The Structure of Single-Track Gray Codes

Moshe Schwartz Tuvi Etzion

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Abstract

Single-track Gray codes are cyclic Gray codes with codewords of length n, such that all the n tracks which correspond to the n distinct coordinates of the codewords are cyclic shifts of the first track. We investigate the structure of such binary codes and show that there is no such code with 2^n codewords when n is a power of 2. This implies that the known codes with $2^n - 2n$ codewords, when n is a power of 2, are optimal. This result is then generalized to codes over $GF(p)$, where p is a prime. A subclass of single-track Gray codes, called single-track Gray codes with evenly spaced heads, is also defined. All known systematic constructions for single-track Gray codes result in codes from this subclass. We investigate this class and show it has a strong connection with two classes of sequences, the full-period necklaces and the full-period self-dual necklaces. We present an iterative construction for binary single-track Gray codes which are asymptotically optimal if an infinite family of asymptotically optimal seed-codes exists. This construction is based on an effective way to generate a large set of nonequivalent necklaces and a merging method for cyclic Gray codes based on necklaces representatives.

Keywords: cyclic Gray codes, feedback shift-register, linear complexity, necklaces, self-dual sequences, single-track codes.

1 Introduction

Gray codes were found by Gray [14] and introduced by Gilbert [13] as a listing of all the binary n-tuples in a list such that any two successive n-tuples in the list differ in exactly one position. Generalization of Gray codes were given during the years. Such generalizations include the arrangements of other combinatorial objects in a such way that any two consecutive elements

in the list differ in some prespecified, usually small way [14, 13]. Other generalizaions include listing subsets of the binary n-tuples in a Gray code manner, in such a way that the list has some more prespecified properties. These properties were usually forced by a specific application for the Gray code. As an example we have the uniformly balanced Gray codes. In certain applications, it is needed that the number of bit changes will be uniformly distributed among the bit positions. Uniformly balanced Gray codes were shown to exist for n which is a power of 2 by Wagner and West [23]. Recently Bhat and Savage [1] have shown that such codes exist for all n. During the years Gray codes and their generalizations have found applications in a variaty of areas such as circuit testing [21], signal encoding [19], ordering of documents on shelves [18], data compression [20], statistics [6], processor allocation in the hypercube [4], codes for certain memory devices [7] hashing [9], information storage and retrieval [3], and puzzles, such as the Chinese Rings and Tower of Hanoi [12]. Finally, for an excellent survey on Gray codes the interested reader is referred to [22].

The classic example of a Gray code is the *reflected Gray code* [14, 13]. This code is a list of the 2^n binary n-tuples in the following way. For $n = 1$ the list consists of the words 0 and 1. Given the list $\mathcal{X}$ of the 2^{n-1} binary $(n-1)$-tuples, we generate the list of the the 2^n binary n-tuples by attaching a ZERO as a prefix to every element of the list $\mathcal{X}$ in its order, and then attaching a ONE as a prefix to every element of the same list $\mathcal{X}$ in reversed order. As an example, for $n = 3$ the list of the reflected Gray code is 000, 001, 011, 010, 110, 111, 101, 100. One of the properties of the reflected Gray code is that there is a change in the last coordinate of every other word. We will use this property later.

In this paper we discuss another class of Gray codes, single-track Gray codes. A single-track Gray code is a list of P binary words of length n, such that each two consecutive words, including the last and the first, differ in exactly one position and when looking at the list as an $P \times n$ array, each column of the array is a cyclic shift of the first column. These codes were defined by Hiltgen, Paterson, and Brandestini [15] who also gave their main application. A length n, period P Gray code can be used to record the absolute angular positions of a rotating wheel by encoding (e.g. optically) the codewords on n concentrically arranged tracks. n reading heads, mounted in parallel across the tracks suffice to recover the codewords. When the heads are nearly aligned with the division between two codewords, any components which change between those words will be in doubt and a spurious position value may result. Such quantization errors are minimized by us-

ing a Gray code encoding, for then exactly one component can be in doubt and the two codewords that could possibly result identifies the positions bordering the division, resulting in a small angular error. When high resolution is required, the need for a large number of concentric tracks results in encoders with large physical dimensions. This poses a problem in the design of small-scale or high speed devices. Single-track Gray codes were proposed in [15] as a way of overcoming these problems. Note, that since all the columns in these codes are cyclic shifts of the first one, it follows that the code is also a uniformly balanced Gray code, which again can be described by a single column. Not many constructions for single-track Gray codes are known. All these constructions are given in [15] and [8]. None of the known constructions is known to produce an infinite family of optimal codes, where by the word optimal we mean that the code has the largest period for a given length n. The main goal of this paper is to study the structure of these codes and to construct codes with period P as large as possible.

In Section 2 we present the formal definitions for single-track Gray codes. Then, we discuss the the known construction methods and structure of single-track Gray codes mainly of those generated by the known construction methods. We discuss all the main known results in this area. In section 3 we present an improvement to one of the known upper bounds, i.e., we show that single-track Gray codes with words of length n and period 2^n do not exist even if n is a power of 2. This proof establishes as a corollary that Etzion and Paterson [8] have constructed an infinite family of optimal single-track Gray codes. In section 4 we present an iterative construction for Gray codes of length nk from specific classes of Gray codes of lengths n and k. This class is infinite and the codes constructed are asymptotically optimal, given an infinite family of asymptotically optimal seed-codes for the construction.

2 The Structure of Single-Track Gray Codes

In this section we present the formal definitions for single-track Gray codes. Then, we present some basic properties of such codes and the idea of the main two known methods to construct such codes. These two methods provide single-track Gray codes with additional special properties. We further investigate these properties. We also outline the results of past work in this area.

Let $W = [w_0, w_1, \ldots, w_{n-1}]$ be a length n word. The *cyclic shift operator*, $\boldsymbol{E}$, is defined by $\boldsymbol{E}W = [w_1, w_2, \ldots, w_{n-1}, w_0]$ and the *complementary cyclic shift operator*, $\overline{\boldsymbol{E}}$, is defined similarly by $\overline{\boldsymbol{E}}W = [w_1, w_2, \ldots, w_{n-1}, \overline{w_0}]$, where $\overline{b}$ is the binary complement of b. Two length n words W_1, W_2 are said to be *equivalent* if there exists an integer i such that $\boldsymbol{E}^i W_1 = W_2$, where $\boldsymbol{E}^i$ is i consecutive applications of $\boldsymbol{E}$. The equivalence classes under the shift operator are called *necklaces*. Efficient algorithms for producing necklaces of a given length are given in [10, 11]. A length $2n$ word $W = [w_0, w_1, \ldots, w_{2n-1}]$ is called *self-dual* if for each i, $0 \leq i \leq n-1$, $w_{n+i} = \overline{w_i}$. Finally, for any two positive integers a and b, $gcd(a, b)$ denotes the greatest common divisor of a and b.

Definition 1 *Let W be a length n word. We define the* cyclic order *of W as*

$$o(W) \stackrel{\triangle}{=} \min\{i \mid \boldsymbol{E}^i W = W, i \geq 1\}$$

and the complementary cyclic order *of W as*

$$\overline{o}(W) \stackrel{\triangle}{=} \min\{i \mid \overline{\boldsymbol{E}}^i W = W, i \geq 1\}$$

If $o(W) = n$ we say that W has full cyclic order *(or W is a full-period word), and if $\overline{o}(W) = 2n$ we say that $W\overline{W}$ is a full-period self-dual word.*

Definition 2 *A* length n period P Gray code *is an ordered list of P distinct binary length n words,*

$$W_0, W_1, \ldots, W_{P-1}$$

such that each two adjacent words differ in exactly one coordinate. If W_{P-1} and W_0 also satisfy this condition, we say the code is cyclic.

Definition 3 *Let C be an ordered list of P length n words,*

$$W_0, W_1, \ldots, W_{P-1}$$

For each $0 \leq i < P$ we mark the components of W_i as,

$$W_i = [w_i^0, w_i^1, \ldots, w_i^{n-1}]$$

*The j-*th track *of C, for $0 \leq j < n$, is defined as*

$$t_j(C) \stackrel{\triangle}{=} [w_0^j, w_1^j, \ldots, w_{P-1}^j]$$

We say that C has the single-track property *if there exist integers,*

$$k_0, k_1, \ldots, k_{n-1}$$

where $k_0 = 0$, such that $t_i(C) = \boldsymbol{E}^{k_i} t_0(C)$ for each $0 \leq i < n$. For each $0 \leq i < n$, k_i is called the position of the i-th head.

Definition 4 *Let C be an ordered list of P length n words,*

$$W_0, W_1, \ldots, W_{P-1}$$

We say that C is a length n, period P single-track Gray code *if C is a cyclic Gray code and C has the single-track property.*

The main goal is now to construct a length n, period P single-track Gray code, where P is as large as possible. Bounds on P are of a special interest and a very straightforward result is the following lemma.

Lemma 1 *(Lemma 2 [15])*
If C is a length n, period P single-track Gray code , then $2n \mid P$ and $2n \leq P \leq 2^n$.

There are only a few constructions for single-track Gray codes [15, 8]. None of them attains the upper bound forced by lemma 1 for infinitely many values of n. Each of these constructions is based on one of the following methods.

Theorem 1 *(Theorem 4 [8])*
Let $S_0, S_1, \ldots, S_{r-1}$ be r length n binary nonequivalent full-period words, such that for each $0 \leq i < r-1$, S_i and S_{i+1} differ in exactly one coordinate, and there also exists an integer l, $\gcd(l, n) = 1$, such that S_{r-1} and $E^l S_0$ differ in exactly one coordinate, then the following words form a length n, period nr single-track Gray code :

$$\begin{array}{llll}
S_0, & S_1, & \ldots & S_{r-1}, \\
\boldsymbol{E}^l S_0, & \boldsymbol{E}^l S_1, & \ldots & \boldsymbol{E}^l S_{r-1}, \\
\boldsymbol{E}^{2l} S_0, & \boldsymbol{E}^{2l} S_1, & \ldots & \boldsymbol{E}^{2l} S_{r-1}, \\
 & \vdots & \vdots & \\
\boldsymbol{E}^{(n-1)l} S_0, & \boldsymbol{E}^{(n-1)l} S_1, & \ldots & \boldsymbol{E}^{(n-1)l} S_{r-1} \ .
\end{array}$$

Theorem 2 *(Theorem 15 [8])*
Let $S_0, S_1, \ldots, S_{r-1}$ r be length $2n$ self-dual full-period nonequivalent words. For each i, $1 \leq i \leq r-1$, let $S_i = [s_i^0, s_i^1, \ldots, s_i^{2n-1}]$ and let

$$F^j S_i = [s_i^j, s_i^{j+1}, \ldots, s_i^{j+n-1}]$$

where superscripts are taken modulo $2n$.

If for each $0 \leq i < r-1$, S_i and S_{i+1} differ in exactly two coordinates, and there also exists an integer l, $\gcd(l, 2n) = 1$, such that S_{r-1} and $E^l S_0$ differ in exactly two coordinates, then the following words form a length n, period $2nr$ single-track Gray code :

$$\begin{array}{llll} F^0 S_0, & F^0 S_1, & \ldots & F^0 S_{r-1}, \\ F^l S_0, & F^l S_1, & \ldots & F^l S_{r-1}, \\ F^{2l} S_0, & F^{2l} S_1, & \ldots & F^{2l} S_{r-1}, \\ & \vdots & \vdots & \\ F^{(2n-1)l} S_0, & F^{(2n-1)l} S_1, & \ldots & F^{(2n-1)l} S_{r-1} \ . \end{array}$$

Now, in order to construct a single-track Gray code we want to order as many as possible full-period necklaces of length n, or full-period self-dual necklaces of length $2n$ in a way which satisfies either Theorem 1 or Theorem 2, respectively. Hiltgen, Paterson, and Brandestini [15] suggested a method for ordering length n full-period necklaces in a way which satisfies the conditions of Theorem 1. Their result is summarized in the following theorem.

Theorem 3 *(Theorem 3 [15])*
If $n \geq 4$, then there exists a length n, period nt single-track Gray code for each even t which satisfies

$$2 \leq t \leq 2^{n - \lceil \sqrt{2(n-3)} \rceil - 1}$$

Etzion and Paterson [8] supplied three iterative constructions. The first construction produces a special arrangement of $2^{n-1}r$ nonequivalent full-period necklaces of length $2n$, which satisfies the conditions of Theorem 1 from a special arrangement of r full-period necklaces of length n which satisfies the same conditions. If p is prime and such arrangement of the $\frac{2^p-2}{p}$ full-period necklaces is known, then the construction produces a length $2p$, period $2^{2p} - 2^{p+1}$ single-track Gray code. This is an optimal code based on Theorem 1, but by using Theorem 2 it might be possible to obtain a length

$2p$ period $2^{2p}-4$ single-track Gray code. This comparison is important as all the known codes are obtained from these two constructions and no code which is not obtained by these construction or a variant of Theorem 2, which will be mentioned later in this section, is known.

The second construction of [8] which is a generalization of the first one in a certain sense produces a length kn, period $rs(2^n-s)^{k-1}(k+1)n$ single-track Gray code, where $n+1 \leq s \leq 2^{n-1}$, from a code of length n and period rn. This code is far from being optimal in any sense. In section 4 we improve this result for most cases, by producing better codes for similar parameter lengths.

The third construction of [8] is based on Theorem 2 and generates an infinite family of asymptotically optimal codes. These codes have length $n=2^m$, $m \geq 3$, and period 2^n-2n. As we will see in the next section this construction actually produces optimal codes since the upper bound given in Lemma 1 on the period of length n, period P single track Gray code, for n which is a power of 2, can be improved. A similar construction can be given for n's which are not powers of 2. Unfortunately, we need some seed-codes with some given properties to obtain better codes for other parameters, and these seed-codes were not found yet.

Definition 5 *Let C be a length n, period P single-track Gray code , and let the head positions be $k_0, k_1, \ldots, k_{n-1}$. We say that C has* evenly k-spaced heads *if*

$$k_{i+1} \equiv k_i + k \pmod{P}$$

for each $0 \leq i \leq n-2$.

It is important to note that all the constructions for single-track Gray codes known today produce codes which are either with evenly spaced heads or with a self-dual generating track which can produce a single-track Gray code with evenly spaced heads, as will be proved later in this section. As a first step we want to show that all evenly spaced heads single-track Gray codes are generated by the construction method of either Theorem 1 or Theorem 2.

Definition 6 *Let C be a set of words. The* cyclic order *and* complementary cyclic order *of the code C are defined as*

$$o(C) \stackrel{\triangle}{=} \min\{o(W) \mid W \in C\}$$
$$\overline{o}(C) \stackrel{\triangle}{=} \min\{\overline{o}(W) \mid W \in C\}$$

Theorem 4 *Let C be a length n, period P single-track Gray code with evenly k-spaced heads.*

- *If k is even then:*
  - $\gcd(k,P) = \frac{P}{o(C)}$
  - $o(W) = o(C) = n$ *for each* $W \in C$.
  - *There exists an ordering of $\frac{P}{o(C)}$ length n necklaces representatives of cyclic order n, which satisfies the requirements of Theorem 1.*
- *If k is odd then:*
  - $\gcd(k,P) = \frac{P}{\overline{o}(C)}$
  - $\overline{o}(W) = \overline{o}(C) = 2n$ *for each* $W \in C$.
  - *There exists an ordering of $\frac{P}{\overline{o}(C)}$ length $2n$ self-dual necklaces representatives of full cyclic order $2n$, which satisfies the requirements of Theorem 2.*

Proof *W.l.o.g. we assume that $k_0 = 0$. Let C be a length n, period P single-track Gray code with evenly k-spaced heads. Let $s = [s_0, s_1, \ldots, s_{P-1}]$ be the generating track of C. The i-th word, W_i, of C has the form $W_i = [s_i, s_{i+k}, s_{i+2k}, \ldots, s_{i+(n-1)k}]$ and hence $W_{i+k} = [s_{i+k}, s_{i+2k}, \ldots, s_{i+(n-1)k}, s_{i+nk}]$. We now distinguish between two cases.*

Case 1: *k is even. Since C is a Gray code, it follows that the parity of W_i and W_{i+k} is the same, and hence $s_i = s_{i+nk}$ and $W_{i+k} = \boldsymbol{E}W_i$. Therefore, for each j_1, j_2, which satisfy $j_1 \equiv j_2 \pmod{\gcd(k,P)}$ there exists an l such that $\boldsymbol{E}^l W_{j_1} = W_{j_2}$. Now, let W_m be a word in C for which $o(W_m) = o(C)$. Since $W_{i+jk} = \boldsymbol{E}^j W_i$, it follows that $W_{m+o(C)k} = \boldsymbol{E}^{o(C)} W_m = W_m$ and $W_{m+jk} = \boldsymbol{E}^j W_m \neq W_m$ for each $0 < j < o(C)$. Since each word appears at most once in the code, it follows that $o(C)k \equiv 0 \pmod{P}$ and hence $\boldsymbol{E}^{o(C)} W = W$ for each $W \in C$, and $\boldsymbol{E}^i W \neq W$ for each $0 < i < o(C)$, which means that $o(W) = o(C)$ and therefore, $\gcd(k,P) \cdot o(C) = P$.*

It is well known that $o(C)$ divides n, and if $o(C) < n$, then the weight of all the words is divisible by $n/o(C) > 1$. Therefore, no two words differ in exactly one coordinate. Thus, $o(C) = n$.

It is obvious that the list $W_0, W_1, \ldots, W_{P/o(C)-1}$ forms a Gray code, and since $\gcd(k,P) = P/o(C)$, it follows that all the words in it are nonequivalent. Moreover, there exists l_1 such that $W_{P/o(C)} = \boldsymbol{E}^{l_1} W_0$. Therefore, there exists l_2 such that $l_1 k = l_2 P + P/o(C)$ which implies $\frac{o(C)k}{P} \cdot l_1 - l_2 \cdot o(C) = 1$.

Since $o(C)k/P$ is an integer, it follows that $\gcd(o(C), l_1) = 1$. Thus, the list $W_0, W_1, \ldots, W_{P/o(C)-1}$ satisfies all the requirements of Theorem 1.
Case 2: *k is odd. The parity of W_i is different from the parity of W_{i+k}, and hence $s_i = \overline{s_{i+nk}}$. The rest of the proof is similar to the one of Case 1, where we use $\overline{o}(\cdot)$ and $\overline{\boldsymbol{E}}$ instead of $o(\cdot)$ and $\boldsymbol{E}$, respectively.* □

Single-track Gray codes with evenly k-spaced heads have some additional properties as the one given in the following lemma.

Lemma 2 *If C is a length n, period P single-track Gray code with evenly k-spaced heads, k odd, then the generating track of the code is self-dual.*

Proof *Let C be a length n, period P single-track Gray code with evenly k-spaced heads, k odd, and s its generating track. From the proof of Theorem 4 there exists an integer l, for which $gcd(l, \overline{o}(C)) = 1$, such that*

$$W_{i+P/\overline{o}(C)} = \overline{\boldsymbol{E}}^{l} W_i$$

for each $0 \leq i < P$. Since $\overline{o}(C)$ is even, it follows that l is odd and therefore,

$$W_{i+P/2} = \overline{\boldsymbol{E}}^{l \cdot \overline{o}(C)/2} W_i = \overline{\boldsymbol{E}}^{\overline{o}(C)/2} W_i = \overline{W_i}$$

and the generating track is self-dual. □

When the generating track of a single-track Gray code is self-dual, many other single-track Gray codes can be generated by selecting any subset of the columns and complementing them. These new single-track Gray codes do not necessarily have evenly spaced heads. These are the only known single-track Gray codes which cannot be constructed directly by the use of either Theorem 1 or Theorem 2. But, they are of course constructed by a straightforward variant of Theorem 2. Moreover, as an immediate consequence from Theorem 4 we can conclude that there is no similar arrangements as in Theorems 1 and 2 of feedback shift registers sequences of order n. It is a very interesting problem to construct single-track Gray codes which do not have evenly spaced heads and are not constructed by this variant of Theorem 2. Note, that if n is odd then by complementing every other column of the code generated by Theorem 2 we obtain a code which can be constructed via Theorem 1.

3 Nonexistence Result

Let C be a single-track Gray code of length n and period P. By Lemma 1, there is a theoretic possibility that $P = 2^n$, but then, necessarily, n is a power of 2. The only known code with these parameters is the length 2 period 4 single-track Gray code. In this section we show that there is no other code with such parameters. The proof will consider the track as a sequence of length 2^n and investigate the polynomial of minimal degree which generates this sequence. In the literature the degree of this polynomial is often called the linear complexity of the sequence. Hence, we first present the necessary definitions for this discussion.

Definition 7 *Let $S = [s_0, s_1, \ldots, s_{r-1}]$ be a length r sequence, and let*

$$S(x) \stackrel{\triangle}{=} \sum_{i=0}^{r-1} s_i x^i$$

be a polynomial. We say the $S(x)$ is the characteristic polynomial *of S, and S is the* characteristic word *of $S(x)$.*

Definition 8 *Let S be a length r sequence over $GF(q)$. The* linear complexity *of S is defined as*

$$c(S) \stackrel{\triangle}{=} \min\{\deg f(x) \mid f(x) \not\equiv 0, f(x) \cdot S(x) \equiv 0 \pmod{x^r - 1}\}$$

The linear complexity as defined here is the same as the degree of the minimal degree linear recursion which generates the sequence. This is the more common definition as given in [17].

Lemma 3 *Let S be a length $r = p^{l_1}$ sequence over $GF(q)$, where $q = p^{l_2}$, p prime. The linear complexity of S is c if and only if*

$$(x-1)^{c-1} S(x) \equiv d(1 + x + x^2 + \ldots + x^{r-1}) \pmod{x^r - 1} \qquad (1)$$

for some $d \neq 0$.

Lemma 4 *(Theorem 2 [2])*
Let S be a length $n = 2^m$ binary sequence. S is self-dual if and only if $c(S) = 2^{m-1} + 1$.

We will prove now a result which is stronger than the nonexistence result that we actually want to prove.

Theorem 5 *There is no ordering of all the 2^n, words of length $n = 2^m$, $m \geq 2$, in a list which satisfies all the following requirements:*

1. *Each two adjacent words have different parity.*
2. *The list has the single-track property.*
3. *Each word appears exactly once.*

Proof *Let us assume the contrary, i.e., let s be the track of a single-track code in which each n-tuple appears exactly once and each two adjacent words have different parity. Let $s(x)$ be the characteristic polynomial of s and c_1 the largest integer for which there exists a polynomial $p_1(x)$ which satisfies,*

$$s(x) \equiv (x+1)^{c_1} p_1(x) \pmod{x^{2^n}+1} . \tag{2}$$

Let $k_0, k_1, \ldots, k_{n-1}$ be the locations of the heads in the list, $h(x) \stackrel{\triangle}{=} \sum_{i=0}^{n-1} x^{k_i}$ the head locator *polynomial of the list, and h the characteristic length 2^n word of $h(x)$. Let c_2 be the largest integer for which there exists a polynomial $p_2(x)$ which satisfies,*

$$h(x) \equiv (x+1)^{c_2} p_2(x) \pmod{x^{2^n}+1} . \tag{3}$$

$x^{2^n}+1 = (x+1)^{2^n}$ over $GF(2)$ and hence $0 \leq c_1,\ c_2 \leq 2^n - 1$. Since each two adjacent words have different parity it follows that

$$(x+1)h(x)s(x) \equiv 1 + x + x^2 + \ldots + x^{2^n-1} \pmod{x^{2^n}+1} . \tag{4}$$

Since $(x+1)^{2^n} = x^{2^n}+1$ and $(x+1)^{2^n-1} = 1 + x + x^2 + \ldots + x^{2^n-1} \pmod{x^{2^n}+1}$, it follows from (1), (2), (3), and (4) that

$$c_1 + c_2 = 2^n - 2 . \tag{5}$$

Equations (1), (2), (3), and (4) also imply the $c_1 + 2$ is the linear complexity of h, and $c_2 + 2$ is the linear complexity of s. Since each word appears in the list exactly once, it follows that s must be of full cyclic order, and hence

$$c_2 \geq 2^{n-1} - 1 . \tag{6}$$

If we assume that h is not a full period word, then

$$\{k_i\}_{i=0}^{n-1} = \{2^{n-1} + k_i\}_{i=0}^{n-1}$$

and the i-th word and the $(i+2^{n-1})$-th word contain exactly the same components of the generating track s. The allzero word appears somewhere in the list, and hence it will appear at least twice, which is a contradiction. Thus, h is of full-period and therefore $c_1 \geq 2^{n-1} - 1$.

Self-dual sequences of length 2^n have weight 2^{n-1} and since h has weight n, it follows that h is not self-dual when $2^n \geq 4$, and hence by Lemma 4 the linear complexity of h is not $2^{n-1} + 1$. Therefore,

$$c_1 \geq 2^{n-1} \ . \tag{7}$$

Summing (6) and (7) we get that

$$c_1 + c_2 \geq 2^n - 1$$

in contradiction to (5). Thus, no such single-track code with track s exists. □

Corollary 1 *There are no single-track Gray codes of length $n \geq 3$ and period 2^n.*

As mentioned in Section 2, Etzion and Paterson [8] have constructed single-track Gray codes of length $n = 2^m$ and period $2^n - 2n$.

Corollary 2 *The single-track Gray codes of length $n = 2^m$ and period $2^n - 2n$ are optimal.*

The nonexistence theorem can be generalized in a very interesting way to single-track Gray codes over $GF(p)$, where p is a prime. Since, we don't discuss nonbinary codes in this paper elsewhere, we present this generalization of the nonexistence theorem in Appendix A.

4 An Iterative Construction

In this section we describe an iterative construction which generates long period single-track Gray codes. We are given two pairs of disjoint Gray codes, of lengths n and k, from nonequivalent necklaces. Each pair satisfies a set of properties needed for the construction. The construction itself is made of five stages. The first stage is an iterative generation of a large amount of nonequivalent, full-period necklaces. The second, is ordering of the necklaces into many Gray codes. The third stage consists of merging

these Gray codes into two sets of Gray codes. In the fourth stage, the Gray codes of each set are concatenated into two cyclic Gray codes which satisfy the properties needed for the construction. The last stage is a simple merging of these two Gray codes into one Gray code.

4.1 Nonequivalent Necklaces Generation

The first step in generating a long single-track Gray code based on necklaces, is to generate a large set of nonequivalent full-period necklaces. This construction should generate the necklaces in such a way that it will be easy to order them into a Gray code. The construction of these nonequivalent necklaces will be iterative, i.e., given two sets of nonequivalent full-period words of length n and length k, respectively, we generate a set of nonequivalent full-period words of length nk. We first partition the set of all binary m-tuples into two sets, those ending in a ZERO and those ending in a ONE, i.e., for each $b \in \{0,1\}$, we define,

$$\mathcal{X}_m^b \stackrel{\triangle}{=} \{[x_0, x_1, \ldots, x_{m-2}, x_{m-1}] \in \{0,1\}^m \mid x_{m-1} = b\} \ .$$

Construction 1 *For each $b \in \{0,1\}$, let*

$$\mathcal{N}_n^b \stackrel{\triangle}{=} \{S_0^b, S_1^b, \ldots, S_{r_b-1}^b\} \subseteq \mathcal{X}_n^b$$
$$\mathcal{N}_k'^b \stackrel{\triangle}{=} \{S_0'^b, S_1'^b, \ldots, S_{r_b'-1}'^b\} \subseteq \mathcal{X}_k^b$$

be sets of nonequivalent, full-period necklaces, such that

$$\mathcal{N}_n^0 \bigcap \bigcup_{i=0}^{n-1} \{E^i S \mid S \in \mathcal{N}_n^1\} = \mathcal{N}_k'^0 \bigcap \bigcup_{i=0}^{k-1} \{E^i S' \mid S' \in \mathcal{N}_k'^1\} = \emptyset$$

From these sets we generate the following set,

$$\mathcal{N}_{n,k} \stackrel{\triangle}{=} \left\{ [X_0, X_1, \ldots, X_{k-2}, S + \sum_{i=0}^{k-2} X_i] \;\middle|\; \begin{array}{l} [b_0, b_1, \ldots, b_{k-1}] \in \mathcal{N}_k'^0 \cup \mathcal{N}_k'^1, \\ X_i \in \mathcal{X}_n^{b_i}, \ 0 \le i < k-1, \\ S \in \mathcal{N}_n^p, p = \sum_{i=0}^{k-1} b_i \end{array} \right\}$$

Theorem 6 *The set $\mathcal{N}_{n,k}$ of Construction 1 contains nonequivalent full-period words of length kn.*

Proof *Let* $Y = [y_0, y_1, \ldots, y_{kn-1}]$ *and* $Z = [z_0, z_1, \ldots, z_{kn-1}]$ *be two words in the defined set* $\mathcal{N}_{n,k}$. *Let us assume that* $\boldsymbol{E}^c Y = Z$ *for some* $0 \leq c < kn$. *Now,*

$$\sum_{j=0}^{k-1} \boldsymbol{E}^{jn} \boldsymbol{E}^c Y = [\boldsymbol{E}^c S_{m_1}, \boldsymbol{E}^c S_{m_1}, \ldots, \boldsymbol{E}^c S_{m_1}]$$

$$\sum_{j=0}^{k-1} \boldsymbol{E}^{jn} Z = [S_{m_2}, S_{m_2}, \ldots, S_{m_2}]$$

where $S_{m_1} \in \mathcal{N}_n^{a_1}$ *and* $S_{m_2} \in \mathcal{N}_n^{a_2}$, $a_1, a_2 \in \{0, 1\}$.

For $a_1 \neq a_2$, S_{m_1} *and* S_{m_2} *are nonequivalent, and hence* $a_1 = a_2$. *For* $m_1 \neq m_2$, S_{m_1} *and* S_{m_2} *are nonequivalent, and therefore* $m_1 = m_2$, *which implies* $S_{m_1} = \boldsymbol{E}^c S_{m_1}$.
S_{m_1} *is a full-period word, and hence* $n \mid c$. *This implies, that if we look at*

$$\begin{aligned} S'_{m'_1} &= [y_{n-1}, y_{2n-1}, \ldots, y_{kn-1}] \\ S'_{m'_2} &= [z_{n-1}, z_{2n-1}, \ldots, z_{kn-1}] \end{aligned}$$

then $\boldsymbol{E}^{c/n} S'_{m'_1} = S'_{m'_2}$. *As before,* $S'_{m'_1}$ *and* $S'_{m'_2}$ *are nonequivalent full-period words, and hence* $k \mid c/n$ *or* $kn \mid c$. *Therefore,* $Y = Z$ *and thus, all the words in* $N_{n,k}$ *are nonequivalent.*

□

Construction 1 produces iteratively a large set of nonequivalent, full-period necklaces. This set is generated in a way which makes it relatively easy to order its necklaces in a cyclic Gray code, provided that the elements of $\mathcal{N}_n^0, \mathcal{N}_n^1, \mathcal{N}_k'^0, \mathcal{N}_k'^1$ can be ordered as a cyclic Gray code.

4.2 Generation and Merging of Gray Codes

Given a Gray code of length n, we will generate many Gray codes of the same period and length nk, for which all words belong to nonequivalent necklaces. Those Gray codes will be then merged into two sets of Gray codes. The generation of one such short Gray code and the merging of some of these Gray codes will be based on the following two trivial lemmas whose proofs are omitted.

Lemma 5 *If the words $S_0, S_1, \ldots, S_{r-1} \in \{0,1\}^n$ form a cyclic Gray code, then the following words form a cyclic Gray code:*

$$\begin{array}{c} [X_0, X_1, \ldots, X_{k-2}, S_0 + \sum_{i=0}^{k-2} X_i] \\ [X_0, X_1, \ldots, X_{k-2}, S_1 + \sum_{i=0}^{k-2} X_i] \\ \vdots \\ [X_0, X_1, \ldots, X_{k-2}, S_{r-1} + \sum_{i=0}^{k-2} X_i] \end{array}$$

where $X_i \in \{0,1\}^n$ for each $0 \leq i < k-1$.

Lemma 6 *Let $X_l \in \{0,1\}^n$ for each $0 \leq l < k-1$ and for some j, $0 \leq j < k-1$, let $X'_j \in \{0,1\}^n$ be a word such that X'_j differs from X_j in exactly the d-th coordinate. Furthermore, the necklaces $S_0, S_1, \ldots, S_{r-1} \in \{0,1\}^n$ form a cyclic Gray code in which, for some i, S_i and S_{i+1} differ in the d-th coordinate. Then, the following necklaces form a cyclic Gray code in which the last and first pair of necklaces differ in the $(d+jn)$-th coordinate:*

$$\begin{array}{c} [X_0, X_1, \ldots, X_j, \ldots, X_{k-2}, S_i + X_0 + X_1 + \ldots + X_j + \ldots + X_{k-2}] \\ [X_0, X_1, \ldots, X'_j, \ldots, X_{k-2}, S_{i+1} + X_0 + X_1 + \ldots + X'_j + \ldots + X_{k-2}] \\ \vdots \\ [X_0, X_1, \ldots, X'_j, \ldots, X_{k-2}, S_{r-1} + X_0 + X_1 + \ldots + X'_j + \ldots + X_{k-2}] \\ [X_0, X_1, \ldots, X'_j, \ldots, X_{k-2}, S_0 + X_0 + X_1 + \ldots + X'_j + \ldots + X_{k-2}] \\ \vdots \\ [X_0, X_1, \ldots, X'_j, \ldots, X_{k-2}, S_i + X_0 + X_1 + \ldots + X'_j + \ldots + X_{k-2}] \\ [X_0, X_1, \ldots, X_j, \ldots, X_{k-2}, S_{i+1} + X_0 + X_1 + \ldots + X_j + \ldots + X_{k-2}] \ . \end{array}$$

4.3 A Set of Properties for the Codes

In order to make an iterative construction of Gray codes based on nonequivalent full-period necklaces we need that our Gray codes will satisfy certain additional properties. One of the important properties concerns the positions in which adjacent words in the code differ.

Definition 9 *Let W_1, W_2 be words of length n which differ only in the i-th coordinate. We define,*

$$\Delta(W_1, W_2) \stackrel{\triangle}{=} i \ .$$

We are now in a position to state the set of properties required for some Gray codes based on nonequivalent necklaces in order to obtain the iterative construction. Let

$$\begin{aligned}
\mathcal{N}_n^0 &= S_0^0, S_1^0, \ldots, S_{r_0-1}^0 \\
\mathcal{N}_n^1 &= S_0^1, S_1^1, \ldots, S_{r_1-1}^1 \\
\mathcal{N}_k'^0 &= S_0'^0, S_1'^0, \ldots, S_{r_0'-1}'^0 \\
\mathcal{N}_k'^1 &= S_0'^1, S_1'^1, \ldots, S_{r_1'-1}'^1
\end{aligned}$$

be cyclic Gray codes such the the following properties are satisfied:
(p.1) The sets of necklaces which belong to $\mathcal{N}_n^0, \mathcal{N}_n^1, \mathcal{N}_k'^0, \mathcal{N}_k'^1$, respectively, satisfy the conditions of Construction 1.
(p.2)

- The words $[0^{n-1}1], [0^{n-2}11]$ are adjacent in $\mathcal{N}_n^1$.
- $[0^{k-1}1] \in \mathcal{N}_k'^1$

(p.3)

- There exist i_0, i_1 such that $S_{i_1}^1$ and $S_{i_0+1}^0$ differ in exactly the last coordinate, and also $S_{i_0}^0$ and $S_{i_1+1}^1$ differ in exactly the last coordinate. We say that i_0 and i_1 are the *bridging indices* of $\mathcal{N}_n^0$ and $\mathcal{N}_n^1$ respectively and that $S_{i_0}^0, S_{i_0+1}^0, S_{i_1}^1, S_{i_1+1}^1$ are the *bridging words* of their respective codes.
- There exist i_0', i_1' such that $S_{i_1'}'^1$ and $S_{i_0'+1}'^0$ differ in exactly the last coordinate, and also $S_{i_0'}'^0$ and $S_{i_1'+1}'^1$ differ in exactly the last coordinate. We say that i_0' and i_1' are the *bridging indices* of $\mathcal{N}_k'^0$ and $\mathcal{N}_k'^1$ respectively and that $S_{i_0'}'^0, S_{i_0'+1}'^0, S_{i_1'}'^1, S_{i_1'+1}'^1$ are the *bridging words* of their respective codes.

(p.4)

- Let j be the index for which $\{S_j^1, S_{j+1}^1\} = \{[0^{n-1}1], [0^{n-2}11]\}$, and let i_0 and i_1 be the bridge indices of $\mathcal{N}_n^0$ and $\mathcal{N}_n^1$ respectively, then

$$\begin{aligned}
&\{\Delta(S_l^0, S_{l+1}^0) \mid 0 \le l < r_0, l \ne i_0\} = \\
&\{\Delta(S_l^1, S_{l+1}^1) \mid 0 \le l < r_1, l \ne i_1, l \ne j\} = \\
&\qquad \{0, 1, 2, \ldots, n-2\}
\end{aligned}$$

- Let i'_0 and i'_1 be the bridge indices of $\mathcal{N}'^0_k$ and $\mathcal{N}'^1_k$ respectively, then

$$\{\Delta(S'^0_l, S'^0_{l+1}) \mid 0 \leq l < r'_0, l \neq i'_0\} =$$
$$\{\Delta(S'^1_l, S'^1_{l+1}) \mid 0 \leq l < r'_1, l \neq i'_1\} =$$
$$\{0, 1, 2, \ldots, k-2\}$$

4.4 Generation of Short Gray Codes

We are given the four cyclic Gray codes $\mathcal{N}^0_n, \mathcal{N}^1_n, \mathcal{N}'^0_k, \mathcal{N}'^1_k$ based on nonequivalent full-period necklaces of period n and k, respectively. We partition the necklaces generated by Construction 1 into disjoint Gray codes, where each Gray code corresponds to a necklace $B = [b_0, b_1, \ldots, b_{k-1}] \in \mathcal{N}'^b_k$ $(b = b_{k-1})$. Let $p \stackrel{\Delta}{=} \sum_{i=0}^{k-1} b_i$ be the parity of B. For any choice of $W_i \in \mathcal{X}^{b_i}_n$, for each $0 \leq i < k-1$, we use Lemma 5 to construct a cyclic Gray code with the words:

$$[W_0, W_1, \ldots, W_{k-2}, S^p_j + \sum_{i=0}^{k-2} W_i] \quad (0 \leq j < r_p) \ .$$

This code is of length kn and period r_p and will be denoted by

$$\mathcal{C}(B, [W_0, W_1, \ldots, W_{k-2}]) \ .$$

By Theorem 6, all the words in this code are nonequivalent, full-period words.

For the given B we continue and merge all its cyclic Gray codes into one Gray code. There are $2^{(n-1)(k-1)}$ Gray codes which are related to B and we want to order them in such a way that it will be simple to merge them in the given order. The merging will be performed as done in Lemma 6. To apply this lemma we need two Gray codes

$$\mathcal{C}(B, [W_0, W_1, \cdots, W_j, \cdots, W_{k-2}])$$

and

$$\mathcal{C}(B, [W_0, W_1, \cdots, W'_j, \cdots, W_{k-2}])$$

such that W_j and W'_j differ in exactly one coordinate. This coordinate is not the last one since the last coordinate is predetermined by B. Thus, we should order the $2^{(n-1)(k-1)}$ sequences of the form

$$W_0, W_1, \ \cdots, \ W_{k-2}$$

in such a way that any two differ in exactly one coordinate. This is a Gray code ordering and most (and usually all) Gray codes are good for this purpose. But, for the simplicity of the construction we will choose the reflected Gray code, which was introduced in the Introduction, and in the appropriate positions we will plug in the predetermined values of B. We call this code, the *merging Gray code* and we require another property from the merging Gray code (and this property can be removed if we request some more properties from the four Gray codes of length n and k, which can be easily obtained). As said in the Introduction half of the changes in a reflected Gray code is in one specified coordinate (usually the last one). We require that this coordinate will not be congruent modulo n to $\Delta(S^1_{i_1}, S^1_{i_1+1})$, or $\Delta(S^0_{i_0}, S^0_{i_0+1})$, or $n-2$. This can be done easily by an appropriate permutation on the code coordinates. After this is done in the generated merging Gray code no two consecutive changes are in a coordinate congruent to $\Delta(S^1_{i_1}, S^1_{i_1+1})$, or $\Delta(S^0_{i_0}, S^0_{i_0+1})$, or $n-2$, modulo n. Our Gray code of length $n(k-1)$ and period $2^{(n-1)(k-1)}$ will be denoted by

$$\mathcal{X}^B \triangleq X^B_0, X^B_1, \ldots, X^B_{2^{(n-1)(k-1)}-1}.$$

We note that in $\mathcal{X}^B$

$$\{\Delta(X^B_i, X^B_{i+1})\}_{i=0}^{2^{(n-1)(k-1)}-1} = \{0,1,2,\ldots,n(k-1)-1\} \setminus \bigcup_{i=1}^{k-1} \{ni-1\} \ .$$

We are now in a position to merge all the Gray codes which are related to B. During this merging we make sure that each two adjacent words of length nk constructed from either bridging words or the words $[0^{n-1}1]$ and $[0^{n-2}11]$ will remain adjacent. The merging starts with the code $\mathcal{C}(B, X^B_0)$ and the code $\mathcal{C}(B, X^B_1)$. Since X^B_0 and X^B_1 differ in exactly one coordinate, say, the d-th coordinate, $(0 \leq d < (k-1)n,\ d \not\equiv n-1 \pmod{n})$ then it is possible to merge the latter into the former using **(p.4)** and Lemma 6. The resulting code will be called the *main code*. In a typical step of the merging we have a main code obtained by merging the following Gray codes,

$$\begin{array}{c} \mathcal{C}(B, X^B_0) \\ \mathcal{C}(B, X^B_1) \\ \vdots \\ \mathcal{C}(B, X^B_l) \end{array}$$

and we merge to it the Gray code $\mathcal{C}(B, X^B_{l+1})$. Let

$$\begin{aligned} d_1 &\stackrel{\triangle}{=} \Delta(X^B_{l-1}, X^B_l) \\ d_2 &\stackrel{\triangle}{=} \Delta(X^B_l, X^B_{l+1}) \ . \end{aligned}$$

$\mathcal{X}^B$ was chosen in a way that if $d_1 \equiv d_2 \pmod n$ then $d_2 \not\equiv \Delta(S^1_{i_1}, S^1_{i_1+1})$, $d_2 \not\equiv \Delta(S^0_{i_0}, S^0_{i_0+1})$, and $d_2 \not\equiv n-2 \pmod n$. Hence, in this case and also when $d_1 \not\equiv d_2 \pmod n$, by **(p.4)** and Lemma 6 there is a pair of adjacent words in $\mathcal{C}(B, X^B_l)$, originated from pair of adjacent words in in $\mathcal{N}^p_n$, which are not the bridging words or the words $[0^{n-1}1], [0^{n-2}11]$. Therefore, $\mathcal{C}(B, X^B_{l+1})$ can be merged by Lemma 6. This merging process ends when all the Gray codes

$$\begin{gathered} \mathcal{C}(B, X^B_0) \\ \mathcal{C}(B, X^B_1) \\ \vdots \\ \mathcal{C}(B, X^B_{2^{(n-1)(k-1)}-1}) \end{gathered}$$

are merged together. The resulting code is called $\mathcal{C}(B)$.

Lemma 7 *For each $B = [b_0, b_1, \ldots, b_{k-1}] \in \mathcal{N}'^0_k \cup \mathcal{N}'^1_k$, $p \stackrel{\triangle}{=} \sum_{i=0}^{k-1} b_i$, the code $\mathcal{C}(B) = Y_0, Y_1, \ldots, Y_{P-1}$ is a cyclic Gray code of length kn and period $P = 2^{(k-1)(n-1)} r_p$ which satisfies:*

$$\{\Delta(Y_i, Y_{i+1}) \mid 0 \le i < P\} = \{0, 1 \ldots, kn-1\} \backslash \bigcup_{i=1}^{k} \{ni - 1\} \ .$$

Proof *It is obvious that $\mathcal{C}(B)$ is of length kn, and since we merged $2^{(k-1)(n-1)}$ codes of period r_p, it follows that the period of the resulting is $P = 2^{(k-1)(n-1)} r_p$. By lemma 6, the resulting code is a cyclic Gray code. Finally, since*

$$\{\Delta(X^B_i, X^B_{i+1})\}_{i=0}^{2^{(n-1)(k-1)}} = \{0, 1, 2, \ldots, n(k-1)-1\} \backslash \bigcup_{i=1}^{k-1} \{ni - 1\},$$

*it follows by Lemma 6, and **(p.4)** that*

$$\{\Delta(Y_i, Y_{i+1}) \mid 0 \le i < P\} = \{0, 1 \ldots, kn-1\} \backslash \bigcup_{i=1}^{k} \{ni - 1\} \ .$$

□

4.5 Concatenation the Short Gray Codes

Now, we have a set of $r'_0 + r'_1$ cyclic Gray codes, each one corresponds to another member of $\mathcal{N}'^0_k \bigcup \mathcal{N}'^1_k$. Recall that the bridging indices of $\mathcal{N}^0_n, \mathcal{N}^1_n, \mathcal{N}'^0_k, \mathcal{N}'^1_k$ are i_0, i_1, i'_0, i'_1 respectively. Let $V_0, V_1, \ldots, V_{k-2} \in \{0,1\}^{n-1}$ be $k-1$ words of length $n-1$ chosen arbitrarily. For each $B = [b_0, b_1, \ldots, b_{k-1}] \in \mathcal{N}'^0_k \cup \mathcal{N}'^1_k$ and $p \stackrel{\triangle}{=} \sum_{i=0}^{k-1} b_i$, we cyclically shift the rows of the cyclic Gray code $C(B, [Z_0, Z_1, \ldots, Z_{k-1}])$, where $Z_i = [V_i, b_i]$ for each $0 \leq i < k-1$, in such a way that the first word will be

$$[Z_0, Z_1, \ldots, Z_{k-2}, S^p_{i_p+1} + \sum_{i=0}^{k-2} Z_i]$$

and the last word will, therefore, be

$$[Z_0, Z_1, \ldots, Z_{k-2}, S^p_{i_p} + \sum_{i=0}^{k-2} Z_i] \ .$$

We look at the following two concatenations of our cyclic Gray codes.

$$\begin{aligned} \mathcal{N}^0_{nk} &= \mathcal{C}(S'^0_{i'_0+1}), \mathcal{C}(S'^0_{i'_0+2}), \ldots, \mathcal{C}(S'^0_{r'_0-1}), \mathcal{C}(S'^0_0), \ldots, \mathcal{C}(S'^0_{i'_0}) \\ \mathcal{N}^1_{nk} &= \mathcal{C}(S'^1_{i'_1+1}), \mathcal{C}(S'^1_{i'_1+2}), \ldots, \mathcal{C}(S'^1_{r'_1-1}), \mathcal{C}(S'^1_0), \ldots, \mathcal{C}(S'^1_{i'_1}) \ . \end{aligned}$$

In the rest of this section and in the next section we will prove that this pair of codes satisfies properties **(p.1)** through **(p.4)** and thus can be used for further iterations of the construction.

Lemma 8 *$\mathcal{N}^0_{nk}$ and $\mathcal{N}^1_{nk}$ are cyclic Gray codes of length kn and period $2^{(n-1)(k-1)-1}(r_0 + r_1)r'_0$ and $2^{(n-1)(k-1)-1}(r_0 + r_1)r'_1$, respectively. Furthermore, $\mathcal{N}^0_{nk}$ and $\mathcal{N}^1_{nk}$ contain nonequivalent full-period words.*

Proof *All the necklaces of $\mathcal{N}^0_{nk}$ and $\mathcal{N}^1_{nk}$ were produced as in Construction 1 and hence by Theorem 6 all the necklaces are nonequivalent full-period and of length nk.*

For a given $B = [b_0, b_1, \ldots, b_{k-1}]$ and $W_0, W_1, \cdots, W_j, \cdots, W_{k-2}$, clearly $\mathcal{C}(B, [W_0, W_1, \cdots, W_j, \cdots, W_{k-2}])$ has period r_p, where p is the parity of B. $\mathcal{C}(B)$ was constructed by merging $2^{(n-1)(k-1)}$ Gray codes of the form $\mathcal{C}(B, [W_0, W_1, \cdots, W_j, \cdots, W_{k-2}])$ and hence its period is $2^{(n-1)(k-1)}r_p$. Since S'^1_j and S'^1_{j+1} have different parity, it follows that $\mathcal{C}(S'^1_j)$ and $\mathcal{C}(S'^1_{j+1})$ together have $2^{(n-1)(k-1)}(r_0+r_1)$ necklaces, and $\mathcal{N}^b_{nk}$ has $2^{(n-1)(k-1)-1}(r_0+r_1)r'_b$ words.

To complete the proof we have to show that $\mathcal{N}^b_{nk}$ is a cyclic Gray code. As said before the last word in $\mathcal{C}(S'^b_j)$ is

$$[Z_0, Z_1, \ldots, Z_{k-2}, S^p_{i_p} + \sum_{i=0}^{k-2} Z_i],$$

where p is the parity of S'^b_j, and the first word in $\mathcal{N}(S'^b_{j+1})$ is

$$[Z'_0, Z'_1, \ldots, Z'_{k-2}, S^{\overline{p}}_{i_{\overline{p}+1}} + \sum_{i=0}^{k-2} Z'_i].$$

*Clearly, these two words differ in exactly one coordinate. If S'^b_j and S'^b_{j+1} differ in the d-th coordinate, $0 \leq d < k-1$, then for each $0 \leq j < k-1$, $j \neq d$, $Z_j = Z'_j$, and Z_d and Z'_d differ in the last coordinate. Since by **(p.3)** $S^p_{i_p}$ and $S^{\overline{p}}_{i_{\overline{p}+1}}$ differ in exactly the last coordinate, it follows that the last word of $\mathcal{C}(S'^b_j)$ and the first word of $\mathcal{C}(S'^b_{j+1})$ differ in exactly the $(n(d+1)-1)$-th coordinate. Thus, $\mathcal{N}^b_{nk}$ is a cyclic Gray code of length nk and period $2^{(n-1)(k-1)-1}(r_0+r_1)r'_b$.* □

4.6 Properties of the Generated Gray Codes

In this Section we will prove that the generated Gray codes $\mathcal{N}^0_{nk}$ and $\mathcal{N}^1_{nk}$ satisfy **(p.1)** through **(p.4)** and therefore can be used for further iterations of the construction. The first lemma is an immediate consequence of Lemma 8.

Lemma 9 *$\mathcal{N}^0_{nk}$ and $\mathcal{N}^1_{nk}$ satisfy **(p.1)**.*

Lemma 10 *The words $[0^{nk-1}1]$ and $[0^{nk-2}11]$ are adjacent in $\mathcal{N}^1_{nk}$.*

Proof *By **(p.2)** we have that $[0^{k-1}1] \in \mathcal{N}'^1_k$ and $[0^{n-1}1]$ and $[0^{n-2}11]$ are adjacent in $\mathcal{N}^1_n$. Therefore, the words $[0^{nk-1}1]$ and $[0^{nk-2}11]$ are adjacent in $\mathcal{C}([0^{k-1}1],[0^{(k-1)n}])$. Since during the merging process we didn't separate these words, it follows that they are also adjacent in $\mathcal{C}([0^{k-1}1])$. To complete the proof we have to show that these two words weren't separated during the concatenation. This is an immediate consequence from the fact that $[0^{n-1}1]$ is not a bridging word since $[0^n]$ is not a full-period word.* □

Lemma 11 *$\mathcal{N}^0_{nk}, \mathcal{N}^1_{nk}$ satisfy **(p.3)**.*

Proof *Observe that the last word of* $\mathcal{N}^1_{nk}$*, which is also the last word of* $\mathcal{C}(S^{\prime 1}_{i^{\prime}_1})$ *is,*

$$[Z_0, Z_1, \ldots, Z_{k-2}, S^p_{i_p} + \sum_{i=0}^{k-2} Z_i],$$

where p *is the parity of* $S^{\prime 1}_{i^{\prime}_1}$*. The first word of* $\mathcal{N}^0_{nk}$*, which is also the first word of* $\mathcal{C}(S^{\prime 0}_{i^{\prime}_0+1})$ *is,*

$$[Z_0, Z_1, \ldots, Z_{k-2}, S^{\overline{p}}_{i_{\overline{p}}+1} + \sum_{i=0}^{k-2} Z_i],$$

where clearly $\overline{p}$ *is the parity of* $S^{\prime 0}_{i^{\prime}_0+1}$*. Since* $S^p_{i_p}$ *and* $S^{\overline{p}}_{i_{\overline{p}}+1}$ *differ in exactly the last coordinate, it follows that these two words differ exactly in the last coordinate. Similarly, the first word of* $\mathcal{N}^1_{nk}$ *and the last word of* $\mathcal{N}^0_{nk}$ *differ in exactly the last coordinate. Therefore, these four words can serve as bridging words of* $\mathcal{N}^0_{nk}$ *and* $\mathcal{N}^1_{nk}$*.*

□

Lemma 12 *Let* $\mathcal{N}^b_{nk} = S^{*b}_0, S^{*b}_1, \ldots, S^{*b}_{P_b-1}$*,* j^* *be the index such that* $\{S^{*1}_j, S^{*1}_{j+1}\} = \{[0^{nk-1}1], [0^{nk-2}11]\}$*, and* i^*_0 *and* i^*_1 *be the bridging indices of* $\mathcal{N}^0_{nk}$ *and* $\mathcal{N}^1_{nk}$*, respectively. Then*

$$\{\Delta(S^{*0}_l, S^{*0}_{l+1}) \mid 0 \le l < r^*_0, l \ne i^*_0\} =$$
$$\{\Delta(S^{*1}_l, S^{*1}_{l+1}) \mid 0 \le l < r^*_1, l \ne i^*_1, l \ne j^*\} =$$
$$\{0, 1, 2, \ldots, nk-2\}$$

Proof *If* $B \in \mathcal{N}^{\prime b}_k$*, then by lemma 7,*

$$\{\Delta(Y_i, Y_{i+1}) \mid 0 \le i < P\} = \{0, 1 \ldots, kn-1\} \setminus \bigcup_{i=1}^{k} \{in-1\},$$

where $\mathcal{C}(B) = Y_0, Y_1, \ldots, Y_{P-1}$*. By the proof of Lemma 8 we obtain the changes in all positions which are congruent to* $n-1$ *modulo* n *in the concatenation of the short Gray codes, and thus by taking into consideration* ***(p.3)*** *for* $\mathcal{N}^{\prime 0}_k$ *and* $\mathcal{N}^{\prime 1}_k$*, and the fact that one change in a coordinate implies at least two changes in the same coordinate we have.*

$$\{\Delta(S^{*b}_l, S^{*b}_{l+1}) \mid 0 \le l < r^*_b, l \ne i^*_b, l \ne j^*\} = \{0, 1, 2, \ldots, nk-2\}\ .$$

□

In Lemmas 9 through 12, we have proved that the pair of codes $\mathcal{N}_{nk}^0, \mathcal{N}_{nk}^1$ satisfy properties **(p.1)** through **(p.4)**. Thus, we can use this pair of codes for another iteration of the construction.

4.7 Optimality of the Code

Lemma 13 *Concatenating the two codes given by the construction,*

$$\mathcal{C}_{nk} \stackrel{\Delta}{=} \mathcal{N}_{nk}^0, \mathcal{N}_{nk}^1$$

produces a Gray code of nonequivalent, full-period words of length kn and period $2^{(k-1)(n-1)-1}(r_0+r_1)(r_0'+r_1')$ which satisfies the conditions of Theorem 1.

Proof *By Lemmas 8 and 11 $\mathcal{C}_{nk}$ is a cyclic Gray code of nonequivalent full-period words of the required parameters. In addition, by Lemma 10 the words $[0^{nk-1}1], [0^{nk-2}11]$ are adjacent in the code. The code can be cyclically shifted so they become the first and last words. Since $[0^{kn-2}11]$ and $\boldsymbol{E}[0^{kn-1}1]$ differ in exactly one coordinate, the conditions of Theorem 1 are satisfied.* □

The code of Lemma 13 can be used in Theorem 1 to produce a single-track Gray code of length kn and period

$$2^{(k-1)(n-1)-1}(r_0 + r_1)(r_0' + r_1')nk$$

In order to use the construction, we need seed-codes which satisfy the conditions of Theorem 1. Such seed-codes exist for $n \geq 9$. A simple computer search has found such seed-codes for length 9 through 13. The seed-codes for $n = 9$, 10, and 11, are presented in appendix B.

If we assume that $r_0 + r_1 = (2^n - c_n)/n$ and $r_0' + r_1' = (2^k - c_k)/k$ then by Lemma 13 one iteration of the construction gives a single-track Gray code of length kn and period

$$P^* = 2^{nk}(1 - c_k 2^{-k} - c_n 2^{-n} + c_k c_n 2^{-(n+k)})$$

If we further assume that

$$\lim_{n\to\infty} \frac{c_n}{2^n} = 0 \quad , \quad \lim_{k\to\infty} \frac{c_k}{2^k} = 0 \tag{8}$$

then the period P^* asymptotically reaches the upper bound of Lemma 1,

$$\lim_{n,k\to\infty} P^* = 2^{nk}.$$

When $P^* = 2^{nk} - c_{nk}$ we have that $c_{nk} = 2^{nk}(c_k 2^{-k} + c_n 2^{-n} - c_k c_n 2^{-(n+k)})$. Under assumption (8), we get, again, that

$$\lim_{n,k\to\infty} \frac{c_{nk}}{2^{nk}} = 0$$

which means that the family of codes generated by any number of iterations of the construction is still asymptotically optimal. Of course, as said before one needs an infinite family of optimal seed codes to make the resulting sequence of codes also optimal. If we start with "good" codes which are not optimal we obtain codes which are usually better than the best known codes.

4.8 Generalization

As mentioned before, seed codes for our construction exist only for length $n \geq 9$. This fact limits the list of lengths for which we can obtain good single-track Gray codes by our construction. We can overcome this limitation by weakening the requirements induced by the properties **(p.1)** through **(p.4)**. Let,

$$\begin{aligned} \mathcal{N}_n^0 &= S_0^0, S_1^0, \ldots, S_{r_0-1}^0 \\ \mathcal{N}_n^1 &= S_0^1, S_1^1, \ldots, S_{r_1-1}^1 \\ \mathcal{N}_k' &= S_0', S_1', \ldots, S_{r'-1}' \end{aligned}$$

be cyclic Gray codes, such that the following properties hold:
(q.1) The sets of sequences which belong to $\mathcal{N}_n^0, \mathcal{N}_n^1$ satisfy the conditions of Construction 1, and $\mathcal{N}_k'$ contains nonequivalent, full-period words.
(q.2)

- The words $[0^{n-1}1], [0^{n-2}11]$ are adjacent in $\mathcal{N}_n^1$.
- $[0^{k-1}1] \in \mathcal{N}_k'$

(q.3) There exist i_0, i_1 such that $S_{i_1}^1$ and $S_{i_0+1}^0$ differ in exactly the last coordinate, and also $S_{i_0}^0$ and $S_{i_1+1}^1$ differ in exactly the last coordinate. We say that i_0 and i_1 are the *bridging indices* of $\mathcal{N}_n^0$ and $\mathcal{N}_n^1$ respectively and that $S_{i_0}^0, S_{i_0+1}^0, S_{i_1}^1, S_{i_1+1}^1$ are the *bridging words* of their respective codes.

(q.4) Let j be the index for which $\{S_j^1, S_{j+1}^1\} = \{[0^{n-1}1], [0^{n-2}11]\}$, and let i_0 and i_1 be the bridging indices of $\mathcal{N}_n^0$ and $\mathcal{N}_n^1$ respectively, then

$$\{\Delta(S_l^0, S_{l+1}^0) \mid 0 \le l < r_0, l \ne i_0\} =$$
$$\{\Delta(S_l^1, S_{l+1}^1) \mid 0 \le l < r_1, l \ne i_1, l \ne j\} =$$
$$\{0, 1, 2, \ldots, n-2\}$$

Unlike the first construction, this one is not symmetric relative to the parameters n and k of the seed-codes. Therefore, we say that $\mathcal{N}_n^0, \mathcal{N}_n^1$ are the *multiplied codes* and $\mathcal{N}_k'$ is the *multiplier code*. The construction process itself is very similar to the first construction. We start by constructing for each $B \in \mathcal{N}_k'$, the code $\mathcal{C}(B)$. As before, we concatenate the codes to get the main code:

$$\mathcal{N}_{nk}' \stackrel{\triangle}{=} \mathcal{C}(S_0'), \mathcal{C}(S_1'), \mathcal{C}(S_2'), \ldots, \mathcal{C}(S_{r'-1}')$$

This code contains nonequivalent full-period words of length kn and satisfies all the properties of a multiplier code. Using Theorem 1 we can obtain a length nk, period

$$2^{(n-1)(k-1)-1}(r_0 + r_1)r'nk$$

single-track Gray code ,when $k > 3$. If $k = 2$ then the only word of length 2 used is [01] and we use only $\mathcal{N}_1$. In this case the construction coincides with the first construction of [8] and we obtain a length $2n$, period $2^n r_1 n$ single-track Gray code. Unlike the first construction, this construction has multiplier seed-codes for length $k \ge 3$ and they are given in Appendix C.

Appendix A

We discuss the generalization of Gray codes over nonbinary alphabets. Let $\mathcal{Z}_a$, $a \ge 2$ be the group of residues $\{0, 1, \cdots, a-1\}$ modulo a, and $\mathcal{Z}_a^n$ the set of a^n n-tuples over $\mathcal{Z}_a$.

Definition 10 *For* $X = [x_0, x_1, \ldots, x_{n-1}]$, $Y = [y_0, y_1, \ldots, y_{n-1}] \in \mathcal{Z}_a^n$ *We define,*

$$d_m(X, Y) = \sum_{i=0}^{n-1} (y_i - x_i)$$

where the subtraction is done in $\mathcal{Z}_a$ *and the addition is an integer addition.*

Definition 11 *A* length n period P Gray code over $\mathcal{Z}_a$ *is an ordered list of* P *distinct length* n *words over* $\mathcal{Z}_a$,

$$W_0, W_1, \ldots, W_{P-1}$$

such that for each $0 \leq i < P-1$, W_i *and* W_{i+1} *differ in exactly one coordinate and* $d_m(W_i, W_{i+1}) = d$, *for a given* $d \in \mathcal{Z}_a$. *If* W_{P-1} *and* W_0 *satisfy this condition, we say that the code is* cyclic.

Single-track Gray codes are cyclic Gray codes which have the single-track property. A single-track Gray code over $\mathcal{Z}_a$, is equivalent to a single-track Gray code over $\mathcal{Z}_{a/\gcd(a,d)}$. For this reason we only consider the case where $\gcd(a,d) = 1$. The following lemma is a straightforward generalization of its binary equivalent.

Lemma 14 *If* C *is a length* n, *period* P *single-track Gray code over* $\mathcal{Z}_a$, *then* $na \mid P$ *and* $na \leq P \leq a^n$.

All the results regarding single-track Gray codes with evenly spaced heads can be easily generalized in a very natural way. The nonexistence theorem can be proved for certain cases, with an interesting generalization of the proof.

Theorem 7 *Except for* $p = 2$ *and* $m = 1$, *there is no ordering of all the* p^n *words of length* $n = p^m$ *over* $GF(p)$, *where* $m \geq 1$ *and* p *is a prime, in a list which satisfies all the following requirements:*

1. *There exists a nonzero constant* $d \in GF(p)$, *such that for any two consecutive words in the list* W_i *and* W_{i+1} *we have* $d_m(W_{i+1}, W_i) = d$, $0 \leq i \leq p^n - 1$.
2. *The list has the single-track property.*
3. *Each word appears exactly once.*

Proof *Let us assume the contrary, i.e., that such a code with a track* s *exists. Let* $s(x)$ *be the characteristic polynomial of* s, *and* c_1 *be the largest integer such that there exists a polynomial* $p_1(x)$ *which satisfies,*

$$s(x) \equiv (x-1)^{c_1} p_1(x) \pmod{x^{p^n} - 1}.$$

Let $k_0, k_1, \ldots, k_{n-1}$ *be the locations of the heads in the list,* $h(x) \stackrel{\triangle}{=} \sum_{i=0}^{n-1} x^{k_i}$ *the* head locator *polynomial of the list, and* h *the characteristic length* p^n

word of $h(x)$. Let c_2 be the largest integer for which there exists a polynomial $p_2(x)$ which satisfies,

$$h(x) \equiv (x-1)^{c_2} p_2(x) \pmod{x^{p^n}-1} .$$

$x^{p^n}-1 = (x-1)^{p^n}$ over $GF(p)$ and hence $0 \le c_1,\ c_2 \le p^n - 1$.

Since the distance between any two adjacent words is d, it follows that

$$(x-1)h(x)s(x) \equiv d(1 + x + x^2 + \ldots + x^{p^n-1}) \pmod{x^{p^n}-1} \qquad (9)$$

and therefore,

$$c_1 + c_2 = p^n - 2 . \qquad (10)$$

Equation (9) also implies that $c_1 + 2$ is the linear complexity of h, and c_2+2 is the linear complexity of s. Since each word appears in the list exactly once, s must be of full cyclic order, and hence

$$c_2 \ge p^{n-1} - 1 . \qquad (11)$$

In order to restrict the linear complexity of h, we notice that

$$\sum_{i=0}^{p-1} x^{i \cdot p^{n-1}} = (x-1)^{(p-1)p^{n-1}} .$$

Now, let us assume that $c_1 < (p-1)p^{n-1} - 1$, i.e.,

$$h(x) \sum_{i=0}^{p-1} x^{i \cdot p^{n-1}} \equiv 0 \pmod{x^{p^n}-1} .$$

Since h contains only zeros and ones, and the calculations are performed over $GF(p)$, it follows that h has the following form

$$h = [\underbrace{AA\ldots A}_{p}] , \quad A \in GF^{p^{n-1}}(p)$$

This means that

$$\{k_i\}_{i=0}^{n-1} = \{p^{n-1} + k_i\}_{i=0}^{n-1}$$

and then, the i-th word and the $(i + p^{n-1})$-th word contain exactly the same components of the generating track s. Since the allzero word appears somewhere in the list, it will appear at least twice, which is a contradiction. Therefore, $c_1 \ge (p-1)p^{n-1} - 1$.

The linear complexity of h cannot be $(p-1)p^{n-1}+1$, otherwise,

$$h(x)\sum_{i=0}^{p-1} x^{i\cdot p^{n-1}} \equiv c\cdot\sum_{i=0}^{p^n-1} x^i \pmod{x^{p^n}-1}$$

for some $c \in GF(p)$, $c \neq 0$. The polynomial on the right has p^n nonzero components. $h(x)$ has exactly p^m nonzero components and hence, the left side has at most p^{m+1} nonzero components. Thus,

$$p^{m+1} \geq p^{p^m-1},$$

but this equation can hold only if $p=2$ and $m=1$. Therefore,

$$c_1 \geq (p-1)p^{n-1}\ . \tag{12}$$

Summing (11) and (12) we get that

$$c_1 + c_2 \geq p^n - 1$$

and this contradicts (10). □

Corollary 3 *There are no single-track Gray codes over $GF(p)$, p prime, of length $n \geq 2$ and period p^n, except for the trivial binary code of length 2 and period 4.*

Appendix B

In this appendix we present the seed-codes for $9 \leq n \leq 11$.

[010001010]	[010001000]	[011001000]	[011011000]	[011010000]
[011110000]	[011111000]	[111111000]	[111111100]	[111101100]
[111101110]	[111111110]	[111111010]	[101111010]	[001111010]
[011111010]	[011011010]	[011011110]	[011010110]	[010010110]
[010110110]	[010111110]	[010011110]	[010001110]	[110001110]
[110001100]	[110001000]	[110001010]	[110001011]	[110001001]
[110000001]	[110100001]	[010100001]	[010000001]	[000000001]
[000000011]	[000001011]	[000001001]	[100001001]	[101001001]
[101001011]	[100001011]	[100101011]	[100101001]	[100111001]
[110111001]	[110101001]	[010101001]	[010101101]	[011101101]
[111101101]	[110101101]	[110001101]	[010001101]	[010001001]
[010001011]				

Seed-codes for n=9

[0110101110] [0111101110] [0111101010] [0101101010] [0101111010]
[0101111110] [0101110110] [0001110110] [0011110110] [0011111110]
[0011111100] [0011110100] [0111110100] [0111010100] [0101010100]
[0101010110] [0101000110] [0101100110] [1101100110] [1101100010]
[1101110010] [1101111010] [1100111010] [0100111010] [0100110010]
[0101110010] [0111110010] [0111100010] [0111100000] [0111100100]
[0111100110] [0110100110] [0110100010] [0110110010] [0010110010]
[0010100010] [0010000010] [0110000010] [1110000010] [1100000010]
[1100001010] [1100001000] [1110001000] [1110001100] [1100001100]
[1000001100] [1000011100] [1000011000] [1010011000] [1010010000]
[1010010010] [1010000010] [1010000110] [1010100110] [1010101110]
[0010101110] [0010101111] [0010100111] [0110100111] [1110100111]
[1110110111] [1110110011] [1110111011] [1110101011] [1111101011]
[1111101111] [1111001111] [1111001101] [1110001101] [0110001101]
[0110001001] [0010001001] [0000001001] [0100001001] [0100000001]
[0000000001] [0000000011] [1000000011] [1010000011] [1010000001]
[1010010001] [1011010001] [1011000001] [1111000001] [1111000101]
[1101000101] [1101001101] [1101011101] [1100011101] [1100001101]
[1100001001] [1100001011] [0100001011] [0100101011] [0110101011]
[0110101111]

Seed-codes for $n = 10$.

[10101110000]	[10101010000]	[10001010000]	[11001010000]	[11001110000]
[11101110000]	[11101010000]	[11111010000]	[11011010000]	[11011110000]
[11011100000]	[11010100000]	[11110100000]	[10110100000]	[10100100000]
[10101100000]	[11101100000]	[11101000000]	[10101000000]	[10111000000]
[10111100000]	[00111100000]	[00111110000]	[00111111000]	[01111111000]
[11111111000]	[11111111100]	[11111111110]	[11011111110]	[11011011110]
[11111011110]	[10111011110]	[10111111110]	[10101111110]	[10101101110]
[10101101010]	[10111101010]	[10111101000]	[10111111000]	[10111111100]
[10011111100]	[10010111100]	[10010111110]	[10010101110]	[10010101010]
[10010111010]	[10010011010]	[10010011110]	[10010010110]	[10010010100]
[10010011100]	[10110011100]	[11110011100]	[11010011100]	[11010111100]
[11010101100]	[11011101100]	[11011001100]	[11111001100]	[11101001100]
[11101011100]	[11101010100]	[11101110100]	[11101111100]	[11101101100]
[11111101100]	[11111101000]	[11111001000]	[11101001000]	[11101101000]
[11001101000]	[11001001000]	[11011001000]	[11011101000]	[11010101000]
[11110101000]	[11110111000]	[10110111000]	[10110011000]	[10100011000]
[10100111000]	[10100101000]	[10110101000]	[10010101000]	[10010111000]
[10010011000]	[10011011000]	[10011111000]	[10011110000]	[10011010000]
[10111010000]	[10111110000]	[10111110001]	[10011110001]	[10001110001]
[10001111001]	[10001111101]	[11001111101]	[11001101101]	[11001101111]
[01001101111]	[01001101011]	[01001111011]	[01001111111]	[01001111101]
[01001101101]	[01101101101]	[01111101101]	[01011101101]	[11011101101]
[11011101111]	[11011101011]	[01011101011]	[01111101011]	[00111101011]
[00111101111]	[00101101111]	[00101101101]	[00101001101]	[00101001111]
[00101011111]	[00101010111]	[00101010011]	[00101110011]	[00101111011]
[00101101011]	[00101001011]	[00101011011]	[00101011001]	[00111011001]
[00110011001]	[00110111001]	[01110111001]	[01110110001]	[01110010001]
[01010010001]	[01010110001]	[01011110001]	[01011010001]	[00011010001]
[00111010001]	[00110010001]	[00010010001]	[00010110001]	[00011110001]
[00001110001]	[00001111001]	[00001111011]	[00001110011]	[00001100011]
[00001101011]	[00001101001]	[00001001001]	[00001011001]	[00001010001]
[00001010011]	[00001011011]	[00001001011]	[00001000011]	[00001000111]
[00000000111]	[00000000101]	[00000001101]	[00000001001]	[00000001011]
[00000000011]	[00000000001]	[00010000001]	[10010000001]	[10110000001]
[00110000001]	[00111000001]	[00101000001]	[00001000001]	[00011000001]
[10011000001]	[10001000001]	[11001000001]	[11001010001]	[11001110001]
[11011110001]	[11010110001]	[10010110001]	[10110110001]	[10100110001]
[10101110001]				

Seed-codes for n=11

Appendix C

In this appendix we present the seed-codes of the second constructions for $3 \leq n \leq 8$.

[001] [011]

Seed-codes for $n = 3$.

[0001] [0011]

Seed-codes for $n = 4$.

[00001] [00011] [00111] [01111] [01101] [00101]

Seed-codes for $n = 5$.

[000001] [000011] [000111] [001111] [011111] [011101] [001101] [000101]

Seed-codes for $n = 6$.

[0000001] [0000101] [0001101] [0001001] [1001001] [1011001]
[1111001] [1111101] [0111101] [0110101] [0110111] [0100111]
[0100101] [1100101] [1000101] [1000111] [0000111] [0000011]

Seed-codes for $n = 7$.

[00000001] [00000011] [00000111] [00010111] [00010011] [00011011]
[00011001] [00011101] [00010101] [00101011] [00100101] [00100111]
[00101111] [00101101] [00111101] [00111111] [00111011] [01101111]
[00110111] [00110101] [01010111] [01011111] [00011111] [00001111]
[00001101] [00001001] [00001011] [00000101]

Seed-codes for $n = 8$.

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