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The Structure of the W_∞ Algebra

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Abstract. We prove rigorously that the structure constants of the leading (highest spin) linear terms in the commutation relations of the conformal chiral operator algebra W_∞ are identical to those of the $\mathrm{Diff}_0^+ \mathbb{R}^2$ algebra generated by area preserving diffeomorphisms of the plane. Moreover, all quadratic terms of the W_N algebra are found to be absent in the limit $N \to \infty$. In particular we show that W_∞ is a central extension of $\mathrm{Diff}_0^+ \mathbb{R}^2$ with non-trivial cocycles appearing only in the commutation relations of its Virasoro subalgebra. We also propose a representation of W_∞ in terms of a single scalar field in $2+1$ dimensions and discuss its significance in the context of quantum field theory.

1. Introduction

The construction of all unitary highest weight representations of the infinite dimensional symmetry algebras that arise in two dimensional conformal field theory has provided a non-perturbative framework for solving a large class of physically interesting quantum field theory models (see for instance [1] and references therein). One of the most striking results in the classification of rational conformal field theories was the realization that simple Lie algebras determine the structure and the operator content of unitary scale invariant 2-dim systems. In the chiral operator approach, rational conformal field theories are described as minimal models of extended conformal symmetry algebras $\mathcal{W}$ generated by the stress-energy tensor $T(z)$ and other holomorphic fields $\{w_s(z), s \in J\}$, which are associated with additional conserved currents in the 2-dim world. Typically, the generators of $\mathcal{W}$-algebras are labeled by the vertices of Dynkin diagrams of simple Lie algebras G, which also determine the conformal weight (spin) s of the chiral fields $w_s(z)$. The Virasoro algebra

$$[T(z), T(z')] = (T(z) + T(z'))\delta_{,z}(z-z') + \frac{c}{12}\delta_{,zzz}(z-z') \tag{1}$$

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is the most elementary member in the family of conformal algebras and is associated with the simplest Lie algebra A_1.

$\mathcal{W}$-algebras are closed (in the sense that they satisfy the Jacobi identity) and have determining relations which are in general quadratic in nature. Nevertheless, their unitary representations can be constructed in analogy with the Verma module representations of the Virasoro and Kac–Moody algebras and correspond to unitary conformal field theories with discrete symmetry, depending on the simple Lie algebra G [2]. Using a suitable generalization of the Feigin–Fuks construction, it follows that the full spectrum of the anomalous dimensions of the $\mathcal{W}$-invariant primary fields for these theories is expressed in terms of the fundamental weights of G and forms a set of rational numbers. In this fashion, the bootstrap approach to quantum field theory yields a systematic way to understand the vacuum structure of string theory as well as construct a periodic table of all types of criticality in 2-dim statistical mechanics, parallel to the Cartan classification of Lie algebras.

In what follows, we focus our attention on the operator algebra W_N which is associated with the Lie algebra A_{N-1} and study the details of its structure in the limit $N \to \infty$. W_N is generated by the stress-energy tensor $T(z)$ and a collection of primary conformal fields $\{w_s(z); s = 3, 4, \ldots, N\}$ with integer spin s. The interest in the large N limit behavior of extended conformal symmetries arose from some earlier preliminary considerations which showed that the leading (highest spin) linear terms in the commutation relations of W_∞ describe an infinite dimensional subalgebra, $\mathrm{Diff}_0^+ \mathbb{R}^2$, of all area preserving diffeomorphisms of the plane [3]. In this paper we present the details of the proof and show furthermore that the quadratic and the rest of the terms in W_N vanish in the limit $N \to \infty$, with the exception of central cocycle terms in the Virasoro subalgebra of W_∞. Our result sheds some new light in the relations between conformal algebras and area preserving diffeomorphisms of 2-manifolds that arise as residual symmetries in the light-cone formulation of membrane theories [4]. It also suggests a conformal field theory approach to the representation theory of $\mathrm{Diff}_0^+ \mathbb{R}^2$ which might be of (some) value in mathematics as well as quantum field theory. The proof we present here relies on the Hamiltonian description of $\mathcal{W}$-algebras using the Gelfand–Dickey algebraic structure of integrable non-linear differential equations of the KdV type.

There are several motivations and objectives in our program. First notice that due to the quadratic nature of the determining relations of W_N, there is no natural geometric interpretation of higher spin fields with $s > 2$. This problem arises in all higher spin theories (see for instance [5]) and makes difficult the construction of consistent self-interacting gauge theories of massless higher spin fields (in any number of dimensions). One possible resolution to the problem is provided by the inclusion of an infinite family of particles with all possible spins. In our case this procedure leads to the operator algebra W_∞, while the connection we find with the symmetry algebra of area preserving diffeomorphisms assigns a definite meaning to the role that 2-dim higher spin chiral fields have in geometry.

Second, as far as strings are concerned, we would very much like to have a non-perturbative framework for studying their quantization. Since there is no such prescription available at the moment, it seems natural to "play it by ear" and adopt some ideas and techniques from the $1/N$ expansion of $SU(N)$ Yang–Mills gauge theories (see for instance [6] and references therein). Motivated by the behavior of $SU(N) \supset SU(3)$ gauge theories in the limit $N \to \infty$ and their description in terms of loop dynamics, it is natural for our purposes to embed conventional (W_2)-strings in a much larger theory (W_N-strings) with $W_N \supset W_2$ as the symmetry algebra on the world sheet. This generalization may accommodate new prospects in building string theories [7] and in analogy with $SU(\infty)$ gauge theories, W_∞-strings could be used for developing a non-perturbative approach to string quantization itself. Certainly, the association of W_∞ with the algebra of area preserving diffeomorphisms we establish in this paper suggests that higher dimensional extended objects (e.g. membranes) could be employed in the formulation of W_∞-string theory. This construction is still far from being completed and constitutes a long term goal in our investigation.

Third, we would like to know whether the bootstrap (operator algebra) approach to quantum field theory can be applied successfully to theories in more than two dimensions. Although we do not have any systematic procedure available in our disposal for $d > 2$, we find that a certain class of 3-dim field theory models can be approximated by 2-dim conformal field theories that possess an infinite collection of additional conserved currents with all integer spins. To be more precise, we think of the Dynkin diagrams that describe the operator content of $\mathcal{W}$-algebras (and only for classical non-exceptional simple Lie algebras) as some suitable discretizations/skeletonizations of a continuous third dimension in space-time. Passing to the limit $N \to \infty$ we effectively obtain a field theory in one dimension higher, provided that the large N limit of the underlying simple Lie algebra (e.g. A_∞ for W_∞) is defined as a continual algebra, in the nomenclature of reference [8]. In this sense, we manage to describe (genuine) higher dimensional theories in terms of lower dimensional physics using infinite dimensional structure groups. We will return to this point later, while discussing the geometric interpretation of W_∞ as a subalgebra of the area preserving diffeomorphisms of $\mathbb{R}^2$.

Incidentally, we point out that the idea to use 2-dim models with infinite dimensional structure groups for the description of higher dimensional theories, has already been adopted by Atiyah in his work on the moduli space of Yang–Mills instantons [9]. In particular, 4-dim self-dual gauge connections with values in G and with topological charge $k \in \mathbb{Z}$, can be thought of as instantons of a 2-dim principal chiral model with values in ΩG, the loop space of G. Also, recent work in general relativity has shown that in many aspects, gravity resembles ordinary gauge theories with infinite dimensional structure groups. For example, the self-dual Einstein equations for gravitational instantons are equivalent to Nahm's equations for BPS monopoles with $G \cong \mathrm{Diff}_0\, \Sigma^{(3)}$, the volume preserving diffeomorphism group of the 3-space $\Sigma^{(3)}$ [10].

We think that all these reasons are very compelling for justifying the study we undertake in this paper. Certainly, the results we obtain in the sequel do not answer in detail all the questions we have raised; they should be thought of as a modest first step toward the general direction we have outlined. Here is a brief description of the way that our material is organized. In Sect. 2 we set up the notation and review the Hamiltonian formulation of extended conformal symmetries in terms of the Gelfand–Dickey Lie–Poisson algebra of formal pseudodifferential operators. We find this framework most appropriate and effective for studying the large N limit behavior of W_N. In Sect. 3 we present the results of somewhat lengthy computations which establish the advertised relation between W_∞ and the algebra of area preserving diffeomorphisms of the plane. In Sect. 4 we continue this investigation and address the problem of constructing unitary representations of $\mathrm{Diff}_0 \mathbb{R}^2$ with the aid of 2-dim $\mathbb{Z}_\infty$-symmetric conformal field theories. In Sect. 5 we concentrate on the 3-dim description of W_∞ using a continual analogue of the Toda field and (Feigin–Fuks) free field representations. Finally, in Sect. 6 we draw our conclusions and indicate some other directions for future work.

2. The Chiral Operator Algebra W_N

Zamolodchikov's spin N operator algebra W_N is generated by the stress-energy tensor $T(z)$ and the additional conserved currents $\{w_s(z)\}$ with spin $s = 3, 4, \ldots, N$ respectively. Introducing Fourier modes, the commutation relations of W_N take the form [2]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{2}$$

$$[L_n, w_s(m)] = [(s-1)n - m]w_s(n+m), \tag{3}$$

$$[w_s(n), w_{s'}(m)] = \sum_{\{s_i \leqq N\}} \sum_{\{k_i\}} C_{ss'}^{s_1 \cdots s_p}(n, m; k_1, \ldots, k_p; c) w_{s_1}(k_1) \cdots w_{s_p}(k_p). \tag{4}$$

Here, L_n denote the Fourier modes of $T(z)$ and $w_s(m)$ those of $w_s(z)$. The identity operator is included among the generators using the identification $I = w_0$ (i.e., $w_0(k) = \delta_{k,0}$) and the integers $\{k_i\}$ are chosen so that the condition of momentum conservation $k_1 + k_2 + \cdots + k_p = n + m$ is always satisfied. Furthermore, the normal ordering prescription is implicitly introduced on the right-hand side of Eq. (4). The central charge c is the only free parameter in the operator algebra W_N, whose defining relations (2)–(4) are non-linear (quadratic or higher polynomial) and satisfy the Jacobi identity.

W_N is clearly a conformal algebra because it contains the Virasoro as subalgebra (see Eq. (2)). Determining the structure constants C for arbitrary N is not an easy task and requires extremely lengthy computations. For any given pair of spins (s, s'), the structure constants C are (in general) different from zero only if $s_1 + s_2 + \cdots + s_p \leqq s + s' - 2$. We also assume that $C(0, 0; 0, \ldots, 0; c) = 0$ for all s, s'. The only terms in (4) which are linear in the generating fields $\{w_s\}$ occur when all $\{s_i\}$ but one (call it s'') are zero. Obviously, we have to demand that $s'' \leqq s + s' - 2$, as well as $s'' \leqq N$. Then, for sufficiently large N, the leading (highest spin) linear term in (4) has $s'' = s + s' - 2$. Later we will show (among other things) that when $N \to \infty$, $C_{ss'}^{s+s'-2}(n, m; n+m; c)$ is independent of c and in fact

$$C_{ss'}^{s+s'-2}(n, m; m+m) = n(s'-1) - m(s-1). \tag{5}$$

We mention for completeness that central charge terms occur in Eq. (4) when $s_1 = s_2 = \cdots = s_p = 0$ and $C_{ss'}^{0 \cdots 0} \neq 0$.

There are several ways to describe the detailed structure of the conformal operator algebra W_N. The most direct method (though not very practical) is provided by the constraint of Jacobi identities in the commutation relations (2)–(4). This leads to an algebraic system of equations for the structure constants $C_{ss'}^{\{s_i\}}$, which turns out to be very complicated to solve when N is sufficiently large. Equivalently, the explicit calculation of the operator product expansion $w_s(z)w_{s'}(z')$ for any two primary conformal fields w_s and $w_{s'}$ requires knowledge of the singular as well as (some) of the non-singular terms in the operator product expansion of fields with spin less than s and s'. Although the closure of the resulting operator algebra puts severe constraints on the values of all possible free parameters (coefficients) that appear, this procedure does not seem to be the most efficient for our purposes.

Alternatively, one may use the free field representation of W-algebras and introduce a multi-component free massless bose field $\Phi(z)$ [1,2]. For W_N, the components of Φ, $\{\phi_a; a = 1, 2, \ldots, N-1\}$, have two-point correlation functions of the form

$$\langle \phi_a(z)\phi_b(0)\rangle = -\delta_{a,b}\log z. \tag{6}$$

The generators $T(z)$, $w_s(z)$ $(s = 3, 4, \ldots, N)$ of W_N are composite fields of Φ, i.e.,

$$T(z) = -\frac{1}{4}\sum_{a=1}^{N-1} :(\partial\phi_a(z))^2: + i\alpha_0\rho\cdot\partial^2\Phi(z), \tag{7}$$

$$w_s(z) = \sum_{\{1\leqq a_i\leqq N-1\}} M^{a_1\cdots a_s}:\partial\phi_{a_1}(z)\cdots\partial\phi_{a_s}(z): + \text{derivative terms}, \tag{8}$$

with appropriately chosen numerical constants $M^{a_1\cdots a_s}$.

The structure of the operator algebra W_N is determined entirely by the Lie algebra characteristics of A_{N-1}. In particular,

$$\rho = \frac{1}{2}\sum_{k=1}^{N}(N+1-2k)h_k \tag{9}$$

is the Weyl vector of A_{N-1} and the N vector h_k $(k = 1, 2, \ldots, N)$ which are defined by

$$\sum_{k=1}^{N} h_k = 0, \quad h_k\cdot h_m = \delta_{k,m} - \frac{1}{N} \tag{10}$$

form an overcomplete system in $(N-1)$-dim Euclidean space. In this notation, $\{h_k - h_{k+1}; k = 1, 2, \ldots, N-1\}$ are the $N-1$ simple roots of A_{N-1} and $\left\{\sum_{k=1}^{i} h_k;\ i = 1, 2, \ldots, N-1\right\}$ the fundamental weights. Therefore, $\rho^2 = \frac{1}{12}(N^3 - N)$. Also, α_0 is a loop counting parameter of the order $-1/\alpha_0^2 \sim \hbar$, i.e. Planck's constant. The asymptotic behavior of the $(N-1)$-component free boson $\Phi(z)$ is $\Phi(z) \sim 2i\alpha_0\rho\log z$, which insures the fall off rate

$$T(z) \sim z^{-4}; \quad w_s(z) \sim z^{-2s}\ (s = 3, \ldots, N) \tag{11}$$

as $z \to \infty$. The latter condition is necessary for conformal symmetry to be unbroken.

Another method for deriving the commutation relations of W_N is provided by the Toda field theory for the Lie algebra A_{N-1} [11]. In this case, one introduces a collection of scalar fields $\phi_a(z, \bar{z})$ $(a = 1, 2, \ldots, N-1)$, which satisfy the coupled system of differential equations

$$\partial\bar{\partial}\phi_a(z,\bar{z}) = \exp\left(\sum_{b=1}^{N-1} K_{ab}\phi_b(z,\bar{z})\right). \tag{12}$$

Here K_{ab} is the Cartan martix of A_{N-1}. There is always a set of characteristic $(N-1)$ chiral fields $w_s^{(+)}(\partial\phi_a, \partial^2\phi_a, \ldots, \partial^s\phi_a)$ and similarly $w_s^{(-)}(\bar{\partial}\phi_a, \bar{\partial}^2\phi_a, \ldots, \bar{\partial}^s\phi_a)$ $(s = 2, 3, \ldots, N)$ associated with the Toda equations (12). The simplest one,

$$T(z) := w_2^{(+)} = -\frac{1}{2}\sum_{a,b} \partial\phi_a(z,\bar{z})K_{ab}\partial\phi_b(z,\bar{z}) + \sum_a \partial^2\phi_a(z,\bar{z}), \tag{13}$$

coincides with the holomorphic component of the improved stress-energy tensor of Toda field theory, while $\bar{T}(\bar{z}) := w_2^{(-)}(\bar{z})$ is defined similarly and yields the antiholomorphic component. The explicit expression of $w_s^{(\pm)}$ with $s = 3, \ldots, N$ is a little bit more complicated and we refer the reader to reference [11] for further details. Nevertheless, it is important to point out that all the characteristic fields $w_s^{(\pm)}$ are conserved, i.e. $\bar{\partial}w_s^{(+)} = 0$ and $\partial w_s^{(-)} = 0$, by virtue of the equations of motion (12). Moreover their conformal weights are $s = 2, 3, \ldots, N$ respectively and generate two (commuting) copies of the operator algebra W_N, one holomorphic and one antiholomorphic, depending on the choice of the + or − signs. Since we are considering chiral operator algebras, it is sufficient to concentrate only on the holomorphic sector generated by $\{w_s^{(+)}(z)\}$.

We intend to return to the free field and Toda field theory representations of W_N when we discuss W_∞ from a 2 + 1 viewpoint using the continual analogue of the algebra A_∞. Meanwhile, for the actual computation of the commutation relations of W_∞, we adopt an alternative (though equivalent) formalism originated from the Hamiltonian description of integrable non-linear differential equations of the KdV type. More specifically, we are going to use the Gelfand–Dickey algebra of formal pseudodifferential operators, $\mathrm{GD}(A_{N-1})$, with the simple Lie algebra A_{N-1} as its label [12], and study the structure of W_N in the large N limit, $N \to \infty$. We point out that Gelfand–Dickey algebras have already been employed quite successfully in the classical description of extended conformal symmetries [13, 14]. In this framework, the calculation of the structure constants $C_{ss'}^{\{s_i\}}$ becomes just a combinatorics problem. Of course, to make exact contact with the corresponding (quantum) chiral operator algebras of 2-dim conformal field theory, one has to normal order all quadratic (and higher) terms that appear in the commutation relations of fields with spin $s \geqq 3$. Although the structure of the algebra remains unchanged and consistent with Jacobi identity, the normal ordering prescription deforms (in general) the classical values of the structure constants C. However, this deformation is insignificant when $N \to \infty$ [15] and the classical (Gelfand–Dickey algebra) calculation of W_∞ yields the exact quantum mechanical answer. We will return to this point later.

Next, we review (briefly) the basic theory of Gelfand–Dickey algebras, $\mathrm{GD}(A_{N-1})$, associated with the A-series of simple Lie algebras (see [12–14] for more details). Let

$$L_N = \partial^N + u_2(z)\partial^{N-2} + \cdots + u_N(z) \tag{14}$$

be an Nth order differential (Lax) operator with $N-1$ coordinate (potential) functions $\{u_i(z);\ i=2,3,\dots,N\}$. We also consider (formal) pseudodifferential operators $A(z)=A_-(z)+A_+(z)$ with

$$A_-(z)=\sum_{k<0}\partial^k a_k(z);\quad A_+(z)=\sum_{k\geqq 0}a_k(z)\partial^k \tag{15}$$

and introduce the notation $\operatorname{res}A=a_{-1}(z)$. Then, to any local functional $f[u_2,\dots,u_N]$, we assign the (formal) operator sum

$$X_f=\sum_{k=1}^{N-1}\partial^{-k}\frac{\delta f}{\delta u_{N+1-k}}+\partial^{-N}x_N(f). \tag{16}$$

The variable $x_N(f)$ is chosen so that the following condition is satisfied: $\operatorname{res}[L_N,X_f]=0$. With this in mind, the Gelfand–Dickey bracket between any two functionals $f[u]$ and $g[u]$ is defined to be

$$\{f,g\}_N=\int\operatorname{res}(V_{X_f}(L_N)X_g), \tag{17}$$

where

$$V_{X_f}(L_N)=L_N(X_fL_N)_+-(L_NX_f)_+L_N. \tag{18}$$

In all formulas, we use Leibniz's rule for the multiplication of operators (both differential and formal). In particular, for all $k>0$, the following identities are true:

$$\partial^k a(z)=\sum_{m=0}^{k}\binom{k}{m}a^{(m)}(z)\partial^{k-m}, \tag{19a}$$

$$\partial^{-k}a(z)=\sum_{m=0}^{\infty}(-)^m\binom{k+m-1}{m}a^{(m)}(z)\partial^{-k-m}. \tag{19b}$$

It was shown in references [13, 14] (see also [2]) that under the GD-bracket (17), the coordinate functionals $u_2,u_3,\dots,u_N$ form a closed conformal algebra with quadratic determining relations. In this case, the Virasoro subalgebra is generated by $u_2(z)$,

$$\{u_2(z),u_2(z')\}_N=(u_2(z)+u_2(z'))\delta_{,z}(z-z')+\tfrac{1}{12}(N^3-N)\delta_{,zzz}(z-z'), \tag{20}$$

and the central charge is $c=12\rho^2$. However, the rest of the coordinate fields $u_s(z)$ $(s=3,\dots,N)$ are not primary. Primary conformal fields $w_s(z)$ with spin $s=3,\dots,N$ are obtained using appropriate (polynomial) combinations of all $\{u_i(z);\ i\leqq s\}$ and their derivatives. They are of the (general) form

$$w_s(z)=\sum_{\{i\},\{k\}}A_{N;k_1\cdots k_p}^{i_1\cdots i_p}u_{i_1}^{(k_1)}(z)\cdots u_{i_p}^{(k_p)}(z), \tag{21}$$

with $k_1+\cdots+k_p+i_1+\cdots+i_p=s\leqq N$. Then, $T(z):=u_2(z)$ and $\{w_s(z);\ s=3,\dots,N\}$ generate the extended conformal symmetry algebra W_N, as desired. All the calculations are fairly straightforward but quite lengthy. In fact, the numerical constants $A_{N;\{k\}}^{\{i\}}$ in Eq. (21) are not easy to compute for arbitrary N. Explicit results are only available for small values of N, e.g. 2, 3, 4. Nevertheless, we find that when $N\to\infty$ the structure of the operator algebra W_N simplifies considerably and exact expressions for all structure constants become available. This is the subject of the next section.

3. The Algebraic Structure of W_∞

According to Eq. (16), the (formal) operators

$$X_{u_s(z')} = \partial^{s-N-1}\delta(z-z') + \partial^{-N}x_N(u_s(z')) \tag{22}$$

are assigned to the coordinate functions u_s with $s = 2, 3, \ldots, N$. The variables $x_N(u_s)$ are not arbitrary. Recall that the definition of the GD(A_{N-1}) algebra imposes the constraint $\operatorname{res}[L_N, X_{u_s}] = 0$, for all values of s. This condition is equivalent to the following differential equation:

$$x'_N(u_s(z')) = \frac{1}{N}\sum_{\rho=0}^{N}\sum_{k=1}^{s}(-)^k\binom{N-s+k}{k}(\delta(z-z')u_\rho(z))^{(k)}\delta_{s,\rho+k}. \tag{23}$$

Here $u_0(z) = 1$, $u_1(z) = 0$ and the derivatives are taken with respect to z. Then, the Gelfand–Dickey bracket (17) between any two (coordinate) fields u_s and $u_{s'}$ is given by

$$\{u_s(z), u_s(z')\}_N = \int d\tilde{z}(A+B+C+D), \tag{24}$$

where

$$A = \sum_{\rho,\rho'=0}^{N}\sum_{k,k'=0}^{\infty}\sum_{m=0}^{s-1-k-\rho}(-)^{k+k'}\binom{N-s+k}{k}\binom{N-s'+k'}{k'}$$
$$\cdot\binom{s-1-\rho-k}{m}\delta_{s+s',\rho+\rho'+k+k'+m+1}\cdot(\delta(z-\tilde{z})u_\rho(\tilde{z}))^{(k)}(\delta(z'-\tilde{z})u_{\rho'}(\tilde{z}))^{(k'+m)} \tag{25a}$$

with ρ, k subject to the restriction $\rho + k \leqq s-1$,

$$B = \sum_{\rho,\rho'=0}^{N}\sum_{k,k'=0}^{\infty}\sum_{m=0}^{s-1-k-\rho}(-)^{k+k'}\binom{N-s+k}{k}\binom{N-1+k'}{k'}$$
$$\cdot\binom{s-1-\rho-k}{m}\delta_{s,\rho+\rho'+k+k'+m}\cdot(\delta(z-\tilde{z})u_\rho(\tilde{z}))^{(k)}(x_N(u_{s'}(z'))u_{\rho'}(\tilde{z}))^{(k'+m)} \tag{25b}$$

with ρ, k subject to the same restriction $\rho + k \leqq s-1$,

$$C = -\sum_{\rho,\rho'=0}^{N}\sum_{k=0}^{s-\rho-1}\binom{s-\rho-1}{k}\delta_{s+s',\rho+\rho'+k+1}u_\rho(\tilde{z})(\delta(z-\tilde{z})u_{\rho'}(\tilde{z}))^{(k)}\delta(z'-\tilde{z}) \tag{25c}$$

with ρ subject to the restriction $\rho \leqq s-1$, and

$$D = -\sum_{\rho,\rho'=0}^{N}\sum_{k=0}^{s-\rho-1}\binom{s-\rho-1}{k}\delta_{s,\rho+\rho'+k}u_\rho(\tilde{z})(\delta(z-\tilde{z})u_{\rho'}(\tilde{z}))^{(k)}x_N(u_{s'}(z')) \tag{25d}$$

with ρ subject to the same restriction $\rho \leqq s-1$. We have $u_0(\tilde{z}) = 1, u_1(\tilde{z}) = 0$ as before and all derivatives in (25a–d) are taken with respect to the integration variable $\tilde{z}$.

With this result in our disposal we study now the leading behavior of the commutation relations (24). Notice that for sufficiently large N (i.e., $N \geqq s + s' - 2$) the spin of all leading linear terms is $s + s' - 2$. The term (25a) will provide a contribution $\sim u_{s+s'-2}$ only if ($\rho = 0; \rho' = s + s' - 2$) or ($\rho = s + s' - 2; \rho' = 0$). Since $\rho + k \leqq s - 1$, the second possibility is immediately ruled out. On the other hand, the constraint $k + k' + m = s + s' - \rho - \rho' - 1 = 1$ implies that either ($k = 1$; $k' = m = 0$) or ($k' = 1; k = m = 0$) or ($m = 1; k = k' = 0$). Therefore, the total highest spin linear contribution from A is

$$\begin{aligned} &-(N-s+1)\delta_{,\tilde{z}}(z-\tilde{z})\delta(z'-\tilde{z})u_{s+s'-2}(\tilde{z}) \\ &-(N-s'+1)\delta(z-\tilde{z})(\delta(z'-\tilde{z})u_{s+s'-2}(\tilde{z}))_{,\tilde{z}} \\ &+(s-1)\delta(z-\tilde{z})(\delta(z'-\tilde{z})u_{s+s'-2}(\tilde{z}))_{,\tilde{z}}. \end{aligned} \tag{26}$$

It is easy to see that the second and fourth terms (25b, d) do not contribute at all to this order, while the only relevant term in (25c) has $\rho = 0$, $\rho' = s + s' - 2, k = 1$ and equals to

$$-(s-1)(\delta(z-\tilde{z})u_{s+s'-2}(\tilde{z}))_{,\tilde{z}}\delta(z'-\tilde{z}). \tag{27}$$

Putting the expressions (26), (27) together and performing the necessary integrations, we obtain the following result (for $N \geqq s + s' - 2$):

$$\begin{aligned} \{u_s(z), u_{s'}(z')\}_N = {} & [(s-1)u_{s+s'-2}(z) + (s'-1)u_{s+s'-2}(z')]\delta_{,z}(z-z') \\ & + (\text{lower spin terms}). \end{aligned} \tag{28}$$

The lower spin terms are local functionals of $\{u_i(z)\}$ with $i < s + s' - 2$. Their structure is determined by all other terms in Eqs. (25a–d) and turns out to be quite complicated indeed.

It is clear that when $N \to \infty$, most of the numerical coefficients that appear in the commutation relations (24) diverge rapidly. For example, the central charge c of the Virasoro algebra (20) blows up to infinity like ∞^3. However, the infinities we encounter here are not characteristic of the chiral operator algebra W_∞. They originate from the normalization of the fields $\{u_s(z); s = 2, 3, \ldots, N\}$ used in the definition of the Gelfand–Dickey algebra GD(A_{N-1}). At this point we realize that the standard choice (14), (18) of the operators L_N and $V_X(L_N)$ is rather special, because the value of the central charge $c = N^3 - N$ is fixed. On the other hand we know that the commutation relations of W_N allow for arbitrary values of c. Therefore, to obtain a (classical) Hamiltonian description of W_N for all values of N and c, it is necessary to modify our definitions by introducing appropriate rescalings in the (basic) variables of the theory.

First we consider the Virasoro subalgebra (20) and define the quantities

$$\tilde{u}_2(z) = \frac{c}{N^3}u_2(z); \quad [\ ,\ ]_N = \frac{N^3}{c}\{\ ,\ \}_N \tag{29}$$

for all $c \neq 0$. It follows that

$$[\tilde{u}_2(z), \tilde{u}_2(z')]_N = (\tilde{u}_2(z) + \tilde{u}_2(z'))\delta_{,z}(z-z') + \frac{N^3 - N}{N^3}\frac{c}{12}\delta_{,zzz}(z-z') \tag{30}$$

with arbitrary central charge for all $N = 2, 3, \ldots, \infty$. For consistency, we also have to rescale the rest of the generating fields $\{u_s(z)\}$. Recall that in the free field representation of W_N, conformal fields of weight s are $\sim (\partial\Phi(z))^s$. Then, for $s = 2$, Eq. (29) implies that $\partial\Phi(z)$ has been modified by a factor of $\sqrt{c/N^3}$. Therefore, it is natural to define

$$\tilde{u}_s(z) = \left(\frac{c}{N^3}\right)^{s/2} u_s(z), \quad \text{for} \quad s = 2, 3, \ldots, N. \tag{31}$$

This is the only consistent prescription applicable to all values of N, including ∞. Moreover, it does not affect the coefficients of the (leading) highest spin linear terms in Eq. (28). From now on, we adopt the rescaled variables $\{\tilde{u}_s(z)\}$ in our description of W_N and its large N limit. For convenience (and to simplify the calculations) we may choose $c = 1$; arbitrary values of c will be considered in the next section.

Our next task is the computation of all terms in the commutation relations $[\tilde{u}_s(z), \tilde{u}_{s'}(z')]_\infty$ with arbitrary s and s'. Using Eqs. (24), (25) and with the definitions (29), (31) in mind we obtain:

$$\begin{aligned}
[\tilde{u}_s(z), \tilde{u}_{s'}(z')]_\infty &= [(s-1)\tilde{u}_{s+s'-2}(z) + (s'-1)\tilde{u}_{s+s'-2}(z')]\delta'(z-z') \\
&+ \tfrac{1}{12}(\tilde{u}_{s-2}(z)\delta_{s',2} + \tilde{u}_{s'-2}(z')\delta_{s,2})\delta'''(z-z') + \tfrac{1}{12}\delta_{s,2}\delta_{s',2}\delta'''(z-z') \\
&- \tfrac{1}{2}(s-1)(\tilde{u}_{s-1}(z)\tilde{u}_{s'-1}(z) + \tilde{u}_{s-1}(z')\tilde{u}_{s'-1}(z'))\delta'(z-z') \\
&+ \tfrac{1}{2}(s-1)(\tilde{u}'_{s-1}(z)\tilde{u}_{s'-1}(z) - \tilde{u}_{s-1}(z)\tilde{u}'_{s'-1}(z))\delta(z-z') \\
&+ \tfrac{1}{12}\tilde{u}_{s-2}(z)(\delta(z'-z)\tilde{u}_{s'-2}(z))''' \\
&+ \frac{1}{2}\sum_{\rho,\rho' \geqq 2}^{\{s \geqq \rho+2\}} (s'-1-\rho)\delta_{s+s'-2,\rho+\rho'}(\tilde{u}_\rho(z)\tilde{u}_{\rho'}(z) \\
&+ \tilde{u}_\rho(z')\tilde{u}_{\rho'}(z'))\delta'(z-z') \\
&- \frac{1}{2}\sum_{\rho,\rho' \geqq 2}^{\{s \geqq \rho+2\}} (s'-1-\rho)\delta_{s+s'-2,\rho+\rho'}(\tilde{u}'_\rho(z)\tilde{u}_{\rho'}(z) - \tilde{u}_\rho(z)\tilde{u}'_{\rho'}(z))\delta(z-z') \\
&- \frac{1}{2}\sum_{\rho,\rho' \geqq 2}^{\{s \geqq \rho'+2\}} (s'-1-\rho)\delta_{s+s'-2,\rho+\rho'}(\tilde{u}_\rho(z)\tilde{u}_{\rho'}(z) \\
&+ \tilde{u}_\rho(z')\tilde{u}_{\rho'}(z'))\delta'(z-z') \\
&+ \frac{1}{2}\sum_{\rho,\rho' \geqq 2}^{\{s \geqq \rho'+2\}} (s'-1-\rho)\delta_{s+s'-2,\rho+\rho'}(\tilde{u}'_\rho(z)\tilde{u}_{\rho'}(z) - \tilde{u}_\rho(z)\tilde{u}'_{\rho'}(z))\delta(z-z').
\end{aligned} \tag{32}$$

Few clarifying remarks are in order. The final result (32) has been organized so that the subscript (spin) of all contributing fields is strictly $\geqq 2$. For example the second term on the right-hand side ($\sim \tilde{u}_{s-2}$) is present only if $s - 2 \geqq 2$; similar restrictions apply to all other terms. The derivatives are always taken with respect to the variable z and not z'. The calculation is straightforward but quite lengthy. We find that many individual terms in (25a–d) diverge when $N \to \infty$ (even after introducing the rescaling (29) and (31)); however, their net contribution to the bracket $[\tilde{u}_s(z), \tilde{u}_{s'}(z')]_\infty$ is zero. We also point out that some of the quadratic terms in Eq. (32) are not manifestly antisymmetric; nevertheless one can easily verify that the total expression we have obtained is in fact antisymmetric, as required. Also the Jacobi identity is automatically satisfied, because Gelfand–Dickey algebras have been designed this way.

The commutation relations (32) provide a Hamiltonian description of the extended conformal algebra W_∞. Setting $s = s' = 2$, we obtain the Virasoro (sub)algebra with $c = 1$. Unfortunately, the generating fields $\{\tilde{u}_s(z);\ s \geqq 3\}$ are not primary (in general). We have

$$[\tilde{u}_2(z), \tilde{u}_s(z')]_\infty = [\tilde{u}_s(z) + (s-1)\tilde{u}_s(z')]\delta'(z-z') + \tfrac{1}{12}\tilde{u}_{s-2}(z')\delta'''(z-z') \qquad (33)$$

for all $s \geqq 4$. Only $\tilde{u}_3(z)$ is a primary field with conformal weight 3. Also, the determining relations (32) are quadratic in nature. The simplest example where quadratic terms occur is provided by $[\tilde{u}_3(z), \tilde{u}_s(z')]_\infty$ with $s \geqq 3$. In this case we have

$$\begin{aligned}[\tilde{u}_3(z), \tilde{u}_s(z')]_\infty &= [2\tilde{u}_{s+1}(z) + (s-1)\tilde{u}_{s+1}(z')]\delta'(z-z') \\ &\quad - (\tilde{u}_2(z)\tilde{u}_{s-1}(z) + \tilde{u}_2(z')\tilde{u}_{s-1}(z'))\delta'(z-z') \\ &\quad + (\tilde{u}_2'(z)\tilde{u}_{s-1}(z) - \tilde{u}_2(z)\tilde{u}_{s-1}'(z))\delta(z-z'). \end{aligned} \qquad (34)$$

Notice that the right-hand side of Eq. (34) involves linear terms with spin higher than s as well as quadratic terms. This is a characteristic behavior of higher spin algebras, in any number of space-time dimensions.

The standard attitude in higher spin theories is that consistent self-interacting gauge theories of massless particles with $s > 2$ require the inclusion of an infinite family of fields with all integer spins $s = 2, 3, \ldots, N \to \infty$. In terms of the corresponding infinite dimensional algebra of gauge transformations this means that all quadratic terms should disappear when $N \to \infty$. On these grounds we expect that only the leading (highest spin) linear terms will actually contribute in the large N limit of W_N. Certainly, this behavior is not manifest in Eq. (32). Nevertheless, we are going to show that all quadratic terms are trivial, in the sense that there exist fields

$$w_s(z) = \tilde{u}_s(z) + \sum_{\{i\},\{k\}} A^{i_1 \cdots i_p}_{k_1 \cdots k_p} \tilde{u}^{(k_1)}_{i_1}(z) \cdots \tilde{u}^{(k_p)}_{i_p}(z) \qquad (35)$$

with linear commutation relations. $A^{\{i\}}_{\{k\}}$ are numerical constants which will be calculated shortly. The integers $\{i\}$ and $\{k\}$ are not arbitrary but satisfy the constraint $k_1 + \cdots + k_p + i_1 + \cdots + i_p = s$, for all $s = 4, 5, \ldots$. We also assume that $A^{\{i\}}_{\{0\}} = 0$ if $p = 1$.

At this point it is instructive to compare Eq. (35) with (21). Recall that the coordinate (Lax) fields $\{u_s(z)\}$ of the Gelfand–Dickey algebra $\mathrm{GD}(A_{N-1})$ are not primary in general. However, for appropriate choices of the structure constants $A^{\{i\}}_{\{k\}}$, the polynomial combinations (21) yield primary conformal fields $w_s(z)$ of weight s. The classical (Hamiltonian) description of W_N (for all $N = 2, 3, \ldots$) is formulated entirely in terms of the field variables $\{w_s\}$ and not $\{u_s\}$ (see [11, 13, 14]). For this reason we need to know which are the appropriate combinations to consider for the generators of W_∞. It is quite remarkable that the choice (35) not only eliminates all non-linear terms, but also provides the primary conformal fields (generators) of W_∞. The calculations are once more a little bit complicated; we only indicate how to proceed and then state the result for the general form of the numerical coefficients $A^{\{i\}}_{\{k\}}$.

The key formula is provided by the commutation relations (34), which play the role of a "generating function" for all $w_s(z)$ with $s \geqq 4$. To be more specific, we start with $w_3(z) := \tilde{u}_3(z)$ which is a primary conformal field when $N \to \infty$. We find that

$$[w_3(z), w_3(z')]_\infty = 2(w_4(z) + w_4(z'))\delta'(z - z'), \tag{36}$$

where

$$w_4(z) = \tilde{u}_4(z) - \tfrac{1}{2}\tilde{u}_2^2(z). \tag{37}$$

Next we calculate the bracket between $w_3(z)$ and $w_4(z')$. We find that

$$[w_3(z), w_4(z')]_\infty = (2w_5(z) + 3w_5(z'))\delta'(z - z'), \tag{38}$$

where

$$w_5(z) = \tilde{u}_5(z) - \tilde{u}_2(z)\tilde{u}_3(z). \tag{39}$$

Iterating this procedure we obtain a whole tower of fields of the form (35). The next few members are given by the following expressions:

$$w_6 = \tilde{u}_6 - \tilde{u}_2\tilde{u}_4 - \tfrac{1}{2}\tilde{u}_3^2 + \tfrac{1}{3}\tilde{u}_2^3, \tag{40}$$

$$w_7 = \tilde{u}_7 - \tilde{u}_3\tilde{u}_4 - \tilde{u}_2\tilde{u}_5 + \tilde{u}_2^2\tilde{u}_3, \tag{41}$$

$$w_8 = \tilde{u}_8 - \tilde{u}_2\tilde{u}_6 - \tilde{u}_3\tilde{u}_5 - \tfrac{1}{2}\tilde{u}_4^2 + \tilde{u}_2\tilde{u}_3^2 + \tilde{u}_2^2\tilde{u}_4 - \tfrac{1}{4}\tilde{u}_2^4, \tag{42}$$

$$w_9 = \tilde{u}_9 - \tilde{u}_2\tilde{u}_7 - \tilde{u}_3\tilde{u}_6 - \tilde{u}_4\tilde{u}_5 + \tilde{u}_2^2\tilde{u}_5 + 2\tilde{u}_2\tilde{u}_3\tilde{u}_4 + \tfrac{1}{3}\tilde{u}_3^3 - \tilde{u}_2^3\tilde{u}_3, \tag{43}$$

$$\begin{aligned} w_{10} = {}& \tilde{u}_{10} - \tilde{u}_2\tilde{u}_8 - \tilde{u}_3\tilde{u}_7 - \tilde{u}_4\tilde{u}_6 - \tfrac{1}{2}\tilde{u}_5^2 + \tilde{u}_2^2\tilde{u}_6 + \tilde{u}_2\tilde{u}_4^2 \\ & + 2\tilde{u}_2\tilde{u}_3\tilde{u}_5 + \tilde{u}_3^2\tilde{u}_4 - \tilde{u}_2^3\tilde{u}_4 - \tfrac{3}{2}\tilde{u}_2^2\tilde{u}_3^2 + \tfrac{1}{5}\tilde{u}_2^5 \end{aligned} \tag{44}$$

and so on. For simplicity we have suppressed the z-dependence of the variables $\tilde{u}$.

All these fields are designed so that for all $s \geqq 3$ we have

$$[w_3(z), w_s(z')]_\infty = [2w_{s+1}(z) + (s-1)w_{s+1}(z')]\delta'(z - z'). \tag{45}$$

It is interesting to notice that no derivative terms appear in the expressions for $\{w_s\}$. Therefore, the structure constants $A^{\{i\}}_{\{k\}}$ in Eq. (35) are independent of $\{k\}$. It is very straightforward now to guess the general formula for w_s with arbitrary $s \geqq 4$. It is given by

$$w_s(z) = \sum_{\{i\},\{n\}} \frac{(-)^{n_1 + \cdots + n_p + 1}}{n_1 + n_2 + \cdots + n_p} \tilde{u}_{i_1}^{n_1}(z)\tilde{u}_{i_2}^{n_2}(z)\cdots\tilde{u}_{i_p}^{n_p}(z), \tag{46}$$

with $n_1 i_1 + n_2 i_2 + \cdots + n_p i_p = s$. We emphasize that here, the integers $n_1, n_2, \ldots, n_p$ denote powers and not derivatives of the fields $\tilde{u}_{i_1}, \tilde{u}_{i_2}, \ldots, \tilde{u}_{i_p}$. Also the subscripts $i_1, i_2, \ldots, i_p$ (spins) are not ordered in any particular way and their values are all $\geqq 2$. For example, to obtain w_7, we partition 7 as $\langle 7\cdot 1\rangle$, $\langle 3\cdot 1, 4\cdot 1\rangle$, $\langle 2\cdot 1, 5\cdot 1\rangle$ and $\langle 2\cdot 2, 3\cdot 1\rangle$. Then, we write down the corresponding polynomial combinations $\tilde{u}_7, \tilde{u}_3\tilde{u}_4, \tilde{u}_4\tilde{u}_3, \tilde{u}_2\tilde{u}_5, \tilde{u}_5\tilde{u}_2, \tilde{u}_2^2\tilde{u}_3, \tilde{u}_2\tilde{u}_3\tilde{u}_2, \tilde{u}_3\tilde{u}_2^2$ and weight them with $+1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, +\frac{1}{3}, +\frac{1}{3}, +\frac{1}{3}$ respectively. Summing up all these terms we recover Eq. (41). Similarly we construct any other field w_s that satisfies the relation (45).

Using the expression (46) and the commutation relations (32) it is possible to show that

$$\begin{aligned} [w_s(z), w_{s'}(z')]_\infty = {}& [(s-1)w_{s+s'-2}(z) + (s'-1)w_{s+s'-2}(z')]\delta'(z - z') \\ & + \tfrac{1}{12}\delta_{s,2}\delta_{s',2}\delta'''(z - z') \end{aligned} \tag{47}$$

is true for all values of s and $s' = 2, 3, 4, \ldots$. Therefore, the structure of the extended conformal algebra W_∞ is linear and the assertion we made earlier is proven. From now on we concentrate on Eq. (47) and discard the intermediate variables $\tilde{u}$ used in the course of this investigation.

The final piece of information comes from a recent paper by Bilal [15]. It was shown that in the $N \to \infty$ limit, the quantum (commutator) algebra of $\{w_s(z); s = 2, 3, \ldots, N\}$ reduces to the classical (Poisson bracket) algebra W_∞. In particular, the Gelfand–Dickey calculation provides the dominant contribution to the structure constants of W_N when N is large. To put it differently, the normal ordering prescription deforms the numerical coefficients in (25a–d) only by a negligible amount which has no effect on the final result for $[w_s(z), w_{s'}(z')]_\infty$. It is then safe to conclude that the chiral operator algebra W_∞ of two dimensional (quantum) conformal field theory is described precisely by the commutation relations (47).

4. Unitary Representations of $\mathbf{Diff}_0^+ \, \mathbb{R}^2$

Introducing Fourier components $w_s(n)$ ($n \in \mathbb{Z}$) for the fields $w_s(z)$ ($s = 2, 3, \ldots$), we obtain

$$[w_s(n), w_{s'}(m)] = [(s' - 1)n - (s - 1)m] w_{s+s'-2}(n + m) + \frac{c}{12} n^3 \delta_{n+m,0} \delta_{s,2} \delta_{s',2}, \quad (48)$$

with $c = 1$. Arbitrary values of the central charge will arise if we make use of Eqs. (29) and (31) with $c \neq 1$ in general. For this reason we may consider Eq. (48) as the (quantum mechanical) commutation relations of the chiral algebra W_∞ with arbitrary central charge c. For convenience we also drop the subscript (∞) of the GD-bracket (47). The observation that the structure constants of W_∞ are given by Eq. (5) was made first in reference [3], where the leading order (highest spin) behavior of W_∞ was studied. The present result completes this investigation and shows that no other terms appear in the commutation relations of W_∞, apart from a central (cocycle) term in the Virasoro subalgebra.

It is quite natural now to look for a geometric interpretation of the infinite dimensional symmetry algebra (48). This is certainly possible because the determining relations of W_∞ are linear. We will see that the right framework for our purposes is provided by the algebra of area preserving diffeomorphisms of the 2-plane, $\mathrm{Diff}_0 \, \mathbb{R}^2$. To be more precise let us consider the Lie algebra of all Hamiltonian (i.e., divergenceless) vector fields on $\mathbb{R}^2$ with commutation relations

$$[\xi_f, \xi_g] = \xi_{\{f,g\}}. \quad (49)$$

Here, $\{f, g\}$ denotes the Poisson bracket between any two functions f, g on $\mathbb{R}^2$, i.e.

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \quad (50)$$

and

$$\xi_f = \frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \quad (51)$$

is the Hamiltonian vector field associated with f. The coordinate functions (x, y) on $\mathbb{R}^2$ have been chosen so that $\{x, y\} = 1$. The symmetry algebra (49) arises in 1-dim classical mechanics and generates all canonical transformations on the phase space $\mathbb{R}^2$. In this setting, the variables x and y represent the position and conjugate momentum (respectively) of a particle with one degree of freedom.

Next, we introduce the basis elements

$$f_{m,n} = x^{m+1} y^{n+1}; \quad m, n \in \mathbb{Z} \tag{52}$$

for the generating functions of $\mathrm{Diff}_0 \mathbb{R}^2$. It is straightforward to verify that in this basis the commutation relations (50) become

$$\{f_{m,n}, f_{m',n'}\} = [(m+1)(n'+1) - (m'+1)(n+1)] f_{m+m',n+n'}. \tag{53}$$

This is a double graded algebra which contains the Virasoro as subalgebra (with central charge $c = 0$). Indeed, the basis elements $\{f_{m,0}; m \in \mathbb{Z}\}$ satisfy the relations

$$\{f_{m,0}, f_{m',0}\} = (m - m') f_{m+m',0}. \tag{54}$$

There are many other interesting subalgebras of (53). Some of them have already been considered in the mathematics literature, in connection with certain problems in (co)homology [16, 17]. Explicit calculations have shown that the homologies of arbitrary fixed dimension of the Lie algebra of polynomial Hamiltonian vector fields on $\mathbb{R}^2$ are finite dimensional. Also, the subalgebra of polynomial vector fields generated by all even functions in x and y has been used extensively in the cohomology theory of differential operator algebras. For our purposes it is natural to consider the subalgebra of Hamiltonian vector fields

$$\mathrm{Diff}_0^+ \mathbb{R}^2 : \{f_{m,n}; m \in \mathbb{Z}, n \geqq 0\} \tag{55}$$

which are polynomial only in the y variable. We will prove shortly that W_∞ is a central extension of $\mathrm{Diff}_0^+ (\mathbb{R}^2)$.

Notice that the infinite dimensional space (55) is a module for the Virasoro algebra whose action is defined as follows:

$$f_{k,0} \cdot f_{m,n} := \{f_{k,0}, f_{m,n}\} = -[(m+1) - (n+1)(k+1)] f_{m+k,n}. \tag{56}$$

It can be readily checked that the central charge c is zero. Equation (56) has an important meaning in conformal field theory. It states that (for fixed n) $f_{m,n}$ are the basis elements of a primary conformal field of weight $\Delta = -(n+1)$. Indeed, in the holomorphic representation $L_k = -z^{k+1} \frac{d}{dz}$, primary fields transform according to the rule

$$L_k (z^{m+1} dz^\Delta) = -[(m+1) + \Delta(k+1)] z^{m+k+1} dz^\Delta. \tag{57}$$

Therefore, $\mathrm{Diff}_0^+ \mathbb{R}^2$ can be viewed as an infinite dimensional higher spin conformal symmetry algebra. The Fourier components of the generating conformal fields $\{w_s\}$ are $f_{m,n}$ and the allowed values for the weight (spin) s are $1 - \Delta = n + 2 = 2, 3, 4, \ldots$. (Recall that Δ and $1 - \Delta$ are dual to each other.) To make exact contact with the commutation relations (48), we introduce the variables

$$w_s(n) = f_{n+s-2,s-2}; \quad n \in \mathbb{Z}, s \geqq 2. \tag{58}$$

Then Eq. (53) yields

$$\{w_s(n), w_{s'}(m)\} = [(s'-1)n - (s-1)m] w_{s+s'-2}(n+m), \tag{59}$$

which is identical to Eq. (48), up to central charge terms. We conclude immediately that W_∞ is a central extension of $\mathrm{Diff}_0^+ \mathbb{R}^2$, with non-trivial cocycle appearing only in the commutation relations of its Virasoro subalgebra.

This result is quite important for the geometric interpretation of higher spin symmetries in two dimensions. Area preserving diffeomorphisms are certainly easy to comprehend geometrically. However, we have no good explanation for their origin at the moment, other than the algebraic identification of W_∞ with $\mathrm{Diff}_0^+ \mathbb{R}^2$. Why area preserving diffeomorphisms and not some other infinite dimensional Lie algebra? We think that 2-dim conformal field theories with an infinite number of additional conserved currents can be interpreted as (2 + 1)-dim quantum field theories. The higher dimensional viewpoint we propose in Sect. 5 is very suggestive and could provide a physical answer to this question.

$\mathrm{Diff}_0^+ \mathbb{R}^2$ is also interesting from a mathematical point of view. We point out that the full algebra of area preserving diffeomorphisms of $\mathbb{R}^2$ admits the following decomposition:

$$\mathrm{Diff}_0 \mathbb{R}^2 = \mathrm{Diff}_0^+ \mathbb{R}^2 \oplus H \oplus \mathrm{Diff}_0^- \mathbb{R}^2. \tag{60}$$

Here, $\mathrm{Diff}_0^+ \mathbb{R}^2$ is generated by $\{f_{m,n}; m\in\mathbb{Z}, n \geqq 0\}$, H by $\{f_{m,-1} = x^{m+1}; m\in\mathbb{Z}\}$ and $\mathrm{Diff}_0^- \mathbb{R}^2$ by $\{f_{m,n}; m\in\mathbb{Z}, n < -1\}$. It is clear that H is an abelian subalgebra of $\mathrm{Diff}_0 \mathbb{R}^2$, i.e. $\{f_{m,-1}, f_{m',-1}\} = 0$. Also, notice that the Belavin–Polyakov–Zamolodchikov duality between conformal fields of weight Δ and $1-\Delta$ [1] implies that

$$(\mathrm{Diff}_0^- \mathbb{R}^2)^* \cong \mathrm{Diff}_0^+ \mathbb{R}^2 \oplus H. \tag{61}$$

However, "self-dual" fields have weight $\Delta^* = \Delta$, i.e. $\Delta = 1/2$, which is not integer. Since the basis element $\{f_{m,-1}; m\in\mathbb{Z}\}$ have $\Delta = 0$, H is not a Cartan subalgebra of $\mathrm{Diff}_0 \mathbb{R}^2$. Also, one may verify directly that $\{H, \mathrm{Diff}_0^\pm \mathbb{R}^2\} \not\subseteq \mathrm{Diff}_0^\pm \mathbb{R}^2$ and so (60) is not a Cartan decomposition of $\mathrm{Diff}_0 \mathbb{R}^2$. In fact we have no such candidate for $\mathrm{Diff}_0 \mathbb{R}^2$. On the other hand, area preserving diffeomorphisms of compact 2-surfaces, e.g. the torus $S^1 \times S^1$, share many common features with simple Lie algebras. As we will see later, $\mathrm{Diff}_0(S^1 \times S^1) \cong A_\infty$.

Unitary representations of area preserving diffeomorphisms pose a difficult problem in mathematics. Of course we already know a unitary representation of $\mathrm{Diff}_0 \mathbb{R}^2$ from elementary quantum mechanics. It is defined by assigning the operators

$$\hat{F} = i\hbar \xi_f + f - y\frac{\partial f}{\partial y} \tag{62}$$

to all classical functions f on $\mathbb{R}^2$. ξ_f is the Hamiltonian vector field (51) and the Hilbert space where the operators (62) act consists of all square integrable functions on $\mathbb{R}^2$, $L^2(\mathbb{R}^2)$. This representation is known as prequantization in the theory of geometric quantization (see for instance [18]). Unfortunately, it is not relevant for quantum field theory because $L^2(\mathbb{R}^2)$ cannot accommodate an infinite number of degrees of freedom. Nevertheless, it is possible to gain some information about the (field theoretic) representations of $\mathrm{Diff}_0^+ \mathbb{R}^2$ using the connection we found with the symmetry algebra W_∞.

It is known that unitary (highest weight) representations of the chiral operator algebras W_N ($N = 2, 3, \ldots$) correspond to 2-dim conformally invariant models with (global) $\mathbb{Z}_N$ symmetry [1,2]. These theories are minimal models of the W_N algebras (in the sense that they have a finite number of conformal building blocks) and fall into classes (series) depending on N. For any (fixed) N, the only allowed values for the central charge c are given by the rational numbers

$$c_p^{(N)} = (N-1)\left[1 - \frac{N(N+1)}{p(p+1)}\right]; \quad p = N+1, N+2, \ldots. \tag{63}$$

Moreover, the spectrum of anomalous dimensions of W_N-invariant fields is

$$h(k_1, \ldots, k_{N-1}; k'_1, \ldots, k'_{N-1}) = \frac{12\left[\sum_{i=1}^{N-1} (pk_i - (p+1)k'_i)\omega_i\right]^2 - N(N^2-1)}{24p(p+1)}, \tag{64}$$

where $\{k_i\}$ and $\{k'_i\}$ are all positive integers subject to the constraints

$$\sum_{i=1}^{N-1} k_i \leqq p; \quad \sum_{i=1}^{N-1} k'_i \leqq p-1 \tag{65}$$

and $\{\omega_i\}$ are the fundamental weights of A_{N-1} satisfying

$$\omega_i \cdot \omega_j = \frac{i(N-j)}{N}, \quad \text{for} \quad i \leqq j. \tag{66}$$

It is clear that when $N \to \infty$, the only values of c allowed by unitarity are

$$c_k = 2k; \quad k = 1, 2, 3, 4, \ldots, \tag{67}$$

and the number of conformal (building) blocks of all W_∞-minimal models is infinite. This procedure suggests that highest weight (Verma) module representations of the centrally extended $\mathrm{Diff}_0^+ \mathbb{R}^2$ algebra (48) can be constructed in analogy with the Virasoro algebra. In particular, we introduce a normalized state $|h\rangle$ (so that $\langle h|h\rangle = 1$), with the property

$$w_s(n)|h\rangle = 0, \quad \text{for} \quad n > 0, \tag{68a}$$

$$w_s(0)|h\rangle = h_s|h\rangle \tag{68b}$$

for all $s \geqq 2$. Then, the Verma module of W_∞ is obtained by successive application of the operators $\{w_s(n); s \geqq 2, n < 0\}$ on $|h\rangle$. For central charge $c_k = 2k$ ($k = 1, 2, 3, \ldots$) the roots of the corresponding Kac determinant ($h_{s=2}$) will be given by Eq. (64) in the limit $N \to \infty$. Notice that many of these numbers are zero. Since $\omega_1 + \cdots + \omega_{N-1} = \rho$, we have that $h(1, 1, \ldots; 1, 1, \ldots) = 0$. Also for certain other choices of $\{k_i\}$ and $\{k'_i\}$, the numerator in (64) diverges (typically) like N and the denominator like N^2. However, it is also possible for some of the h's to be infinite.

All structural details in the representations of W_N are determined by the simple Lie algebra A_{N-1}. When $N \to \infty$, the number of vertices in the corresponding Dynkin diagram becomes infinite and so does the dimensionality of the Cartan matrix. Then it is natural to trade the infinite number of Lie algebra labels with one continuous parameter. But this is equivalent to introducing an extra dimension in space-time and reformulating the underlying field theory in $2+1$ dimensions. Of course, we have to make additional regularizations in quantum theory to avoid divergencies from the A_∞-degrees of freedom. For instance, in the free field representation for the stress-energy tensor $T(z)$, the summation over a has to be regulated appropriately when $N \to \infty$.

It is (more or less) obvious now that the systematic study of W_∞ and its unitary field theoretic representations require a higher dimensional viewpoint. In the next section we present a general framework for this study and derive some (partial) results. However, the complete picture is still lacking.

5. W_∞ From a 2 + 1 Viewpoint

The key idea for the rest of our investigation comes from the recent work of Saveliev and collaborators [8]. These authors proposed a new class of infinite dimensional Lie algebras–the continual generalizations of $\mathbb{Z}$-graded Lie algebras, which have an infinite dimensional Cartan subalgebra and a continuous set of roots. Their approach incorporates many physically interesting symmetries, including arbitrary current algebras, the algebra of Poisson brackets and the algebra of vector fields on a manifold (diffeomorphisms). However, for our needs, we only have to consider the large N limit of A_N and its continual realization. Somewhat different descriptions of A_∞ have also been discussed in refs. [19–21]. For notational purposes we review first the main constructions of ref. [8] and then apply them to W_∞. This way we set the basis for a new representation of W_∞ in terms of a single scalar field in 2 + 1 dimensions.

Let X_i, H_i, Y_i $(i = 1, 2, \ldots, N-1)$ be the system of Chevalley–Weyl generators for A_{N-1} with commutation relations (see for instance [22])

$$[H_i, H_j] = 0; \qquad [X_i, Y_j] = \delta_{ij} H_i, \tag{69a}$$

$$[H_i, X_j] = K_{ij} X_j; \quad [H_i, Y_j] = -K_{ij} Y_j. \tag{69b}$$

Of course, there is no summation over repeated indices in the equations above and K is the (non-degenerate) $(N-1) \times (N-1)$ Cartan matrix of A_{N-1},

$$K = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}. \tag{70}$$

It is convenient to introduce a parameter Δt (which depends on N) and rescale the generators $\{X_i, H_i, Y_i\}$ as follows:

$$\tilde{H}_i = \frac{H_i}{\Delta t^3}; \quad \tilde{X}_i = \frac{X_i}{\Delta t^2}; \quad \tilde{Y}_i = \frac{Y_i}{\Delta t^2}. \tag{71}$$

Then the structure of the defining relations (69) remains the same, with the only difference that δ_{ij} and K_{ij} are replaced by

$$\tilde{\delta}_{ij} = \frac{\delta_{ij}}{\Delta t}; \quad \tilde{K}_{ij} = \frac{K_{ij}}{\Delta t^3}. \tag{72}$$

When $N \to \infty$, it is possible to choose Δt appropriately so that $\tilde{\delta}$ and $\tilde{K}$ become distributions of some continuous variable t. In this case we interpret Eqs. (72) as discretized versions of the generalized functions $\delta(t-t')$ and $\mathscr{K}(t,t') = -\delta''(t-t')$ respectively. With this prescription in mind, it is natural to introduce a one-parameter family of generators $\{\mathscr{X}(t), \mathscr{H}(t), \mathscr{Y}(t)\}$ which satisfy the commutation relations

$$[\mathscr{H}(t), \mathscr{H}(t')] = 0; \quad [\mathscr{X}(t), \mathscr{Y}(t')] = \delta(t-t')\mathscr{H}(t), \tag{73a}$$

$$[\mathscr{H}(t), \mathscr{X}(t')] = \mathscr{K}(t,t')\mathscr{X}(t'); \quad [\mathscr{H}(t), \mathscr{Y}(t')] = -\mathscr{K}(t-t')\mathscr{Y}(t'). \tag{73b}$$

The set of generators (71) is a discretization of $\{\mathscr{X}(t), \mathscr{H}(t), \mathscr{Y}(t)\}$ and Eqs. (73) can be thought as the defining (Weyl) relations of A_∞. In analogy with finite dimensional simple Lie algebras, the rest of the A_∞ generators are obtained by taking successive commutators of the $\mathscr{X}$'s and $\mathscr{Y}$'s and imposing Serre's condition.

The limiting procedure we adopt here is well motivated by (certain) physical considerations in elementary quantum mechanics. To be more precise, let $S[q] = \frac{1}{2}\int dt\,\dot{q}(t)^2$ be the action of a 1-dim free particle propagating in time from t_0 to t_f. The quantum mechanical propagator of this system, $\langle q_0, t_0 | q_f, t_f \rangle$, is computed in a standard way by introducing a time skeletonization $(t_0, t_1 = t_0 + \Delta t, \ldots, t_f = t_0 + (N-2)\Delta t)$. In other words, we think of the Dynkin diagram of A_{N-1}

$$\begin{array}{ccccc} 1 & 2 & 3 & & N-1 \\ 0 \!-\!\!-\!\!- & 0 \!-\!\!-\!\!- & 0 \!-\!\!-\!\!- & \cdots \!-\!\!-\!\!- & 0 \\ (t_0) & & & & (t_f) \end{array} \tag{74}$$

as representing the propagation of a free particle in $N-2$ discrete time steps Δt from the original to the final configuration. Introducing small fluctuations $\xi(t)$ around the classical path of the system, $q(t) = q_{\text{cl}}(t) + \xi(t)$ one finds that $\langle q_0, t_0 | q_f, t_f \rangle = C \exp\left\{\frac{i}{\hbar} S_{\text{cl}}[q]\right\}$, where C is a constant written in terms of the determinant of $\mathscr{K}(t,t') := \dfrac{\delta^2 S}{\delta q(t) \delta q(t')} = -\delta''(t-t')$. When N is finite, $\det \mathscr{K}$ is approximated by $(\Delta t)^{-3} \det K$, the determinant of the Cartan matrix (70). To obtain the complete quantum mechanical answer for the propagator, we pass to the continuum limit $\Delta t \to 0$ by letting $N \to \infty$. Since $\mathscr{K}(t,t')$ is precisely the defining distribution of the commutation relations (73), the analogy with techniques used in the path integral quantization of a free particle shows that Saveliev's description of A_∞ as a continual Lie algebra is very natural physically.

It is advantageous to adopt the notion of continual algebras and try to reformulate conformal field theories with an infinite collection of additional (higher spin) conserved currents in terms of 3-dim physics. In this case, as we have already indicated, the extra (third) dimension of space-time is associated with the continuous set of roots of A_∞. The picture we envision here is best described by the field theoretic representations (6)–(13) of W_N in the limit $N \to \infty$. Notice that in the free field as well as the Toda field representations of W_N, the components of the scalar field Φ, $\{\phi_a; a = 1, 2, \ldots, N-1\}$, are labeled by the roots of the simple Lie algebra A_{N-1}. Therefore, in the large N limit, it is appropriate to replace the infinite collection of fields $\{\phi_a\}$ by a single scalar field $\Phi(t)$ which depends on a continuous variable t. Of course the z and/or $\bar{z}$ dependence of $\Phi(t)$ has been suppressed to simplify the notation.

It is necessary now to construct the continual analogue of the free and Toda field theory representations of W_∞. This way we hope to obtain an intrinsic 3-dim description of $\mathrm{Diff}_0^+ \mathbb{R}^2$ and its unitary representations. Using a single scalar field $\Phi(z,\bar{z},t)$, the Toda field equations (12) for A_∞ assume the following form:

$$\partial\bar{\partial}\Phi(z,\bar{z},t) = \exp\left(\int dt' \mathscr{K}(t,t')\Phi(z,\bar{z},t')\right) = \exp\left(-\Phi''(z,\bar{z},t)\right). \tag{75}$$

Then it is straightforward to verify that the current

$$w_2(z) := \int dt\,dt' \left[-\tfrac{1}{2}\partial\Phi(z,\bar{z},t)\mathscr{K}(t,t')\partial\Phi(z,\bar{z},t') + \delta(t-t')\partial^2\Phi(z,\bar{z},t)\right] \tag{76}$$

is conserved, i.e. $\bar{\partial}w_2(z) = 0$, as a consequence of Eq. (75). Comparison with Eq. (13) shows that $w_2(z)$ can be regarded as the (improved) stress-energy tensor $T(z)$ of the continual A_∞ Toda field theory with action

$$S[\Phi] = \int dz\,d\bar{z}\,dt\left[\tfrac{1}{2}\partial\Phi(z,\bar{z},t)\hat{\mathscr{K}}\bar{\partial}\Phi(z,\bar{z},t) + e^{\hat{\mathscr{K}}\Phi(z,\bar{z},t)}\right]. \tag{77}$$

Clearly, $S[\Phi]$ describes a 3-dim field theory whose equations of motion are given by (75). Here we have used implicitly the (short-hand) notation

$$\hat{\mathscr{K}}\Phi(z,\bar{z},t) = \int dt' \mathscr{K}(t,t')\Phi(z,\bar{z},t'). \tag{78}$$

In analogy with the results obtained in ref. [11], it is possible to construct an infinite list of higher (characteristic) fields $\{w_s(\partial\Phi(t), \partial^2\Phi(t), \ldots, \partial^s\Phi(t)); s = 2, 3, \ldots\}$ with spin s, so that $\bar{\partial}w_s(z) = 0$ for all values of s. These fields will certainly provide the representation of W_∞ we are looking for. Since their existence is guaranteed by the general theory of Toda models, it is reasonable to expect that $\{w_s(z)\}$ are s-fold integrals of the form

$$w_s(z) = \int dt_1 \cdots dt_s \mathscr{M}(t_1, \ldots, t_s)\partial\Phi(z,\bar{z},t_1)\cdots\partial\Phi(z,\bar{z},t_s) + \text{higher derivative terms}. \tag{79}$$

$\mathscr{M}(t_1, \ldots, t_s)$ and the coefficients of all other terms are generalized functions written in terms of $\delta(t)$ and its derivatives. They are the continual analogue of the formulas derived by Bilal and Gervais in [11].

It is unfortunate that the explicit expressions for the generating fields $\{w_s(z); s \leqq N\}$ of W_N are complicated and quite difficult to derive. Nevertheless we think that the Toda field representation of W_N simplifies considerably in the limit $N \to \infty$. In fact we only need to know the continual analogue of the chiral field $w_3(z)$, because the rest of the generators $\{w_s(z); s \geqq 4\}$ can be obtained by successive application of the commutation relations (45). Recall from Sect. 3 that Eq. (45) acts as a recursive relation for all generating higher spin fields of the infinite dimensional Lie algebra W_∞. This procedure provides a practical and efficient way to construct any field theoretic representation of W_∞, once the explicit (continual) expressions for $w_2(z)$ and $w_3(z)$ are known. However, the results we have obtained so far are incomplete and we postpone their presentation for future publication [23].

It is clear now that similar considerations apply to the free field representation of W_∞. For this reason we do not repeat the arguments we gave earlier. However, there are a few additional remarks we would like to make next. It is known that the free field representation of W_N (for $N=2,3,\ldots$) is described in terms of the Heisenberg (oscillator) algebra

$$[A_a(n), A_b(m)] = n\delta_{n+m,0}\delta_{a,b}; \quad n,m\in\mathbb{Z}; \quad a,b=1,2,\ldots,N-1. \tag{80}$$

The operators $A_a(n)$ are the Fourier components of $\partial\phi_a(z)$ and the system of conformal fields $\{w_s(z); s=2,3,\ldots,N\}$ belongs in the universal enveloping algebra of (80) [1,2]. Therefore, when $N\to\infty$, there is an infinite number of Lie algebra (A_∞) degrees of freedom and additional regularization are required in field theory. This point has to be taken into account when we construct the continual analogue of the Feigin–Fuks (free field) representation, or any other representation of W_∞.

Finally, it would be interesting to develop a BRST quantization of field theories with W_∞ as the underlying symmetry algebra. It is known that the BRST operator Q of the chiral operator algebra W_N ($N=2,3,\ldots$) is nilpotent, provided that the central charge of the theory is [24]

$$c = 2\sum_{s=2}^{N}(6s^2-6s+1). \tag{81}$$

Notice that when $N\to\infty$, c diverges like ∞^3 which is not very satisfactory. In this case we have an infinite collection of ghost and conjugate-ghost fields $(b_s(z), c_s(z))$ and each one contributes $2(6s^2-6s+1)$ to the total value of the central charge (81). Here, we also expect that appropriate regularization of the A_∞-algebra degrees of freedom will renormalize the unwanted divergencies in the ghost sector of the theory and produce a finite answer for the "critical" value of c. This problem will be addressed elsewhere.

6. Conclusions

We have shown that the infinite dimensional symmetry algebra W_∞ of 2-dim conformal field theory is a central extension of $\mathrm{Diff}_0^+\,\mathbb{R}^2$. We also proposed representations of W_∞ in terms of 3-dim scalar field theories. This description, although incomplete for the moment, could help us understand why area preserving diffeomorphisms arise in the large N limit of extended conformal symmetries. It is crucial to realize that in this case, the symmetry algebra of area preserving diffeomorphisms does not refer to the 2-dim world M on which chiral W_∞-conformal field theories are defined. If that were the case, the transformations $(z,\bar{z})\to(\sigma_1(z,\bar{z}),\sigma_2(z,\bar{z}))$ with Jacobian $J(\sigma_1,\sigma_2)=1$ would mix the (local) light-cone coordinates $z,\bar{z}$ and violate chirality. Two dimensional theories in the fixed area gauge break scale invariance and therefore fail to be conformal. In order to give a geometric interpretation to the algebra of area preserving diffeomorphisms associated with W_∞, one has to introduce an auxiliary surface (membrane) and reformulate the theory in $2+1$ dimensions. Then, according to Eq. (54), point canonical transformations on the membrane induce conformal transformations on M.

It is quite remarkable that both infinite dimensional Lie algebras W_∞ and A_∞ are described in terms of area preserving diffeomorphisms of 2-surfaces. The latter is identified with $\mathrm{Diff}_0(S^1\times S^1)$, the algebra of area preserving diffeomorphisms of the torus [8,20,21]. Although we have no good explanation for this occurrence, it is tempting to speculate that membrane dynamics could provide a non-perturbative framework for string quantization. In particular, it would be very interesting to construct a consistent theory of strings with W_∞ as the conformal symmetry algebra on the 2-dim world-sheet and study its relation with ordinary W_2-string theory. Work toward this direction is in progress.

Finally, we point out that there are several integrable systems in $2+1$ dimensions that may be viewed as 2-dim models with infinite dimensional structure algebras. Some examples (including the A_∞-Toda field theory (75)) have already been considered in ref. [25]. However, an intrinsic 3-dim description of their quantization is still lacking. We think that a systematic study of these problems will improve our present understanding of the relations between 2 and 3 dimensional physics. It could also lead to new classes of exactly solvable quantum field theories, which are quite different from those considered by Witten [26].

References

1. Itzykson, C., Saleur, H., Zuber, J. B. (eds.): Conformal invariance and applications to statistical mechanics. Singapore: World Scientific 1988
2. Zamolodchikov, A. B.: Theor. Math. Phys. **65**, 1205 (1985); Zamolodchikov, A. B., Fateev, V. A.: Nucl. Phys. **B280[FS18]**, 644 (1987); Fateev, V. A., Lykyanov, S. L.: Int. J. Mod. Phys. **A3**, 507 (1988); Bais, F. A., Bouwknegt, P., Surridge, M., Schoutens, K.: Nucl. Phys. **B304**, 348 (1988); **B304**, 371 (1988)
3. Bakas, I.: Phys. Lett. **B228**, 57 (1989); In the proceedings of the "Trieste Conference on Supermembranes and Physics in 2 + 1 Dimensions," Duff, M., Pope, C., Sezgin, E. (eds.), to be published by World Scientific
4. Duff, M.: Class. Quant. Grav. **5**, 189 (1988); Hoppe, J.: In: Workshop on constraint's theory and relativistic dynamics, Longhi, G., Lusanna, L. (eds.); Singapore: World Scientific 1987; de Wit, B., Hoppe, J., Nicolai, H.: Nucl. Phys. **B305[FS23]**, 545 (1988); Bars, I., Pope, C., Sezgin, E.: Phys. Lett. **B210**, 85 (1988)
5. Vasil'ev, M. A., Fradkin, E. S.: Ann. Phys. **177**, 63 (1987); Berends, F. A., Burgers, G. J. H., Van Dam, H.: Nucl. Phys. **B260**, 295 (1985)
6. Migdal, A.: Phys. Rep. **102**, 199 (1983); Polyakov, A.: Gauge fields and strings, New York: Harwood Academic Press 1987
7. Bilal, A., Gervais, J.-L.: Nucl. Phys. **B326**, 222 (1989)
8. Saveliev, M.: Commun. Math. Phys. **121**, 283 (1989); Saveliev, M., Vershik, A.: Commun. Math. Phys. **126**, 387 (1989); Phys. Lett. **A143**, 121 (1990)
9. Atiyah, M.: Commun. Math. Phys. **93**, 437 (1984)
10. Ashtekar, A., Jacobson, T., Smolin, L.: Commun. Math. Phys. **115**, 631 (1988); Mason, L., Newman, E. T.: Commun. Math. Phys. **121**, 659 (1989)
11. Bilal, A., Gervais, J.-L.: Phys. Lett. **B206**, 412 (1988); Nucl. Phys. **B314**, 646 (1989); Nucl. Phys. **B318**, 579 (1989)
12. Gelfand, I. M., Dorfman, I.: Funct. Anal. Appl. **15**, 173 (1981); Dickey, L. A.: Commun. Math. Phys. **87**, 127 (1982); Manin, Yu. I.: J. Sov. Math. **11**, 1 (1979); Drinfeld, V. G., Sokolov, V. V.: J. Sov. Math. **30**, 1975 (1985); Khovanova, T. G.: Funct. Anal. Appl. **20**, 332 (1986)
13. Bakas, I.: Commun. Math. Phys. **123**, 627 (1989); Phys. Lett. **B213**, 313 (1988); Phys. Lett. **B219**, 283 (1989); In the proceedings of the "18th International Conference on Differential Geometric Methods in Physics," Chau, L.-L., Nahm, W. (eds.), to be published by New York: Plenum Press
14. Yamagishi, K.: Phys. Lett. **B205**, 466 (1988); Mathieu, P.: Phys. Lett. **B208**, 101 (1988); Lykyanov, S.: Funct. Anal. Appl. **22**, 255 (1988)
15. Bilal, A.: Phys. Lett. **B227**, 406 (1989)
16. Fuks, D. B.: Cohomology of infinite dimensional Lie algebras. New York: Consultants Bureau, Plenum Press 1986

17. Fuks, D. B.: Funct. Anal. Appl. **19**, 305 (1985); Feigin, B. L.: Russ. Math. Surv. **43:2**, 169 (1988)
18. Abraham, R., Marsden, J. E.: Foundations of mechanics. Reading, Massachusetts: The Benjamin/Cummings Publishing 1978; Woodhouse, N.: Geometric Quantization. Oxford: Clarendon Press 1980
19. Kac, V. G.: Infinite dimensonal Lie algebras, second edition. Cambridge: Cambridge University Press 1985
20. Hoppe, J.: see ref. [4]; Phys. Lett. **B215**, 706 (1988); Hoppe, J., Schaller, P.: Karlsruhe preprint KA-THEP-16-89 (1989); Pope, C., Stelle, K.: Phys. Lett. **B226**, 257 (1989)
21. Floratos, E., Iliopoulos, J., Tiktopoulos, G.: Phys. Lett. **B217**, 285 (1989); Fairlie, D., Fletcher, P., Zachos, C.: Phys. Lett. **B218**, 203 (1989); Fairlie, D., Zachos, C.: Phys. Lett. **B224**, 101 (1989)
22. Serre, J.-P.: Complex semisimple Lie algebras. Berlin, Heidelberg, New York: Springer 1987
23. Bakas, I.: in preparation
24. Thierry-Mieg, J.: Phys. Lett. **B197**, 368 (1987); Bais, F. A. et al.: see ref. [2]; Bershadsky, M., Ooguri, H.: Commun. Math. Phys. **126**, 49 (1989)
25. Bogoyavlenskii, O. I.: Math. USSR Izv. **31**, 47 (1988)
26. Witten, E.: Commun. Math. Phys. **121**, 351 (1989)
27. Pope, C., Romans, L., Shen, X.: Phys. Lett. **B236**, 173 (1990); Nucl. Phys. **B339**, 191 (1990)
28. Bakas, I., Kiritsis, E.: Nucl. Phys. **B343**, 185 (1990); Grassmannian coset models and unitary representations of W_∞. Mod. Phys. Lett. **A** (to appear)

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Note added in proof. After this work was completed I realized that the tower of higher spin fields (46) can be obtained from the generating function

$$w(t) = \log(1 + \tilde{u}(t)),$$

where

$$w(t) = \sum_{s=2}^{\infty} w_s(z)t^s, \quad \tilde{u}(t) = \sum_{s=2}^{\infty} \tilde{u}_s(z)t^s$$

and $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$.

It also became clear that the large N limit of W_N algebras is not uniquely defined. It turns out that the commutation relations of W_∞ depend on the background charge α_0 used in the Feigin–Fuks free field realization of the W-generators. However, different limiting procedures for the quantum commutation relations of W_N at large N (depending on the choice of α_0) do not affect the leading structure (59), but only the subleading terms associated with derivatives of the fields $w_s(z)$ and central terms. This is so, because the structure constants of the purely polynomial u-terms are independent of the background charge and do not deform upon quantization. Since the rescaling (29), (31) does not change the structure constants of purely polynomial terms, the results of our analysis are valid for them in all cases.

If the background charge is not independent of N but is chosen so that $\alpha_0 \sim 1/N$, in the limit $N \to \infty$ the resulting W_∞ algebra will be well-defined and no rescaling of the generators is necessary. In this case, the quantum commutation relations will be given by Eq. (59), up to subleading terms associated with derivatives of the fields $w_s(z)$ as well as derivatives of δ-functions. However, it is quite difficult to compute the form of these deformation terms directly using the Feigin–Fuks representation. Recently, Pope, Romans and Shen made an ansätz for the complete quantum structure of W_∞ which admits central terms in the commutation relations of all higher spin fields, by requiring linearity and compatibility with the Jacobi identities [27]. Subsequently, it was shown that their algebra provides an adequate description of W_∞ (with $\alpha_0 \sim 1/N$, as $N \to \infty$) using the large N limit of W_N minimal models (Z_∞ parafermions) [28].