ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 1, Spring 1992

THE STRUCTURE OF THE SET OF IDEMPOTENTS IN A BANACH ALGEBRA

BY

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Abstract

We study here the algebraic, geometric, and analytic structure of the set of idempotent elements in a real or complex Banach algebra. A neighborhood of each idempotent in the set of idempotents forms the set of idempotents in a Rees product subsemigroup of the Banach algebra. Each nontrivial connected component of the set of idempotents is shown to be a generalized saddle, a type of analytic manifold. Each component is also shown to be the quotient of a (possibly infinite dimensional) Lie group by a Lie subgroup.

Introduction

By a Banach algebra we mean a real or complex Banach space X together with an associative continuous bilinear multiplication function $\cdot : X \times X \to X$. For example, X could be the algebra of continuous linear transformations on some Banach space Y and $\cdot$ could be composition. Denote by E the set to which e belongs if and only if $e \cdot e = e$. This is the set of idempotent elements of X. If $e \in E$ we denote by $C(e)$ the connected component of E which contains e.

Zemánek in [8] showed that each component $C(e)$ is arcwise connected in the case of a complex Banach algebra and showed how to connect any two members of $C(e)$ by an analytic arc in $C(e)$. He did this by showing that the component of the identity element in the group of invertible elements in X acts transitively on $C(e)$, if X has an identity element, via the action

$$g : \to g^{-1} \cdot e \cdot g.$$

This implies via standard results in the finite dimensional case that $C(e)$ is homeomorphic with the manifold G/H where G is the group of invertible elements and H is the isotrophy subgroup of e. This is not a theorem in the

Received May 29, 1990
1980 Mathematics Subject Classification (1985 Revision). Primary 46H05. Secondary 46H20, 22E65.

infinite dimensional case, but we show here that in our particular situation G/H has a natural manifold structure and that the natural map of G/H onto $C(e)$ is a homeomorphism.

Zemánek also notes that $C(e)$ is a singleton if and only if e is in the center of X. This could also be inferred from results in [3] concerning the more general setting of a semigroup with differentiable operation.

Aupetit [1] transfered Zemánek's results to the setting of a real Banach algebra. Esterle [2] showed that each pair in $C(e)$ can be connected by a polynomial arc in $C(e)$. Tremon [7] examined the degree of these polynomial paths and showed how to construct one of degree 3 between e and f if $e - f$ is invertible. He also showed that in the case of the algebra of $n \times n$ real or complex matrices any two members of $C(e)$ can be connected by a degree 3 polynomial arc in $C(e)$.

In [3], [4], [5] the set of idempotent elements in a semigroup with differentiable multiplication function is examined and it is shown that each $C(e)$ is a differentiable submanifold of the semigroup. This of course implies that each component is arcwise connected. Here, we specialize these results to the Banach algebra setting. We identify the tangent space to $C(e)$ at e with a splitting closed subspace of X and provide, for each such tangent space, a degree 3 polynomial map from the space into $C(e)$ which is a local homeomorphism from a neighborhood of 0 onto a neighborhood of e. Using these charts, we show that $C(e)$ is a generalized saddle in the sense that it is locally the union over an affine subspace of smoothly varying disjoint affine subspaces. Tremon [7] points this out in the case of the algebra of 2×2 matrices.

Algebraic description of $C(e)$

Suppose X is a Banach algebra and E is the set of idempotent elements of X. For each element e of E denote by $C(e)$ the connected component of E which contains e. We will now show that the members of $C(e)$ live locally in subparagroups of X and we will see how $C(e)$ interacts algebraically with nearby members of X. Define projections P_e and Q_e on X by

$$P_e(x) = xe - exe$$

and

$$Q_e(x) = ex - exe.$$

THEOREM 1. *The affine subspaces $e + \operatorname{im}(P_e)$ and $e + \operatorname{im}(Q_e)$ are subsets of $C(e)$. Moreover,*

$$L_e \equiv e + \operatorname{im}(P_e) = \{l : el = e \text{ and } le = l\}$$

and

$$R_e \equiv e + \mathrm{im}(Q_e) = \{r : er = r \text{ and } re = e\}.$$

Finally, $P_e \circ Q_e = Q_e \circ P_e = 0$ *so* $P_e + Q_e$ *is a projection of* X *onto a closed linear subspace of* X.

Proof. It is clear since $e \cdot P_e(x) = Q_e(x) \cdot e = 0$, $P_e(x) \cdot e = P_e(x)$, and $e \cdot Q_e(x) = Q_e(x)$ that each of P_e and Q_e is a projection of X onto a closed linear subspace of X. Moreover each of $\mathrm{im}(P_e)$ and $\mathrm{im}(Q_e)$ is a subalgebra of X with trivial multiplication. These remarks show that the members of $e + \mathrm{im}(P_e)$ and $e + \mathrm{im}(Q_e)$ are idempotents. But since these are connected sets of idempotent elements containing e they must be contained in $C(e)$. Moreover, they show that if $l = e + P_e(x)$ then $le = l$ and $el = e$ and if $r = e + Q_e(x)$ then $re = e$ and $er = r$.

Suppose $le = l$ and $el = e$. Note $P_e(l - e) = P_e(l) = le - ele = l - e$. A similar argument justifies the rest of the second assertion.

Applying P_e to $Q_e(x)$ involves multiplying on the right by e. This annialates $Q_e(x)$ and hence $P_e \circ Q_e = 0$. Similarly, $Q_e \circ P_e = 0$.

Define the function F on $\mathrm{im}(P_e + Q_e)$ by

$$F(x) = e + Q_e(x) + P_{e+Q_e(x)}(P_e(x)).$$

Using the fact that if $x \in \mathrm{im}(P_e + Q_e)$ then $P_e(x) = xe$, $Q_e(x) = ex$, and $exe = 0$ we see that

$$F(x) = e + x + xex - ex^2e - ex^2ex.$$

By Theorem 1, F maps into the set of idempotents. Its image is connected and contains e and hence is contained in $C(e)$. The next few results are aimed at showing that F is a local homeomorphism onto a neighborhood of e in $C(e)$. Since the restriction of F to each of $\mathrm{im}(P_e)$ and $\mathrm{im}(Q_e)$ is an affine homeomorphism we will thus arrive at a justification of our assertion that $C(e)$ is a generalized saddle.

We first derive results from [5] which are needed here. The arguments here are special to the present setting and are more accessible than those in [5].

THEOREM 2. *Let the functions* G_P *and* G_Q *on* $X \times \mathrm{im}(P_e)$ *and* $X \times \mathrm{im}(Q_e)$ *respectively be defined by*

$$G_P(x, y) = P_{y+e}(x)$$

and

$$G_Q(x, y) = Q_{y+e}(x).$$

Then G_P *and* G_Q *map into* $\mathrm{im}(P_e)$ *and* $\mathrm{im}(Q_e)$ *respectively. There are neighborhoods* U *of* e *in* X *and* V *and* W *of* 0 *in* $\mathrm{im}(P_e)$ *and* $\mathrm{im}(Q_e)$ *respectively so that the equations*

$$G_P(x, y) = G_Q(x, z) = 0$$

have unique solutions $(x, y) \in U \times V$ *and* $(x, z) \in U \times W$. *Finally, let* ψ *and* ϕ *be the functions defined implicitly by these equation respectively. The functions* $\Psi : x \to \psi(x) + e$ *and* $\Phi : x \to \phi(x) + e$ *are analytic retractions onto neighborhoods of* e *in* L_e *and* R_e *respectively.*

Proof. First, since the condition $l \cdot e = l$ and $e \cdot l = e$ is symmetric in e and l, by Theorem 1 it is clear that

$$\mathrm{im}(P_e) = \mathrm{im}(P_{y+e})$$

if $y \in \mathrm{im}(P_e)$. Similarly we have

$$\mathrm{im}(Q_e) = \mathrm{im}(Q_{y+e})$$

if $y \in \mathrm{im}(Q_e)$ so indeed, G_P maps into $\mathrm{im}(P_e)$ and G_Q maps into $\mathrm{im}(Q_e)$.

Since

$$D_2G_P(x, y)(z) = xz - zx(y + e) - (y + e)xz$$

we have

$$D_2G_P(e, 0)(z) = ez - ze - ez = -z \quad \textit{if } z \in \mathrm{im}(P_e).$$

Thus, since $G_p(e, 0) = P_e(e) = 0$, the assertion about the existence of U, V, and ψ is simply an application of the implicit function theorem. In fact, the iteration scheme

$$y_0 = 0 \quad \textit{and} \quad y_{n+1} = y_n + G_P(x, y_n)$$

converges to $\psi(x)$ if x is sufficiently close to e. The argument for the existence of ϕ is similar.

That the functions ψ and ϕ are analytic follows from the implicit function theorem. On the other hand, it is clear that if $y \in \mathrm{dom}(\psi) \cap \mathrm{im}(P_e)$ then, since $y + e \in L_e$ we have $G_P(y + e, y) = P_{y+e}(y + e) = 0$, we must have $\psi(y + e) + e = y + e$ by the uniqueness part of the conclusion of the implicit function theorem. Thus, Ψ is a retraction. The argument for Φ is similar.

If $e \in E$ denote by $H(e)$ the largest multiplicative subgroup of X which contains e. This is exactly the group of invertible elements in eXe, the subalgebra of X consisting of those elements for which e acts as an identity element.

NOTE 3. *The function which sends g in $H(e)$ to lg for $l \in L_e$ is an isomorphism of $H(e)$ onto $H(l)$, and the one which sends g in $H(e)$ to gr for $r \in R_e$ is an isomorphism of $H(e)$ onto $H(r)$.*

Proof. Suppose $g_i \in H(e)$ for $i = 1, 2$ and $l \in L_e$. Note that

$$lg_1g_2 = lg_1(eg_2) = lg_1(el)g_2 = lg_1lg_2$$

so $g : \to lg$ is a homomorphism. But clearly $g : \to eg$ for $g \in H(l)$ is the inverse of this homomorphism so the function in question is an isomorphism. The rest is similar.

Again let Φ and Ψ be defined on the common domain of ϕ and ψ by $\Phi(x) = \phi(x) + e$ and $\Psi(x) = \psi(x) + e$.

THEOREM 4. *There is an open set U of X containing e so that if $x \in U$ then $\Phi(x) \cdot \Psi(x) \in H(e)$ and the function θ defined by*

$$\theta(x) = \Psi(x)(\Phi(x)\Psi(x))^{-1}\Phi(x)$$

is an analytic retraction onto a neighborhood of e in $C(e)$ and $x\theta(x) = \theta(x)x \in H(\theta(x))$ if $x \in U$.

Proof. Since $H(e)$ is open in eXe, $\Phi(x)\Psi(x) \in R_e \cdot L_e \subset eXe$, and each of Ψ and Φ is continuous and sends e to e there is a neighborhood U of e in X so that if $x \in U$ then $\Phi(x) \cdot \Psi(x) \in H(e)$.

If $f \in E$ and $x \in fXf$ is within 1 of f then $x \in H(f)$ since the geometric series

$$f + (f - x) + (f - x)^2 + \cdots$$

converges to the inverse of x relative to f. By making the set U chosen above smaller we can insure that the distances from $x\Psi(x)$ to $\Psi(x)$ and from $\Phi(x)x$ to $\Phi(x)$ are both less than 1 and hence that $x\Psi(x) \in H(\Psi(x))$ and $\Phi(x)x \in H(\Phi(x))$ if $x \in U$. If we also require that U be connected we have a set which suffices for the conclusion.

To see this, suppose $x \in U$ and let $r = \Phi(x)$ and $l = \Psi(x)$. By Theorem 2, $r \in R_e$ and $l \in L_e$. By choice of U we have $rl \in H(e)$. It is clear from arithmetic that $\theta(x) = l(rl)^{-1}r \in E$. Since θ is a combination of analytic functions it is analytic. Thus, the image of θ is contained in $C(e)$.

Let $f = \theta(x)$. Then $l \cdot f = f$ and $f \cdot l = l$ so by Theorem 1, $l \in R_f = R_l$. Similarly, $L_r = L_f$.

By construction we have $xl = lxl \in H(l)$ and $rx = rxr \in H(r)$ so by note 3, $xf = x(lf) = (xl)f$ is in $H(f)$ and similarly, $fx \in H(f)$. Hence $xf = fxf = fx$ is in $H(f)$.

Now consider the situation in which x is in the domain of θ and is in E and is within 1 of $\theta(x)$. Since x commutes with $\theta(x)$ we have $x \cdot \theta(x)$ is an idempotent in both $H(x)$ and $H(\theta(x))$. Thus, $x = x\theta(x) = \theta(x)$ and the image of θ fills a neighborhood of e in $C(e)$. It follows from this that θ is a retraction.

Remark. Esterle in [1] showed how to use Kovarik's construction from [6] to construct from r and l, arbitrary idempotents within 1 of each other, an idempotent e so that $r \in R_e$ and $l \in L_e$. The construction is based on functional calculus and spectral theory.

In this connection we also note that under suitable circumstances the idempotent $\theta(x)$ is exactly the one obtained by integrating the resolvant of x around a small circle in the complex plane which encloses the number 1 but not the number 0.

One way to construct a semigroup with differentiable multiplication is the following. Suppose that G is a Lie group, each of L and R is a differentiable manifold, and s is a differentiable function from $R \times L$ into G. Then the multiplication

$$(l, g, r) \cdot (\hat{l}, \hat{g}, \hat{r}) \equiv (l, gs(r, \hat{l})\hat{g}, \hat{r})$$

is associative and differentiable on the differentiable manifold $L \times G \times R$. The semigroup constructed this way is called a paragroup or Rees product semigroup. The construction naturally occurs in the present situation.

The group $H(e)$ is an open subset of eXe. It is clear that $R_e \cdot L_e$ is contained in eXe so the set $L_e(eXe)R_e$ is closed under multiplication. Since multiplication is continuous, there are open sets U_R and U_L of R_e and L_e respectively containing e so that $U_R \cdot U_L \subset H(e)$.

THEOREM 5. *If U_L and U_R are as above then $S = U_L \cdot H(e) \cdot U_R$ forms a subsemigroup of X. There is a neighborhood U of e in X so that $U \cdot e \cdot U \subset S$. The subsemigroup S has the structure of a paragroup. The idempotents of S are exactly those of the form $f = l(rl)^{-1}r$ for $l \in U_L$ and $r \in U_R$. These idempotents cover a neighborhood of e in $C(e)$.*

Proof. The fact that S is a subsemigroup follows from the observation that

$$(lgr)(\hat{l}\hat{g}\hat{r}) = l(g(r\hat{l})\hat{g})\hat{r}$$

remains in S if each of l, $\hat{l} \in U_L$, $g, \hat{g} \in H(e)$, and $r, \hat{r} \in U_R$.

To see the next part of the assertion, suppose each of x and y is in X and is sufficiently close to e to insure that each of exe and eye is in $H(e)$. Then

$$xey = lgr$$

where $l = xey(exeye)^{-1}$, $g = exeye$, and $r = (exeye)^{-1}xey$. The fact that $l \in L_e$ and $r \in R_e$ follows from the obvious $le = l$, $el = e$, $re = e$, $er = r$, and Theorem 1.

Since $e(lgr)e = g$, $lgrg^{-1} = l$, and $g^{-1}lgr = r$ it is clear that $(l, g, r) \to l \cdot g \cdot r$ is a homeomorphism of $U_L \times H(e) \times U_R$ onto $U_L \cdot H(e) \cdot U_R$. If $s : U_R \times U_L \to H(e)$ via $s(r, l) = rl$ this homeomorphism is an isomorphism with the paragroup determined by s.

To see that the idempotents of S are of the form $f = l(rl)^{-1}r$ for $l \in L_e$ and $r \in R_e$ note that if $f = lgr$ for $g \in H(e)$ and l and r as before then $lgrlgr = lgr$ implies $grlg = g$ and hence $g = (rl)^{-1}$. By Theorem 4 all members of $C(e)$ near e are of this form for l and r near e.

Geometric structure of $C(e)$

We will now show that $C(e)$ is a manifold, the tangent space to $C(e)$ at e is $\operatorname{im}(P_e + Q_e)$, and the functions F_e defined by

$$F_e(x) = e + Q_e(x) + P_{e+Q_e(x)}(P_e(x)) = e + x + xex - ex^2e - ex^2ex$$

form an analytically compatible collection of charts for $C(e)$. As stated before this shows that $C(e)$ is a generalized saddle.

We need the following tool which we include for completeness.

LEMMA 6. *Suppose Y is a Banach space and $X = L(Y)$ is the Banach algebra of continuous linear operators on Y with the operator norm. If $e, f \in E$ and $\|e - f\| < 1$ then $(f|\operatorname{im}(e))$ is a linear homeomorphism of* $\operatorname{im}(e)$ *onto* $\operatorname{im}(f)$.

Proof. Since $\|e - f\| < 1$ we have $I + (e - f)$ invertible and hence

$$X = (I + e - f)(X) = (I - f)(X) + e(X).$$

Thus,

$$f(X) = f(I-f)(X) + f(e(X)) = f(e(X)),$$

and $(f|\mathrm{im}(e))$ is onto. If $x \in \mathrm{im}(e)$ then $\|x\| - \|f(x)\| \le \|e(x) - f(x)\| \le \|e - f\|\,\|x\|$ so

$$\|f(x)\| \ge (1 - \|e - f\|)\|x\|$$

and f is one to one on $\mathrm{im}(e)$.

We are now prepared to show that the F_e's are local homeomorphisms. We use the obvious fact that $e \to P_e$ and $e \to Q_e$ are continuous into $L(X)$.

THEOREM 7. *For each $e \in E$ the function F_e is a homeomorphism from a neighborhood of 0 in $T_e \equiv \mathrm{im}(P_e + Q_e)$ onto a neighborhood of e in $C(e)$.*

Proof. We have already observed that T_e is a splitting subspace of X and that F_e maps T_e into $C(e)$. Thus, it remains to show that F_e is a local homeomorphism at 0 and that a neighborhood of e in $C(e)$ is covered by a neighborhood of 0 under F_e.

Since F_e is a degree three polynomial, it is analytic. Moreover,

$$F_e'(x)(y) = y + yex + xey - exye - eyxe - ex^2ey - eyxex - exyex$$

and hence

$$F_e'(0)(y) = y.$$

Choose the open set U of T_e containing 0 so that if each of x and y is in U then

$$\|F_e(x) - F_e(y) - (x-y)\| \le \tfrac{1}{2}\|x - y\|.$$

It follows that for each of x and y in U we have

$$\tfrac{1}{2}\|x - y\| \le \|F_e(x) - F_e(y)\| \le \tfrac{3}{2}\|x - y\|.$$

Thus, F_e is one to one on U and $(F_e|U)^{-1}$ is continuous on its domain.

We know from Theorem 5 that for each f near e in $C(e)$ there is an $l \in L_e$ near e and an $r \in R_e$ near e so that $f = l(rl)^{-1}r$. Thus, from Theorem 1, since $f \in L_r$ we have $f = r + P_r(x)$ for some x in $\mathrm{im}(P_r)$. But, $r = Q_e(r - e) + e$ and if r is close to e then P_r maps the image of P_e onto $\mathrm{im}(P_r)$. Thus, $f = r + P_r(z)$ for some z near 0 in $\mathrm{im}(P_e)$. That is to say, $f \in \mathrm{im}(F_e)$.

This shows that as r ranges along the affine subspace R_e near e the affine spaces L_r are mutually disjoint and sweep out a neighborhood of e in $C(e)$. This is our justification for the terminology *generalized saddle* in describing $C(e)$.

We will now show that local homeomorphisms F_e are analytically compatible and hence serve as an atlas of charts for $C(e)$.

THEOREM 8. *If $e \in C(e)$ there is a neighborhood U of e in $C(e)$ so that if $f \in U$ then there are neighborhoods A and B of 0 in T_e and T_f respectively so that $F_f^{-1} \circ F_e$ is an analytic homeomorphism of A onto B.*

Proof. Suppose $e \in E$ and $f \in C(e)$. Consider the function H defined on $T_e \times T_f$ into T_e by

$$H(x, y) = (P_e + Q_e)\big(e + F_e(x) - F_f(y)\big).$$

It is clear that H is analytic and

$$D_2H(x, y)(z) = (P_e + Q_e)\big(-F_f'(y)(z)\big).$$

If $y = 0$ we have

$$D_2H(x, 0)(z) = -(P_e + Q_e)(z).$$

Thus, if f is sufficiently close to e to insure that $P_e + Q_e$ is a homeomorphism of T_f onto T_e, we have the existence of an implicitly defined analytic function u with domain an open set of T_e containing x into T_f so that $u(x) = 0$ and $H(z, u(z)) = H(x, 0)$ for all $z \in \operatorname{dom}(u)$.

Let θ be the retraction of a neighborhood of e in X onto a neighborhood of e in $C(e)$ constructed in Theorem 4. Choose A open in T_e so that F_e is a homeomorphism from A onto a neighborhood of e in $C(e)$ which is contained in the image of θ and so that if $f \in F_e(A)$ then $P_e + Q_e$ is a homeomorphism of T_f onto T_e. Suppose $x_0 \in A$ and let $f = F_e(x_0)$. Consider the function H based on this choice of x and f. We have

$$H(x_0, 0) = (P_e + Q_e)\big(e + F_e(x_0) - F_f(0)\big) = (P_e + Q_e)(e + f - f) = 0$$

so our implicitly defined function u satisfies

$$(P_e + Q_e)(F_e(x)) - F_f(u(x)) = 0$$

for each $x \in \operatorname{dom}(u)$.

Now, since θ is a retraction and $\theta(e) = e$, $\theta'(e)$ is an idempotent linear operator on X. Since the image of θ is a neighborhood of e in $C(e)$, for $l \in L_e$ and near e we have $\theta(l) = l$. Similarly, for $r \in R_e$ and near e we have $\theta(r) = r$. It follows that $\theta'(e)(l - e) = l - e$ for each $l \in L_e$ and $\theta'(e)(r - e) = r - e$ for each $r \in R_e$ since

$$\|\theta(l) - \theta(e) - (l - e)\| = 0$$

for l near e in L_e and

$$\|\theta(r) - \theta(e) - (r - e)\| = 0$$

for r near e in R_e. On the other hand, if $ex = xe$ and x is near e then $\Phi(x) = \Psi(x) = e$ by Theorem 2. Thus, $\theta(x) = \theta(e) = e$ and hence

$$\|\theta(x) - \theta(e) - 0(x - e)\| = 0.$$

It follows that $\theta'(e)(x - e) = 0$ for such x. Thus, if $xe = ex$ then $\theta'(e)(x) = 0$. Thus the kernel of $\theta'(e)$ contains the set of x such that $xe = ex$. Hence, we see that $\theta'(e)$ leaves the image of $P_e + Q_e$ fixed and maps $\ker(P_e + Q_e)$ to 0. Hence, $\theta'(e) = P_e + Q_e$.

Choose the open set V of X containing e so that if $x, y \in V$ and $x \neq y$ then

$$\|\theta(x) - \theta(y) - \theta'(e)(x - y)\| < \|x - y\|.$$

For $x \in \operatorname{dom}(u)$ we have

$$\begin{aligned} &\big\|F_e(x) - F_f(u(x)) - (P_e + Q_e)\big(F_e(x) - F_f(u(x))\big)\big\| \\ &\quad = \big\|F_e(x) - F_f(u(x))\big\|. \end{aligned}$$

Thus, recalling that $\theta(F_e(x)) = F_e(x)$ and $\theta(F_f(u(x))) = F_f(u(x))$, we have

$$F_e(x) - F_f(u(x)) = 0$$

if each of $F_e(x)$ and $F_f(u(x))$ is in V. By making A smaller, we can guarantee this happens for all x in a neighborhood of x_0. Thus, there is an open set A containing 0 in T_e so that for each $x \in A$ there is an analytic function u from a neighborhood of x in T_e into a neighborhood of 0 in $T_{F_e(x)}$ so that $F_e(z) = F_{F_e(x)}(u(z))$ on the domain of u. That is to say, so that $u = F_{F_e(x)}^{-1} \circ F_e$ is analytic on $\operatorname{dom}(u)$.

$C(e)$ as a homogenous space

Suppose X contains an identity element 1. We now turn to an examination of Zemánek's result that G, the component of 1 in $H(1)$, acts transitively on $C(e)$ via $x \to g^{-1}xg$. The isotropy subgroup of an idempotent e is the subgroup H defined by

$$H = \{g \in G : g^{-1}eg = e\}.$$

This is exactly the set of elements of G which commute with e. The function which sends $gH \to g^{-1}eg$ is a natural one to one correspondence between the members of G/H and $C(e)$. The space G/H of left cosets of H in G with the quotient topology is naturally topologically homogeneous since the left translations by members of G move the cosets among themselves and are homeomorphisms. We will show that the correspondence between G/H and $C(e)$ is a local homeomorphism from a neighborhood of H onto a neighborhood of e.

If we regard X in the usual way as a Lie algebra under the commutator product $[x, y] = xy - yx$ then X is the Lie algebra of G. The exponential map is a local homeomorphism from a neighborhood of 0 in X onto a neighborhood of 1 in G. The tangent set to H at 1 is exactly the set h of members of X which commute with e since H is the intersection of this subspace h with G. It is clear that $\exp(h) \subset H$ since $\exp(x)$ commutes with e if x does. Moreover, h is a sub Lie algebra of X.

The Lie subalgebra h splits in X because it is exactly the kernel of the projection $P_e + Q_e$. The complementary subspace T_e in turn splits into the linear but not Lie algebraic direct sum of the two subalgebras $P_e(X)$ and $Q_e(X)$. On these subalgebras the multiplication function of X is the trivial $xy = 0$.

LEMMA 9. *The function M defined by the equation*

$$M(x) = \exp(P_e(-x))\exp(Q_e(x))\exp((I - P_e - Q_e)(x))$$

has domain containing a neighborhood of 0 *in X on which M is a homeomorphism onto a neighborhood of* 1 *in G.*

Proof. This is just an application of the inverse function theorem, given the facts that $\operatorname{dom}(M) = X$, M is analytic, and

$$M'(0) = -P_e + Q_e + 1 - P_e - Q_e = (1 - P_e) - P_e$$

is invertible.

The following maps are also of interest:

$$\pi : G \to G/H \quad \text{via} \quad \pi(x) = xH$$
$$\beta : G/H \to C(e) \quad \text{via} \quad \beta(xH) = x^{-1}ex$$
$$N : T_e \to G/H \quad \text{via} \quad N(x) = \exp(-P_e(x))\exp(Q_e(x))H.$$

Note that since each of $P_e(X)$ and $Q_e(X)$ is a subalgebra with trivial multiplication we have

$$\exp(x) = 1 + x \text{ for } x \in P_e(X) \cup Q_e(X).$$

Hence $\exp(\mathrm{im}(P_e)) = 1 + P_e(X)$ and $\exp(\mathrm{im}(Q_e)) = 1 + Q_e(X)$ are abelian subgroups of G, each intersecting the other and H at $\{1\}$. Lemma 9 shows that $(1 + Q_e(X))(1 + P_e(X))H$ is a direct product of sorts and that it contains a neighborhood of 1.

If $x \in T_e$ then there are unique $l \in L_e$ and $r \in R_e$ so that $P_e(x) = (l - e)$ and $Q_e(x) = (r - e)$. The composition $\beta \circ N$ thus is given at x by

$$\begin{aligned}
\beta(N(x)) &= (1 - (r - e))(1 + (l - e))e(1 - (l - e))(1 + (r - e)) \\
&= (1 - (r - e))l(1 + (r - e)) \\
&= (1 - (r - e))lr \\
&= lr - rlr + r \\
&= F_e(x).
\end{aligned}$$

Choose an open set W containing 0 so that $(M|W)$ is a homeomorphism. Suppose U is an open set in $\mathrm{im}(F_e)$ and is sufficiently close to e to make $F_e^{-1}(U) + V \subset W$ for some open set V of H. The set $M(F_e^{-1}(U) + V) \cdot H$ is open in G since it is the product of an open set and a set. Thus

$$N\big(F_e^{-1}(U)\big) = \beta^{-1}(U)$$

is open in G/H since

$$\pi^{-1}\big(\beta^{-1}(U)\big) = M\big(F_e^{-1}(U) + V\big) \cdot H$$

is open in G. Thus, β is continuous on a neighborhood of H in G/H and maps this neighborhood onto a neighborhood of e in $C(e)$.

Since $N = \pi \circ (M|T_e)$ we have N continuous. Because of the above factorization of F_e we have that N is one to one from an open set of T_e onto an open set of G/H and maps open sets to open sets. Thus, β itself is a homeomorphism from an open set of G/H onto a neighborhood of e in $C(e)$.

These remarks constitute a proof of the following theorem.

THEOREM 10. *The homogenous space G/H is locally homeomorphic with T_e and the function β is a local homeomorphism from a neighborhood of H in G/H onto a neighborhood of e in $C(e)$.*

We remark that the polynomial paths constructed by Esterle [2] could be thought of as using the basic idea of parametrizing G/H as $\exp(-Q_e(x))\exp(-P_e(x)) \cdot H$ for $x \in T_e$ then shooting T_e into $C(e)$ by composing with β. What is missing there is the fact that this parametrizes a *neighborhood* of e in $C(e)$.

Final remarks

In the example in which $X = L(Y)$ for some Banach space Y it is easy to see that

$$L_e = \{l : \ker(l) = \ker(e)\} \text{ and } R_e = \{r : \operatorname{im}(r) = \operatorname{im}(e)\}.$$

As we remarked after the proof of Theorem 4, one can use spectral theory to construct the retraction θ on a neighborhood of e in X. The existence of θ implies that if x is close enough to e then x leaves the kernel and image of the nearby idempotent $\theta(x)$ invariant. The construction of Ψ and Φ by successive approximations yields the nearby $l = \Psi(x)$ with the same kernel as e and whose image is left invariant by x and the nearby $r = \Phi(x)$ whose image is the same as that of e and whose kernel is invariant under x. There doesn't seem to be a natural construction based on spectral theory for the idempotents l and r.

The charts F_e arise naturally in the homogeneous space through the parametrization N of G/H and the natural map of G/H onto $C(e)$. They also arise through the parametrization of $C(e)$ obtained by following R_e to r then L_r to the (it turns out) typical idempotent near e. Of course, another natural collection of charts arise via first following L_e to l then R_l to the typical idempotent. Among the parametrizations of $C(e)$ obtainable from natural homeomorphisms from T_e into G/H these two seem to have the lowest degree.

We close with a question. Must the functions F_e be one to one? Must they be homeomorphisms?

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