real numbers.

Since $p > -1$ and $q < 1$, and by virtue of Lemma 2.2 in [4], the above equations possess unique solutions $v(r; a, A)$ and $w(r; b, B)$, respectively. Furthermore, we also have

$$\begin{cases} v(1; a, A) = a + O(e^{2a})\,, \\ v'(1; a, A) = A + O(e^{2a}) \end{cases}$$

for $a \le -M$ and

$$\begin{cases} v(1; a, A) = -a + C_1 + O(e^{-\mu a})\,, \\ v'(1; a, A) = -(A + 2 + 2p) + O(e^{-\mu a}) \end{cases}$$

for $a \ge M$, where M is large, C_1 is a constant independent of a and $0 < \mu \le 2$;

$$\begin{cases} w(1; b, B) = b + O(e^{2b})\,, \\ w'(1; b, B) = -(2 + B) + O(e^{2b}) \end{cases}$$

for $b \le -N$ and

$$\begin{cases} w(1; b, B) = -b + C_2 + O(e^{-\nu b})\,, \\ w'(1; b, B) = B - 2q + O(e^{-\nu b}) \end{cases}$$

for $b \ge N$, where N is large, C_2 is a constant independent of b and $0 < \nu \le 2$. Therefore, we obtain the asymptotic behaviors of curves as follows:

$$(3.1) \qquad \begin{cases} \lim_{a \to -\infty} \gamma_1(a) = (-\infty, 0)\,, & \lim_{a \to +\infty} \gamma_1(a) = (-\infty, -2(1 + p))\,, \\ \lim_{b \to -\infty} \gamma_2^-(b) = (-\infty, -2)\,, & \lim_{b \to +\infty} \gamma_2^-(b) = (-\infty, -2q)\,, \\ \lim_{c \to -\infty} \gamma_2^+(c) = (-\infty, 2)\,, & \lim_{c \to +\infty} \gamma_2^+(c) = (-\infty, -2(q + 2))\,, \\ \lim_{d \to -\infty} \gamma_3(d) = (-\infty, 0)\,, & \lim_{d \to +\infty} \gamma_3(d) = (-\infty, -2(q + 1))\,. \end{cases}$$

REMARK 3.1. (3.1) shows that γ_1, $\gamma_2^{\pm}$ and γ_3 do not possess limit points in (θ, η)-plane as parameters tending to plus and minus infinity.

Now, by combining the facts confirmed in Section 2, we are in a position to demonstrate our main consequences.

PROOF OF THEOREM 1.1. By integrating (2.5) over $[0, r]$, we see that for any solution $u(r)$ of (1.11), i.e., $u(r)$ being of Type R-$*$,

$$P(r; u) = G(r)e^{2u} - 2\int_0^r G(s)e^{2u}u'(s)ds\,, \quad r > 0\,,$$

where $P(r; u) = P(r; u; 0, 0)$ and $G(r)$ are defined as in (2.4) and (1.14), respectively. Since $G(r)$ is nonnegative for $r > 0$, we have

$$P(r; u) > -2\int_0^r G(s)e^{2u}u'(s)ds > 0\,, \quad r > 0\,,$$

which implies

$$\begin{aligned} c_\infty(c_\infty + 2) &= \lim_{r\to\infty} P(r; u) \\ &\geq -2\int_0^\infty G(s)e^{2u}u'(s)ds \\ &> 0\,, \end{aligned}$$

where c_∞ is set as in Lemma 2.1(b) with respect to $u(r)$, i.e., $c_\infty = \lim_{r\to\infty} ru'(r)$. Then $u(r)$ is not of Type $*$-R$^-$ since otherwise c_∞ must equal -2, and hence (a) is proved. Moreover, since $0 \leq q < 1$ and by Remark 2.1, we assure that any solution of (1.9) goes to minus infinity as $r \to \infty$. Finally, from Remark 2.3 and combining Propositions 2.1 and 2.3, we obtain (b), (c) and (d). □

PROOF OF THEOREM 1.2. Using the similar arguments as in the proof of Theorem 1.1, we obtain, for any solution $u(r)$ of (1.11),

$$c_\infty(c_\infty + 2) < 0\,,$$

where c_∞ is selected as in Lemma 2.1(b) with respect to $u(r)$. Then $u(r)$ can not be of Type $*$-R$^-$ because $c_\infty \neq -2$. Hence (a) is proved. The proofs for the remaining assertions follow by the same way as those of Theorem 1.1. □

PROOFS OF THEOREMS 1.3 THROUGH 1.6. First, by combining (3.1) with functions $G(r)$ and $P(r; u)$, we can determine whether Γ_1 intersects the other three curves or not. In particular, Γ_1 and Γ_2^- are identical if $K(r)$ is a constant function. Finally, by virtue of Propositions 2.1 through 2.3 and Remark 2.3, the structure of solutions for (1.9) can be clarified completely case by case. □

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