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MANUEL ALFARO GARCIA, MARK CONGER AND KENNETH HODGES

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THE STRUCTURE OF SINGULARITIES IN Φ-MINIMIZING NETWORKS IN $\mathbf{R}^2$

MANUEL ALFARO, MARK CONGER, KENNETH HODGES, ADAM LEVY, RAJIV KOCHAR, LISA KUKLINSKI, ZIA MAHMOOD, AND KAREN VON HAAM

It is well known that length-minimizing networks in $\mathbf{R}^2$ consist of segments meeting only in threes. This paper considers uniformly convex norms Φ more general than length. The first theorem says that for any such smooth Φ, minimizing networks still meet only in threes. The second theorem shows that for some *piecewise* smooth Φ, segments can meet in fours (although never in fives or more).

1. Introduction. Length-minimizing networks in $\mathbf{R}^2$ consist of straight line segments meeting only in threes. Soap films meet in threes for exactly the same reasons. (See Figures 1.0.1 and 1.0.2 and [**CR**, pp. 354–356].)

This paper studies the structure of minimizing networks for elliptic integrands Φ, which depend on direction and thus are more general than length. (The surface energy of most crystals, unlike that of soap films, depends on orientation as well as area.)

THEOREM (3.3). *Let Φ be a smooth, elliptic integrand. Then, segments in Φ-minimizing networks meet only in threes.*

THEOREM (3.4). *There is a piecewise smooth, elliptic integrand Φ_0 for which the X is Φ_0-minimizing* (*see Figure* 1.0.3).

Theorem 3.3 is proved by showing that conjunctions of more than three segments are unstable. The proof of Theorem 3.4 uses symmetry arguments to reduce the analysis to a one-dimensional calculus problem. The result holds for an infinite family of elliptic integrands with unit balls that are perturbations of the square. (The unit ball is the set of all points reachable from the origin with an energy no greater than one.) (See Figure 1.0.4.) The square itself is the unit ball of the rotated "Manhattan Metric," Φ_M, for which our result would be trivial; however, Φ_M is not elliptic because the square is not *uniformly* convex (see §2).

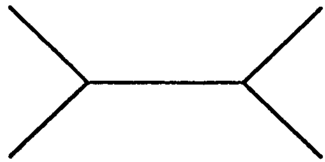


FIGURE 1.0.1
Segments in length-minimizing networks only meet in threes

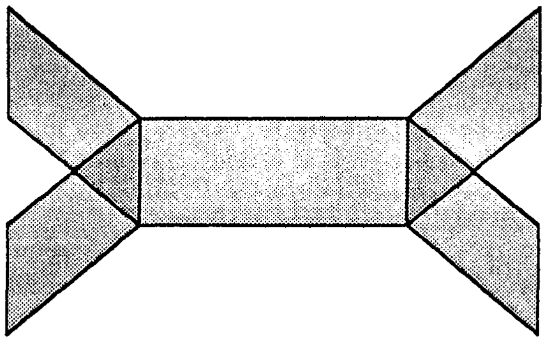


FIGURE 1.0.2
Soap films also meet in threes

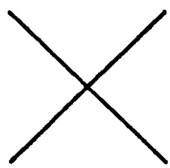
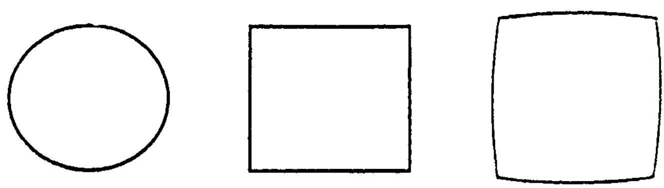


FIGURE 1.0.3
The X is Φ_0-minimizing for a piecewise-smooth elliptic integrand Φ_0



(a) (b) (c)

FIGURE 1.0.4
(a) The unit ball for the length integrand. All directions have equal cost. (b) The unit ball for the Manhattan Metric. Diagonal directions are favored. (c) The unit-ball for our integrand Φ_0, a perturbation of the square. Diagonal directions are favored. Theorem 3.4 shows the X is Φ_0-minimizing.

Theorem 3.3 is the main result of a senior thesis by Adam Levy [L] at Williams College under the supervision of Professor Frank Morgan.

Theorem 3.4 is the work of the Geometry Group of the Williams College SMALL Undergraduate Research Project, Summer 1988. For

a period of ten weeks, each of fifteen Williams students worked in two of the five groups that comprised the Project. The Geometry Group consisted of the following members: Manuel Alfaro, Mark Conger, Kenneth Hodges, Rajiv Kochar, Lisa Kuklinski, Zia Mahmood, and Karen von Haam. Adam Levy was the student leader and Professor Frank Morgan was the supervisor of this group.

General background can be found in [**T**], [**M1**], [**M2**], and [**F**]. Earlier related results were obtained by J. Abrahamson [**Ab**], R. Bassini [**B**], and M. McCutchan [**Mc**].

Support for the project was provided by grants from the National Science Foundation (including add-on student stipends from the Research Experiences for Undergraduates Program), the Ford Foundation, G.T.E., Shell, the PEW Charitable Trusts, and the Bronfman Science Center at Williams College. The NSF awards were given to Colin Adams (DMS-8711495), Colin Adams and Frank Morgan (DMS-8802266), Deborah Bergstrand (DMS-8808695), and Frank Morgan (DMS-8504029).

2. Definitions. For a given, finite set of boundary points in $\mathbf{R}^2$, a *network* S is a finite collection of smooth curves, intersecting only at endpoints and connected as a graph, whose endpoints include the boundary points. The other endpoints are called *nodes*. For example, the network of Figure 1.0.1 has four boundary points and two nodes.

An *integrand* is a positive, continuous function $\Phi(t)$ which assigns to each unit direction vector t a cost associated with that direction. Since we work with only unoriented networks, we require Φ to be even. Φ assigns to any network S an *energy*

$$E(S) = \int_S \Phi(t)\, ds,$$

where t is a unit vector tangent to S. We will often write Φ as a function of θ, the angle of t in polar coordinates. A network is called *Φ-minimizing* if no other network for the same boundary has less energy.

An integrand $\Phi(t)$ is *elliptic* if the unit Φ-ball

$$\{rt\colon r\Phi(t) \leq 1\}$$

is uniformly convex. An elliptic integrand has the property that straight line segments are uniquely Φ-minimizing. See [**L**, Lemma 2.3] or [**F**, 5.1.2].

3. The existence and structure of energy-minimizing networks. Here we present our main results, which first appeared in [**L**] and [**Al**].

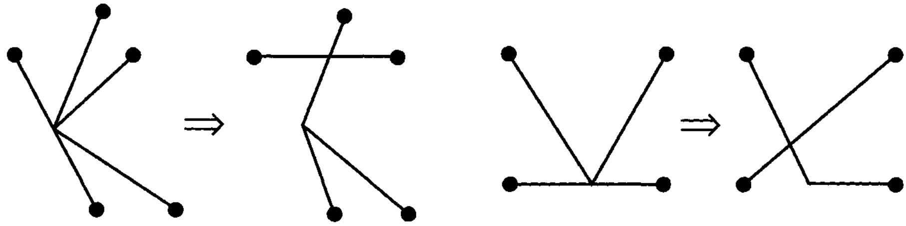


FIGURE 3.2.1
Segments can never meet in fives or more—whenever three intersecting segments lie in a half-plane, a network of less energy can be produced as shown. Furthermore, if segments meet in fours, opposite segments must form straight lines

3.1. PROPOSITION [L, *Prop.* 2.81]. *Let* Φ *be an elliptic integrand in* $\mathbf{R}^2$. *For a given, finite set of boundary points, there exists a* Φ-*minimizing network.*

REMARK. This result holds in $\mathbf{R}^n$ as well.

Proof. Since Φ is elliptic, one need only consider acyclic networks of straight line segments and obtain bounds on the number of nodes and segments. The result then follows by a standard compactness argument.

The following theorem classifies all singularities in Φ-minimizing networks. Theorem 3.4 will show that the juncture of four segments occurs.

3.2. PROPOSITION [L, *Prop.* 3.1]. *Let* Φ *be an elliptic integrand in* $\mathbf{R}^2$. *A* Φ-*minimizing network consists of straight line segments which never meet at nodes in fives or more. If they meet in fours, opposite segments form a straight line.*

Proof. Since Φ is elliptic, of course, a minimizing network must consist of straight line segments. A network which includes three segments meeting at a point and contained within a half-plane cannot be energy-minimizing, since the outer two of these line segments can be replaced by a straight line segment: see Figure 3.2.1. Therefore, segments cannot meet in fives or more, and if they meet in fours, opposite segments must form a straight line, since straight lines are uniquely energy-minimizing for elliptic integrands.

The following theorem shows that the standard regularity for length-minimizing networks also holds for any smooth, elliptic integrand.

3.3. THEOREM [L, *Theorem* 3.5]. *Let* Φ *be a smooth, elliptic integrand in* $\mathbf{R}^2$. *In a minimizing network, segments meet at a node only in threes.*

REMARK. This result applies as well to "variable-coefficient" integrands $\Phi(x, t)$ by a limit argument which is not difficult.

Proof. By Proposition 3.2, we need only eliminate the possibility of two lines crossing. Suppose there is a case where such a network is energy minimizing. We can apply a linear transformation to the integrand to produce a case in which the "X" is minimizing; i.e., the two lines crossing are orthogonal.

We consider the variations suggested by Figure 3.3.2, in which the boundary points are kept fixed and one of the intersection points is moved slightly away from the center of the square, thus either increasing θ (perturbation 1) or decreasing θ (perturbation 2). The energies of the right half or perturbation 1 and the top half of perturbation 2 are given by

$$E_1 = \left(\frac{1}{2} - \frac{1}{2\tan\theta}\right)\Phi(0) + \left(\frac{1}{2\sin\theta}\right)\Phi(\theta) + \left(\frac{1}{2\sin(-\theta)}\right)\Phi(-\theta), \tag{1}$$

$$E_2 = \left(\frac{1}{2} - \frac{\tan\theta}{2}\right)\Phi\left(\frac{\pi}{2}\right) + \left(\frac{1}{2\cos\theta}\right)\Phi(\theta) + \left(\frac{1}{2\cos\theta}\right)\Phi(\pi-\theta). \tag{2}$$

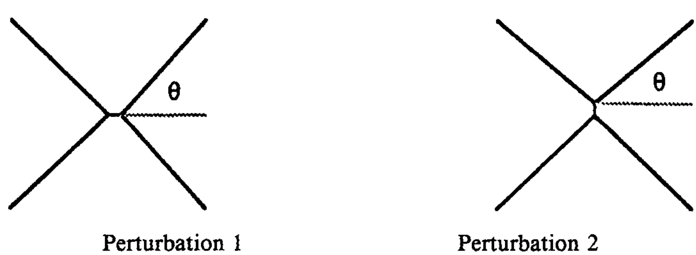


FIGURE 3.3.2

If the X is the minimizing network, then Perturbations 1 and 2 must have more energy than the X

Taking the appropriate one-sided derivatives of these energies gives

(3) $$\left.\frac{dE_1}{d\theta^+}\right]_{\theta=\frac{\pi}{4}} = 2\Phi(0) - \sqrt{2}\left[\Phi\left(\frac{\pi}{4}\right) - \Phi'\left(\frac{\pi}{4}\right) + \Phi\left(-\frac{\pi}{4}\right) + \Phi'\left(-\frac{\pi}{4}\right)\right] \geq 0,$$

(4) $$\left.\frac{dE_2}{d\theta^-}\right]_{\theta=\frac{\pi}{4}} = -2\Phi\left(\frac{\pi}{2}\right) + \sqrt{2}\left[\Phi\left(\frac{\pi}{4}\right) + \Phi'\left(\frac{\pi}{4}\right) + \Phi\left(-\frac{\pi}{4}\right) - \Phi'\left(-\frac{\pi}{4}\right)\right] \leq 0.$$

The inequalities follow from the assumption that the X is energy-minimizing.

Now, since Φ is elliptic, the curvature κ of the unit ball of Φ is positive. The border of the unit ball is given in polar coordinates by

$$r(\theta) = \frac{1}{\Phi(\theta)}$$

and the formula for curvature in polar coordinates is

$$\kappa = \frac{f^2(\theta) - f(\theta)f''(\theta) + 2[f'(\theta)]^2}{([f'(\theta)]^2 + [f(\theta)]^2)^{3/2}} \quad \text{for } r = f(\theta).$$

Thus, the curvature of the unit ball of Φ is

$$\kappa = \left(\frac{\Phi(\theta)}{\sqrt{[\Phi(\theta)]^2 + [\Phi'(\theta)]^2}}\right)^3 [\Phi(\theta) + \Phi''(\theta)]$$

which, since $\kappa > 0$, means

$$\Phi(\theta) + \Phi''(\theta) > 0.$$

If $\Phi(\theta) + \Phi''(\theta) = 0$, then $\Phi(\theta) = \Phi(0)\cos\theta + \Phi'(0)\sin\theta$. Since we have a strict inequality, we can conclude

(5) $$\Phi(\theta) > \Phi(0)\cos\theta + \Phi'(0)\sin\theta \quad \text{for all } \theta \neq 0,$$

and similarly,

(6) $$\Phi(\theta) > \Phi\left(\frac{\pi}{2}\right)\sin\theta - \Phi'\left(\frac{\pi}{2}\right)\cos\theta \quad \text{for all } \theta \neq \frac{\pi}{2}.$$

But (5) and (6) yield

$$2\sqrt{2}\left[\Phi\left(\frac{\pi}{4}\right) + \Phi\left(-\frac{\pi}{4}\right)\right] > 2\Phi(0) + 2\Phi\left(\frac{\pi}{2}\right);$$

while (3) and (4) yield

$$2\Phi(0) + 2\Phi\left(\frac{\pi}{2}\right) - 2\sqrt{2}\left[\Phi\left(\frac{\pi}{4}\right) + \Phi\left(-\frac{\pi}{4}\right)\right] \geq 0,$$

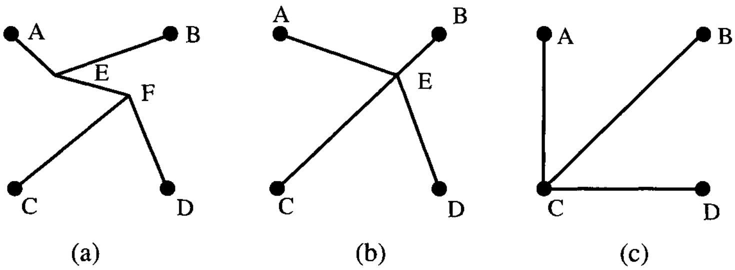


FIGURE 3.4.1
(a) A double-Y with nodes E and F. (b) A double-Y with two coincident nodes at E. (c) A double-Y with two nodes coincident at the boundary point C

clearly a contradiction. So for no smooth elliptic integrand can four segments meet at a point in a Φ-minimizing network.

The following theorem shows that if the smoothness of Φ is relaxed to piecewise smoothness, four segments can meet in a Φ-minimizing network. The particular example X is the diagonals of a unit square. We show the X is minimizing for any elliptic integrand whose unit ball is a suitably symmetric small perturbation of the square. (See Figure 1.0.4.)

3.4. THEOREM. *There is a piecewise smooth elliptic integrand* Φ_0 *in* $\mathbf{R}^2$ *such that the* X *is* Φ_0*-minimizing.*

Proof. For an elliptic integrand Φ, let S be a Φ-minimizing network having the four corners of a square as boundary. It is easy to show that a network with four boundary points has at most two nodes. We may say that any potential length-minimizing network has two nodes, each one connected to two adjacent boundary points and the other node, if we allow the possibility that the nodes are coincident with each other or boundary points. We call these networks "double-Y" networks (see Figure 3.4.1), and say they are made up of two "V's" (the segments connecting a node to boundary points) and a "bar" connecting the two nodes.

If Φ is symmetric about the x- and y-axes, we can show that any double-Y can be improved by placing both nodes on the horizontal or vertical bisector of the square. We decompose the double-Y $ABCDEF$ of Figure 3.4.1(a) into the two V's (AEB and CFD) and the bar (EF), and we draw lines l and m parallel to an axis of symmetry. (See Figure 3.4.2.)

We reflect AEB around l to form the network $ABEA'B'$ (see Figure 3.4.3). By ellipticity, we know AGB' has less energy than AEB'

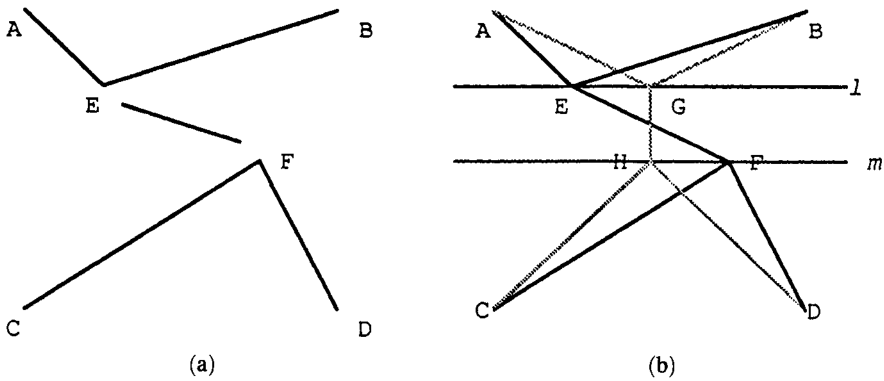


FIGURE 3.4.2
To show the double-Y $ABCDEF$ is not minimizing, (a) we decompose it into the subnetworks AEB, EF, and CFD. We then (b) draw lines l and m containing the points E and G and H and F, respectively; by reflecting the subnetworks around these lines we will show $ABCDGH$ has less energy than $ABCDEF$

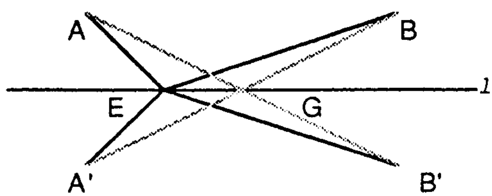


FIGURE 3.4.3
Reflecting AEB creates the network $ABEA'B'$. Since, with elliptic integrands, the least-energy path is a straight line, $ABGA'B'$ has less energy than $ABEA'B'$

and BGA' has less energy than BEA'. Therefore, $E(ABGA'B') < E(ABEA'B')$. Since Φ is symmetric, $E(ABGA'B') = 2E(AGB)$ and $E(ABEA'B') = 2E(AEB)$, and so $E(AGB) < E(AEB)$. Reflecting CFD about m and using an identical argument shows $E(CHD) < E(CFD)$. Finally, reflecting EF about either line shows that GH has less energy than EF (see Figure 3.4.4). Clearly $E(EJE') < E(EFE')$. $E(EJE') = 2E(EJ)$, $E(EFE') = 2E(EF)$, and $E(EJ) = E(GH)$; therefore, $E(GH) < E(EF)$. Therefore, $ABCDGH$ has less energy than $ABCDEF$.

S therefore must consist of two symmetric V's joined by a bar (of length ≥ 0) on the horizontal or vertical bisector of the square (Figure 3.4.5).

In each of the networks (a) and (b) in Figure 3.4.5, clearly one of the angles of the minimizers must be greater than $\pi/2$. A similar computation to that in the proof of Theorem 3.3 shows that the rate

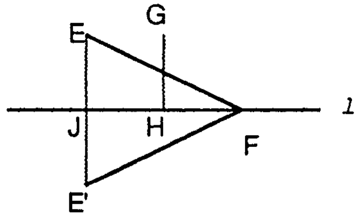


FIGURE 3.4.4
Reflecting EF to get the network EFE' shows clearly that GH has less energy than EF

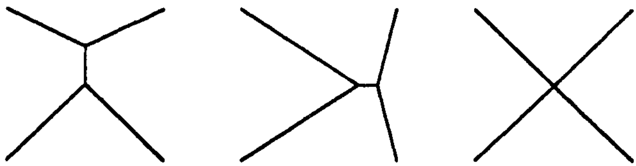


FIGURE 3.4.5
"Symmetric" double-Y 's with their nodes on the horizontal or vertical bisectors of the square. (c) is a degenerate case. Only symmetric double-Y s are potential minimizers for any Φ symmetric about both the x- and y-axes

of change of energy as that angle decreases is negative, provided that

$$(7)\quad \Phi\left(\frac{\pi}{2}\right) - 2\Phi(\theta)\sin\theta - 2\Phi'(\theta)\cos\theta > 0 \quad \text{if } 0 \le \theta < \frac{\pi}{4} \quad \text{and}$$
$$\Phi(0) - 2\Phi(\theta)\cos\theta + 2\Phi'(\theta)\sin\theta > 0 \quad \text{if } \frac{\pi}{4} < \theta \le \frac{\pi}{2}.$$

Therefore, if condition (7) is satisfied, S must be of form (c) of Figure 3.4.1 (i.e., it must have a bar of length 0), because forms (a) and (b) are unstable. Since Φ is elliptic, opposite segments of S must form a straight line (by Proposition 3.2), hence S is just the diagonals of the square.

There are many elliptic integrands with the required symmetry satisfying equation (7). For example, one could take the family of integrands

$$\Phi_\varepsilon(\theta) = (1-\varepsilon)|\cos\theta| + \frac{\varepsilon}{\sqrt{2}} \quad \text{when } -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4},$$
$$\Phi_\varepsilon(\theta) = (1-\varepsilon)|\sin\theta| + \frac{\varepsilon}{\sqrt{2}} \quad \text{when } \frac{\pi}{4} < \theta < \frac{3\pi}{4} \text{ or } \frac{5\pi}{4} < \theta < \frac{7\pi}{4}$$

as long as $0 < \varepsilon \le (4+\sqrt{2})/7$.

Incidentally, if $\varepsilon = 0$, Φ_ε has the square as its unit ball.

Added in proof. Finding a length-minimizing network is often called the Steiner problem. E. Cockayne [C] considered general integrands or norms Φ but did not discuss the dependence on the differentiability of Φ. M. Hanan [H] noted that for the nonelliptic "Manhattan" or "rectilinear" integrand, minimizing networks can meet in fours.

More recently, M. Conger [**Con**] has proved a result analogous to Theorem 3.4 for six vectors along the axes in $\mathbf{R}^3$. G. Lawlor and F. Morgan [LM] have generalized Theorem 3.3 to differentiable norms Φ on $\mathbf{R}^n$, showing $n+1$ a sharp bound on the number of segments that can meet at a node.

A survey appears in [**M3**, Chapter 10].

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