

# THE STRUCTURE OF THE FREE BOUNDARY FOR LOWER DIMENSIONAL OBSTACLE PROBLEMS

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ABSTRACT. We study the regularity of the “free surface” in boundary obstacle problems. We show that near a non-degenerate point the free boundary is a  $C^{1,\alpha}$   $(n-2)$ -dimensional surface in  $\mathbb{R}^{n-1}$ .

## 1. INTRODUCTION

The purpose of this paper is to study the structure and regularity of the free boundary in “boundary obstacle” problems. Boundary obstacle problems concern the following question.

We are given a smooth  $\Omega$  in  $\mathbb{R}^n$ ,  $n \geq 3$ , and seek a function  $u$  that:

- a) In the interior of  $\Omega$ ,  $u$  satisfies a nice, elliptic equation, say  $\Delta u = f$ .
- b) Along the boundary of  $\Omega$ , instead of giving Dirichlet or Neumann conditions we prescribe “complementary conditions” of the following type.

As long as  $u$  is bigger than some prescribed function  $\varphi$ , there is no flux across  $\partial\Omega$ :  $u_\nu = 0$ . But as soon as  $u$  becomes equal to  $\varphi$ , boundary flux,  $u_\nu$ , is turned on ( $u_\nu > 0$ ) to keep  $u$  above  $\varphi$ .

This type of problem arises in elasticity (the Signorini problem) when an elastic body is at rest, partially laying on a surface, in optimal control of temperature across a surface (see [F], [A]), in the modelling of semipermeable membranes where some saline concentration can flow through the membrane only in one direction (see Duvaut-Lions [DL]) and in financial math when the random variation of underlying asset changes in a discontinuous fashion (a Levi process) (see [S] and references there).

There is considerable literature on the regularity properties of the solution (see [F], [C], [R], [U]). In particular, two of the authors proved recently (see [AC2]) the optimal regularity of solutions to such a problem.

This opens the way to study the properties of the interface by using geometric P.D.E. techniques. This is precisely what we develop in this paper. We show that there is one basic global non-degenerate profile after blow up, and that in a neighborhood of a point that has this profile the free boundary is a  $C^{1,\alpha}$  “curve” on the boundary (i.e., an  $n-2$  dimensional graph on the  $n-1$  dimensional boundary).

Simple examples show that singular free boundary points and degenerate profiles are unavoidable. For simplicity, in this paper we only treat the case in which  $\partial\Omega$  is locally a hyperplane and  $f, \varphi \equiv 0$ .

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## 2. DESCRIPTION OF THE PROBLEM AND KNOWN RESULTS

In this section we explain exactly which kind of problem we shall deal with and we recall some known results.

Let  $B_1 = B_1(0)$  the unit ball in  $\mathbb{R}^n$ ,  $n \geq 2$ ; we write points  $x \in \mathbb{R}^n$  as  $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  and denote by  $\Pi$  the hyperplane  $\{(x', x_n) : x_n = 0\}$ .

Given a smooth function  $\varphi$  on  $\partial B_1$  we look at the unique minimizer  $u$  of the Dirichlet integral

$$J(v) = \int_{B_1} |\nabla v|^2$$

over the closed convex set

$$\mathbb{K} = \{v \in H^1(B_1), v = \varphi \text{ on } \partial B_1, v(x', 0) \geq 0\}$$

The minimizer  $u$  can be constructed also as the least superharmonic function in  $\mathbb{K}$ . To have a nontrivial coincidence set  $\Lambda(u) = \{(x', 0) : u(x', 0) = 0\}$  and a nontrivial free boundary  $F(u)$ , the boundary of the set  $\{u \geq \varphi\}$  on  $\Pi \cap B_1$ , we assume that  $\varphi$  changes sign and that  $\varphi(\theta', 0) > 0$ ,  $\theta' \in \partial B_1' = \Pi \cap \partial B_1$ . Without losing generality we can choose  $\varphi$  symmetric with respect to the hyperplane  $\Pi$  so that,  $u$  also is symmetric with respect to  $\Pi$  (otherwise we can symmetrize without changing the coincidence set).

The solution  $u$  is harmonic in  $B_1 \setminus \Lambda(u)$ , it is globally Lipschitz continuous, and (see [C])

$$\|u\|_{\text{Lip}(B_{1/2})} \leq C \|u\|_{L^2(B_1)} \quad (1)$$

Moreover,

$$\inf_{B_{1/2}} u_{\tau\tau} \geq -C \|u\|_{L^2(B_1)} \quad (2)$$

for every direction  $\tau$  on  $\Pi$ . We will call *tangential* such directions. Inequality (2) expresses semiconvexity of  $u$  along tangential directions.

The optimal regularity of  $u$ , proven in [AC2], is  $C^{1,1/2}$  on either side of  $\Pi$ , and

$$\|u\|_{C^{1,1/2}(B_{1/2}^\pm)} \leq C \|u\|_{L^2(B_1)}. \quad (3)$$

Furthermore,  $u_{x_n} = 0$  on  $\{(x', 0) : u(x', 0) > 0\}$  and  $u_{x_n}(x', 0+) \leq 0$  on  $\Lambda(u)$ .

In this paper we want to examine the structure of the free boundary  $F(u)$  (clearly in dimension  $n \geq 3$ ) through the analysis of asymptotic profiles around one of its points, that we assume to be the origin.

It turns out that only in correspondence to a specific asymptotic profile (that we call nondegenerate) it is possible to achieve smoothness of  $F(u)$ . To get a clue of what happens let us start with an observation of Hans Lewy in dimension 2.

The complex function  $w = u_x - iu_y$  ( $x_n = y$ ) is analytic outside  $\Lambda(u)$ , thus

$$u_x u_y = -\frac{1}{2} \text{Im}(w^2)$$

is harmonic and vanishes on  $y = 0$ . Thus  $u_x u_y$  has a harmonic odd extension across  $y = 0$  and  $w^2$  has an analytic extension. Then  $w$  is  $C^{1/2}$  and  $u \in C^{1,1/2}$ , which is indeed the optimal regularity. Accordingly, the first admissible nontrivial global solution is  $u_0(x) = \rho^{3/2} \cos \frac{3}{2}\theta$  and this is the typical nondegenerate asymptotic profile. On the other hand there are solutions like  $\rho^{k+1/2} \cos((k+1/2)\theta)$ ,  $k \in \mathbb{N}$ ,  $k > 1$ , or  $\rho^{2k} \cos 2k\theta$ ,  $k \geq 1$ , with higher order asymptotic behavior.

In correspondence to points with these asymptotic profiles the free boundary could be very narrow or a singular point. Notice that these 2-dimensional solutions can be considered as  $n$ -dimensional solutions, constant with respect to the other  $n-2$  variables, so that analogous considerations can be made in any dimension.

### 3. MONOTONICITY FORMULAS

In this section we prove some monotonicity formulas that play a crucial role in the identification of limiting blow-up profiles.

**Lemma 1** (Almgren's frequency formula). *Let  $u$  be a continuous function on  $\bar{B}_r$ , harmonic in  $B_r \setminus \Lambda(u)$ ,  $u(0) = 0$ ,  $u(x', 0) \cdot u_{x_n}(x', 0) = 0$ . Define, for  $0 < r < 1$ ,*

$$D_r(u) = r \frac{\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2 d\sigma} \equiv r \frac{V_r}{S_r}.$$

Then, for  $0 < r \leq \frac{1}{2}$ ,  $D'_r(u) \geq 0$  ( $' = \frac{d}{dr}$ ). Moreover, let

$$\mu = \lim_{r \rightarrow 0^+} D_r(u).$$

then  $D'_r(u) \equiv 0$  in  $(0, \frac{1}{2})$  if and only if

$$u(x) = |x|^\mu g(\theta) \quad \theta \in \partial B_1$$

and  $\mu \geq \frac{3}{2}$ .

*Proof.* We have

$$\log D_r = \log r + \log V_r - \log S_r$$

and

$$\frac{d}{dr} \log D_r = \frac{1}{r} + \frac{V'_r}{V_r} - \frac{S'_r}{S_r}.$$

By rescaling, it is enough to show that

$$1 + \frac{V'_1}{V_1} - \frac{S'_1}{S_1} \geq 0 \tag{4}$$

that is ( $\nu$  exterior normal)

$$2 - n + \frac{\int_{\partial B_1} |\nabla u|^2 d\sigma}{\int_{B_1} |\nabla u|^2} - 2 \frac{\int_{\partial B_1} u u_\nu d\sigma}{\int_{\partial B_1} u^2 d\sigma} \geq 0. \tag{5}$$

Since  $u(x', 0) = u_{x_n}(x', 0)$  we get, after an integration by parts,

$$\int_{B_1} |\nabla u|^2 = \frac{1}{2} \int_{B_1} \Delta(u^2) = \int_{\partial B_1} u u_\nu d\sigma.$$

To control  $\int_{\partial B_1} |\nabla u|^2 d\sigma$  we use the divergence theorem in  $B_1 \setminus \Lambda(u)$ . Let

$$h(x) = \operatorname{div}[x|\nabla u|^2 - 2(x \cdot \nabla u)\nabla u].$$

Notice that, in our case

$$h(x) = (n-2)|\nabla u|^2.$$

From Gauss formula, we have (using that on  $\Lambda$   $u_\nu$  vanishes continuously)

$$(n-2) \int_{B_1} |\nabla u|^2 = \int_{B_1} h = \int_{\partial B_1} |\nabla u|^2 d\sigma - 2 \int_{\partial B_1} u_\nu^2 d\sigma. \tag{6}$$

By inserting (6) into (5) we obtain

$$1 + \frac{V'_1}{V_1} - \frac{S'_1}{S_1} = 2 \frac{\int_{\partial B_1} u_\nu^2 d\sigma}{\int_{\partial B_1} u u_\nu d\sigma} - 2 \frac{\int_{\partial B_1} u u_\nu d\sigma}{\int_{\partial B_1} u^2 d\sigma} \geq 0$$

by Schwarz inequality. The equality sign in  $D'_r(u) = 0$  holds for  $0 < r \leq \frac{1}{2}$  if and only if  $u$  is proportional to  $u_\nu$  on  $\partial B_r$  for every  $r$ , which implies  $u$  is of the form

$$u(x) = h(|x|)g(\theta) \quad \theta \in \partial B_1 .$$

From the radial formula of the Laplace operator, in a neighborhood of any point where  $u \neq 0$ , it must be

$$h(|x|) = |x|^\mu .$$

In fact, by unique continuation,  $\mu$  must be the same for all components of  $B_1 \setminus \Lambda(u)$  where  $g$  has constant sign. Thus, each connected component of the region where  $u$  is harmonic is a cone, generated by the support of  $g$ . Finally, from optimal regularity, it must be  $\mu \geq \frac{3}{2}$ .  $\square$

An important consequence is the following result.

**Lemma 2.** *Let  $u$  and  $\mu$  as in Lemma 1, and*

$$\varphi(r) = \int_{\partial B_r} u^2 d\sigma \quad 0 < r \leq 1 .$$

*Then we have:*

(a) *The function*

$$r \mapsto r^{-2\mu} \varphi(r)$$

*is increasing; moreover*

$$\frac{d}{dr} [r^{-2\mu} \varphi(r)] \equiv 0$$

*in  $(0, 1)$  if and only if*

$$u(x) = |x|^\mu g(\theta) \quad \theta \in \partial B_1$$

*with  $\mu \geq \frac{3}{2}$ .*

(b) *Let  $0 < r < R \leq 1$ ; given  $\varepsilon > 0$ , for  $r \leq r_0(\varepsilon)$*

$$\varphi(R) \leq \left(\frac{R}{r}\right)^{2(\mu+\varepsilon)} \varphi(r) . \tag{7}$$

*Proof.* (a) We have

$$\varphi'(r) = \frac{d}{dr} \int_{\partial B_r} u^2 = 2 \int_{\partial B_r} u u_\nu d\sigma = 2r \int_{B_r} |\nabla u|^2 \tag{8}$$

so that

$$\frac{d}{dr} [r^{-2\mu} \varphi(r)] = 2r^{-2\mu-n} \left\{ r \int_{B_r} |\nabla u|^2 - \mu \int_{\partial B_r} u^2 d\sigma \right\}$$

and (a) follows from the frequency formula.

(b) Let  $r_0 = r_0(\varepsilon)$  such that  $D_r(u) \leq \mu + \varepsilon$ . From (8)

$$D_r(u) = \frac{r}{2} \frac{d}{dr} \log \varphi(r) \leq \mu + \varepsilon$$

and (7) follows by integrating over  $(r, R)$ .  $\square$

## 4. LIMITING PROFILES

Given a solution  $u$  of our thin obstacle problem, we consider the blow-up family

$$v_r(x) = \frac{u(rx)}{(\int_{\partial B_r} u^2)^{1/2}} .$$

If  $\mu = \lim_{r \rightarrow 0^+} D_r(u)$ , our purpose is to identify the limit of  $v_r$  as  $r \rightarrow 0$  when  $\frac{3}{2} \leq \mu \leq 2$ .

Observe that

$$\|v_r\|_{L^2(\partial B_1)} = 1 \tag{9}$$

and, from Lemma 2

$$\|v_r\|_{L^2(B_R)} \leq R^{(\mu+\varepsilon)}$$

for every  $R > 1$  and every small  $r$  such that  $rR \leq r_0(\varepsilon)$ . Thus, there exists a sequence  $v_j = v_{r_j}$  such that

$$v_j \rightarrow v_0 \tag{10}$$

in  $L^2$  and uniformly on every compact set in  $\mathbb{R}^n$ . Because of (9),  $v_0$  is a nontrivial global solution.

Since

$$D_{r_j}(u) = D_1(v_{r_j}) \rightarrow D_1(v_0) = \mu$$

as  $r_j \rightarrow 0$ , from Lemma 1 we deduce that

$$v_0(x) = |x|^\mu g(\theta) \quad \theta \in \partial B_1 .$$

**Theorem 3** (blow up limits). *Let  $u$  be a solution of our thin obstacle problem,  $\mu = \lim_{r \rightarrow 0^+} D_r(u)$  and  $v_0$  the global solution defined above. The following hold:*

- (a) *Assume  $\frac{3}{2} \leq \mu < 2$ . Then, up to a multiplicative constant, in a suitable system of coordinates*

$$v_0(x) = \rho^{3/2} \cos \frac{3}{2} \psi$$

where  $\rho^2 = x_{n-1}^2 + x_n^2$  and  $\tan \psi = x_n/x_{n-1}$ .

- (b) *Assume  $\mu = 2$ . Then  $v_0$  is a quadratic polynomial*

$$v_0(x) = \sum_{i < n} a_i x_i^2 - c x_n^2, \quad a_i \geq 0 .$$

*Proof.* Let  $\frac{3}{2} \leq \mu < 2$ . From the tangential quasi-convexity property of  $u$ , we have, for every tangential direction  $\tau$ :

$$D_{\tau\tau} v_{r_j} \geq -c \frac{r_j^2}{(\int_{\partial B_{r_j}} u^2)^{1/2}} . \tag{11}$$

In Lemma 2(b), choose  $\varepsilon$  such that  $\mu + \varepsilon < 2$ . Then, letting  $r_j \rightarrow 0$  in (11) we obtain from (7)

$$D_{\tau\tau} v_0 \geq 0$$

so that  $v_0$  is tangentially convex and  $\Lambda(v_0)$  is a convex cone. We first observe that on  $\Lambda(v_0)$ ,  $v_0 \equiv 0$ , and  $D_n v_0 \leq 0$  and for  $x_n \geq 0$ ,  $D_{nn} v_0 \leq 0$ . This implies that  $v_0(x', x_n) \leq 0$  if  $(x', 0)$  belongs to  $\Lambda_0$ . Assume now that the vector  $-e_{n-1}$  belongs to  $(\Lambda(v_0))^0$ . For any point  $x$ , consider the line  $L_x = \{x + te_{n-1}\}$ . For  $t$  negative enough the function  $v_0(x + te_{n-1})$  becomes negative from the remark above.

Since  $v_0$  is convex along  $L_x$ , it follows that  $w = D_{e_{n-1}} v$  cannot be negative anywhere on  $L_x$ . In particular, since  $x$  is arbitrary  $w \geq 0$  in  $\mathbb{R}^n$ . On the other hand,  $w = 0$  on  $\Lambda(v_0)$

and  $w_{x_n} = 0$  on  $\{x_n = 0\} \setminus \Lambda(v_0)$  (by symmetry). Thus, the restriction of  $w$  to the unit sphere must be the first eigenfunction of the Dirichlet problem for the spherical Laplacian, with zero data on  $\partial B_1 \cap \Lambda(v_0)$ . Now, if  $\Lambda(v_0)$  is not a half-plane, and thus from convexity, is strictly contained in half a plane. Then the homogeneity degree of  $w$  should be less than  $1/2$  (see [AC2]), since homogeneity  $1/2$  corresponds to the case in which half a plane is removed, contradicting  $\mu \geq \frac{3}{2}$ . Therefore  $\Lambda(v_0)$  is a half-plane,  $w(x) = \rho^{1/2} \cos \frac{\psi}{2}$  where  $\rho^2 = x_{n-1}^2 + x_n^2$ , and  $\tan \psi = x_n/x_{n-1}$ . This implies

$$v_0(x) = \frac{2}{3} \rho^{3/2} \cos \frac{3}{2} \psi . \quad (12)$$

Observe that if  $\tau = \alpha e_{n-1} + \beta e$ , where  $e$  is tangential,  $e \perp e_{n-1}$  and  $\alpha^2 + \beta^2 = 1$ ,  $\alpha > 0$ , then outside a  $\eta$ -strip,  $|x_n| < \eta$ , we have in  $B_1$ , say

$$D_\tau v_0(x) \geq C(\alpha, \eta) > 0 . \quad (13)$$

Let now  $\mu = 2$ . The limiting profile is of the form  $v_0(x) = |x|^2 g(\theta)$ ,  $\theta \in \partial B_1$  and  $\Lambda(v_0)$  is a cone. Consider  $w = D_{x_n} v_0$ ;  $w$  is linearly homogeneous and  $w = 0$  on  $\{x_n = 0\} \setminus \Lambda(v_0)$ . We reflect evenly with respect to the hyperplane  $x_n = 0$ , defining

$$\tilde{w}(x) = \begin{cases} w(x', x_n) & x_n > 0 \\ w(x', -x_n) & x_n < 0 \end{cases} .$$

Suppose  $\tilde{w}$  changes sign. Then, since  $\tilde{w}$  is harmonic on its support and  $w(0) = 0$ , we can apply the monotonicity formula in [CS, Corollary 12.4], to  $w^+$  and  $w^-$ . According to this formula, the homogeneity and the linear behavior of  $\tilde{w}$  forces  $\tilde{w}$  to be a two plane solution with respect to a direction transversal to the plane  $x_n = 0$ , say,  $\tilde{w}(x) = \alpha x_{n-1}^+ - \beta x_{n-1}^-$ , due to the even symmetry of  $\tilde{w}$ . This is a contradiction since along  $x_n = 0$ ,  $w$  is negative on  $\Lambda_0$  and zero otherwise and therefore  $\tilde{w}$  cannot change sign. Suppose now that  $\Lambda(v_0)$  has non-empty interior. Then  $\tilde{w}$  is the first eigenfunction for the spherical Laplacian, with zero boundary data on  $(\{x_n = 0\} \setminus \Lambda(v_0)) \cap \partial B_1$ . This forces a superlinear behavior of  $\tilde{w}$  at the origin since linear behavior corresponds to a half sphere and we reach again a contradiction. Thus,  $\Lambda(v_0)$  has empty interior,  $v_0$  is harmonic across  $\Lambda(v_0)$  and therefore  $v_0$  must coincide with a quadratic polynomial ( $v_0(x) = \sum_{i < n} a_i x_i^2 - C x_n^2$ ,  $a_i \geq 0$ ).  $\square$

## 5. LIPSCHITZ CONTINUITY OF THE FREE BOUNDARY ( $\mu < 2$ )

Through the identification of the limiting profile in section 4, we can prove that, when  $\frac{3}{2} \leq \mu < 2$ , the free boundary  $F(u)$  is locally a Lipschitz graph. Precisely:

**Lemma 4.** *Let  $u$  be a solution of the thin obstacle problem in  $B_1$ . Assume that  $\frac{3}{2} \leq \mu < 2$ . Then, there exists a neighborhood of the origin  $B_\rho$  and a cone of tangential directions  $\Gamma'(e_{n-1}, \theta)$ , with axis  $e_{n-1}$  and opening  $\theta \geq \frac{\pi}{3}$  (say), such that, for every  $\tau \in \Gamma'(e_{n-1}, \theta)$ , we have*

$$D_\tau u \geq 0 .$$

*In particular, in that neighborhood,  $F(u)$  is the graph of a Lipschitz function  $x_{n-1} = f(x_{n-1}, \dots, x_{n-2})$ .*

*Proof.* From the definition in section 4,

$$v_{r_j}(x) = \frac{u(r_j x)}{\int_{\partial B_{r_j}} (u^2)^{1/2}} .$$

We know that  $v_{r_j}(x) \rightarrow v_0(x)$  with  $v_0$  given by (12), uniformly on compact sets. Fix  $\alpha \geq \frac{1}{2}$  (say) and let  $\tau = \alpha e_{n-1} + \beta e$  be a tangential direction ( $\alpha^2 + \beta^2 = 1$ ). For  $\sigma > 0$ , small, and  $r_j \leq r_0(\sigma)$ , we deduce from (13) that  $D_\tau v_{r_j}$  enjoys the following properties in  $B_{5/6}$ .

- (i)  $D_\tau v_{r_j} \geq 0$  outside the strip  $|x_n| < \sigma$ ;
- (ii)  $D_\tau v_{r_j} \geq c_0 > 0$  for  $|x_n| \geq \frac{1}{2}$ ;
- (iii)  $D_\tau v_{r_j} \geq -c\sigma^{1/2}$  in the strip  $|x_n| < \sigma$

(from (i) and optimal regularity).

Then, we conclude the proof by applying to  $h = D_\tau v_{r_j}$  the following approximation Lemma.  $\square$

**Lemma 5.** *Let  $u$  be a solution of the thin obstacle problem in  $B_1$ . Suppose  $h$  is a continuous function with the following properties:*

- (i)  $\Delta h \leq 0$  in  $B_1 \setminus \Lambda(u)$ ;
- (ii)  $h \geq 0$  for  $|x_n| \geq \sigma$ ,  $h = 0$  on  $\Lambda(u)$ , with  $\sigma > 0$ , small;
- (iii)  $h \geq c_0 > 0$  for  $|x_n| \geq \frac{1}{8(n-1)}$ ;
- (iv)  $h \geq -\omega(\sigma)$ , where  $\omega$  is the modulus of continuity of  $h$ , for  $|x_n| < \sigma$ .

There exists  $\sigma_0 = \sigma_0(n, c_0, \omega)$  such that, if  $\sigma \leq \sigma_0$  then  $h \geq 0$  in  $B_{1/2}$ .

*Proof.* Suppose  $z = (z', z_n) \in B_{1/2}$  and  $h(z) < 0$ . Let

$$Q = \left\{ (x', x_n) : |x' - z'| \leq \frac{1}{3}, |x_n| < \frac{1}{4(n-1)} \right\}$$

and

$$P(x', x_n) = |x' - z'|^2 - (n-1)x_n^2.$$

Define

$$v(x) = h(x) + \delta P(x)$$

where  $\delta > 0$  is to be chosen later. We have

- (a)  $v(z) = h(z) - \delta(n-1)z_n^2 < 0$
- (b)  $\Delta v \leq 0$  outside  $\Lambda(u)$
- (c)  $v \geq 0$  on  $\Lambda(u)$ , since  $h \geq 0$ ,  $P \geq 0$  there.

Thus,  $v$  must have a negative minimum on  $\partial Q$ .

On  $\partial Q \cap \{|x_n| > 1/8(n-1)\}$ ,

$$v \geq c_0 - \frac{\delta}{16(n-1)} \geq 0$$

if  $\delta \leq 16(n-1)c_0$ .

On  $|x' - z'| = 1/3$ ,  $\sigma \leq |x_n| \leq 1/8(n-1)$ , we have  $h \geq 0$  so that

$$v \geq \delta \left[ \frac{1}{9} - \frac{1}{64(n-1)} \right] \geq 0.$$

Finally, on  $|x' - z'| = 1/3$ ,  $|x_n| < \sigma$ , we have

$$v \geq -c\omega(\sigma) + \delta \left[ \frac{1}{9} - (n-1)\sigma^2 \right] \geq 0$$

if  $\sigma$  is small enough.

Hence,  $v \geq 0$  on  $\partial Q$  and we have reached a contradiction. Therefore  $h \geq 0$  in  $B_{1/2}$ .  $\square$

6. BOUNDARY HARNACK PRINCIPLES AND THE  $C^{1,\alpha}$  REGULARITY OF THE FREE  
BOUNDARY ( $\mu < 2$ )

We are now in position to show that the free boundary is locally a  $C^{1,\alpha}$  graph, if  $\mu < 2$ . Precisely, our main result is the following.

**Theorem 6.** *Let  $u$  be a solution of the thin obstacle problem in  $B_1$ . If  $\mu = \lim_{r \rightarrow 0^+} D_r(u) < 2$  then the free boundary  $F(u)$  is given in a neighborhood of the origin by the graph of a  $C^{1,\alpha}$  function  $x_{n-1} = f(x_1, \dots, x_{n-2})$ .*

One way to prove the theorem is to use the results in [AC1]. Through a bilipschitz transformation, a neighborhood of the origin in  $B_1 \setminus \Lambda(u)$  is mapped onto the upper half ball, say,  $B^+ = \{|z| < 1, z_{n-1} > 0\}$ , and the Laplace operator is transformed into a uniformly elliptic divergence form operator. Each tangential derivative  $D_\tau u$ , with  $\tau$  belonging to the cone  $\Gamma'(e_{n-1}, \theta)$  of monotone directions, is mapped onto a positive solution of  $\mathcal{L}v = 0$  in  $B^+$ , vanishing on  $\{z_{n-1} = 0\}$ . An application of Corollary 1 in [AC1] concludes the proof.

On the other hand, there is a more direct proof based on the following result, that could be of interest in itself.

Let  $D$  be a subdomain of  $B_1$  and let  $\Omega = \partial D \cap B_1$ . We denote by  $d_g(x, y)$  the geodesic distance in  $D$  of the points  $x, y$ . We will assume that the following properties hold:

- (1) For every  $x, y \in D$ ,  $d_g(x, y)$  is finite.
- (2) **Non tangential ball condition.** Let  $Q \in \Omega$ . There exist positive numbers  $r_0 = r_0(D, Q)$  and  $\eta = \eta(D)$  such that, for every  $r \leq r_0$  there is a point  $A_r(Q) \in B_r(Q)$  such that

$$B_{\eta r}(A_r(Q)) \subset B_r(Q) \cap D .$$

- (3) **Harnack chain condition.** There exists a constant  $M = M(D)$  such that, for all  $x, y \in D$ ,  $\varepsilon > 0$  and  $k \in \mathbb{N}$  satisfying

$$d(x, \Omega) > \varepsilon , \quad d(y, \Omega) > \varepsilon , \quad d_g(x, y) < 2^k \varepsilon ,$$

there is a sequence of  $Mk$  balls  $B_{r_1}, \dots, B_{r_{Mk}} \subset D$  with

$$x \in B_{r_1}, \quad y \in B_{r_{Mk}}, \quad B_{r_j} \cap B_{r_{j+1}} \neq \emptyset \quad (j = 1, \dots, Mk - 1)$$

and

$$\frac{1}{2}r_j < d(B_{r_j}, \Omega) < 4r_j \quad (j = 1, \dots, Mk) .$$

- (4) **Uniform capacity condition.** Let  $Q \in \Omega$ . There exist positive numbers  $r_0 = r_0(D, Q)$  and  $\gamma = \gamma(D)$  such that, for every  $r \leq r_0$ ,

$$\text{cap}_\Delta [(B_r(Q) \setminus B_{r/2}(Q)) \cap \Omega] \geq \gamma r^{n-2}$$

where  $\text{cap}_\Delta(K)$  is the capacity of  $K$  in  $B_1$ , with respect to the Laplace operator.

Conditions (2) and (3) appear in the notion of non tangentially accessible domain (see [JK]). Condition (4) replaces the exterior tangential ball property in that definition. Since condition (4) is related to the Laplace operator we call a domain  $D$  with properties (1)–(4) a  $\Delta$ -N.T.A. domain. A simple example of  $\Delta$ -N.T.A. domain is an  $(n-1)$ -dimensional smooth manifold with Lipschitz boundary.

Let now  $\mathcal{L}$  be a uniformly elliptic operator with ellipticity constant  $\lambda$  and bounded measurable coefficients. Recall that  $\text{cap}_{\mathcal{L}}(K) \sim \text{cap}_\Delta(K)$  with constant depending only on  $\lambda$  and  $n$ , so that the notion of  $\Delta$ -N.T.A. domains is actually related to an entire class of operators. The following result holds.



**Theorem 7.** (Boundary Harnack Principles). *Let  $D \subset B_1$  be a  $\Delta$ -N.T.A. domain. Suppose  $v$  and  $w$  are positive functions in  $D$ , continuously vanishing on  $\Omega$ , satisfying  $\mathcal{L}v = \mathcal{L}w = 0$  in  $D$ . Assume  $x_0 \in D \cap B_{2/3}$ ,  $d(x_0, \Omega) = d_0 > 0$  and  $v(x_0) = w(x_0) = 1$ . Then:*

(a) For every  $Q \in \Omega \cap B_{1/2}$

$$\sup_{D \cap B_{1/3}(Q)} v \leq C(u, d, d_0, D) \quad (\text{Carleson estimate})$$

and

$$\sup_{D \cap B_{1/3}(Q)} \frac{v}{w} \leq C(u, \lambda, d_0, D) .$$

(b)  $\frac{v}{w}$  is Hölder continuous in  $B_{1/2} \cap D$  up to  $\Omega$ .

*Proof.* the proof follows by now standard lines (see for instance [CS, section 11.2] and [JK]). We sketch the main steps emphasizing the main differences.

(a) Fix  $Q \in \Omega \cap B_{1/2}$  and let

$$v(y_0) = N = \sup_{B_{1/3}Q \cap D} v .$$

The interior ball condition and the Harnack chain condition plus the interior Harnack inequality imply that if  $N$  is large,  $d(y_0, \Omega) \equiv |y_0 - Q_0| \leq N^{-\varepsilon}$  where  $\varepsilon = \varepsilon(n, \lambda, d_0, D) > 0$ . Let  $r_0 = d(y_0, \Omega)$ .

The uniform capacity condition implies that

$$\sup_{B_{2r_0}(Q_0)} v \equiv v(y_1) \geq CN$$

where  $C = C(n, \lambda, D) > 1$ . Iterating the process, one constructs a sequence of points  $y_k$ , satisfying

- i)  $v(y_k) \geq C^k N$
- ii)  $d(y_k, \Omega) \leq (C^k N)^{-\varepsilon}$
- iii)  $|y_k - y_{k+1}| \leq 4(C^k N)^{-\varepsilon}$ .

If  $N$  is large enough, we can make

$$\sum |y_k - y_{k+1}| \leq \frac{1}{16}$$

and we get a contradiction. This proves (a). To prove (b), let  $P \in B_{1/3}(Q) \cap \Omega$  and  $R_0 = d(x_0, P) + \frac{d_0}{2}$ . Notice that  $\frac{3}{2}d_0 \leq R_0 \leq 1$ . Define

$$\psi_{R_0}(P) = B_{R_0}(P) \cap D$$

and

$$\Sigma_0 = \partial B_{R_0}(P) \cap B_{d_0}(x_0) .$$

Observe that  $\Sigma_0 \subset D$ . We first control the Green's function  $G(x, x_0)$  for  $\mathcal{L}$  in  $\psi_{R_0}(P)$  from above by the  $\mathcal{L}$ -harmonic measure  $\omega_{\mathcal{L}}^x(\Sigma_0)$ , in  $\psi_{R_0}(P) \setminus B_{d_0/3}(x_0)$ . This follows from the maximum principle. In fact, on  $\partial B_{d_0/3}(x_0)$  we have, from Hölder continuity,

$$\omega_{\mathcal{L}}^x(\Sigma_0) \geq c > 0$$

and

$$G(x, x_0) \leq c d_0^{2-n} .$$

On the other hand, on  $\partial\psi_{R_0}(P)$ , we have  $G(x, x_0) = 0$  and  $\omega_{\mathcal{L}}^x(\Sigma_0) \geq 0$ . Therefore, outside  $B_{d_0/3}(x_0)$  we get

$$G(x, x_0) \leq c d^{2-n} \omega_{\mathcal{L}}^x(\Sigma_0) . \quad (14)$$

Let now  $\Sigma_1 = \partial\psi_{R_0}(P) \setminus \Omega$  and let  $\varphi$  be a  $C^\infty$  cut-off function such that  $\varphi \equiv 0$  in  $B_{R_0/4}(P)$ ,  $\varphi \equiv 1$  outside  $B_{R_0/2}(P)$  and  $0 \leq \varphi \leq 1$  in  $B_{R_0/2}(P) \setminus B_{R_0/4}(P) \equiv C_{R_0}(P)$ .

We have

$$\omega_{\mathcal{L}}^x(\Sigma_1) \leq \int_{\partial\psi_{R_0}(P)} \varphi d\omega_{\mathcal{L}}^x .$$

Fix  $x \in B_{R_0/8}(P) \cap D$ . Then

$$0 = \varphi(x) = \int_{\partial\psi_{R_0}(P)} \varphi d\omega_{\mathcal{L}}^x - \int_{C_{R_0}(P) \cap D} a_{ij}(x) D_{y_i} G(x, y) D_{y_j} \varphi(y) dy .$$

Therefore, from Caccioppoli estimate and Carleson estimate, we have, in  $B_{R_0/8}(P) \cap D$ ,

$$\begin{aligned} \omega_{\mathcal{L}}^*(\Sigma_1) &= c(u, \lambda, d_0) \left( \int_{C_{R_0}(P) \cap D} |\nabla_y G(x, y)|^2 dy \right)^{1/2} \\ &\leq c(u, \lambda, d_0) \left( \int_{C_{R_0}(P) \cap D} G^2(x, y) dy \right)^{1/2} \\ &\leq c(u, \lambda, d_0) G(x, x_0) . \end{aligned} \quad (15)$$

From (14) and (15) we obtain the following doubling condition for the  $\mathcal{L}$ -harmonic measure:

$$\omega_{\mathcal{L}}^x(\Sigma_1) \leq c(u, \lambda, d_0) \omega_{\mathcal{L}}^x(\Sigma_0)$$

for every  $x \in B_{R_0/8}(P) \cap D$ .

The rest of the proof of (a) and the proof of (b) follow now, for instance, as in [CS, section 11.2].  $\square$

*Proof of Theorem 6.* We apply Theorem 7 with  $\Omega = \Lambda(u)$ ,  $D = B_1 \setminus \Lambda(u)$  and  $v = D_\tau u$ ,  $w = D_{e_{n-1}} u$  where  $\tau \in \Gamma'(e_{n-1}, \theta)$ . We obtain, in particular, that on  $\{x_n = 0\} \setminus \Lambda(u)$ , the quotient  $D_\tau u / D_{e_{n-1}} u$  is Hölder continuous up to  $F(u)$  in a neighborhood of the origin. This implies that the level sets in  $\mathbb{R}^{n-1}$  of  $u$  are  $C^{1,\alpha}$  surfaces and, in particular, the  $C^{1,\alpha}$  regularity of  $F(u)$  in  $B_{1/2}$ .  $\square$

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