*The Structure of Shock Waves in the Continuum Theory of Fluids**

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1. Introduction and summary. The purpose of this paper is to explore the consequences and potentialities of continuum methods in the theory of the steady one-dimensional shock profile. The problem of shock structure is to present a quantitative description of the narrow transition region in the shock wave. Special interest attaches to the problem because this region is so narrow as to call into question the basic concepts of continuum mechanics and the validity of the continuum equations. It will be our conclusion to reject as unfounded on the basis of present evidence the frequent assertion that the equations of fluid dynamics are either of little value or in some sense inapplicable to the shock problem.[1]

From the time of Becker's fundamental paper [2] which seemed to show that the Navier-Stokes equations predict shock thicknesses so small as to invalidate continuum methods the belief has grown that only the kinetic theory is capable of a correct account of the shock structure. While other investigators, starting with Thomas [3], have emphasized the considerable effect on the shock thickness of temperature dependence of viscosity and thermal conductivity—these effects having been neglected in Becker's work—the feeling has persisted that

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[1] As examples of such assertions, GRAD ([1], p. 261) refers to the "breakdown of the Navier-Stokes equations", TSIEN (Mathematical Reviews **13**, p. 196) states that "early investigators . . . pointed out the inadequacy of the Navier-Stokes equations," LIGHTHILL (Mathematical Reviews **12**, p. 139) comments that "within stronger shocks the Navier-Stokes equations are too inaccurate . . . ," and OSWATITSCH (Zentralblatt für Mathematik **44**, p. 216), commenting on shock calculations with the Navier-Stokes equations, states "In Anbetracht, daß es sich aber dabei bekanntlich nicht mehr um ein Problem der Kontinuitätsphysik handelt, sind solche Rechnungen kaum von praktischer Bedeutung."

the Navier-Stokes equations predict values for the shock thickness which are unrealistically small and that these equations are basically inadequate to deal with the shock problem.

Let us examine in some detail the criticisms of the Navier-Stokes equations from the point of view of molecular theory. To begin with there is the fundamental objection that the shock transition is only several molecular mean free paths in length and that over such a small interval of rapid change the various continuum concepts are no longer meaningful. This argument has considerable force, but ultimately it stands or falls on the comparison of theory with observation. While it is true that we look intuitively upon continuum concepts such as density, temperature, and pressure as "averaged" in some sense over regions "large" compared with molecular distances, it does not follow that the equations founded on these concepts *necessarily* yield incorrect results when applied over small dimensions. Equations have often been successful beyond the limits of their original derivation, and indeed this type of success is one of the hallmarks of a great theory. Thus the same criticism can be and has been leveled at the Navier-Stokes equations with regard to the problem of ultrasonic absorption, yet here we find these equations giving exact agreement with experiments on monatomic gases down to wavelengths of two to three mean free paths (*cf.* Greenspan [4], Truesdell [5]), whereas attacks on the same problem by means of kinetic theory have been quite unsuccessful. No amount of discussion alone, either pro or con, can settle whether the above *a priori* objection to the Navier-Stokes equations is really pertinent in the shock problem; as stated before, this is an issue which must ultimately be decided by an appeal to observation, and unfortunately the experimental data available at present do not nearly suffice for such a decision.[2] For lack of such information we shall content ourselves with investigating the resources of the continuum theory and with showing that its results are not significantly different from those produced thus far by the kinetic theory and are in some respects better.

A second objection to the Navier-Stokes equations from the molecular viewpoint stems from the special character of the kinetic theory formalism. In all systematic treatments of the shock problem by kinetic theory methods (such as Wang Chang [8], Mott-Smith [9], Zoller [10], and Grad [1]) the starting point is the Boltzmann equation: it is a solution of this equation, satisfying the shock wave boundary conditions, which is sought. According to the existing approximation methods for solving the Boltzmann equation (Chapman-Enskog [11], Grad [1]) the Navier-Stokes equations emerge as low order approximations in a perturbation scheme which is presumed to yield a solution, and it is in the spirit of such methods that any later approximation is regarded as preferable, or closer

[2] The experiments of GREENE, COWAN, & HORNIG and of GREENE & HORNIG [7] are a first and important step toward measuring shock thicknesses. However, their results are still fragmentary, and in addition the indirect type of measurement involved in the experiments raises a serious question of interpretation of the data.

to a correct solution, than an earlier one. However, there is at present no real mathematical evidence to support this view.[3] Certainly this predilection for the higher kinetic theory approximations is not justified by the experimental evidence in the problem of ultrasonic absorption (*cf.* [**4**], [**5**]).

In this paper we first determine the solutions of the Navier-Stokes equations representing shock waves in monatomic gases. The resulting shock thicknesses we use for comparison with kinetic theory values. With the exception of Grad, all of the kinetic theory writers on the subject, although primarily concerned with monatomic gases, have based comparisons with the continuum theory on the results of Thomas and Becker, whose calculations were intended for air. These comparisons are unrealistic for several important reasons, first because the assumed temperature dependence of viscosity—constant for Becker and $T^{\frac{1}{2}}$ for Thomas—is less than that for real gases, and then because the Prandtl number[4] and ratio of specific heats, respectively $\frac{2}{3}$ and in $\frac{5}{3}$ monatomic gases, were taken equal to $\frac{3}{4}$ and $\frac{7}{5}$. The result is that the Becker and Thomas values of shock thickness are substantially smaller than those given by the Navier-Stokes equations for real monatomic gases (see Figure 2). Here we perform calculations on two monatomic gases, Helium and Argon, using the empirical values of the viscosity and Prandtl number, as required by the continuum theory. For comparison purposes we include results for monatomic gases obeying a $T^{\frac{1}{2}}$ and T^{1} law of viscosity. The results of the calculations on shock thickness are shown in Figure 2 where they are compared with the kinetic theory values of Mott-Smith [**9**] and Zoller [**10**].

To draw conclusions from such data we must bear in mind that in the kinetic theory the law of viscosity, and therefore the shock thickness, is determined by the type of molecule making up the gas (*e.g.*, elastic sphere, Maxwellian, *etc.*), so that all gases described by a given molecular model have the same shock thickness in units of molecular mean free path.[5] The choice of molecular model is essentially arbitrary, so that a particular kinetic theory formalism might be said to predict a range of values for shock thickness of a given monatomic gas, this range depending on the class of molecular models considered proper for the gas. This situation is to be contrasted with the continuum theory, in which any specific fluid has a definite experimentally given value for the viscosity and the other fluid parameters, and therefore a definite value for shock thickness.

The values of shock thickness calculated by Mott-Smith and Zoller, each

[3] Existence and uniqueness of the solution of the Boltzmann equation is an open problem. There are also those who doubt the convergence of the approximation schemes (*cf.* MOTT-SMITH [**9**], for example) and still others who question the validity of the Boltzmann equation itself as the correct mathematical embodiment of the physical principles of the kinetic theory of non-equilibrium states (*cf.* H. JEFFREYS, *Proc. Roy. Soc. London* (A) **160**, 325–348 (1937)).

[4] Prandtl number $= c_p\mu/\lambda$, where c_p is the specific heat at constant pressure, μ is the viscosity, and λ the thermal conductivity.

[5] The shock strength is assumed fixed in this discussion.

using his own method, are based, respectively, on the elastic sphere molecule, for which the kinetic theory gives $\mu \sim T^{\frac{1}{2}}$, and on the Maxwellian molecule, for which $\mu \sim T^{1}$. The Navier-Stokes values for Helium and Argon (at larger Mach numbers) fall between the two kinetic theory values, so that from one point of view the Navier-Stokes equations can be said to predict for these gases shock thicknesses larger than Mott-Smith's but smaller than Zoller's. On the other hand, if Mott-Smith's and Zoller's values are compared with those obtained from the Navier-Stokes equations for gases obeying respectively the $T^{\frac{1}{2}}$ and T^{1} law of viscosity, then the kinetic theory thicknesses are in each case larger than those given by the continuum equations. We can summarize by stating that the variation in both Navier-Stokes and kinetic theory thicknesses due to the law of temperature dependence of viscosity is large enough to make the two sets of values overlap. This statement includes Grad's results as well.[6]

When we turn from monatomic to polyatomic gases the point of view of continuum fluid dynamics is basically unaltered, namely, the shock profile is to be calculated by inserting in the Navier-Stokes equations the empirical values of the various fluid parameters (such as viscosity, *etc.*). Whereas the kinetic theory requires a new formal apparatus to take into account the internal degrees of freedom of polyatomic molecules in transport processes—a theory that is still largely undeveloped—continuum mechanics takes molecular structure into account only indirectly, through the values of the fluid parameters.[7] Thus, while the theory of monatomic gases rests on the Stokes relation, $2\mu_1 + 3\mu_2 = 0$, connecting shear viscosity μ_1 and compression viscosity μ_2, in polyatomic gases we are led for theoretical and experimental reasons to discard the Stokes relation and to consider the effect of an independent compression viscosity. In this way fluid dynamics introduces an additional parameter which reflects the internal structure and state of the fluid. To observe the effect on shock structure we have performed calculations using the few available experimental results on compression viscosity in polyatomic gases, and from these it appears that the theoretical values of shock thickness are substantially larger than those which result when the Stokes relation is assumed.

In §5 we construct the theory of the shock profile in a fluid with nonlinear viscosity. Up to this point we have considered only flows governed by the Navier-Stokes equations. The linear relation between viscous stress and rate of deformation which lies at the basis of these equations seems to be only approximately correct for certain types of phenomena (*cf.* Truesdell [**6**]). In particular, where

[6] Interpretation of the data on shock thickness is further complicated by the question of reference mean free path. For this see the concluding paragraph in §3 and GRAD ([**1**], p. 297).

[7] This simple view can yield incorrect results in case relaxation or quantum effects play an important role. However, we must guard against the attitude which insists upon specifically molecular influences in all phenomena and therefore rejects any other mode of description.

large deformation rates occur, as in shock waves, we can expect the influence of nonlinear terms in the relation between stress and rate of deformation to be of real importance. In order to estimate the effect of nonlinearity on the shock profile we fix our attention on a *Stokesian fluid*, in which the viscous stress may be an essentially arbitrary function of deformation rate subject to the condition that it approximates the usual linear law in case of small rates. For a general fluid of this character we are able to show, as for the Navier-Stokes equations, the existence of a shock wave between any pair of end states satisfying the Rankine-Hugoniot shock conditions. This general result is of additional interest in view of the fact that the flow equations for the post-Navier-Stokes approximations of the kinetic theory, which in effect describe fluids with nonlinear viscosity, break down and do not yield solutions beyond a certain value of shock strength; this is the case in the work of Grad [1], Zoller [10], and Wang Chang [8].

Specializing to an appropriately selected quadratic law of viscous stress, we find in calculations on Helium that the nonlinear theory provides higher values for the shock thickness than the Navier-Stokes equations, with a percentage increase over the linear theory of 16 % at Mach number 2 and of 30 % at Mach number 4. Although these results indicate only roughly the effect of nonlinear viscosity, they show that the resources of the continuum theory are by no means limited to the Navier-Stokes equations and, if expanded to include nonlinear stress, are capable of providing shock thicknesses comparable with the largest obtained from the kinetic theory thus far.

In the course of our calculations we have kept the usual maximum slope definition of shock thickness.[8] We have done so largely for convenience in calculation and in order to make comparisons with previous writers on the subject. However, we are in agreement with Grad ([1], p. 258) in his objections to the maximum slope definition on the grounds that it depends on a purely local feature of the shock profile and that in the general case this measure of shock thickness is unrealistically small. Although the thickness is the most evident parameter to associate with the shock wave, the difficulties in its direct measurement and even in its precise meaning make it desirable to have more sensitive quantities by which to study the shock profile.

2. Mathematics of the shock profile. In this section we recapitulate for sake of reference the mathematical theory of the shock profile.[9] We consider the

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[8] $$\text{Thickness} \equiv \frac{u_0 - u_1}{\left|\frac{du}{dx}\right|_{\max}}$$

where u_0, u_1 are respectively the initial and final values of the flow speed, and the denominator is the maximum slope (in magnitude) of the velocity profile.

[9] See GILBARG [12] and GRAD [1] for more details.

steady one-dimensional motion of a fluid in the absence of external forces. The equations of motion are:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} = \frac{d\tau}{dx} \tag{1}$$

$$\rho u \frac{d}{dx}(e + \tfrac{1}{2}u^2) = \frac{d}{dx}(\tau u) - \frac{dq}{dx}.$$

These express, in order, the conservation of mass, momentum, and energy. The quantities in the equations, all considered functions of x, are the density ρ, velocity u, x-component of stress τ, specific internal energy e, and rate of heat flow q in the x direction. The internal energy is supposed known as function of thermodynamic state, say, in terms of ρ and temperature T.

Throughout the following we shall assume the classical Fourier law of heat conduction, namely,

$$q = -\lambda \frac{dT}{dx} \tag{2}$$

where λ, the thermal conductivity, is a positive function of thermodynamic state. To complete the description of the fluid we must specify the stress τ. In this and the following two sections we shall adopt the linear relation,

$$\tau = -p + \mu \frac{du}{dx}, \tag{3}$$

where p is the pressure in the fluid ($p = p(\rho, T)$) and μ is a combined viscosity. This relation is the specialization to one dimension of the well known linear law connecting stress and rate of deformation in three dimensional isotropic fluids; namely, if $\tau_{\alpha\beta}$ are the stress components ($\alpha, \beta = 1, 2, 3$) and $d_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$ are the components of the rate of deformation tensor, where the u_α are the velocity components and $u_{\alpha,\beta}$ their derivatives, then

$$\tau_{\alpha\beta} = -p\delta_{\alpha\beta} + 2\mu_1 d_{\alpha\beta} + \mu_2 d_{\gamma\gamma}\delta_{\alpha\beta} \qquad \left(\delta_{\alpha\beta} = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta \end{cases}\right); \tag{4}$$

in this expression μ_1 is the *shear viscosity*, or simply the viscosity, and μ_2 is the *compression viscosity* (often called "second viscosity"); both are functions of thermodynamic state. For one-dimensional flows ($u_2, u_3 = 0$, $u_1 = u$, $\tau_{11} = \tau$) we obtain from (4)

$$\tau = -p + (\mu_2 + 2\mu_1)\frac{du}{dx} \tag{5}$$

so that in (3),

$$\mu = 2\mu_1 + \mu_2. \tag{6}$$

In particular, if the *Stokes relation* holds,

$$2\mu_1 + 3\mu_2 = 0, \tag{7}$$

we have

$$\mu = \tfrac{4}{3}\mu_1. \tag{8}$$

It is this expression for μ, rather than the more general (6), which usually appears in the literature.

The equations of motion (1) have the immediate first integrals,

$$\begin{aligned} \rho u &= m \\ \rho u^2 - \tau &= P \\ \rho u(e + \tfrac{1}{2}u^2) - \tau u - q &= E; \end{aligned} \tag{9}$$

and when (2) and (3) are inserted, we obtain

$$\begin{aligned} \rho u &= m \\ \rho u^2 + p - \mu \frac{du}{dx} &= P \\ \rho u\left(e + \tfrac{1}{2}u^2 + \frac{p}{\rho}\right) - \mu \frac{du}{dx} - \lambda \frac{dT}{dx} &= E. \end{aligned} \tag{10}$$

To bring these equations into the form we shall use, we first eliminate ρ from $(10)_2$ and $(10)_3$ by means of $(10)_1$, thus obtaining the system of equations,

$$\begin{aligned} \mu \frac{du}{dx} &= p + b(u - a) \\ \lambda \frac{dT}{dx} &= b(e - (u - a)^2 - c) \end{aligned} \tag{11}$$

where

$$a = \frac{P}{m}, \qquad b = m, \qquad c = \frac{E}{m} - \frac{P^2}{2m^2},$$

and e and p are known functions of u and T.

We see from (10) that any pair of values (u_0, T_0) and (u_1, T_1) making the right

members of equations (11) simultaneously zero determine states that satisfy the Rankine-Hugoniot shock conditions,

$$\rho_0 u_0 = \rho_1 u_1 = m$$

$$p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2 = P \tag{12}$$

$$e_0 + \tfrac{1}{2}u_0^2 + \frac{p_0}{\rho_0} = e_1 + \tfrac{1}{2}u_1^2 + \frac{p_1}{\rho} = E;$$

the other quantities appearing in these equations are obtained from the constants a, b, c in (11) and from the equations of state, $p = p(\rho, T)$, $e = e(\rho, T)$. Conversely, any pair of states satisfying the shock conditions make the right members of (11) simultaneously zero. It is possible to study the shock profile by means of the system (11) in general classes of fluids, as in [12]. However, since all our applications here are to perfect gases, we specialize at once to these simpler fluids, for which we have

$$p = RT\rho, \qquad e = c_v T, \qquad R = c_p - c_v;$$

the constants c_p, c_v are the specific heats per unit mass at constant pressure and volume, respectively. We set $\gamma = c_p/c_v$ as usual, then define

$$\delta = \tfrac{1}{2}(\gamma - 1).$$

Following Becker, we introduce dimensionless variables in (11) by means of the substitutions

$$\omega = \frac{m}{P}u, \qquad \phi = \frac{p}{P}, \qquad \theta = \frac{m^2 RT}{P^2} = \phi\omega, \tag{13}$$

and also define

$$\alpha = \frac{2Em^2}{P^2} - 1, \qquad \bar{\mu} = \frac{\mu}{m}, \qquad \bar{\lambda} = \frac{\lambda}{c_v m}. \tag{14}$$

After these substitutions equations (11) become

$$\bar{\mu}\frac{d\omega}{dx} = \omega + \frac{\theta}{\omega} - 1 \equiv M(\omega, \theta)$$

$$\bar{\lambda}\frac{d\theta}{dx} = \theta - \delta[(1 - \omega)^2 + \alpha] \equiv L(\omega, \theta). \tag{15}$$

The curves $L(\omega, \theta) = 0$, $M(\omega, \theta) = 0$ are the parabolas

$$\theta = \delta[(1 - \omega)^2 + \alpha], \qquad \theta = \tfrac{1}{4} - (\tfrac{1}{2} - \omega)^2,$$

which intersect in the points $Z_0 = (\omega_0, \theta_0)$, $Z_1 = (\omega_1, \theta_1)$ given by

$$\left.\begin{matrix}\omega_0\\ \omega_1\end{matrix}\right\} = \frac{1}{2(\delta + 1)}\,[2\delta + 1 \pm \sqrt{1 - 4\delta(\delta + 1)\alpha}]$$

(16)

$$\left.\begin{matrix}\theta_0\\ \theta_1\end{matrix}\right\} = \frac{\delta}{2(\delta + 1)^2}\,[1 + 2(\delta + 1)\alpha \mp \sqrt{1 - 4\delta(\delta + 1)\alpha}];$$

from $\theta = \varphi\omega$ we have also

$$\left.\begin{matrix}\phi_0\\ \phi_1\end{matrix}\right\} = \frac{1}{2(\delta + 1)}\,[1 \mp \sqrt{1 - 4\delta(\delta + 1)\alpha}].$$

These are the non-dimensional values for the initial and final states of a shock wave and, as we see, they are completely described (except for the fluid constant δ) in terms of the single parameter α. The latter in turn is fixed in terms of the pressure ratio $\eta = p_1/p_0$ and the incident Mach number M_0 by the relations

$$\alpha = \frac{1}{\delta(\delta + 1)}\,\frac{\eta}{(\eta + 1)^2}, \qquad \eta = \frac{2\delta + 1}{\delta + 1}\,M_0^2 - \frac{\delta}{\delta + 1}.$$

Let $Z_0 = (\omega_0, \theta_0)$ and $Z_1 = (\omega_1, \theta_1)$ be the initial and final states of a discontinuous shock wave. The continuous shock wave, or shock profile, is a solution $(\omega(x), \theta(x))$ of (15) such that

$$(\omega(x), \theta(x)) \to (\omega_0, \theta_0) \text{ as } x \to -\infty, \qquad (\omega(x), \theta(x)) \to (\omega_1, \theta_1) \text{ as } x \to +\infty.$$

Thus to construct a shock wave joining Z_0 and Z_1 we must find an integral curve of (15) connecting these points; this curve, considered in the (ω, θ) plane, we shall call the *shock curve*. As we have seen, Z_0 and Z_1 make the right members of (15) vanish simultaneously and therefore are singular points of the system. It is now well known that a shock curve can be found because Z_1 is a saddle point and Z_0 a nodal point of (15), and because the direction field determined by the equations has an especially simple character. The qualitative pattern of the family of integral curves of (15) is shown in Figure 1, where the arrows indicate the direction in which the curves are traversed as x increases.

The nature of the singular points at Z_0 and Z_1 is determined by the characteristic equation of the system,

$$(17)\qquad 0 = \begin{vmatrix} \frac{M_\omega}{\bar{\mu}} - \zeta & \frac{M_\theta}{\bar{\mu}} \\ \frac{L_\omega}{\bar{\lambda}} & \frac{L_\theta}{\bar{\lambda}} - \zeta \end{vmatrix}_{z=z_0,z_1}$$

$$= \zeta^2 - \left(\frac{M_\omega}{\bar{\mu}} + \frac{L_\theta}{\bar{\lambda}}\right)\zeta + \left(\frac{M_\theta L_\theta}{\bar{\lambda}\bar{\mu}}\right)\left(\frac{M_\omega}{M_\theta} - \frac{L_\omega}{L_\theta}\right) \qquad (Z = Z_0, Z_1).$$

(The subscripts ω, θ denote partial differentiation.) This is obtained in the usual way from the linearized form of equations (15) at the singular points. By in-

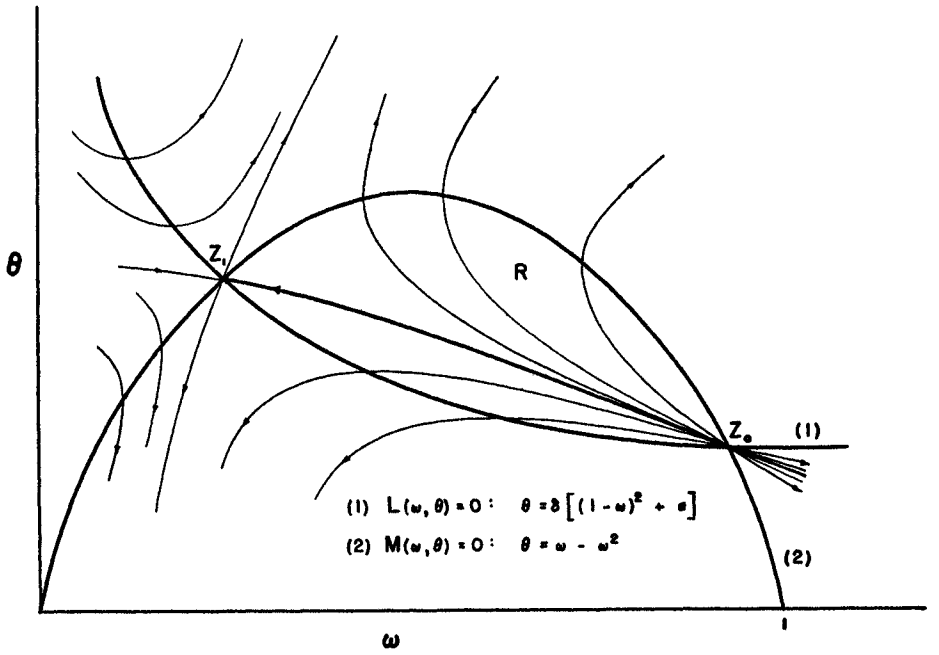


FIGURE 1. Direction field in (ω, θ) plane

serting the functions $M(\omega, \theta)$ and $L(\omega, \theta)$ it is not difficult to see that the roots of the characteristic equation are real and of opposite sign at Z_1, and both positive at Z_0. Thus Z_1 is a saddle point and Z_0 an unstable node. There are exactly two integral curves that approach the saddle Z_1 as $x \to +\infty$ and exactly two that approach it as $x \to -\infty$, these pairs corresponding to the negative and

positive roots, respectively, of the characteristic equation (17). The two members of each pair have the same slope but approach it from opposite directions. The slopes are given by

$$-\frac{L_\omega}{L_\theta - \zeta\bar{\lambda}}(Z_1) = -\frac{2\delta(1-\omega_1)}{1-\zeta\bar{\lambda}_1}, \tag{18}$$

and since $\omega_1 < 1$, this slope is negative for ζ negative. One of the integral curves, call it S, approaches Z_1 as $x \to +\infty$ from within the finite region R bounded by the two parabolas $M(\omega, \theta) = 0$ and $L(\omega, \theta) = 0$. For the parabolas divide the neighborhood of Z_1 into four regions in which the slopes of the direction field are alternately positive and negative (see Figure 1); in R the slope is negative, and in the direction of the second quadrant since $L(\omega, \theta) > 0$, $M(\omega, \theta) < 0$ in R. We observe now by considering the signs of $M(\omega, \theta)$ and $L(\omega, \theta)$ that on the boundary of R all integral curves are directed out of the region as x increases and therefore into the region as x decreases. Hence if we follow the integral curve S backwards from Z_1 in the direction of decreasing x (and increasing ω), we see that it can never leave R and must approach the other singular point as $x \to -\infty$.

This method clearly provides us with a practical procedure for constructing the shock profile. From (18) we know the slope of the shock curve at Z_1. We therefore select a starting point close to Z_1 on the straight line having this slope and then integrate the equation

$$\frac{d\theta}{d\omega} = \frac{\bar{\mu}L(\omega, \theta)}{\bar{\lambda}M(\omega, \theta)} = \frac{\bar{\mu}}{\bar{\lambda}} \cdot \frac{\omega\{\theta - \delta[(1-\omega)^2 + \alpha]\}}{\omega^2 - \omega + \theta} \tag{19}$$

numerically from the starting point up to ω_0. This provides us, as accurately as desired, the shock curve $\theta = \theta(\omega)$ joining Z_0 and Z_1 in the (ω, θ) plane. The shock profile itself, that is, $(\omega(x), \theta(x))$, is then obtained by quadrature from $(15)_1$ and $(15)_2$.

An altogether different method for constructing the shock curve has been developed by Grad [1]. His procedure involves considerably less labor than numerical integration when used in a series of computations in which the shock strength is varied, and is certainly preferably for systematic computations.

Once the shock curve $\theta = \theta(\omega)$ is known the shock thickness can be computed easily. We take as definition of shock thickness the usual one based on maximum slope of the velocity profile,[10] namely,

[10] Grad [1] adopts a definition of shock thickness which is preferable in that it depends on the shock profile in the large. He also considers the effect of using the temperature profile in the definition and finds slightly larger thicknesses thereby. The maximum slope definition gives a lower bound for the shock thickness, and gives unrealistically small values if, for example, the viscosity is very small compared to thermal conductivity; for in this case the maximum slope can be arbitrarily large, although the profile does not approximate the discontinuous shock wave of ideal fluid theory; (*cf.* [1], p. 267).

$$\Delta = \frac{u_0 - u_1}{\left|\frac{du}{dx}\right|_{\max}} = \frac{\omega_0 - \omega_1}{\left|\frac{d\omega}{dx}\right|_{\max}} . \tag{20}$$

The denominator is computed by inserting the function $\theta = \theta(\omega)$ in the right member of

$$\frac{d\omega}{dx} = \frac{M(\omega, \theta)}{\bar{\mu}}$$

and observing the maximum absolute value of the latter as ω varies from ω_0 to ω_1.

3. Shock thickness in monatomic gases. The theory of the shock profile is simplest and most complete for monatomic gases. For these fluids the existing experimental evidence and kinetic theory arguments agree on the whole in support of the Stokes relation

$$2\mu_1 + 3\mu_2 = 0, \tag{7}$$

so that the combined viscosity μ in the preceding section is given by

$$\mu = \tfrac{4}{3}\mu_1 . \tag{8}$$

In the remainder of this section we shall omit the subscript 1 from the symbol for shear viscosity and therefore write $\frac{4}{3}\mu$ for μ. A further simplification, holding for most gases and in particular for the monatomic ones, is the approximate constancy for each gas of the ratio $f = \lambda/\mu c_v$. This is well supported by theory and partially confirmed by experiment (*cf.* [11], Ch. 13). In monatomic gases we have also that f approximates 2.5 very closely whatever the gas, and since $\gamma = \frac{5}{3}$ it follows that

$$\text{Prandtl number} = \frac{c_p \mu}{\lambda} = \tfrac{2}{3} \tag{21}$$

for monatomic gases. Thus since

$$\frac{\bar{\mu}}{\bar{\lambda}} = \tfrac{4}{3}\frac{\mu c_v}{\lambda} = \tfrac{8}{15}$$

and $\delta = \frac{1}{2}(\frac{5}{3} - 1) = \frac{1}{3}$, we see from (19) that the equation of the shock curve for an arbitrary monatomic gas is

$$\frac{d\theta}{d\omega} = \tfrac{8}{15} \cdot \frac{\omega\{\theta - \frac{1}{3}[(1-\omega)^2 + \alpha]\}}{\omega^2 - \omega + \theta} , \tag{22}$$

and hence the shock curve for given strength of shock (or equivalently, for given α) is the same for all monatomic gases, irrespective of the law of viscosity.

To find the shock profile in its dependence on x we must still specify the law of viscosity. We shall assume that the temperature dependence of viscosity is given by a power law,

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^s = \mu_0 \left(\frac{\theta}{\theta_0}\right)^s, \qquad \mu_0 = \text{const.}; \tag{23}$$

although this dependence is not exact, it is sufficiently accurate for most purposes (*cf.* [11], Ch. 12).

In order to obtain shock thicknesses in units of mean free path before the shock (*i.e.* at $x = -\infty$) it is convenient to write the independent variable in these units. The Maxwell mean free path before the shock is given by the formula

$$l_0 = \tfrac{16}{5} \frac{\mu_0}{\rho_0 \sqrt{2\pi R T_0}} \tag{24}$$

where μ_0, ρ_0, T_0 are the initial values of the respective quantities. We introduce

$$z = x/l_0$$

in equations (15), and recalling (8), (14), (21), (23) and (24), we obtain

$$\begin{aligned} \frac{\beta}{M_0}\left(\frac{\theta}{\theta_0}\right)^s \frac{d\omega}{dz} &= \omega + \frac{\theta}{\omega} - 1 \\ \tfrac{5}{2}\frac{\beta}{M_0}\left(\frac{\theta}{\theta_0}\right)^s \frac{d\theta}{dz} &= \theta - \tfrac{1}{3}[(1-\omega)^2 + \alpha] \end{aligned} \tag{25}$$

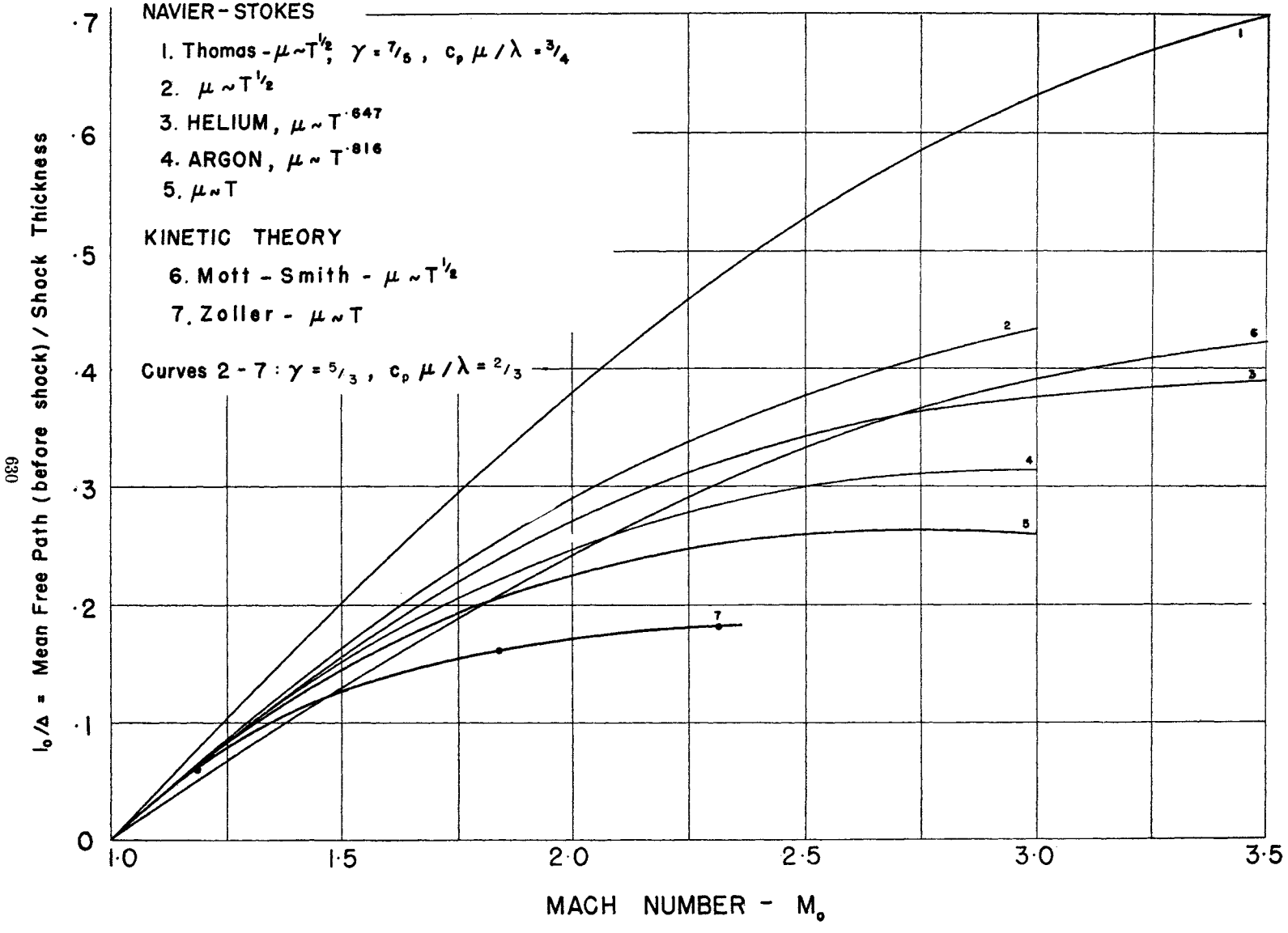
where $\beta = \sqrt{30\pi}/12$ and M_0 is the incident Mach number

$$M_0 = \frac{u_0}{\sqrt{\frac{5}{3}RT_0}}.$$

Since θ_0, M_0, α are specified by the shock strength alone we see that for fixed strength the thickness is completely determined by the exponent s in the law of viscosity. The expression for shock thickness in units of mean free path is simply

$$\frac{\Delta}{l_0} = \frac{\omega_0 - \omega_1}{\left|\frac{d\omega}{dz}\right|_{\max}}.$$

In order to see what the theory gives for representative monatomic gases, we have chosen Helium and Argon as subjects of calculation. In the law of viscosity we take $s = .647$ for Helium and $s = .816$ for Argon (*cf.* [11], Ch. 12). These values of the exponent s lie between the two most frequently used values provided by the kinetic theory, namely, $s = \frac{1}{2}$ and $s = 1$, which correspond, re-


NAVIER-STOKES
1. Thomas - $\mu \sim T^{1/2}$; $\gamma = 7/5$, $c_p \mu / \lambda = 3/4$
2. $\mu \sim T^{1/2}$
3. HELIUM, $\mu \sim T^{.647}$
4. ARGON, $\mu \sim T^{.816}$
5. $\mu \sim T$
KINETIC THEORY
6. Mott - Smith - $\mu \sim T^{1/2}$
7. Zoller - $\mu \sim T$
Curves 2 - 7 : $\gamma = 5/3$, $c_p \mu / \lambda = 2/3$
l_o/Δ = Mean Free Path (before shock) / Shock Thickness
MACH NUMBER - M_o
0
·1
·2
·3
·4
·5
·6
·7
1·0
1·5
2·0
2·5
3·0
3·5

spectively, to the elastic sphere and Maxwellian molecule. For comparison we have therefore made computations based on the $T^{\frac{1}{2}}$ and T^1 law of viscosity as well. The results in all four cases are shown in Figure 2 where l_0/Δ, the reciprocal shock thickness in units of mean free path before the shock, is plotted as function of incident Mach number. The tabulated values are given below. The four curves for $s = \frac{1}{2}$, .647, .816, and 1 show the dependence of shock thickness, in a monatomic gas having Prandtl number $\frac{2}{3}$, on the exponent s in the law of viscosity. At Mach number 2 the shock thickness for $s = 1$ is about 30% larger than for $s = \frac{1}{2}$, and at Mach number 3 about 67% larger. The four curves have the same slope at $M_0 = 1$ and differ but slightly up to $M_0 = 1.2$.

Included in Figure 2 is also Thomas' curve of l_0/Δ for a gas obeying the $T^{\frac{1}{2}}$ law, but in which the Prandtl number is $\frac{3}{4}$ rather than $\frac{2}{3}$ and $\gamma = \frac{7}{5}$ instead of $\frac{5}{3}$. As appears by comparison with the $s = \frac{1}{2}$ curve (curve 2 in the figure), the shock thickness based on correct monatomic gas data is considerably larger than Thomas' value—about 30% larger at $M_0 = 2$ and about 45% at $M_0 = 3$—while the difference is even greater for the other monatomic gas curves. Becker's values of l_0/Δ all lie above the Thomas curve, and are not included.

TABLE: l_0/Δ *vs.* M_0 for $\mu \sim T^s$

M_0 \ s	1/2	.647 (Helium)	.816 (Argon)	1
1.2	.070	.068	.067	.067
1.4	.136	.132	.128	.122
1.7	.222	.210	.199	.187
2.0	.292	.270	.248	.224
2.5	.381	.344	.303	.264
3.0	.437	.377	.314	.261
4.0	—	.410	—	—

In Figure 2 we also compare the Navier-Stokes shock thickness with the kinetic theory values of Mott-Smith and Zoller. The former's results are based on an approximate solution of the Boltzmann equation constructed by superposition of two Maxwellian distributions and so chosen as to satisfy a transport equation.[11] This approximation is not yet part of a larger systematic theory. Mott-Smith expresses the opinion that his thicknesses are an upper bound for the correct values. Zoller's work is based on the Burnett equations of the Chapman-Enskog approximation sequence. His calculated shock profiles exhibit (as yet experimentally unobserved) damped oscillations at Mach numbers above 1.23, and his analysis shows that the existence of the shock wave for Mach numbers greater than 2.36 is in great doubt, suggesting that Burnett's equations break down, as far as the shock problem is concerned, at about this value of the Mach number. Since Zoller's data consist of only three computed points we

[11] The MOTT-SMITH curve shown in Figure 2 is obtained from the u^3 equation of transport in a gas of elastic sphere molecules; it is curve number 4, Figure 1 in [9].

have indicated them on the smooth curve drawn in Figure 2. In both Mott-Smith's and Zoller's calculations, as in ours with the Navier-Stokes equations, the values of γ and Prandtl number are respectively $\frac{5}{3}$ and $\frac{2}{3}$.

We have not included comparisons with Wang Chang [8], whose method, because of convergence difficulties, gives reasonable results only for very weak shocks.[12] For comparison between the Navier-Stokes equations and Grad's thirteen-moment approximation we refer to [1]. The latter theory breaks down at $M_0 = 1.65$ and fails to yield shock wave solutions for greater M_0, but up to this value it gives larger shock thicknesses than all other methods thus far under comparable conditions. However, the total variation in Navier-Stokes thicknesses due to the exponent in the law of viscosity is still sufficient to overlap the set of values obtained from the thirteen-moment method.

The comparison in Figure 2 between the Navier-Stokes and kinetic theory values of shock thickness shows first that the difference between the two theories is smaller than is commonly supposed from previous comparisons with the values of Becker and Thomas. The thicknesses in Helium and Argon exceed Mott-Smith's values;[13] these in turn are larger than for a Navier-Stokes gas in which $\mu \sim T^{\frac{1}{2}}$, but this difference does not exceed 10% at $M_0 = 3$. Hence it appears that the Mott-Smith and Navier-Stokes values are rather close. Zoller's values of thickness, on the other hand, differ from the continuum ones by a greater amount; they exceed the latter in a Maxwellian gas ($s = 1$) by as much as 35% in the range in which the Burnett equations possess an appropriate solution.

The one experimental value of shock thickness in Argon obtained by Greene & Hornig [7] at high enough shock strength to be useful for comparison with theory shows $l_0/\Delta = .15$ at $M_0 = 2.09$, with an experimental error that the authors estimate could easily be $\pm 25\%$. The stated value lies closest to the Zoller curve.

It is imperative now to make certain observations essential to proper interpretation of shock data. We have followed the customary procedure of measuring shock thickness in units of mean free path before the shock. However, the choice of reference mean free path is clearly arbitrary; since the mean free path varies along the shock, and since furthermore it varies in a manner dependent on the shock strength and law of viscosity, results of different apparent physical significance follow from different choices. Grad has shown that it is always possible to select within the shock front a reference mean free path in terms of which the shock thickness is bounded from below, away from zero, whatever the viscosity law, and has used this reference length for presenting his results (*cf.* [1], pp. 295–298). However, there is a large degree of arbitrariness in his choice also, and for this reason we do not adopt it.

If, as is customary and as we have done here, the mean free path before the

[12] GRAD [1], p. 300) states that if WANG CHANG's expansion of the shock thickness in powers of $M_0 - 1$ is transformed to one in ϵ, where $M_0^2 = (1 + \frac{1}{5}\epsilon)/(1 - \frac{1}{5}\epsilon)$, then the convergence is greatly improved and gives better results for stronger shocks.

[13] Exception must be made at smaller Mach numbers, where MOTT-SMITH's model is known to give too small a slope to the l_0/Δ curve (*cf.* [9], p. 886).

shock is chosen as reference, then the thickness in these units may approach zero or infinity for strong shocks depending on the exponent in the law of viscosity; namely, if $\mu \sim T^s$, then according as $s < \frac{1}{2}$, $> \frac{1}{2}$, or $= \frac{1}{2}$ the thickness approaches zero, infinity, or a constant >0 as the shock strength becomes infinite.[14] In physical units, of course, the shock thickness decreases from infinity at zero shock strength to zero at maximum strength. To see how dissimilar the same results can appear because of different choices of reference unit, we observe that the dependence of Navier-Stokes shock thicknesses on the law of viscosity is exactly reversed when we go from the conventional reference path to Grad's; in other words, in his units the thickness *increases* as the exponent in the viscosity law *decreases*. And whereas the Navier-Stokes thicknesses for Helium and Argon are greater than Mott-Smith's values in the units used here, they would be smaller in the units adopted by Grad; on the other hand, in the latter units the Becker and Thomas values and also the values in monatomic gases having constant viscosity would exceed those of Mott-Smith. These facts emphasize the difficulty in judging the merits of a theory of shock structure by means of theoretical data on thicknesses alone.

4. Effect of compression viscosity. When we turn our attention from monatomic to polyatomic gases we find that definite results concerning the shock structure are harder to achieve. In the kinetic theory the difficulty is basic and stems from lack of a formalism that takes into account adequately the internal degrees of freedom. In the continuum theory, on the other hand, the basic method and equations remain unchanged, the complications due to molecular structure entering only indirectly through the determination of the fluid moduli (μ_1, μ_2, λ, *etc.*). Hence if these parameters are known, either, from experiment or theory, shock calculations as outlined in §2 can proceed without further hitch. The chief difficulty is that dependable values of the compression viscosity μ_2 are not well known for polyatomic gases.

It is the general acceptance of the Stokes relation which has prevented more widespread realization that the ordinary hydrodynamical theory offers in a sense another degree of freedom in the attack on the shock problem. Although the validity of the Stokes relation is still a matter of controversy there seems to be ample reason, both experimental and theoretical, for discarding it and for considering the two viscosities as independent.[15] Our procedure here is to use the existing experimental data on compression viscosity as a rough basis for calculating shock thicknesses from the Navier-Stokes equations, the results so obtained being intended to provide estimates of the effect of this additional flexibility of the Navier-Stokes equations arising from compression viscosity.

[14] We are thus led to suspect that much of the current opinion denying the validity of the Navier-Stokes equations would be less emphatic if BECKER's original calculations for strong shocks had been made on a gas obeying a T^1 rather than a T^0 law of viscosity.

[15] See TRUESDELL ([6], pp. 228–231) and cited references for a comprehensive discussion of the theoretical and empirical evidence on the Stokes relation.

Deviation from the Stokes relation is measured by the *bulk viscosity*

$$\kappa = \tfrac{2}{3}\mu_1 + \mu_2;$$

from thermodynamic considerations this quantity must be non-negative, and its vanishing is equivalent to the Stokes relation. Our computations will be based on the two extremes of empirical data obtained on bulk viscosity. The experimental values cited by Tisza [**13**] and also by Karim & Rosenhead [**14**] show κ/μ_1 to be very large in some fluids. For example, Tisza cites experiments that give $\kappa/\mu_1 \approx 2000$ in CO_2 and N_2O, a value that is probably much too high; Karim & Rosenhead present values as high as 200 for some fluids. At the other extreme, Tisza states that in air κ and μ_1 are of the same order of magnitude, and such comparative values of κ and μ_1 are mentioned also by Karim & Rosenhead. However, in all fluids mentioned by these authors the compression viscosity has turned out *positive*, in direct opposition to the Stokes relation.

Returning now to the equations of motion, we find for the combined viscosity coefficient (6)

$$\mu = 2\mu_1 + \mu_2 = \tfrac{4}{3}\mu_1 + \kappa.$$

According to Chapman & Cowling ([**11**], p. 241) we may assume that $f = \lambda/\mu_1 c_v$ is constant for polyatomic as well as for monatomic gases, while experiments on liquids (*cf.* [**14**], p. 113) lead us to assume that μ_2/μ_1, and therefore also κ/μ_1, is a constant independent of temperature. Hence equation (19) for the shock curve now becomes

$$\frac{d\theta}{d\omega} = \sigma \cdot \frac{\omega\{\theta - \delta[(1-\omega)^2 + \alpha]\}}{\theta - \omega + \omega^2} \tag{26}$$

where σ is a constant:

$$\sigma = \frac{\bar{\mu}}{\bar{\lambda}} = \frac{c_v(\frac{4}{3}\mu_1 + \kappa)}{\lambda} = \frac{1}{f}\left(\tfrac{4}{3} + \frac{\kappa}{\mu_1}\right).$$

We retain also the power law of viscosity, $\mu_1 \sim T^s$, which is accurate enough for our purposes (*cf.* [**11**], Ch. 12). We see then that calculations of the shock profile differ only inessentially from those for monatomic gases.

We shall compare shock thicknesses when $\kappa \neq 0$ with the values obtained under the same conditions from the Stokes relation, $\kappa = 0$. First let us consider a gas in which $\kappa/\mu_1 \gg 1$. In this case the ratio $\bar{\mu}/\bar{\lambda} = \sigma$ is large, so that the shock curve solution of (26) is close to the parabola $\theta = \delta[(1-\omega)^2 + \alpha]$, and can be replaced by it with small error; this error approaches zero as $\sigma \to \infty$.[16] In other words the shock profile can be computed as if thermal conductivity were not present. From this fact and (15) we have

$$\bar{\mu}\frac{d\omega}{dx} = \omega + \frac{\theta}{\omega} - 1, \qquad \theta = \delta[(1-\omega)^2 + \alpha]$$

[16] For a rigorous proof see [**12**].

whence

$$\bar{\mu}\frac{d\omega}{dx} = (1+\delta)\frac{(\omega-\omega_0)(\omega-\omega_1)}{\omega}. \tag{27}$$

In this equation

$$\bar{\mu} = \frac{\mu}{m} = \frac{1}{m}\left(\tfrac{4}{3}\mu_1 + \kappa\right) = \frac{\mu_0}{m}\left(\tfrac{4}{3} + \frac{\kappa}{\mu_1}\right)\left(\frac{\theta}{\theta_0}\right)^s, \qquad \left(\mu_1 = \mu_0\left(\frac{\theta}{\theta_0}\right)^s\right). \tag{28}$$

The shock thickness (20) is therefore approximately proportional to $\kappa/\mu_1 + \frac{4}{3}$ when κ/μ_1 is large.[17] It can be computed either explicitly or numerically from (27) and (28).

We apply these considerations now to the special case of CO_2, for which, according to Tisza, $\kappa/\mu_1 \approx 2000 \gg 1$. Let us designate by $\Delta(\kappa)$ the shock thickness for bulk viscosity κ, and by $\Delta(0)$ its value for $\kappa = 0$, corresponding to the Stokes relation. (Of course, $\Delta(0)$ must be computed without the benefit of the above approximation, which is not applicable when κ/μ_1 is small.) Using the data for CO_2 taken from [11],

$$c_v = .153, \qquad \gamma = 1.3, \qquad \mu_1 \sim T^{.935}, \qquad f = 1.67,$$

we obtain for large values of κ/μ_1 (say in excess of 10) the following results:

$$\text{at } M_0 = 2, \qquad \frac{\Delta(\kappa)}{\Delta(0)} = .50\left(\frac{\kappa}{\mu_1} + \tfrac{4}{3}\right);$$

$$\text{at } M_0 = 4, \qquad \frac{\Delta(\kappa)}{\Delta(0)} = .40\left(\frac{\kappa}{\mu_1} + \tfrac{4}{3}\right).$$

Thus if Tisza's value of 2000 for κ/μ_1 is approximately correct, the shock thickness in CO_2 is about 1000 times larger at Mach number 2 than that given by the Stokes relation. Even if—as is more likely— κ/μ_1 is considerably less than 2000, but still large, we see that the increase in shock thickness due to bulk viscosity remains very great.

At the other extreme in the empirical data on bulk viscosity are values comparable with shear viscosity, as for example in air. The minimum value of κ consistent with the experimental observation that compression viscosity is never negative in polyatomic gases is $\kappa = \frac{2}{3}\mu_1$ ($\mu_2 = 0$). We adopt this value in calculations of shock thickness in air and obtain the results.[18]

$$\text{at } M_0 = 2, \qquad \frac{\Delta(\kappa)}{\Delta(0)} = 1.36;$$

[17] In a typical instance the error in calculating shock thickness by this method is about 6% for $\kappa/\mu_1 = 10$, and decreases for larger values.

[18] These computations are based on the following data for air, taken from [11]: $c_v = .171$, $\gamma = 1.4$, $\mu_1 = 1.709 \times 10^{-4}(T/273)^{.768}$, $\lambda = 5.80 \times 10^{-5}$.

$$\text{at} \quad M_0 = 4, \qquad \frac{\Delta(\kappa)}{\Delta(0)} = 1.26.$$

In other words, the effect of bulk viscosity in the extreme case that $\mu_2 = 0$ is an increase in thickness of 36% (26%) at Mach number 2 (4) over the values which result when the Stokes relation is adopted. It would thus appear from the available information on bulk viscosity that such effects and much larger ones are not at all exceptional.

It should be possible to subject the preceding results to partial test in shock tube experiments. It would be of interest to check whether polyatomic gases tend to have larger shock thicknesses than monatomic gases, as predicted by the theory, and also whether gases believed from other experiments to have large values of bulk viscosity show signs of exceptionally large shock thicknesses.

5. Nonlinear viscosity. In the preceding sections analysis of the shock structure has been based on the Navier-Stokes equation and the underlying linear law (3) connecting viscous stress and rate of deformation. While the successes of the Navier-Stokes equations have been impressive over a wide range of phenomena, there is reason to expect, both for heuristic and observational reasons, that the linear law is but a first approximation to a more general nonlinear one whose effects are felt only when the rate of deformation is sufficiently large. Such a generalization is natural if we pursue the analogy with the theory of finite strain in elasticity or, from another point of view, if we look upon the higher order kinetic theory approximations as describing fluids with stress and energy flux dependent nonlinearly on velocity and thermal gradients.[19] We propose here to study the effect of nonlinear viscosity on the theory of the shock profile, an application naturally suggested by the large velocity gradient present in the shock front.

Of the possible generalizations to fluids obeying a nonlinear stress law we select one in which the viscous stress may be a nonlinear function of the velocity gradient, but remains independent of temperature gradients.[20] Although the law of heat conduction can be generalized in a similar way, we shall retain the classical linear law (2) unchanged. To fix our ideas we now define the stress τ appearing in the equations of motion (1) by

$$\tau = -p + v(u', \rho, T) \qquad (u' = du/dx), \tag{29}$$

where $v(u', \rho, T)$, the viscous stress, is an arbitrary function with the following properties:

$$(a) \qquad v(0, \rho, T) = 0 \qquad \text{for all } \rho, T;$$

[19] For an extensive discussion of fluids with nonlinear characteristics we refer to [6] and to the references therein.

[20] For the general theory of these *Stokesian fluids* see TRUESDELL [6], pp. 232–235, also *J. Math. Pures Appl.* (9)**29**, 215–244 (1950).

$$(b) \qquad \frac{\partial v}{\partial u'}(0, \rho, T) = \mu(\rho, T) > 0 \qquad \text{for all } \rho, T.$$

Property (a) asserts the essential characteristic of fluids that viscous stress vanishes when there is no velocity gradient. Property (b) states that for small velocity gradients the stress is approximately given by the linear law (3), and the fluid becomes essentially the classical one described by the Navier-Stokes equations. In addition to (a) and (b) we shall later make one more formal mathematical assumption concerning the function v.

With $v(u', \rho, T)$ in place of $\mu\, du/dx$ in the equations of motion (10) we obtain the analogue of (11) in the form,

$$(30) \qquad \begin{aligned} v\left(\frac{du}{dx}, u, T\right) &= p + b(u - a) \\ \lambda \frac{dT}{dx} &= b[e - (u - a)^2 - c], \end{aligned}$$

where we identify $v(u', u, T)$ with $v(u', mu, T) \equiv v(u', \rho, T)$. This is an implicit system of differential equations for u and T. We shall now show, in close analogy with the results and methods already established for the Navier-Stokes equations, that the above system has a solution joining arbitrary shock wave end states. For convenience in exposition but without essential loss of generality we limit the discussion to perfect gases,[21] for which the equations corresponding to (15) may be written in the form,

$$(31) \qquad \begin{aligned} \varphi\left(\frac{d\omega}{dx}, \omega, \theta\right) &= \omega + \frac{\theta}{\omega} - 1 \equiv M(\omega, \theta) \\ \bar{\lambda} \frac{d\theta}{dx} &= \theta - \delta[(1 - \omega)^2 + \alpha] \equiv L(\omega, \theta). \end{aligned}$$

In the first equation $\varphi(\omega', \omega, \theta)$ is the function of the indicated variables which is obtained on the left when ω and θ are introduced in place of u, p, T, and e in (30). This function carries over the properties of $v(u', u, T)$, namely,

$$(a') \qquad \varphi(0, \omega, \theta) = 0 \qquad \text{for all } \omega, \theta;$$

$$(b') \qquad \frac{\partial \varphi}{\partial \omega'}(0, \omega, \theta) = \bar{\mu}(\omega, \theta) > 0.$$

By the implicit function theorem these properties have as immediate consequence the existence of a function $\psi(\omega, \theta)$ such that

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[21] The same proof carries over as well to the fluids obeying the general equations of state considered in [12].

$$\omega' = \omega'(M(\omega, \theta), \omega, \theta) \equiv \psi(\omega, \theta) \tag{32}$$

in some neighborhood of the parabola $M(\omega, \theta) = 0$, and such that $\psi = 0$ on this curve. Naturally the extent of the domain of definition of $\psi(\omega, \theta)$ cannot be deduced from purely local properties such as (a'), (b').

In the (ω, θ) plane let the two parabolas $M(\omega, \theta) = 0$ and $L(\omega, \theta) = 0$ intersect in points $Z_0 = (\omega_0, \theta_0)$ and $Z_1 = (\omega_1, \theta_1)$ corresponding to initial and final states of a shock wave. To show the existence of a shock curve joining these end points and lying in the finite region R bounded by the parabolas (Figure 1), we now make the further assumption that $\psi(\omega, \theta)$ is defined throughout R. Since $\psi = 0$ implies $\omega' = 0$, which in turn implies $M(\omega, \theta) = 0$ (property (a')), we observe that $\psi = 0$ only on the parabola $M(\omega, \theta) = 0$. Therefore $\psi(\omega, \theta)$ is of one sign throughout R, and this sign is negative since

$$\left.\frac{\partial \psi}{\partial \theta}\right|_{M=0} = -\left.\frac{\varphi_\theta - M_\theta}{\varphi_{\omega'}}\right|_{\omega'=0, M=0} = \left.\frac{M_\theta}{\varphi_{\omega'}}\right|_{\omega'=0, M=0} > 0.$$

If we prove the existence of an integral curve of the system

$$\begin{aligned} \frac{d\omega}{dx} &= \psi(\omega, \theta) \\ \frac{d\theta}{dx} &= \frac{L(\omega, \theta)}{\bar{\lambda}} \end{aligned} \tag{32}$$

joining the singular points Z_0 and Z_1 then obviously we have succeeded in proving it also for the system (31). The proof, patterned after the one for equations (15) in Section 2, rests on the following facts: (i) Z_0 and Z_1 are, respectively, node and saddle point of (32); (ii) there is a unique integral curve S of (32) that approaches the saddle Z_1 from within R as $x \to \infty$; (iii) S has negative slope in R and cannot intersect the boundary of R; (iv) S cannot terminate in the interior of R and must therefore approach Z_0. Steps (iii) and (iv) follow simply, just as for the Navier-Stokes equations, from our knowledge of the signs of $\psi(\omega, \theta)$ and $L(\omega, \theta)$ in and on the boundary of R. It remains only to discuss the nature of the singular points at Z_0 and Z_1 and thereby to establish (i) and (ii). The characteristic equation of (32) at Z_0 and Z_1 is

$$\left| \begin{array}{cc} \dfrac{\partial \psi}{\partial \omega} - \zeta & \dfrac{\partial \psi}{\partial \theta} \\ \dfrac{\partial}{\partial \omega}\left(\dfrac{L}{\bar{\lambda}}\right) & \dfrac{\partial}{\partial \theta}\left(\dfrac{L}{\bar{\lambda}}\right) - \zeta \end{array} \right|_{Z=Z_0, Z_1} = 0.$$

We observe that

$$\frac{\partial \psi}{\partial \omega}(Z_i) = -\frac{\varphi_\omega(0, Z_i) - M_\omega(Z_i)}{\varphi_{\omega'}(0, Z_i)} = \frac{M_\omega(Z_i)}{\bar{\mu}(Z_i)}, \qquad i = 0, 1,$$

and similarly,

$$\frac{\partial \psi}{\partial \theta}(Z_i) = \frac{M_\theta(Z_i)}{\bar{\mu}(Z_i)},$$

while

$$\frac{\partial}{\partial \omega}\left(\frac{L}{\bar{\lambda}}\right)_{z=z_i} = \frac{L_\omega(Z_i)}{\bar{\lambda}(Z_i)},$$

and

$$\frac{\partial}{\partial \theta}\left(\frac{L}{\bar{\lambda}}\right)_{z=z_i} = \frac{L_\theta(Z_i)}{\bar{\lambda}(Z_i)}.$$

Consequently the characteristic equation is precisely the same as (17) which results from the Navier-Stokes equations, and therefore all the earlier remarks about the singular points, in particular (*i*) and (*ii*) above, follow immediately. We have thus established the existence of a shock wave for our more general class of fluids. The result can be further generalized without difficulty to fluids in which the heat flux is a nonlinear function of the thermal gradient and also to fluids in which thermal and velocity gradients appear simultaneously in the expression both for viscous stress and for heat flux. The method requires essential alteration, however, when derivatives of order higher than the first appear, for in this case a plane direction field no longer suffices to describe the solutions. The approximations of the Chapman-Enskog sequence beyond that yielding the Navier-Stokes equations, and also the equations of Truesdell's more general "Maxwellian fluid", are of this character (*cf.* Zoller [**10**]).

The preceding argument shows that our earlier numerical procedure for constructing the shock profile can be carried over without essential change. However, the additional detail that now the function $\psi(\omega, \theta)$ must first be constructed can sometimes present a disagreeable computational problem.

The numerical calculation of shock thickness in our Stokesian fluid requires exact knowledge of the viscous stress $v(u', \rho, T)$, but at present there are no real fluids for which this function is known beyond the linear approximation. In an effort to arrive at an estimate of the influence of nonlinear viscosity on the shock profile we consider a gas for which we assume a quadratic viscous stress in the form,

$$v = \tfrac{4}{3}\mu u' + \alpha(\rho, T)u'^2, \tag{33}$$

where μ is the usual shear viscosity and $\alpha(\rho, T)$ is a coefficient still to be determined. For lack of specific empirical information concerning $\alpha(\rho, T)$ we

arbitrarily take a value suggested by the Burnett approximation of the kinetic theory. In this approximation the complete expression for stress in one-dimensional form in the absence of external forces is[22]

$$\tau = -p + \tfrac{4}{3}\mu u' - \frac{\mu^2}{p}\left(\tfrac{2}{3}\varpi_1 - \tfrac{14}{9}\varpi_2 + \tfrac{8}{27}\varpi_6\right)u'^2$$

$$- \frac{\mu^2}{p}\left[\tfrac{2}{3}\varpi_2(p'/\rho)' + \tfrac{2}{3}\varpi_3 T'' + \tfrac{2}{3}\varpi_4 p'T' + \tfrac{2}{3}\varpi_5 T'^2\right].$$

The coefficients $\varpi_1, \cdots, \varpi_6$ are dimensionless quantities whose values depend on the molecular model. In a general flow it is only the terms in the first line that contribute to (33); from these we get

$$v = \tfrac{4}{3}\mu u' - \frac{\mu^2}{p}\left(\tfrac{2}{3}\varpi_1 - \tfrac{14}{9}\varpi_2 + \tfrac{8}{27}\varpi_6\right)u'^2.$$

To serve as an example, let us take the values for the coefficients $\varpi_1, \varpi_2, \varpi_6$ given by Chapman & Cowling ([11], p. 268);

$$\varpi_1 = \tfrac{4}{3}\left(\tfrac{7}{2} - \frac{T}{\mu}\frac{d\mu}{dT}\right), \qquad \varpi_2 = 2, \qquad \varpi_6 = 8,$$

from which we get

$$v = \tfrac{4}{3}\mu u' + \frac{\mu^2}{p}\left(\tfrac{8}{9}\frac{T}{\mu}\frac{d\mu}{dT} - \tfrac{64}{27}\right)u'^2.$$

When this is inserted in equations (30), which we then make non-dimensional by the substitutions (13), we have

$$\bar{\mu}\frac{d\omega}{dx} + A\bar{\mu}^2\frac{\omega}{\theta}\left(\frac{d\omega}{dx}\right)^2 = \omega + \frac{\theta}{\omega} - 1 = M(\omega, \theta)$$

(34)

$$\bar{\lambda}\frac{d\theta}{dx} = \theta - \delta[(1-\omega)^2 + \alpha],$$

where

$$\bar{\mu} = \tfrac{4}{3}\frac{\mu}{m}, \qquad A = \left(\tfrac{1}{2}\frac{\theta}{\bar{\mu}}\frac{d\bar{\mu}}{d\theta} - \tfrac{4}{3}\right).$$

The left member of the first equation is the function $\varphi(\omega', \omega, \theta)$ of (31). Solving the quadratic equation for $\bar{\mu}\omega'$, we obtain the system,

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[22] *Cf.* [11], pp. 265–269, also WANG CHANG & UHLENBECK, "On the Transport Phenomena in Rarified Gases," Appl. Phys. Lab. Rep't No. APL/JHU, CM-443, 1948.

$$\bar{\mu}\frac{d\omega}{dx} = \frac{\sqrt{1 + 4A\omega M(\omega, \theta)/\theta} - 1}{2A\omega/\theta}$$

(35)

$$\bar{\lambda}\frac{d\theta}{dx} = \theta - \delta[(1 - \omega)^2 + \alpha].$$

The positive square root is taken in the expression for $\bar{\mu}\omega'$ in order that $\omega' = 0$ imply $M(\omega, \theta) = 0$. For purposes of calculating the shock profile this system can now be treated as was that resulting from the Navier-Stokes equations. If $\bar{\mu} \sim \theta^s$, then the expression for A is simply

$$A = \tfrac{1}{2}s - \tfrac{4}{3}.$$

We observe that as long as A is negative the square root in $(35)_1$ is defined throughout R, and therefore the conditions for the existence theorem are all satisfied.

In order to compare with previous results obtained from the Navier-Stokes equations, we specialize to the case of Helium, for which (*cf.* §3) $s = .647$ and $\bar{\mu}/\bar{\lambda} = \frac{8}{15}$. The computations differ only as regards the right member of $(35)_1$. Let us designate by $\Delta_{\text{lin.}}$ and $\Delta_{\text{quad.}}$ the shock thickness according to the linear and nonlinear theory, respectively. Then our sample computations show the following results:

$$\text{at} \quad M_0 = 2, \qquad \frac{\Delta_{\text{quad.}}}{\Delta_{\text{lin.}}} = 1.16$$

$$\text{at} \quad M_0 = 4, \qquad \frac{\Delta_{\text{quad.}}}{\Delta_{\text{lin.}}} = 1.30.$$

Thus the shock thickness increases in going from the linear to nonlinear theory. At $M_0 = 1.2$ the percentage difference between $\Delta_{\text{quad.}}$ and $\Delta_{\text{lin.}}$ is small, as is to be expected. Of course, these results have no absolute significance since the choice of quadratic term in $(34)_1$ was quite arbitrary. In a real fluid the effect of nonlinearity might be greater or smaller depending on the empirical value of the coefficient of the quadratic term. However, the discussion shows that the potentialities of the continuum theory are by no means limited to the Navier-Stokes equations, and that by taking account of nonlinear viscosity we can come close to the largest shock thicknesses calculated from kinetic theory. In a fluid possessing both positive bulk viscosity and nonlinear viscous stresses, shock thickness will be still greater than results from either separately.

A concluding remark. While the evidence from kinetic theory and the theory of nonlinear viscosity indicate that the Navier-Stokes equations yield values of shock thickness that are perhaps smaller than the actual ones, there is nothing in the evidence to suggest that the classical theory is far wrong. When we consider

the additional possibilities inherent in bulk viscosity and in nonlinear stress, the rejection of continuum methods on *a priori* grounds seems altogether unjustified. Further research, both theoretical and experimental, is evidently necessary before the limits of the continuum theory can become clearly defined.

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