The Structure of the American Economy

Eric O'N. Fisher and Kathryn G. Marshall

Abstract

We explore the relationship between input–output accounts and the national revenue function. The generalized inverse of an economy's technology matrix carries information relating changes in endowments with changes in outputs; its transpose relates output prices and factor prices. Our primary theoretical contribution is to derive an economy's revenue function for an arbitrary Leontief technology. Our main empirical contribution is to compute the national revenue function for the American economy in 2003 and to describe its properties. We implement our ideas using two different models: one where all factors are mobile and another with sector-specific capital.

1. Introduction

Presenting a completely novel approach to analyzing the supply side of an economy, we show that an input–output table contains all the information needed to describe an economy's Rybczynski matrix. These effects relate marginal increases in an economy's resources with marginal changes in its vector of net outputs, when goods prices and thus factor uses are fixed. The same information can be interpreted as Stolper–Samuelson effects: the link between output prices and factor rewards, when endowments are fixed. We develop the theory, present leading examples, and implement our ideas using data from the American economy in 2003.

We characterize the supply side of an economy using the national revenue function. For a fixed technology, this function maps an economy's endowments and an output price vector into its maximal revenue. Its Hessian is an economy's Rybczynski matrix.

Our main theoretical contribution is to derive the revenue function for an arbitrary Leontief technology. This function is smooth with respect to its two arguments, endowments and output prices. Hence, it is very well behaved, its properties are easy to describe, and one can compute the exact Rybczynski derivatives for any economy that reports an input–output table and conformable data on factor uses. In the usual case with more goods than factors, we characterize the supply correspondence completely.

We make two broad empirical contributions. First, we look at the United States economy in 2003 disaggregated into 63 sectors and six factors, capital, and five broad types of labor, that are mobile across all sectors. We show detailed Rybczynski effects and also estimate shadow values for six factors. Our second application uses a Ricardo–Viner model with five broad types of mobile labor. Our real empirical contribution is to show how easy it is to implement our theory and to derive plausible empirical effects that describe the details of the American economy. Since our theory is based upon an arbitrary Leontief structure, it can handle any degree of aggregation and any model that the researcher might find appealing. Our theory is simple, so its applications are broad.

We would not have begun this work if Leontief (1951) had not devised input–output accounting; indeed the title of our paper pays blunt homage to his influence. We bring Moore's (1920) and Penrose's (1955) powerful mathematical tools to applied general-equilibrium theory. These authors created a technique to characterize all the solutions to a system of linear equations, even when the set of equations is underdetermined or only "approximately" correct. Input–output accounting was invented to calculate the necessary resources for a marginal increase in an economy's output. Trade theory explores how a marginal increase in resources affects an economy's outputs. Hence input–output accounting and international trade look at similar phenomena from opposite ends, and it is not surprising that the same data shed light on both lines of inquiry.

There is a large literature in empirical trade that explores the relationship between endowments and outputs. Using flexible forms, Kohli (1991) and Harrigan (1997) estimate national revenue functions; both tie theory and empiricism together carefully. These estimates are an important step in describing the sources of comparative advantage. Fitzgerald and Hallak (2004) and Schaur et al. (2008) estimate reduced-form Rybczynski equations. Fitzgerald and Hallak claim that failure to control for productivity differences produces biased estimates, and Schaur et al. state that the average effects across all local industries are positive. Our work renders all this statistical analysis moot; we show how to *compute—not estimate*—exact Rybczynski derivatives for any country that has an input–output table and conformable uses of factors.We find much richer patterns than those in the literature and reap a bonus: our exact effects are also Stolper–Samuelson (directional) derivatives.[1]

Lloyd (2000) extends the Stolper–Samuelson theorem to a model with a general pattern of factor ownership, and Beaulieu (2002) shows that congressional voting patterns can be explained in part by the distribution of factor ownership among constituencies. Using a Ricardo–Viner model where the number of factors exceeds the number of goods, we compute the exact Stolper–Samuelson effects of output price changes on factor rewards at a highly disaggregated level. Our technique will provide new empirical detail for further study of income distribution and trade policy.

2. Theory
The Revenue Function
Let v be the f x 1 vector of aggregate inputs of primary factors that are in fixed supply and y be the l x 1 vector of outputs. Technological considerations are summarized by a set of feasible combinations of outputs and inputs:

$$F \subset \mathbb{R}^{l+f}.$$

We assume that this set is compact for fixed inputs v. Producers take the f x 1 vector of output prices p as given. The revenue function is

$$) \qquad \{r \qquad) \quad \}r$$

The main theoretical advantage of the national revenue function is that it allows one to summarize the relationship between endogenous and exogenous variables succinctly. For example, the output vector is the gradient of the national revenue function:[2]

$$)r$$

The revenue function is appealing because it is so general. The Rybczynski matrix

$$)r \begin{pmatrix} 2r & \cdots r & 2r \\ 2r_{lr} & \cdots r & 2r_{lr} \end{pmatrix} r$$

is the focus of this paper. Its canonical element $\partial^2 r/\partial p_i \partial v_j$ shows how the output of good i changes with respect to a marginal increment in the endowment of factor j, if one holds factor prices and thus factor requirements constant.

The transpose of the Rybczynski matrix shows the Stolper–Samuelson effects:

$$) \qquad)r$$

Each element of this matrix describes the marginal effect of a change in the price of good j on the reward to factor i. Since factor rewards are homogeneous of degree one in prices, the Stolper–Samuelson effects satisfy an important restriction:

$$w \qquad)r.$$

This equation states that the sums of the Stolper–Samuelson effects, weighted by the price of output in each sector, are the shadow values of the factors in the economy, a fact that we will use in our empirical analysis. If there are constant returns to scale, then outputs are homogeneous of degree one in v. In this case, the Rybczynski matrix also satisfies:

$$)r$$

This equation states that the sums of the Rybczynski effects, weighted by the quantities of the the economy's fixed endowments, are the elements of the economy's output vector. Again, we will use this fact that in our empirical work below.

The Factor Pricing Equations and the Resource Constraints
The usual relationship between factor prices and goods prices is given by:

$$Aw \geq p, \ y \geq 0, \text{ with complementary slackness,}$$

where a_{ij} is the unit input requirement of factor j in the output of good i. Because we will be interested in marginal changes in our empirical work, we will restrict our attention to strict equalities without loss of generality. If the ith good's unit cost exceeds its price, then it will not be produced. Then we will set $a_{ij} = 0$ for $j = 1, \dots, f$ and also write $p_i = 0$. In this case, the following equality is true:

$$Aw = p.$$

Any factor rewards w that solve this modified system will also satisfy the original equations, and any solution of the original system will give factor prices that also solve the modified system. Also, since $yi = 0$ in the original system, the modified technology matrix will automatically satisfy the resource constraints in the original system.

The full employment equations are:

$A^T y \leq v$, $w \geq 0$, with complementary slackness.

If the jth factor is in excess supply, then its reward $w_j = 0$. Now we set $a_{ij} = 0$ for $i = 1,...$, l and also write $v_j = 0$. Then

$A^T y = v$,

and each solution to this modified system corresponds to a solution in the original one. Likewise, every vector of outputs in the original system will solve the modified one.

The national revenue function can also be defined as the minimum value of payments to factors of production that is consistent with the zero-profit conditions:

$r(p, v) = \min_w \{w^T v / Aw \geq p\}$.

This approach is helpful if one is interested in using Shephard's lemma to derive aggregate factor demands. For example, it predicts that factor prices are given by the gradient of the unit isoquant evaluated at the endowment vector in a model with one good and several factors. In our empirical work, we use the fact that fixed factor prices entail a restriction on admissible endowment changes if there are more factors than goods.

The Moore–Penrose Pseudo-Inverse
Let A be an $l \times f$ matrix. Then its Moore–Penrose pseudo-inverse is the unique $f \times l$ matrix A^+ that satisfies these four properties:

(P1) $AA^+A=A$;
(P2) $A^+AA^+ = A+$;
(P3) $A+A = (A^+A)^T$;
(P4) $AA^+ = (AA^+)^T$.

If A is square and has full rank, then $A^+ = A^{-1}$. If A^TA has full rank, then $A^+ = (A^TA)^{-1}A^T$ can be computed easily. Every matrix has such a pseudo-inverse.[3]

The primary advantage of the Moore–Penrose pseudo-inverse is that it gives the complete set of solutions to the system of equations $Ax = b$. This set is:

$x = A^+b + (I - A^+A)\,z$,

where z is an arbitrary $f \times 1$ vector. In fact, this pseudo-inverse even gives a solution to an overdetermined and inconsistent system $Ax \approx b$. Then $x = A^+b$ is the vector of coefficients of the least squares estimates from the regression of b on the columns of A.

If a row of a non-null matrix A consists of zeros, then the corresponding column of A^+ does also. This fact justifies our restrictions that $Aw = p$ and $A^Tv = y$ hold with equality, as long as one

works with a modified matrix that replaces the appropriate row or column of the original technology matrix with zeros whenever a constraint is slack.

The Stolper–Samuelson and Rybczynski Matrix

The production function for a fixed coefficients technology is:

$$y_i = \min\{v_{i1}/a_{i1}, \ldots, v_{if}/a_{if}\},$$

where v_{ij} is the input of factor j into sector i. Let A be the $l \times f$ matrix of (direct and indirect) factor requirements that are observed in the data. Assume that the $l \times 1$ vector of output prices p is given. Then the complete solution for factor prices is:

$$w = A^+ p + (I - A^+ A)z,$$

where z is an arbitrary $f \times 1$ vector. The matrix $I - A^+ A$ projects z onto the null space of A. This expression gives all factor prices consistent with perfect competition.

Since factor payments exhaust revenues,

$$r(p, v) = v^T w = v^T A^+ p + v^T (I - A^+ A)z.$$

The full-employment conditions imply that the endowment vector is in the row space of the technology matrix. Since this space is the orthogonal complement of its null space, $v^T(I - A^+ A)z = 0$ for any z.[4] Then factor prices are:

$$w = r_v(p,v) = A^+ p + (I - A^+ A)z,$$

where z is arbitrary. This expression gives the *set of all factor prices* that are consistent with the zero-profit conditions. It is typical in the literature to explain that factor prices are not tied down when there are more factors than goods and that they are derived from other extraneous considerations—such as the full-employment conditions—that have nothing to do with unit costs. Of course, that argument does not work for the case of a fixed coefficients technology or for any other where each output is not differentiable with respect to every input.

It is constructive to derive the national revenue function in an analogous manner to the economy's resource constraint:

$$A^T y = v.$$

The complete solution for the output correspondence is:

$$y = (A^T)^+ v + (I - (A^T)^+ A^T)\, z,$$

where z is now an arbitrary $l \times 1$ vector. Since the value of output is national revenue,

$$r(p, v) = p^T y = p^T (A^T)^+ v + p^T (I - (A^T)^+ A^T)\, z.$$

Since $I - (A^T)^+ A^T$ projects onto the null space of A^T and prices p lie in the column space of A, we conclude that $p^T(I - (A^T)^+ A^T) = 0$. Since $(AT)+ = (A+)T$, this formula is simply the transpose of the one derived using the income approach. These results are significant enough to state formally.

THEOREM. *Consider an economy with a Leontief technology. Assume that all resources are fully employed and that all goods are produced. Let p be the $\ell \times 1$ vector of goods prices, v be the $f \times 1$ vector of factor endowments, and A be the $\ell \times f$ matrix of unit input requirements. Then the revenue function is the quadratic form (1):*

$$r(p,v) = v^T A^+ p + v^T(I - A^+ A)z_f + z_\ell^T(I - AA^+)p, \qquad (1)$$

where $z_f \in \mathbb{R}^f$ and $z_\ell \in \mathbb{R}^\ell$ are arbitrary.

PROOF. The full-employment condition implies that $v^T(I - A^+A) = 0$, and the zero-profit condition implies that $(I - AA^+)p = 0$. Hence the particular solution in (1) is the value of national revenue since each quadratic form involving a homogeneous term has value zero. By construction, the gradient of (1) with respect to p gives the gives the supply correspondence, and its gradient with respect to v gives the set of factor prices consistent with the zero-profit conditions. Further, (1) satisfies all the requisite homogeneity restrictions with respect to its two arguments. ❑

The theorem has two immediate implications. If there are at least as many goods as factors and the technology matrix has full (column) rank, then $I - A^+A = 0$ and factor prices are completely determinate. If there are at least as many factors as goods and the technology matrix has full (row) rank, then $I - AA^+ = 0$ and the output vector is determinate. There are interesting cases—of significant empirical relevance—where the technology matrix does not have full rank. For example, there are many models in macroeconomics with more (differentiated) goods than factors, but all sectors have identical factor intensities. Then the first homogeneous term in (1) is not null, and the gradients of the national revenue function show all factor prices that satisfy the zero-profit conditions, and it also gives the entire supply correspondence. Likewise, there are some models with at least as many factors as goods where the technology matrix does not have full rank; this situation arises when at least two sectors have identical factor intensities. In this case, the gradients of (1) again give the supply and factor–price correspondences.

The theorem shows that the national revenue function is (infinitely) differentiable with respect to both of its arguments, output prices and endowments. From our perspective, the most important of its implications is the following corollary.

COROLLARY. *Under the theorem's assumptions, the economy's Stolper–Samuelson matrix is A^+ and its Rybczynski matrix is A^{+T}.*

This result is controversial at first blush. Assume there are strictly more goods than factors. Trade theorists allege that the Rybczynski effect is not defined in this case. But it is obvious that the only part of the supply correspondence that depends upon endowments is the particular solution in (1). Fix output prices and thus factor uses and consider a marginal change in endowments. We will show in section 3 that any resultant change in outputs can be decomposed into two parts: (1) a change that is orthogonal to the economy's production possibility frontier;

and (2) a movement along one of its flats. Only the former has any effect on national revenue, and that is why it is properly defined as the Rybczynski effect. Indeed, the beauty of (1) is that it solves the indeterminacy that has plagued the empirical literature.

The typical regression that estimates a "Rybczynski effect" actually captures the demand-side effects of wealth changes. Some of the more careful researchers note that the Rybczynski effect is indeterminate in the usual case where there are many goods and few factors. Some assume an (infinitely differentiable) translog approximation to the national revenue function and estimate its parameters, imposing symmetry and homogeneity restrictions. In essence, one notes the problem in theory and then assumes it away blithely in empirical work. Not only are such regressions based upon a potentially misleading approximation, they are not even identified in theory.

There is an even better reason to identify A^{+T} as *the* Rybczynski matrix. Its transpose gives the unique solution for factor prices $w = A^+p$, where we have used that $I - A^+A = 0$ when $l \geq f$ and the technology matrix has full rank. Since all goods are produced, $Aw = p$. In this case, $w = A^+p = A^+Aw$ is an identity. It is obvious that A^+ is the only candidate for a Stolper–Samuelson matrix when input requirements are such that several sectors are active. (This is exactly what one would expect in the long run when local techniques adjust so that several sectors are competitive at prevailing world prices.) Hence, defining A^{+T} as the Rybczynski matrix maintains the duality between Rybczynski and Stolper–Samuelson effects at the heart of classical trade theory.

3. Two Leading Examples

We sketch out two simple examples that illustrate the underlying theory.

EXAMPLE 1. The Solow model is the simplest case where the number of factors exceeds the number of commodities. The vector of endowments is $v = [K \quad L]^T$, and technology is described by an aggregate production function $Y=F(K, L)$ that exhibits constant returns to scale. The unit input requirements depend upon factor prices:

$$A(w, r) = [a_K(w, r) \quad a_L(w, r)]^T$$

where w is the wage rate and r is the rentals rate. Of course, factor prices are not even locally independent of endowments. The Stolper–Samuelson matrix is:

$$\begin{bmatrix} r & 2r & 2) \\ & 2r & L) \\ L & & Lv \end{bmatrix} N,$$

where we have suppressed the dependence on factor prices for notational convenience. Three points are in order. First, for fixed endowments, this matrix allows any change in output prices (in the trivial one-dimensional space in which they lie). Second, the Stolper–Samuelson matrix *does not* consist of the marginal products of capital and labor; it is instead collinear with the average products of these factors. Third, within the strict framework of a Leontief technology

where aggregate output $F(K, L) = \min\{K/a_K, L/a_L\}$, the Stolper–Samuelson matrix can be construed as a theory of factor prices. Among all strictly positive factor rewards that satisfy the zero-profit conditions $\{(w, r) \in R_+^2 \mid p = a_K r + a_L w\}$, it picks the wage/rentals ratio a_L/a_K that corresponds with the economy's aggregate capital/labor ratio.

Figure 1 shows the Stolper–Samuelson effects in this case. The horizontal axis measures the first factor price and the vertical axis measures the second one. Let the price of aggregate output change by an arbitrary amount Δp. Then any observed change in factor prices $\Delta w = A^+ \Delta p + u$ can be decomposed into two orthogonal parts. The first part $A^+ \Delta p$ is orthogonal to the unit cost functions, and in the only direction that affects national income. The second part u (not drawn) has no effect on aggregate factor costs and thus no impact on national income. The first part comes from the particular solution in (1), and u lies in the linear space defined by the first homogeneous term in that equation. In the empirical analysis in section 4, we use the properties of the Stolper–Samuelson derivatives to compute the shadow values of factors in the national economy $w = r_v(p, v)^T = A^+ p$ in a model where there are more goods than factors. We are able to compute these shadow values even though output prices are not observable in our data.

The interpretation of the Rybczynski matrix A^{+T} in this case is subtle. Endowments are constrained to lie in the linear subspace generated by the economy's capital/labor ratio. Only marginal changes $[dK/K \quad dL/L]^T$ of equal proportions can maintain full employment at the factor prices that are assumed fixed. Then the elements of the Rybczynski matrix are components of a directional derivative that explain the change in aggregate output by attributing weights $K^2/(K^2 + L^2)$ and $L^2/(K^2 + L^2)$ to the changes in capital and labor, respectively.

EXAMPLE 2. The Ricardian model is the simplest case where the number of goods exceeds the number of factors. The vector of endowments is simply $v = L$. Technology is summarized by the production possibility frontier $\{ (y_1, y_2) \in R_+^2 \mid a_1 y_1 + a_2 y_2 = L\}$, where a_i is a sector's labor coefficient.

Let $A = [a1 \quad a2]^T$ be unit labor requirements observed in the data. The Stolper– Samuelson matrix is:

$$[r \quad ^2 \quad \tfrac{2}{2})r \quad _{2r} \quad ^2 \quad \tfrac{2}{2v})]r$$

Now its interpretation is subtle since output prices are constrained to lie in the linear subspace generated by A. Only marginal changes of equal proportions $[dp1/p1 \quad dp2/p2]^T$ can assure positive outputs of both goods. Then the elements of the Stolper–Samuelson matrix are components of a directional derivative that explain the change in the wage rate by giving weights $^2 \quad _{2r} \quad \tfrac{2}{2})r$ and $\tfrac{2}{2r} \quad ^2 \quad \tfrac{2}{2v})$ to the changes in the prices of the first and second good, respectively.

Endowments are free to move in any (trivial) direction, but the Rybczynski $A+T$ matrix chooses one element of the supply correspondence. Indeed, only movements in the direction A^{+T} affect national revenue. In any other direction, a feasible change in outputs trades off one good against another according to the fixed marginal rate of transformation inherent in this economy. This tradeoff has no effect on revenue.

Figure 2 shows the Rybczynski effects in this case. Now the horizontal axis measures output of the first good and the vertical axis measures that of the second one. Let the endowment of labor change by an arbitrary amount Δv. Then any observed change in output $\Delta y = A^{+T}\Delta v + u$ can be decomposed into two orthogonal components. The first part $A^{+T}\Delta v$ is orthogonal to the economy's production possibility frontier, and it is the only direction that affects national revenue. The second part u (again not drawn) has no effect on the value of output and thus no impact on GDP; it lies in the linear space defined by the second homogeneous term in (1). In section 4, we use the properties of the Rybczynski derivatives to predict the output effects of an arbitrary endowment vector $y = r_p (p, v)^T = A^{+T}v$ in an economy where there are more goods than factors.

4. Empirical Analyses

Input–output data consist of values denominated in current dollars. Hence we cannot observe prices and quantities independently. We follow the convention established by Leontief (1951, p. 72) himself, who noted, "In order to obtain the corresponding physical amounts of all commodities and services, we simply define the unit of physical measurement of every particular type of product so as to make it equal to that amount of the commodity which can be purchased for one dollar at prevailing prices." The direct factor uses in each sector are measured in person-years for different categories of labor and in current dollars for the stocks of capital. Hence we observe physical quantities of labor, but we do not observe factor prices. We measure capital as the stock of fixed assets in each sector, measured in current dollars; hence this measure is fundamentally different from that for labor since it depends upon current prices. Again, we observe stocks of capital but not rates of return.

The input–output data are published by the Department of Commerce's Bureau of Economic Analysis (BEA). We use data that are disaggregated into 63 sectors.[5] The sum of each column of the input–output matrix is the gross industry output in each industry measured in millions of current dollars. The data on direct factor uses for capital are from the BEA and those for labor are from the Department of Labor's Bureau of Labor Statistics (BLS).[6] We normalize these data by dividing every element in a column by gross industry output. Hence we measure the direct and indirect factor requirements needed for one million dollars of industry output. In essence, we have set the price of each unit of output to \$1 million or the corresponding physical quantity to the amount that can be purchased for \$1 million. The distinction between price and quantity is important since we consider separately the impact of endowment changes on the quantity of outputs, and the impact of price changes on factor payments.

Let B be the f x l matrix of direct factor inputs per unit of gross output, and C be the corresponding l x l matrix of intermediate inputs. $A = [B(I — - C)^{-1}]^T$ is the l x f matrix of direct and indirect factor inputs used in our empirical analysis. The zero-profit condition implies that factor payments w satisfy:

$$Aw = 1_{l \times 1}.$$

There is an interesting interpretation of this condition when $l > f$. In this case, the unit vector

will almost surely not lie in the column space of *A*. In fact, this is exactly the situation of an econometrician trying to find the best fit for a left-hand variable (our unit vector of assumed output prices) onto the column space of the explanatory variables (the direct and indirect factor uses in every sector). We will use this intuition in the next subsection to describe how the model with six mobile factors fits the data.

Since factor prices satisfy $w = A^+1_{lx1} + (I - A^+A)z$, we may write

$$1_{lxr}$$

This vector is the best estimate of the prices that are consistent with the zero-profit conditions that underpin our analysis. The projection matrix AA^+ maps any l x 1 vector into the closest vector in the column space of the direct and indirect factor requirements. Hence, if $l > f$, then our best estimate of the national revenue function is

$$)r \hfill (2)$$

and the shadow of the factors is $r_v (P, v)^T = \quad = A^+1_{l\times1}$ since $A^+AA^+ = A^+$.

All Factors are Mobile

We first analyze a model where all factors are mobile and the number of sectors $l = 63$ exceeds the number of factors $= 6$. Since changes in endowments are unrestricted in this case, we prefer to interpret the elements of A^{+T} as the Rybczynski derivatives. Each column of this matrix reports the impact of an increase in an endowment on the economy's vector of outputs under the assumption that goods prices and thus factor rewards are constant. Each column sum (of this transposed matrix) gives the shadow value of the factor in question.

For example, consider adding one additional Management and Technical person-year to the economy's fixed resources. The column sum of the Rybczynski matrix shows that the shadow value of this worker is $0.145 x 10^6$, and the logic inherent in (2) indicates that this sum is an estimated annual salary. The services of our hypothetical new worker will be distributed throughout the economy, and other factors will be reallocated to maintain constant factor proportions within sectors. The reported change in output in each sector then reflects this complete reallocation of resources. In the Heckscher–Ohlin theory, these effects are an indication of revealed comparative advantage. If domestic absorption is a fixed share of world production, those sectors whose outputs increase most will also contribute most to net exports. Rybczynski effects thus capture the impact of changes in endowments on the pattern of trade.

There is no easy way to report a table of $l \, x \, f = 63$ x 6 = 378 numbers. In fact, no one has ever calculated an actual Rybczynski matrix before. Table 1 follows the tradition in trade theory and reports the strongest positive effect for each factor.

A million-dollar increase in capital will increase output in real estate by $39,000, its strongest effect in any sector. Indeed, the capital intensity of real estate is the highest across all sectors in the economy; it employs $9 of capital per dollar of output.

We present further detail on the Rybczynski effects within the 19 manufacturing industries in our data. These detailed effects show a much richer and more varied picture than is typical in

the literature. Hence, our work stands in stark contrast to the usual approach that reports econometric estimates of output effects; good examples of this kind of work are Leamer (1984) and Harrigan (1995). Such studies face serious data limitations; hence they focus on a more narrow range of sectors, usually only manufacturing outputs. Both Harrigan and Leamer conclude that capital has a positive effect on all manufacturing sectors. Table 2 presents our fndings, and it identifes eight out of 19 sectors whose output actually decreases. For example, an extra million dollars of capital decreases the output of furniture and related products by $4000.

It is also interesting to examine the impact of an increase of one person-year of highly skilled labor (Professional Occupations) and unskilled labor (Production and Transportation Occupations) on manufacturing output. These effects are described in Table 3. In contrast to the limited impacts reported by Harrigan (1995), we find a positive impact on 14 industries. This empirical finding suggests that the United States has a comparative advantage in these sectors if indeed it is relatively abundantly endowed with highly skilled labor. Notice that many manufacturing industries—such as apparel and furniture—actually are more strongly affected by unskilled labor than skilled labor. Again, these rich Rybczynski effects show the importance of human capital even in traditional manufacturing sectors.

The Model's "Statistical" Fit

We now draw our attention to the model's overall fit. As we have emphasized, there are important theoretical and empirical consequences from assuming that that $l > f$. Since the Stolper–Samuelson matrix is the transpose of the Rybczynski matrix, Table 1 also reports the impacts on the factor payments of a change in the price of that sector's output. However, if we consider an arbitrary price change in a single sector, we need to map it into the column space of the technology matrix using the idempotent (projection) matrix AA^+. Again, price changes are restricted to be directional derivatives that lie in the column space of A. This is exactly the situation faced by an econometrician who is trying to fit an arbitrary vector of data onto the column space of some explanatory variables. Our "data" are the assumed price vector $p = 1_{lx1}$ and our explanatory variables are the factor uses in every sector, without a constant term. Our estimated coefficients are the shadow values of the factors we are analyzing.

Table 4 presents the results of this simple "estimation." Capital is measured in millions of dollars, so the "estimated reward" of $136,408 represents an economy-wide gross rate of return of 13.6%. All other factors are measured as person-years, so the estimated coefficients are annual salaries. We find all six shadow values are significantly different from zero for a test of size 5%. However, Education and Healthcare Occupations has an estimated wage that is negative.

This negative shadow value is the best indication that something is amiss in reconciling the *direct* factor requirements for the different sectors with the input–output data on intermediate inputs. Table 4 gives the factor prices that best fit output process under the assumption that payments to capital and the five types of labor exhaust value-added. If we had not tried to use

an exhaustive list of factors, then the zero-profit conditions that are at the heart of the estimation would not be germane. We could not predict any factor price because the unobserved vector of costs imposes no discipline on a model that is based on the definition of unit-value isoquants. Much of the literature in trade theory that measures factor content makes this mistake.

A theoretical purist might assert stubbornly that there is nothing wrong with a model that predicts a negative shadow value for some factor. The value-added in each sector includes information about indirect business taxes. It is easy to show in a simple model where all producer prices are positive that some factor rewards may be negative because the Stolper–Samuelson affects magnify tax wedges. The importance of our analysis is to show that the negative shadow value for education and healthcare occupations is no statistical fluke.

In fact, our technique of deriving the national revenue function and then estimating the shadow value of a factor is a way of confirming the validity of an important aspect of national income accounting. It is a commonplace that one cannot use the product approach to measure the services produced by many public sector employees. We tell our students in introductory courses in macroeconomics that the "output" of a policeman or a public school teacher corresponds exactly to what that worker earns. This accounting fiction maintains the identity between the income and the product approaches in national accounts. But it is quite a different exercise to ask the question, "What is the value of another public service employee, taking all factor prices including the pattern of indirect taxation as given?" The ruthless logic of the Rybczynski theorem reminds us that every extra employee must draw off resources from other sectors. Given how direct factor requirements have been measured, there is no guarantee that the overall effect of the reallocation of resources in the economy will be positive too.

Sector-Specific Capital

This subsection allows us to show the empirical power of our general theoretical approach. The specific factors model has an important place in trade theory and in applied general-equilibrium studies. It is particularly apt for doing comparative statics because the national revenue function is well behaved. It is also used in the study of the political economy of taxation since the effects of distorting taxes are simple to model.

Now the transpose of the technology matrix has this form:

$$
\begin{bmatrix}
b &) & & \\
& & & \\
& & b & {}_{l)\,lr} \\
b & \cdots & b & {}_{lr} \\
& & & \\
Nb & \cdots & b & {}_{lr}
\end{bmatrix} N
$$

where $b_{x(i),i}$ is the direct unit input requirement of specific factor $K(i)$ in the ith sector for i $\{1, \ldots, 63\}$, b_{ij} is the direct unit input requirement of mobile factor i in sector j, and C is again the economy's input–output matrix. The five mobile factors are the different kinds of labor that

we are analyzing, and the 63 specific factors are the measured uses of capital in each sector. Thus there is no such thing as an economy-wide rate of return on capital. Notice that this model has more factors than goods since $f = 68 > 63 = l$. Now our preferred interpretation of A^+ is as a Stolper–Samuelson matrix. It has f x l = 68 x 63 = 4284 elements. Each measures the effect of increasing the price of some good on a factor's reward.

The properties of the pure Ricardo–Viner model are well known.[7] For example, an increase in the price of the ith good will raise the return of the specific factor in that sector. But there is an important subtlety in empirical work. The technology matrix actually incorporates the direct and indirect uses of all factors. So every sector requires the use of every factor—mobile and specific—because of the effect that intermediate inputs have on factor content. Hence, it is not the case that an increase in the price of a sector will automatically lower the return to the specific factors used in all the other sectors. In fact, it is not even true that an increase in the price of one sector will have its strongest impact on that sector's specific factor. In our data, an increase in the price of real estate actually has its strongest effect on the reward for "capital used in educational services."[8] In every other case, the strongest effect of a price increase is on the specific factor used in that sector. Table 5 presents selected Stolper–Samuelson effects from a change in price of output in each of the the five largest sectors in the American economy. It reports the fixed factor that experiences the strongest positive and strongest negative Stolper–Samuelson effect.

We would like to reiterate an important theoretical observation that arose in the discussion of the Solow model as the leading example of the case where there are more factors than goods. The Stolper–Samuelson effects we report *are not* the derivatives of the national revenue function of a model with mobile factors and a smooth neoclassical production function in each sector; hence, they do not correspond to the textbook treatment of the comparative statics of this model. We have been very explicit about holding factor uses constant when output prices change; that is why we derived the national revenue function for a Leontief technology. In the usual treatment of the Ricardo–Viner model, the mobile factor flows into the sector whose price has increased.[9] If output in that sector changes at all, then the unit input requirements of the fixed factor must change too. Hence, the textbook model really captures two effects: (1) that of a price change; and (2) that of an ancillary reallocation of resources between sectors. Only the first one is a true Stolper–Samuelson effect, and that is what we report here. Also, only the first effect has a natural interpretation as the dual of a Rybczynski derivative. We feel that our analysis is in keeping with the spirit of traditional trade theory.

5. Conclusion

This paper has made two main contributions. The first was theoretical, and the second was empirical. Our theoretical contribution was to show that the input–output accounts contain all the information necessary to describe the relationship between factor endowments and output supplies. Since the Rybczynski effects have to do with quantities and assume fixed output prices and factor rewards, this result is not so much of a surprise when it is stated at this level

of generality. But it is startling that input–output accounts also contain complete information about the relationship between output prices and factor rewards. The duality between the Rybczynski and Stolper–Samuelson matrices is well understood, but no one has shown an explicit form for the revenue function before.

Our empirical contributions were to adumbrate some of the details of the supply side of the American economy in 2003. No one has ever used the Moore–Penrose generalized inverse in applied general-equilibrium theory before. Most pieces of mathematical software have an easy function that readily computes the unique Moore–Penrose pseudo-inverse of any non-null matrix. Our work has advanced input–output accounting significantly by showing the exact relationship between these accounts and the national revenue function. Thus any scholar in macroeconomics interested in the wealth effects of supply-side shocks will find ready use for the techniques that we have developed here. For example, a factor-specific technology shock in a small open economy can be modeled as a parametric change in endowments. Then the national revenue function will show the exact output effects for a fixed vector of prices. In sum, our work and the empirical tools we have created will have broad appeal to trade theorists, to macroeconomists, to development economists, to labor economists, and to public finance economists. Any field in our discipline that needs to explore the relationship between factor prices and factor rewards or that between resources and output supplies can build on our methods.

References

Albert, Arthur, *Regression and the Moore–Penrose Pseudoinverse*, New York: Academic Press (1972).

Beaulieu, Eugene, "The Stolper–Samuelson Theorem Faces Congress," *Review of International Economics* 10 (2002):343–60.

Choi, E. Kwan, "Implications of Many Industries in the Heckscher–Ohlin Model," in E. Kwan Choi and James Harrigan (eds), *Handbook of International Trade*, Oxford: Blackwell (2003): 32–59.

Feenstra, Robert C., *Advanced International Trade: Theory and Evidence*, Princeton, NJ: Princeton University Press (2003).

Fitzgerald, Doireann and Juan Carlos Hallak, "Specialization, Factor Accumulation, and Development," *Journal of International Economics* 64 (2004):277–302.

Harrigan, James, "Factor Endowments and the International Location of Production: Econometric Evidence for the OECD, 1970–1985," *Journal of International Economics* 39 (1995):123–41.

———,"Technology, Factor Supplies, and International Specialization: Estimating the Neoclassical Model," *American Economic Review* 87 (1997):475–94.

Kohli, Ulrich, *Technology, Duality, and Foreign Trade: The GNP Function Approach to Modeling Imports and Exports*, Ann Arbor, MI: University of Michigan Press (1991).

Leamer, Edward E., *Sources of International Comparative Advantage: Theory and Evidence*,

Cambridge, MA: MIT Press (1984).

Leontief, Wassily, *The Structure of the American Economy, 1919–1939: An EmpiricalApplication of Equilibrium Analysis*, Oxford: Oxford University Press (1951).

Lloyd, Peter, "Generalizing the Stolper–Samuelson Theorem: A Tale of Two Matrices," *Review of International Economics* 8 (2000):597–613.

Moore, E. H., "Abstract," *Bulletin of the American Mathematical Society* 26 (1920):394–95. Penrose, Roger, "A Generalized Inverse for Matrices," *Proceedings of the Cambridge Philosophical Society* 51 (1955):406–13.

Rassekh, Farhad and Henry Thompson, "Adjustment in General Equilibrium: Some Industrial Evidence," *Review of International Economics* 5 (1997):20–31.

Schaur, Georg, Chong Xiang, and Anya Savikhin, "Factor Uses and the Pattern of Specialization," *Review of International Economics* 16 (2008): 368–82.

Notes

1. Choi (2003) underscores the importance of prices adjusting endogenously so that many sectors are still competitive in a model with more goods than factors. Our emphasis on the interpretation of the Stolper–Samuelson effects as the elements of a directional derivative formalizes this idea.

2. In this paper, all gradients are row vectors. We use the notation $r_p(p, v) = [\partial r/\partial p_1 \ldots \partial r/\partial p_l]$. In this subsection, for ease of exposition, we are quite blithe in assuming that all functions are differentiable. They typically are not, and that is why this elegant theoretical approach has had limited practical appeal for empiricists. We will show below how to compute the subgradient that is the supply correspondence when there are more goods than factors and also how to compute the complete set of factor prices consistent with perfect competition when there are more factors than goods.

3. Albert (1972) gives a very good exposition of the properties of the Moore–Penrose generalized inverse.

4. A simple way to see this fact is to note that $v^T = y^T A$ and $(I - A^+A) = 0$.

5. The BEA publishes annual input–output tables disaggregated into 65 sectors. Data on factor uses are published in a consistent manner with a few exceptions. The input–output data describe four government sectors, including government enterprises for federal and for state and local government. These four sectors were merged into two by combining the general government and government enterprises at the federal level and at the state and local level. We use 63 sectors, including two government sectors. The data are available at this URL: http://www.bea.gov/industry/io_annual.htm.

6. Data on labor inputs are taken from the BLS November 2003 Occupational Employment and Wage Estimates at URL http://www.bls.gov/oes/oes_2003_n.htm. These are categorized by two-, three-, and four-digit NAICS industry, but do not include the self-employed. The Occupational Employment Statistics survey puts workers into about 770 standard occupation categories which are aggregated into 22 broad two-digit classifications. To make these extensive data more man-

ageable we have further aggregated 22 two-digit categories into five categories as follows: Professional Occupations, 11 through 19; Education and Healthcare Occupations, 21 through 29; Food Service and Maintenance Occupations, 31 through 39; Sales, Clerical, and Construction Occupations, 41 through 49; and Production and Transportation Occupations, 51 through 53.

7. Feenstra (2003) has a very nice exposition.

8. A plausible conjecture is that local school districts capture the effects that pricier real estate has on the tax base for local expenditures.

9. Rassekh and Thompson (1997) estimate a Ricardo–Viner model along these lines.
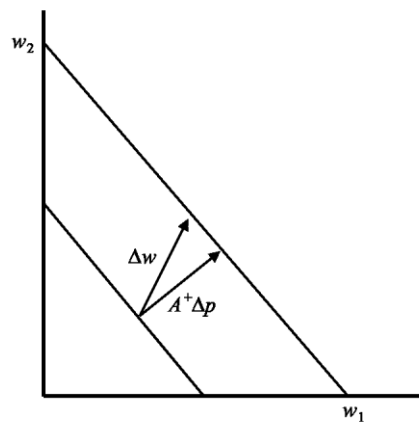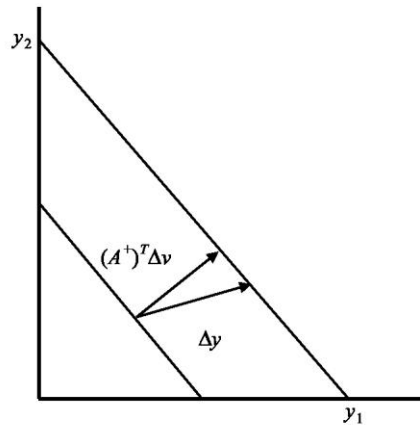
Images

Figure 1. Stolper–Samuelson Effects

Figure 2. Rybczynski Effects

Table 1. Strongest Positive Rybczynski Effects

An increase in one unit of this factor	Increases output most in	Change
Capital	Real estate	39
Management and Technical Occupations	Computer systems design and related services	75
Education and Healthcare Occupations	Educational services	21
Food Service and Maintenance Occupations	Food services and drinking places	31
Sales and Clerical Occupations	Retail trade	37
Production and Transportation Occupations	Transit and ground passenger transportation	20

Notes: Capital is measured in millions of dollars. All other factors are measured in person-years. Output effects are in thousands of dollars per year.

Table 2. Capital's Rybczynski Effects on the Manufacturing Sectors

Manufacturing sector	Increase
Food and beverage and tobacco products	3
Textile mills and textile product mills	0
Apparel and leather and allied products	–4
Wood products	–2
Paper products	2
Printing and related support activities	–2
Petroleum and coal products	14
Chemical products	2
Plastics and rubber products	0
Nonmetallic mineral products	1
Primary metals	3
Fabricated metal products	–1
Machinery	0
Computer and electronic products	0
Electrical equipment, appliances, and components	–1
Motor vehicles, bodies and trailers, and parts	0
Other transportation equipment	–1
Furniture and related products	–4
Miscellaneous manufacturing	–2

Notes: Capital is measured in millions of dollars. Output effects are in thousands of dollars per year.

Table 3. Labor's Rybczynski Effects on the Manufacturing Sectors

Manufacturing sector	Skilled labor	Unskilled labor
Food and beverage and tobacco products	0	3
Textile mills and textile product mills	1	10
Apparel and leather and allied products	−1	16
Wood products	−2	9
Paper products	1	5
Printing and related support activities	1	8
Petroleum and coal products	1	−2
Chemical products	8	1
Plastics and rubber products	5	7
Nonmetallic mineral products	0	7
Primary metals	0	6
Fabricated metal products	5	9
Machinery	11	6
Computer and electronic products	28	1
Electrical equipment, appliances, and components	9	6
Motor vehicles, bodies and trailers, and parts	6	5
Other transportation equipment	22	3
Furniture and related products	1	12
Miscellaneous manufacturing	6	6

Notes: Factors are measured in person-years. Output effects are in thousands of dollars per year.

Table 4. OLS Estimates of Factor Rewards

Factor	Reward	Newey–West standard errors
Capital	$136,408***	$25,939
Professional Occupations	$145,019**	$56,833
Education and Healthcare Occupations	−$28,744***	$6,315
Food Service and Maintenance Occupations	$21,458**	$8,908
Sales and Clerical Occupations	$64,733***	$21,540
Production and Transportation Occupations	$46,829***	$9,696

Notes: Capital is measured in millions of dollars. The standard errors use the Newey–West correction with $k = 3$. All other factors are measured in person-years. The regression $R^2 = 0.935$, and the number of observations $n = 63$.
** Denotes significance at 5%; *** denotes significance at 1%.

Table 5. Selected Stolper–Samuelson Effects on Some Specific Factors

This sector (GDP share)	Maximal effect on capital specific to this sector	Reward	Minimal effect on capital specific to this sector	Reward
Real estate (11%)	Educational services	205	Legal services	−293
State and local government (10%)	State and local government	304	Miscellaneous professional, scientific, and technical services	−4
Retail trade (8%)	Retail trade	1340	Construction	−489
Food, beverage, and tobacco products (3%)	Food, beverage, and tobacco products	2743	Furniture and related products	−208
Petroleum and coal products (1%)	Petroleum and coal products	2299	Truck transportation	−128

Notes: Output prices are measured in millions of dollars. Factor rewards are measured in thousands of dollars per year.