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The $\mathfrak{su}(2|2)$ dynamic S-matrix

Niklas Beisert*

Joseph Henry Laboratories, Princeton University,
Princeton, NJ 08544, USA
nbeisert@aei.mpg.de

Abstract

We derive and investigate the S-matrix for the $\mathfrak{su}(2|3)$ dynamic spin chain and for planar $\mathcal{N}=4$ super Yang–Mills. Due to the large amount of residual symmetry in the excitation picture, the S-matrix turns out to be fully constrained up to an overall phase. We carry on by diagonalizing it and obtain Bethe equations for periodic states. This proves an earlier proposal for the asymptotic Bethe equations for the $\mathfrak{su}(2|3)$ dynamic spin chain and for $\mathcal{N}=4$ SYM.

1 Introduction and conclusions

In general, computations in perturbative field theories are notoriously intricate. Recently, the discovery and application of integrable structures in planar four-dimensional gauge theories, primarily in conformal $\mathcal{N}=4$ super Yang–Mills theory, has led to drastic simplifications in determining some quantities. In particular, planar anomalous dimensions of local operators can be mapped to energies of quantum spin chain states thus establishing some relation to topics of condensed matter physics. The Hamiltonian of this system is completely integrable at one loop [1, 2] and apparently even

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*Max-Planck-Institut für Gravitationphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Postdam, Germany

at higher loops [3, 4], cf. the reviews [5–7]. This remarkable feature shows promise that the planar spectrum might be described *exactly* by some sort of Bethe equation. Bethe equations at the one-loop level were given in [2]. At higher loops some similarity of the exact gauge theory result [3, 4, 8] with the Inozemtsev spin chain [9] being observed and Bethe equations for the $\mathfrak{su}(2)$ sector up to three loops were found in [10]. They were then generalized to the other two rank-one sectors, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$, in [11]. All-loop asymptotic Bethe equations for the $\mathfrak{su}(2)$ sector with some more desirable features for $\mathcal{N}=4$ SYM were proposed in [12]. Putting together all these pieces of a puzzle, asymptotic Bethe equations for the complete model were finally proposed in [13].

Bethe equations have since proved very fruitful for the study of the AdS/CFT correspondence [14] and certain limits of it involving large spins [15, 16]. On the string theory side of the correspondence integrability has been established for the classical theory in [17] and evidence for quantum integrability exists [18, 19]. The results for spinning strings [20] and near plane wave strings [21] have led to new insights into the correspondence, see the reviews [5–7, 22, 23] for details and further references.

The Bethe equations for $\mathcal{N}=4$ SYM mentioned earlier have many desired features and they seem to work, but it is fair to say that their origin remains obscure. At the one-loop level the Hamiltonian involves nearest-neighbour interactions only. One can therefore resort to the well-known R-matrix formalism to derive and study the Bethe equations. At higher loops the interactions of the Hamiltonian become more complex: Their range increases with the loop order [3]. Moreover, the length of the spin chain starts to fluctuate, sites are created or destroyed dynamically [4]. These types of spin chains have not been considered extensively and there is no theoretical framework (yet); the higher-loop Bethe equations are at best well-tested conjectures. The situation improved somewhat with the proposal of [11]. By applying the asymptotic coordinate space Bethe ansatz [24], one may extract a two-particle *S-matrix* from the perturbative Hamiltonian. Assuming factorized scattering, this S-matrix is, like the R-matrix, a nearest-neighbour operator. At this stage one can therefore revert to the familiar framework. The resulting asymptotic Bethe equations turn out to reproduce the spectrum accurately [11].

The perturbative S-matrices for all three rank-one sectors, $\mathfrak{su}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$, were derived in [11] up to three loops. The S-matrix in the $\mathfrak{su}(2)$ sector coincides with the all-loop conjecture of [12] which can be read off directly from the asymptotic Bethe equations. Corresponding all-loop conjectures for the other two rank-one sectors were set up in [13]; they have a similarly concise form. All these rank-one sectors can be joined into one larger sector with $\mathfrak{su}(1,1|2)$ symmetry for which an all-loop S-matrix was also conjectured. This conjecture agrees with the Hamiltonian derived in [4] up to three loops in the subsector where both results apply.

It is the purpose of the present investigation to find the complete S-matrix for planar $\mathcal{N}=4$ SYM. This will allow to put the asymptotic Bethe equations conjectured in [13] on a solid footing and hopefully give us a better understanding of the asymptotic Bethe ansatz as well as the integrable structures in gauge theory in general. The partial results mentioned here as well as the resulting Bethe equations suggest that also the complete S-matrix might have a simple form valid to all perturbative orders. A major problem that one has to deal with in finding the S-matrix is that the complete spin chain is dynamic [4], its length fluctuates. In the excitation picture this might appear not to be a problem as the number of excitations is preserved, but even there one finds flavour fluctuations which may appear problematic [13].

An important property of S-matrices is their symmetry. Often they can be constructed from symmetry considerations and a few additional properties. Also the S-matrices appearing in sectors of planar $\mathcal{N}=4$ SYM are largely constrained by their symmetry. A somewhat unusual feature of these particular S-matrices is that the representations in which the excitations transform obey a dispersion relation [25]. This can be related to the fact that the Hamiltonian is part of the symmetry algebra and not some central generator as for most spin chain models. For instance, in the $\mathfrak{su}(1|2)$ sector the all-loop form of the S-matrix has manifest $\mathfrak{su}(1|1)$ symmetry. The full symmetry algebra of $\mathcal{N}=4$ SYM is $\mathfrak{psu}(2,2|4)$. The S-matrix in the excitation picture, however, is manifestly invariant only under a residual algebra which preserves the excitation number. In this case the residual algebra is $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}$, cf. [5]. The excitations transform in a $(\mathbf{2}|\mathbf{2})$ representation under each $\mathfrak{psu}(2|2)$ factor. Both factors share a common central charge $\mathfrak{C}$ which takes the role of the Hamiltonian. To be precise, we will introduce two further unphysical central charges related to the dynamic nature of the spin chain.

For the construction of the S-matrix it turns out to be very helpful that the algebra splits into two (equal) parts: The complete S-matrix can be constructed as a product of two S-matrices, each transforming only under one of the subalgebras. Moreover, as the particle representations of both subalgebras are isomorphic, it is sufficient to construct only one S-matrix with 4^4 components instead of $(4^2)^4$. We can therefore work with a reduced set of $(2|2)$ excitations and an S-matrix transforming under the reduced algebra

$\mathfrak{su}(2|2)$. Incidentally, this coincides with the S-matrix of the maximally compact sector of $\mathcal{N}=4$ SYM which is the $\mathfrak{su}(2|3)$ dynamic spin chain investigated in [4].

As a first step towards the S-matrix, we investigate the residual algebra in Section 2 and find a suitable representation for the excitations. On the one hand, the representation $(\mathbf{2}|\mathbf{2})$ is almost the fundamental of $\mathfrak{su}(2|2)$, but it requires a trivial central charge $C=\pm\frac{1}{2}$. On the other hand, the central charge of $\mathfrak{su}(2|2)$ represents the energy and we know that it is not quantized in units of $\frac{1}{2}$. To circumvent this seeming paradox we enlarge the algebra by two central charges $\mathfrak{P},\mathfrak{K}$.[1] This is indeed possible and allows for a non-trivial $(\mathbf{2}|\mathbf{2})$ representation with one free continuous degree of freedom. We construct this representation subsequently. The two additional central charges can be related to gauge transformations which act non-trivially on individual fields; nevertheless they must annihilate gauge invariant combinations of fields and therefore we can return to $\mathfrak{su}(2|2)$ as the global symmetry.

Having understood the representation of the symmetry algebra, we construct the S-matrix as an invariant permutation operator on two-excitation states in Section 3. Astonishingly, the S-matrix is uniquely determined up to an overall phase. This fact may be attributed to the uniqueness of $\mathcal{N}=4$ SYM. An unconstrained overall phase is a common problem of constructive methods. In fact, the model in [4] leaves some degrees of freedom which are reflected by this phase [13]. We then study the properties of the S-matrix and find that it naturally satisfies the Yang–Baxter equation. This is a necessary condition for factorized scattering and integrability. Assuming that integrability holds, we outline the construction of eigenstates of the Hamiltonian.

In Section 4 we perform the nested Bethe ansatz [26] on this S-matrix. This leads to a completely diagonalized S-matrix which can be employed for the asymptotic Bethe equations. We then study the symmetry properties of the equations and the remaining phase. It is also straightforward to "square" the S-matrix and obtain Bethe equations for $\mathcal{N}=4$ SYM, cf. Section 5. We can thus prove the validity of the conjectured asymptotic Bethe equations of [13] (up to the unknown abelian phase and under the assumption of integrability). Among other things, this represents a further piece of evidence for the correctness of the conjecture for the three-loop planar anomalous dimensions of twist-two operators [27]. The conjecture was based on an explicit three-loop computation in QCD [28] and a lift to $\mathcal{N}=4$ SYM

[1] The letters $\mathfrak{J},\mathfrak{R},\mathfrak{L},\mathfrak{Q},\mathfrak{S},\mathfrak{C},\mathfrak{P},\mathfrak{K}$ of the \mathfrak alphabet correspond to $\mathrm{J},\mathrm{R},\mathrm{L},\mathrm{Q},\mathrm{S},\mathrm{C},\mathrm{P},\mathrm{K}$.

by means of “transcendentality” counting. They were subsequently reproduced in the asymptotic Bethe ansatz for the $\mathfrak{sl}(2)$ sector [11]. The derivation of the latter required a relation to hold between the S-matrices of rank-one sectors; here we can identify the group theoretical origin of this relation.

The only missing piece of information for the complete S-matrix is its abelian phase. Its determination is prevented here because it is neither constrained by representation theory nor by the Yang–Baxter relation. A frequently employed constraint in two-dimensional integrable sigma models, see e.g., [29], is a crossing relation for the S-matrix whose existence remains obscure here. Furthermore, the pole structure of the S-matrix might lead to some constraints. The results in Appendix D concerning a curious singlet state represent some (failed) attempts in this direction; it is not (yet) clear how to make sense of them.

A possible direction for future research is to perform a similar investigation for the S-matrix of the IIB string theory on $AdS_5 \times S^5$. The classical diagonalized S-matrix elements can be read off from an integral representation of the classical spectral curve in [30] and the proposed Bethe equations for quantum strings [18,13]. Clearly, the actual non-diagonalized S-matrix is important as the underlying structure of the Bethe ansatz, cf. [31] for some results in this direction. Due to the AdS/CFT correspondence, one might expect the S-matrix to have the same or at least a very similar form and an explicit derivation would be very valuable. Unless there are more powerful constraints here, we should again expect an undetermined phase. The phase can be determined perturbatively by comparison to spinning string states, cf. [16] and the reviews [22, 7]. Some leading quantum corrections to these states and methods to deal with them in the Bethe ansatz have recently become available [32]. A somewhat different approach (for a somewhat different model) might also lead to Bethe equations for quantum strings [33].

Another possible application for the current results is plane wave matrix theory [15, 34]. This theory leads to a very similar spin chain model [35], which is however not completely integrable beyond leading order [36]. Nevertheless, it has an $\mathfrak{su}(3|2)$ sector and the present result about the two-particle S-matrix certainly does apply. This S-matrix satisfies the YBE, factorized scattering is thus self-consistent. The important question however is whether the multi-particle S-matrix does indeed factorize; is the $\mathfrak{su}(3|2)$ sector of PWMT integrable?

Finally, we should point out that the current analyses are justified only in the asymptotic region: At high orders in perturbation theory there are interactions whose range may exceed the length of a spin chain state, the so-called wrapping interactions [4, 10, 12]. The asymptotic Bethe equations should only be trusted up to to this perturbative order which depends on the length of the chain (which itself if somewhat ill-defined in dynamic chains). Unfortunately, it is very hard to make precise statements because wrapping interactions are practically inaccessible by constructive methods of the planar Hamiltonian (and four-loop field theory computations are somewhat beyond our current possibilities). Nevertheless when considering the finite-N algebra, they should be incorporated naturally [37]. It is very likely that the asymptotic Bethe equations receive corrections, either in the form of corrections to the undetermined phase in an effective field theory sense or, preferably, by improved equations. The thermodynamic Bethe ansatz may provide a suitable framework here [38].

2 The asymptotic $\mathfrak{su}(2|2)$ algebra

In the following we introduce the spin chain model. We then consider asymptotic states of an infinitely long spin chain and investigate the residual symmetry which preserves the number of excitations of states.

2.1 The $\mathfrak{su}(2|3)$ dynamic spin chain model

In [4] a spin chain with $\mathfrak{su}(2|3)$ symmetry and fundamental matter was considered. This spin chain arises as a sector of perturbative $\mathrm{U}(N)$ $\mathcal{N}=4$ super Yang–Mills theory in the large-N limit. The spin $\mathcal{X}$ at each site can take one out of five orientations, $\mathcal{X} \in \{\mathcal{Z}, \phi^1, \phi^2 | \psi^1, \psi^2\}$. The first three are bosonic states, the remaining two are fermions; in a $\mathcal{N}=1$ notation they represent the three scalar fields and the two spin orientations of the gluino. A generic state $|\Psi\rangle$ is a linear combination of basic states, e.g.,

$$|\Psi\rangle = *|\mathcal{Z}\phi^1\mathcal{Z}\mathcal{Z}\psi^2\mathcal{Z}\ldots\phi^1\rangle + *|\psi^1\phi^2\mathcal{Z}\mathcal{Z}\psi^2\ldots\mathcal{Z}\rangle + \cdots . \tag{2.1}$$

Such a state represents a single-trace gauge invariant local operator. The spin chain is closed and physical states are cyclic, they must be invariant under cyclic permutations of the spin sites taking into account the statistics of the fields. This corresponds to cyclicity of the trace in gauge theory.

The states transform under a symmetry algebra $\mathfrak{su}(2|3)$ which is a subalgebra of the superconformal algebra $\mathfrak{psu}(2,2|4)$ of $\mathcal{N}=4\,\mathrm{SYM}$. The $\mathfrak{gl}(1)$ generator of this algebra is associated with the energy, we shall call it the Hamiltonian; in $\mathcal{N}=4\,\mathrm{SYM}$ it is related to the dilatation generator. Thus, finding the spectrum of this operator is physically interesting, it contains the planar anomalous dimensions of the local operators in the $\mathfrak{su}(2|3)$ sector. A family of representations of $\mathfrak{su}(2|3)$ on spin chain states was constructed in [4]. The family was parametrized by the coupling constant g related to the 't Hooft coupling constant by

$$g^2 = \frac{\lambda}{8\pi^2} = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}. \tag{2.2}$$

At $g=0$ the representation is merely the tensor product of fundamental representations. The deformations around this point can be constructed in perturbation theory. This was done in [4] up to fourth order for all generators and up to sixth order for the Hamiltonian. The constraining property of the representation was that the generators must act locally on the spin chain with a maximum range determined by the order in g. At a finite value of g, the action is therefore *long-ranged*. The action is also *dynamic*, the generators are allowed to change the number of spin chain sites L: the length L fluctuates.

2.2 Asymptotic states

Let us define a vacuum state composed from only $\mathcal{Z}$s. We shall start with an infinitely long vacuum

$$|0\rangle^{\mathrm{I}} = |\ldots\mathcal{Z}\mathcal{Z}\ldots\mathcal{Z}\mathcal{Z}\ldots\rangle. \tag{2.3}$$

In fact, physical states have a finite length and are periodically identified. As pointed out in [11], it is however sufficient to consider periodic states on an infinite chain to obtain the correct spectrum up to a certain accuracy. This is what will be called the *asymptotic regime*. We might then consider a generic *asymptotic state* as an excitation of the vacuum, such as

$$|\mathcal{X}_1\ldots\mathcal{X}''_K\rangle^{\mathrm{I}} = \sum_{n_1\ll\ldots\ll n_K} e^{ip_1n_1}\ldots e^{ip_Kn_K} \times |\ldots\mathcal{Z}\mathcal{Z}\ldots\overset{\overset{n_1}{\downarrow}}{\mathcal{X}}\ldots\overset{\overset{\ldots}{\downarrow}}{\mathcal{X}'}\ldots\overset{\overset{n_K}{\downarrow}}{\mathcal{X}''}\ldots\mathcal{Z}\mathcal{Z}\ldots\rangle. \tag{2.4}$$

The superscript "I" of the state implies that we have screened out all vacuum fields $\mathcal{Z}$. Here "I" refers to the first level of screening; later, at higher levels, more fields will be screened. The excitations $\mathcal{X}\in\{\phi^1,\phi^2|\psi^1,\psi^2\}$ have the same order with which they appear in the original spin chain. The subscript $k=1,\ldots,K$ of an excitation indicates that $\mathcal{X}_k$ carries a definite momentum p_k along the original spin chain.

In (2.4) we have assumed that the excitations are well-separated, $n_k\ll n_{k+1}$, so that the range of interactions is always smaller than the minimum separation. Then the interactions act on only one excitation at a time which is a major simplification; this is our notion of asymptotic states. Of course also the states with nearby excitations are important, but for the determination of asymptotic eigenstates and energies their contribution can be summarized by the S-matrix which will be considered in Section 3.

2.3 The algebra

The spin chain states transform under the full symmetry algebra $\mathfrak{su}(2|3)$ and so do the asymptotic states. However, the number of excitations, K, is not preserved. It is only preserved by a subalgebra of $\mathfrak{su}(2|3)$, namely $\mathfrak{su}(2|2)$, let us therefore restrict to it. This algebra $\mathfrak{su}(2|2)$ consists of the $\mathfrak{su}(2)\times\mathfrak{su}(2)$ rotation generators $\mathfrak{R}^a{}_b$, $\mathfrak{L}^\alpha{}_\beta$, the supersymmetry generators $\mathfrak{Q}^\alpha{}_b$, $\mathfrak{S}^a{}_\beta$ and the central charge $\mathfrak{C}$. The non-trivial commutators are

$$\begin{aligned}
[\mathfrak{R}^a{}_b, \mathfrak{J}^c] &= \delta^c_b \mathfrak{J}^a - \tfrac{1}{2}\delta^a_b \mathfrak{J}^c,\\
[\mathfrak{L}^\alpha{}_\beta, \mathfrak{J}^\gamma] &= \delta^\gamma_\beta \mathfrak{J}^\alpha - \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{J}^\gamma,\\
\left\{\mathfrak{Q}^\alpha{}_a, \mathfrak{S}^b{}_\beta\right\} &= \delta^b_a \mathfrak{L}^\alpha{}_\beta + \delta^\alpha_\beta \mathfrak{R}^b{}_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C},
\end{aligned} \tag{2.5}$$

where $\mathfrak{J}$ represents any generator with the appropriate index. For later convenience we enlarge the algebra by two central charges[2] $\mathfrak{P}, \mathfrak{K}$ [39] to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$

$$\begin{aligned}
\left\{\mathfrak{Q}^\alpha{}_a, \mathfrak{Q}^\beta{}_b\right\} &= \varepsilon^{\alpha\beta}\varepsilon_{ab}\mathfrak{P},\\
\left\{\mathfrak{S}^a{}_\alpha, \mathfrak{S}^b{}_\beta\right\} &= \varepsilon^{ab}\varepsilon_{\alpha\beta}\mathfrak{K}.
\end{aligned} \tag{2.6}$$

These shall have zero eigenvalue on physical states and thus the algebra on physical states is effectively $\mathfrak{su}(2|2)$. The extension is necessary because the representations of $\mathfrak{su}(2,2)$ are too restrictive for the excitation picture.

The enlarged algebra $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3$ is a contraction of the exceptional superalgebra $\mathfrak{d}(2,1;\epsilon,\mathbb{R})$ with $\epsilon\to 0$. The triplet of central charges $\mathfrak{P}$, $\mathfrak{K}$ and $\mathfrak{C}$ is the contraction of the $\mathfrak{sp}(2,\mathbb{R})$ factor while the rotation generators $\mathfrak{R}, \mathfrak{L}$ form the $\mathfrak{so}(4) = \mathfrak{su}(2)^2$ part. See Appendix A for details of this construction.

2.4 The representation

Let us represent $\mathfrak{su}(2|2)$ on a $2|2$-dimensional space. We label the states by $\left|\phi^a\right\rangle^{\mathrm{I}}$ and $\left|\psi^\alpha\right\rangle^{\mathrm{I}}$. These should be considered single excitations (2.4) of the

[2]Central extensions of Lie superalgebras were investigated in [40]. The $\mathfrak{psu}(2|2)$ algebra constitutes a special case with up to three central extensions. I thank F. Spill for pointing out this reference to me.

level-I vacuum $|0\rangle^{\mathrm{I}}$ in (2.3). Each $\mathfrak{su}(2)$ factor should act canonically on either of the two-dimensional subspaces[3]

$$\mathfrak{R}^a{}_b|\phi^c\rangle^{\mathrm{I}} = \delta^c_b|\phi^a\rangle^{\mathrm{I}} - \tfrac{1}{2}\delta^a_b|\phi^c\rangle^{\mathrm{I}},$$
$$\mathfrak{L}^\alpha{}_\beta|\psi^\gamma\rangle^{\mathrm{I}} = \delta^\gamma_\beta|\psi^\alpha\rangle^{\mathrm{I}} - \tfrac{1}{2}\delta^\alpha_\beta|\psi^\gamma\rangle^{\mathrm{I}}. \tag{2.7}$$

The supersymmetry generators should also act in a manifestly $\mathfrak{su}(2)\times\mathfrak{su}(2)$ covariant way. The most general transformation rules are thus

$$\mathfrak{Q}^\alpha{}_a|\phi^b\rangle^{\mathrm{I}} = a\,\delta^b_a|\psi^\alpha\rangle^{\mathrm{I}},$$
$$\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle^{\mathrm{I}} = b\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\mathcal{Z}^+\rangle^{\mathrm{I}},$$
$$\mathfrak{S}^a{}_\alpha|\phi^b\rangle^{\mathrm{I}} = c\,\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta\mathcal{Z}^-\rangle^{\mathrm{I}},$$
$$\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle^{\mathrm{I}} = d\,\delta^\beta_\alpha|\phi^a\rangle^{\mathrm{I}}. \tag{2.8}$$

For the moment we shall ignore the symbols $\mathcal{Z}^\pm$ inserted into the states. We find that the closure of $\{\mathfrak{Q},\mathfrak{S}\} = \ldots$ (2.5,2.6) requires $ad-bc=1$. The central charge is then given by

$$\mathfrak{C}|\mathcal{X}\rangle^{\mathrm{I}} = C|\mathcal{X}\rangle^{\mathrm{I}} = \tfrac{1}{2}(ad+bc)\,|\mathcal{X}\rangle^{\mathrm{I}} \tag{2.9}$$

where $|\mathcal{X}\rangle^{\mathrm{I}}$ is any of the states $|\phi^a\rangle^{\mathrm{I}}$ or $|\psi^\alpha\rangle^{\mathrm{I}}$. For $\mathfrak{su}(2|2)$ we should furthermore impose $\{\mathfrak{Q},\mathfrak{Q}\}=\{\mathfrak{S},\mathfrak{S}\}=0$ which fixes $ab=0$ and $cd=0$. The two solutions to these equations lead to a central charge $C=\pm\frac{1}{2}$ and correspond to the fundamental representations of $\mathfrak{su}(2|2)$. This would lead to the model introduced in [41] which is the correct description of gauge theory at leading order, but not at higher loops.

In order to find more interesting solutions with non-trivial central charge we relax the condition $\{\mathfrak{Q},\mathfrak{Q}\}=\{\mathfrak{S},\mathfrak{S}\}=0$ and allow for non-trivial central charges $\mathfrak{P},\mathfrak{K}$. Closure of the symmetry algebra requires the action of the additional generators to be

$$\mathfrak{P}|\mathcal{X}\rangle = ab\,|\mathcal{X}\mathcal{Z}^+\rangle,$$
$$\mathfrak{K}|\mathcal{X}\rangle = cd\,|\mathcal{X}\mathcal{Z}^-\rangle. \tag{2.10}$$

Of course we are interested in representations of the original $\mathfrak{su}(2|2)$ algebra and not of some enlarged one. Therefore we are bound to constrain the action of $\mathfrak{P}$ and $\mathfrak{K}$ to zero. For this representation we are back at where we started and there is only the fundamental representation. The improvement of this point of view comes about when we consider tensor products. Then, only the action of the *overall* generators $\mathfrak{P}$ and $\mathfrak{K}$ must be zero leaving some degrees of freedom among the individual representations.

[3]The $\mathfrak{su}(2)$ algebra generates a compact group whose unitary/finite-dimensional representations cannot be deformed continuously.

2.5 Dynamic spin chains

To match the representation to excitations of the dynamic $\mathfrak{su}(2|3)$ spin chain [4], we note that $\mathcal{Z}^+$ should be considered as the insertion of a field $\mathcal{Z}$ into the original chain; likewise $\mathcal{Z}^-$ removes a field. Let us consider an excitation with a definite momentum on an infinite spin chain

$$\big|\mathcal{X}\big\rangle^{\mathrm{I}} = \sum_n e^{ipn} \big|\ldots\mathcal{Z}\mathcal{Z}\ldots\overset{\overset{n}{\downarrow}}{\mathcal{X}}\ldots\mathcal{Z}\mathcal{Z}\ldots\big\rangle. \tag{2.11}$$

When we insert or remove a background field $\mathcal{Z}$ in front of the excitation we obtain

$$\begin{aligned}\big|\mathcal{Z}^{\pm}\mathcal{X}\big\rangle^{\mathrm{I}} &= \sum\nolimits_n e^{ipn} \big|\ldots\mathcal{Z}\mathcal{Z}\ldots\overset{\overset{n\pm1}{\downarrow}}{\mathcal{X}}\ldots\mathcal{Z}\mathcal{Z}\ldots\big\rangle \\ &= \sum_n e^{ipn\mp ip} \big|\ldots\mathcal{Z}\mathcal{Z}\ldots\overset{\overset{n}{\downarrow}}{\mathcal{X}}\ldots\mathcal{Z}\mathcal{Z}\ldots\big\rangle,\end{aligned} \tag{2.12}$$

i.e., we can always shift the operation $\mathcal{Z}^{\pm}$ to the very right of the asymptotic state and pick up factors of $\exp(\mp ip)$

$$\big|\mathcal{Z}^{\pm}\mathcal{X}\big\rangle^{\mathrm{I}} = e^{\mp ip}\,\big|\mathcal{X}\mathcal{Z}^{\pm}\big\rangle^{\mathrm{I}}. \tag{2.13}$$

The action of $\mathfrak{P}$ on a tensor product gives

$$\mathfrak{P}\big|\mathcal{X}_1\ldots\mathcal{X}_K\big\rangle^{\mathrm{I}} = P\big|\mathcal{X}_1\ldots\mathcal{X}_K\mathcal{Z}^+\big\rangle^{\mathrm{I}}, \quad P = \sum_{k=1}^{K} a_k b_k \prod_{l=k+1}^{K} e^{-ip_l} \tag{2.14}$$

and should vanish on physical states. Physical states are thus defined by the condition that the central charge P vanishes. On the other hand we know that physical states are cyclic, they have zero total momentum. Indeed $P=0$ coincides with the zero momentum condition provided that we set $a_k b_k = \alpha(e^{-ip_k}-1)$. Then the sum telescopes and becomes

$$P = \alpha\sum_{k=1}^{K}(e^{-ip_k}-1)\prod_{l=k+1}^{K} e^{-ip_l} = \alpha\left(\prod_{k=1}^{K} e^{-ip_k}-1\right). \tag{2.15}$$

The first term is the eigenvalue of the right shift operator. When we set $c_k d_k = \beta(e^{ip_k}-1)$ we obtain the same constraint from a vanishing action of $\mathfrak{K}$

$$\mathfrak{K}\big|\mathcal{X}_1\ldots\mathcal{X}_K\big\rangle^{\mathrm{I}} = \beta\left(\prod_{k=1}^{K} e^{ip_k}-1\right)\big|\mathcal{X}_1\ldots\mathcal{X}_K\mathcal{Z}^-\big\rangle^{\mathrm{I}}. \tag{2.16}$$

We can now write the action of $\mathfrak{P}, \mathfrak{K}$ in (2.10) as

$$\mathfrak{P}|\mathcal{X}\rangle^{\mathrm{I}} = \alpha|\mathcal{Z}^{+}\mathcal{X}\rangle^{\mathrm{I}} - \alpha\,|\mathcal{X}\mathcal{Z}^{+}\rangle^{\mathrm{I}},$$
$$\mathfrak{K}|\mathcal{X}\rangle^{\mathrm{I}} = \beta|\mathcal{Z}^{-}\mathcal{X}\rangle^{\mathrm{I}} - \beta\,|\mathcal{X}\mathcal{Z}^{-}\rangle^{\mathrm{I}}. \tag{2.17}$$

Note that this reveals their nature as a gauge transformation, $\mathfrak{P}$ generates the transformation $\Psi \mapsto \alpha\,[\mathcal{Z},\Psi]$. Similarly, $\mathfrak{K}$ generates a somewhat unusual transformation $\Psi \mapsto \beta\,[\mathcal{Z}^{-},\Psi]$, which removes a field $\mathcal{Z}$. Of course, physical states are gauge invariant and therefore should be annihilated by $\mathfrak{P}$ and $\mathfrak{K}$.

2.6 Solution for the coefficients

Next we solve the central charge in terms of the momenta and obtain

$$C = \sum_{k=1}^{K} C_k, \quad C_k = \pm\tfrac{1}{2}\sqrt{1 + 16\alpha\beta\sin^2(\tfrac{1}{2}p_k)}. \tag{2.18}$$

The central charge is the energy and consequently we have derived the BMN-like energy formula [15] up to the value of the product $\alpha\beta$ which should play the role of the coupling constant.[4] To adjust to the correct coupling constant for $\mathcal{N}=4$ SYM and the one used in [4] we set $\beta = g^2/2\alpha$. We introduce new variables x_k^+, x_k^- to replace the momenta p_k and solve[5]

$$a_k = \gamma_k, \quad b_k = -\frac{\alpha}{\gamma_k x_k^+}(x_k^+ - x_k^-), \quad c_k = \frac{ig^2\gamma_k}{2\alpha x_k^-}, \quad d_k = -\frac{i}{\gamma_k}(x_k^+ - x_k^-). \tag{2.19}$$

For a hermitian representation we should choose

$$|\gamma_k| = \left|ix_k^- - ix_k^+\right|^{1/2}, \quad |\alpha| = \sqrt{g^2/2}. \tag{2.20}$$

The condition $a_k d_k - b_k c_k = 1$ for the closure of the algebra translates to

$$x_k^+ + \frac{g^2}{2x_k^+} - x_k^- - \frac{g^2}{2x_k^-} = i. \tag{2.21}$$

Finally, the momentum and central charge are given by

$$e^{ip_k} = \frac{x_k^+}{x_k^-}, \quad C_k = \frac{1}{2} + \frac{ig^2}{2x_k^+} - \frac{ig^2}{2x_k^-} = -ix_k^+ + ix_k^- - \frac{1}{2}. \tag{2.22}$$

[4] This derivation of the energy formula should be similar to the one in [42].

[5] The parameter γ_k corresponds to a (momentum dependent) relative rescaling of $|\phi^a\rangle$ and $|\psi^\alpha\rangle$ whereas α corresponds to a rescaling of $\mathcal{Z}$.

The physicality constraint $\mathfrak{P}|\Psi\rangle = \mathfrak{K}|\Psi\rangle = 0$ is the zero-momentum condition

$$1 = \prod_{k=1}^{K} e^{ip_k} = \prod_{k=1}^{K} \frac{x_k^+}{x_k^-}\,. \qquad (2.23)$$

Interestingly, the dispersion relation (2.18) admits two solutions with a given momentum but opposite energies. This is a common feature of *relativistic* quantum mechanics: the two solutions can be interpreted as a regular particle and a conjugate one propagating backwards in time. The conjugate excitation can be obtained from a regular one by the substitution $x_k^\pm \mapsto -g^2/2x_k^\mp$ (or by $x_k^\pm \mapsto g^2/2x_k^\pm$ which inverts the momentum as well).

We might solve (2.21) by [12]

$$x_k^\pm = x(u_k \pm \tfrac{i}{2}), \quad x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1 - 2g^2/u^2}\,, \quad u(x) = x + \frac{g^2}{2x}. \quad (2.24)$$

This may appear to yield only the positive energy solution, it is however not possible to exclude the negative energy solution rigorously: In general $x^\pm$ are complex variables and the negative energy solution will always sneak in as the other branch of (2.24). The branch cut may only be avoided in the *non-relativistic* regime at $g \approx 0$, where the perturbative gauge theory and the underlying spin chain are to be found.

It seems that the appearance of conjugate excitations is related to the puzzle observed in [43]: the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM does not have a direct counterpart in string theory, but it is merely embedded in a larger $\mathfrak{su}(2) \times \mathfrak{su}(2)$ sector representing the isometry algebra of an S^3. This larger sector has excitations corresponding to a second $\mathfrak{su}(2)$ which are related to the original ones by the map $x \mapsto g^2/2x$. The reason why the conjugate excitations do not appear in gauge theory is related to perturbation theory. They would have a non-vanishing anomalous dimension -2 at $g = 0$ which is in conflict with the perturbative setup. An interesting application of the conjugate excitations is presented in Appendix D where a peculiar composite of a regular excitation and its conjugate is investigated.

3 The S-matrix

So far we have concentrated on asymptotic states (IR) and discarded the contributions from states with nearby excitations (UV). The latter become important when considering eigenstates of the central charge $\mathfrak{C}$ alias the Hamiltonian. Luckily their inclusion can be summarized in the S-matrix of the model.

3.1 Sewing eigenstates

The symmetry algebra acts on the asymptotic states (2.4) as a tensor product representation: all excitations are treated individually and do not influence each other. This can however be true only in an asymptotic sense; there are additional contributions from the boundaries of the asymptotic regions where excitations come too close. When interested in the exact action of the algebra we must take these into account. This is achieved by sewing together the asymptotic regions in a way compatible with the algebra, e.g.,

$$|\Psi\rangle = a\big|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\big\rangle^{\mathrm{I}} + b\big|\ldots(\mathcal{X}\mathcal{X})_{kl}\ldots\big\rangle^{\mathrm{I}} + c\big|\ldots\mathcal{X}''_l\mathcal{X}'''_k\ldots\big\rangle^{\mathrm{I}}. \tag{3.1}$$

Here the left-hand state is some asymptotic state, the middle one represents contributions with nearby excitations k,l and in the right-hand state the momenta of the excitations k,l are interchanged. There may be various linear combinations of different flavours $\mathcal{X},\mathcal{X}',\ldots$ which we do not specify here. Clearly, the exact algebra transforms the coefficients a,c independently according to the asymptotic rules in Section 2.4. In addition, b must be adjusted so that it yields the correct contributions to the boundaries of the asymptotic regions. This relates b to a *and* b to c and therefore a *with* c. This means that asymptotic states can be completed to exact states in a unique way compatible with the algebra. In particular, the coefficients of all asymptotic regions, a,c in the example, are related among each other. As soon as this relation is known, it is no longer necessary to consider the non-asymptotic contributions.

The completion of asymptotic states can be performed by the S-matrix. The S-matrix $\mathcal{S}^{\mathrm{I}}_{kl}$ is an operator which interchanges two adjacent sites of the spin chain at level I. The affected sites are labelled by their momenta p_k,p_l which are exchanged by $\mathcal{S}^{\mathrm{I}}_{kl}$

$$\mathcal{S}^{\mathrm{I}}_{kl}\big|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\big\rangle^{\mathrm{I}} \mapsto *\big|\ldots\mathcal{X}''_l\mathcal{X}'''_k\ldots\big\rangle^{\mathrm{I}}. \tag{3.2}$$

The consistent completion of this asymptotic state is then

$$|\Psi\rangle = a\Big(\big|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\big\rangle^{\mathrm{I}} + \text{non-asymp.} + \mathcal{S}^{\mathrm{I}}_{kl}\big|\ldots\mathcal{X}_k\mathcal{X}'_l\ldots\big\rangle^{\mathrm{I}}\Big). \tag{3.3}$$

The requirement for asymptotic consistency is that the S-matrix commutes with the algebra, $\big[\mathfrak{J}_k+\mathfrak{J}_l,\mathcal{S}^{\mathrm{I}}_{kl}\big]=0$, where $\mathfrak{J}_k$ is a generator of $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ acting on site k.

3.2 Invariance

Let us now construct the S-matrix by acting on the state $\big|\mathcal{X}_1\mathcal{X}'_2\big\rangle$ with all possible combinations of spins $\mathcal{X},\mathcal{X}'$. We demand the exact invariance under $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$

$$\big[\mathfrak{J}_1 + \mathfrak{J}_2, \mathcal{S}^{\mathrm{I}}_{12}\big] = 0. \tag{3.4}$$

The commutators with the central charges $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$ are automatically satisfied. From commutators with the kinematic generators $\mathfrak{R}, \mathfrak{L}$ the S-matrix takes a generic form determined by ten coefficient functions $A_{12} = A(x_1, x_2)$ to L_{12}. The form is presented at the top of table 1. Remarkably, invariance

Table 1: The dynamic $\mathfrak{su}(2|2)$ S-matrix

$$\mathcal{S}^{\mathrm{I}}_{12}\big|\phi^a_1\phi^b_2\big\rangle^{\mathrm{I}} = A_{12}\big|\phi^{\{a}_2\phi^{b\}}_1\big\rangle^{\mathrm{I}} + B_{12}\big|\phi^{[a}_2\phi^{b]}_1\big\rangle^{\mathrm{I}} + \tfrac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}\big|\psi^\alpha_2\psi^\beta_1\mathcal{Z}^-\big\rangle^{\mathrm{I}},$$

$$\mathcal{S}^{\mathrm{I}}_{12}\big|\psi^\alpha_1\psi^\beta_2\big\rangle^{\mathrm{I}} = D_{12}\big|\psi^{\{\alpha}_2\psi^{\beta\}}_1\big\rangle^{\mathrm{I}} + E_{12}\big|\psi^{[\alpha}_2\psi^{\beta]}_1\big\rangle^{\mathrm{I}} + \tfrac{1}{2}F_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}\big|\phi^a_2\phi^b_1\mathcal{Z}^+\big\rangle^{\mathrm{I}},$$

$$\mathcal{S}^{\mathrm{I}}_{12}\big|\phi^a_1\psi^\beta_2\big\rangle^{\mathrm{I}} = G_{12}\big|\psi^\beta_2\phi^a_1\big\rangle^{\mathrm{I}} + H_{12}\big|\phi^a_2\psi^\beta_1\big\rangle^{\mathrm{I}},$$

$$\mathcal{S}^{\mathrm{I}}_{12}\big|\psi^\alpha_1\phi^b_2\big\rangle^{\mathrm{I}} = K_{12}\big|\psi^\alpha_2\phi^b_1\big\rangle^{\mathrm{I}} + L_{12}\big|\phi^b_2\psi^\alpha_1\big\rangle^{\mathrm{I}}.$$

$$A_{12} = S^0_{12}\,\frac{x^+_2 - x^-_1}{x^-_2 - x^+_1},$$

$$B_{12} = S^0_{12}\,\frac{x^+_2 - x^-_1}{x^-_2 - x^+_1}\left(1 - 2\,\frac{1 - g^2/2x^-_2x^+_1}{1 - g^2/2x^-_2x^-_1}\,\frac{x^+_2 - x^+_1}{x^+_2 - x^-_1}\right),$$

$$C_{12} = S^0_{12}\,\frac{g^2\gamma_2\gamma_1}{\alpha x^-_2x^-_1}\,\frac{1}{1 - g^2/2x^-_2x^-_1}\,\frac{x^+_2 - x^+_1}{x^-_2 - x^+_1},$$

$$D_{12} = -S^0_{12},$$

$$E_{12} = -S^0_{12}\left(1 - 2\,\frac{1 - g^2/2x^+_2x^-_1}{1 - g^2/2x^+_2x^+_1}\,\frac{x^-_2 - x^-_1}{x^-_2 - x^+_1}\right),$$

$$F_{12} = -S^0_{12}\,\frac{2\alpha(x^+_2 - x^-_2)(x^+_1 - x^-_1)}{\gamma_2\gamma_1x^+_2x^+_1}\,\frac{1}{1 - g^2/2x^+_2x^+_1}\,\frac{x^-_2 - x^-_1}{x^-_2 - x^+_1},$$

$$G_{12} = S^0_{12}\,\frac{x^+_2 - x^+_1}{x^-_2 - x^+_1},$$

$$H_{12} = S^0_{12}\,\frac{\gamma_1}{\gamma_2}\,\frac{x^+_2 - x^-_2}{x^-_2 - x^+_1},$$

$$K_{12} = S^0_{12}\,\frac{\gamma_2}{\gamma_1}\,\frac{x^+_1 - x^-_1}{x^-_2 - x^+_1},$$

$$L_{12} = S^0_{12}\,\frac{x^-_2 - x^-_1}{x^-_2 - x^+_1}.$$

under the dynamic generators $\mathfrak{Q}, \mathfrak{S}$ leads to a *unique* solution up to an undetermined overall function S^0_{12}. To obtain the solution is straightforward but somewhat laborious; we merely state the final result at the bottom of table 1.

One may wonder why this S-matrix is uniquely determined. It intertwines two modules and one should expect one degree of freedom for each irreducible module in the tensor product. Intriguingly, it appears that the tensor product is indeed irreducible. This may be the case because both factors are short (atypical). Their tensor product on the other hand has 8|8 components which is the smallest typical multiplet. Note that the usual symmetrizations cannot be applied here, because both factors transform in distinct representations labelled by their momenta p_k. In verifying the invariance of the S-matrix, the following identities have proved useful

$$\begin{aligned}
\frac{x_1^+ - x_2^+}{1 - g^2/2x_1^- x_2^-} &= \frac{x_1^- - x_2^-}{1 - g^2/2x_1^+ x_2^+}, \\
\frac{x_2^+ - x_2^- - x_1^+ + x_1^-}{x_1^+ x_2^+ - x_1^- x_2^-} &= \frac{g^2}{2x_1^+ x_1^- x_2^+ x_2^-} \, \frac{x_1^+ - x_2^+}{1 - g^2/2x_1^- x_2^-}, \\
\frac{B_{12}}{S^0_{12}} &= -1 + \frac{g^2}{2x_1^+ x_1^- x_2^+ x_2^-} \, \frac{x_1^+ x_2^+ - 2x_1^- x_2^+ + x_1^- x_2^-}{1 - g^2/2x_1^- x_2^-} \, \frac{x_1^+ - x_2^+}{x_2^- - x_1^+}, \\
\frac{E_{12}}{S^0_{12}} &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} - \frac{g^2}{2x_1^+ x_1^- x_2^+ x_2^-} \\
&\quad \times \frac{x_1^+ x_2^+ - 2x_1^+ x_2^- + x_1^- x_2^-}{1 - g^2/2x_1^- x_2^-} \, \frac{x_1^+ - x_2^+}{x_2^- - x_1^+}.
\end{aligned} \tag{3.5}$$

They can be derived from the quadratic constraint (2.21) between x^+ and x^-.

Let us compare to the results in [13] for the S-matrix in the $\mathfrak{su}(1|2)$ sector of $\mathcal{N}=4$ SYM. The S-matrix has manifest $\mathfrak{su}(1|1)$ symmetry as explained in [25] and we obtain it by restricting to the spin components $a, b, \alpha, \beta = 1$. Then only the elements A, D, G, H, K, L in table 1 are relevant and the S-matrix agrees with [13, 25].

3.3 Properties

We have already made use of the invariance of the S-matrix in its construction

$$\left[\mathfrak{J}_1 + \mathfrak{J}_2, \mathcal{S}^{\mathrm{I}}_{12}\right] = 0. \tag{3.6}$$

It however obeys a host of other important identities. First of all, it is an involution

$$\mathcal{S}^{\mathrm{I}}_{12}\mathcal{S}^{\mathrm{I}}_{21} = 1 \tag{3.7}$$

assuming that the undetermined phase obeys $S^0(x_1, x_2)S^0(x_2, x_1) = 1$. We have also verified that it satisfies the Yang–Baxter equation[6]

$$\mathcal{S}^{\mathrm{I}}_{12}\mathcal{S}^{\mathrm{I}}_{13}\mathcal{S}^{\mathrm{I}}_{23} = \mathcal{S}^{\mathrm{I}}_{23}\mathcal{S}^{\mathrm{I}}_{13}\mathcal{S}^{\mathrm{I}}_{12}. \tag{3.8}$$

This is very tedious and we have made use of `Mathematica` to evaluate (3.8) on all three-particle states. Note that the appearance of $\mathcal{Z}^{\pm}$ in table 1 can lead to additional phases due to (2.13), e.g.,

$$\mathcal{S}^{\mathrm{I}}_{12}\big|\phi_1\phi_2\psi_3\big\rangle^{\mathrm{I}} \to \big|\psi_2\psi_1\mathcal{Z}^-\psi_3\big\rangle^{\mathrm{I}} + \cdots = \frac{x_3^+}{x_3^-}\,\big|\psi_2\psi_1\psi_3\mathcal{Z}^-\big\rangle^{\mathrm{I}} + \cdots . \tag{3.9}$$

It is also worth considering the $g = 0$ limit corresponding to one-loop gauge theory. Here all the particle representations have central charge $C = \frac{1}{2}$ and transform as fundamentals under $\mathfrak{su}(2|2)$. When we set $\alpha = \mathcal{O}(g)$, it is easy to see that

$$\mathcal{S}^{\mathrm{I}}_{12}\big|_{g=0} = \mathcal{P}^u_{12}S^0_{12}\left(\frac{u_2 - u_1}{u_2 - u_1 - i} + \frac{i}{u_2 - u_1 - i}\,\mathcal{P}_{12}\right) \tag{3.10}$$

where $\mathcal{P}_{12}$ is a graded permutation of the spin labels a, b, α, β and $\mathcal{P}^u_{12}$ interchanges the spectral parameters u_1, u_2. This agrees with the well-known S-matrix in the fundamental representation of $\mathfrak{su}(2|2)$. We recover the model found in [41].

3.4 Eigenstates

A generic eigenstate $\big|\Psi\big\rangle$ of the spin chain can now be represented by a set of numbers $\{x_1, \ldots, x_K\}$ and a residual wave function $\big|\Psi^{\mathrm{I}}\big\rangle$. This residual wave function is given as a state of a new inhomogeneous spin chain with only four spin states $\{\phi^1, \phi^2|\,\psi^1, \psi^2\}$ such that spin site k has momentum $p_k = p(x_k)$ along the original spin chain. The eigenstate is

$$\big|\Psi\big\rangle = \mathcal{S}^{\mathrm{I}}\big|\Psi^{\mathrm{I}}\big\rangle. \tag{3.11}$$

Here $\mathcal{S}^{\mathrm{I}}$ is the multi-particle S-matrix at level I. In the case of infinitely many conserved charges, the set of momenta is preserved in the scattering process,

[6] M. Staudacher has confirmed that the YBE is satisfied at the first few perturbative orders in g.

i.e., only the momenta can be permuted. Indications that this might be true were found in [4, 44]. The S-matrix can thus be written as

$$\mathcal{S}^{\mathrm{I}} = \sum_{\pi \in S_K} \mathcal{S}^{\mathrm{I}}_{\pi}. \tag{3.12}$$

The S-matrix $\mathcal{S}_\pi$ interchanges the sites and momenta of the spin chain $|\Psi^{\mathrm{I}}\rangle$ according to the permutation π. If the S-matrix factorizes, it can be written as a product over pairwise permutations of adjacent excitations

$$\mathcal{S}^{\mathrm{I}}_{\pi} = \prod_{(k,l)\in\pi} \mathcal{S}^{\mathrm{I}}_{kl}. \tag{3.13}$$

Due to the YBE (3.8) this product can be defined self-consistently. Let us therefore *assume* that the S-matrix factorizes and that the Hamiltonian $\mathfrak{C}$ is integrable. An indirect verification of this assumption is that the resulting Bethe equations indeed reproduce several energies correctly [13]. This solves the problem of finding asymptotic eigenstates $|\Psi\rangle$ of the infinite spin chain.

4 Diagonalizing the S-Matrix

The previous solution for the infinite chain is complete, but it requires a residual wave function $|\Psi^{\mathrm{I}}\rangle$ to be specified. In other words, we have replaced the level-0 wave function $|\Psi\rangle$ by a set of parameters $\{x_1, \ldots, x_K\}$ and a level-I wave function $|\Psi^{\mathrm{I}}\rangle$. We can now try to repeat this process and represent the spin chain $|\Psi^{\mathrm{I}}\rangle$ by a set of parameters $\{y_1, \ldots, y_{K'}\}$ and a level-II wave function $|\Psi^{\mathrm{II}}\rangle$. This is the so-called nested Bethe ansatz [26].

4.1 Vacuum

We start by choosing a level-II vacuum state consisting only of ϕ^1s

$$|0\rangle^{\mathrm{II}} = |\phi^1_1 \ldots \phi^1_K\rangle^{\mathrm{I}}. \tag{4.1}$$

For any permutation π, the S-matrix $\mathcal{S}^{\mathrm{I}}_\pi$ yields a total phase S^{I}_π times a vacuum of the inhomogeneous chain with permuted momenta

$$\mathcal{S}^{\mathrm{I}}_{\pi}|0\rangle^{\mathrm{II}} = S^{\mathrm{I}}_{\pi}|0\rangle^{\mathrm{II}}_{\pi}, \quad |0\rangle^{\mathrm{II}}_{\pi} = |\phi^1_{\pi(1)} \ldots \phi^1_{\pi(K)}\rangle^{\mathrm{I}}. \tag{4.2}$$

The total phase is given by a product over two-particle phases

$$S^{\mathrm{I}}_\pi = \prod_{(k,l)\in\pi} S^{\mathrm{I,I}}(x_k, x_l), \quad S^{\mathrm{I,I}}(x_k, x_l) = A(x_k, x_l)$$
$$= S_0(x_k, x_l)\,\frac{x_k^- - x_l^+}{x_k^+ - x_l^-}. \tag{4.3}$$

4.2 Propagation

Now let us insert one excitation which might be of type ψ^1, ψ^2 or ϕ^2. If it is of type ψ^1 or ψ^2, an action of the S-matrix shifts this excitation around. If it is of type ϕ^2, however, the S-matrix can shift it around, but it can also convert it into one excitation of type ψ^1 and ψ^2 each. Subsequently, these two will be propagated by the S-matrix on an individual basis. Therefore ϕ^2 is a non-elementary double excitation whereas ψ^1, ψ^2 are the only two elementary excitations of the vacuum $\big|0\big\rangle^{\mathrm{II}}$.

A generic one-excitation state is given by

$$\big|\psi^\alpha\big\rangle^{\mathrm{II}} = \sum_{k=1}^{K} \Psi_k(y)\,\big|\phi^1_1 \ldots \psi^\alpha_k \ldots \phi^1_K\big\rangle^{\mathrm{I}} \tag{4.4}$$

with some wave function $\Psi_k(y)$. For this wave function we make a plane wave ansatz in the inhomogeneous background which is determined through the x_ls

$$\Psi_k(y) = f(y, x_k) \prod_{l=1}^{k-1} S^{\mathrm{II,I}}(y, x_l). \tag{4.5}$$

Here $S^{\mathrm{II,I}}(y, x_{k'})$ represents the phase when permuting the excitation past a background field and $f(y, x_k)$ is a factor for the combination of the excitation with the background field.

We demand compatibility of the wave function with the S-matrix. This means that $\mathcal{S}^{\mathrm{I}}_\pi$ merely multiplies the state by the previous S^{I}_π in (4.3) and permutes the momenta

$$\mathcal{S}^{\mathrm{I}}_\pi\big|\psi^\alpha\big\rangle^{\mathrm{II}} = S^{\mathrm{I}}_\pi\big|\psi^\alpha\big\rangle^{\mathrm{II}}_\pi, \quad \big|\psi^\alpha\big\rangle^{\mathrm{II}}_\pi = \sum_{k=1}^{K} \Psi_{\pi,k}(y)\,\big|\phi^1_{\pi(1)} \ldots \psi^\alpha_{\pi(k)} \ldots \phi^1_{\pi(K)}\big\rangle^{\mathrm{I}}. \tag{4.6}$$

with

$$\Psi_{\pi,k}(y) = f(y, x_{\pi(k)}) \prod_{l=1}^{k-1} S^{\mathrm{II,I}}(y, x_{\pi(l)}). \tag{4.7}$$

To solve this problem, it is sufficient to consider a spin chain with only two sites

$$\begin{aligned} \left|\psi^\alpha\right\rangle^{\mathrm{II}} &= f(y,x_1)\left|\psi_1^\alpha\phi_2^1\right\rangle^{\mathrm{I}} + f(y,x_2)S^{\mathrm{II,I}}(y,x_1)\left|\phi_1^1\psi_2^\alpha\right\rangle^{\mathrm{I}}, \\ \left|\psi^\alpha\right\rangle_\pi^{\mathrm{II}} &= f(y,x_2)\left|\psi_2^\alpha\phi_1^1\right\rangle^{\mathrm{I}} + f(y,x_1)S^{\mathrm{II,I}}(y,x_2)\left|\phi_2^1\psi_1^\alpha\right\rangle^{\mathrm{I}}. \end{aligned} \tag{4.8}$$

We thus demand

$$\mathcal{S}_{12}^{\mathrm{I}}\left|\psi^\alpha\right\rangle^{\mathrm{II}} = S^{\mathrm{I,I}}(x_1,x_2)\left|\psi^\alpha\right\rangle_\pi^{\mathrm{II}} \tag{4.9}$$

which amounts to

$$\begin{aligned} & f(y,x_1)\,K(x_1,x_2) + f(y,x_2)\,S^{\mathrm{II,I}}(y,x_1)\,G(x_1,x_2) = f(y,x_2)\,A(x_1,x_2), \\ & f(y,x_1)\,L(x_1,x_2) + f(y,x_2)\,S^{\mathrm{II,I}}(y,x_1)\,H(x_1,x_2) \\ & \quad = f(y,x_1)\,S^{\mathrm{II,I}}(y,x_2)\,A(x_1,x_2). \end{aligned} \tag{4.10}$$

These two equations are solved by

$$S^{\mathrm{II,I}}(y,x_k) = \frac{y-x_k^-}{y-x_k^+}, \quad f(y,x_k) = \frac{y\gamma_k}{y-x_k^+}. \tag{4.11}$$

4.3 Scattering

For a two-excitation state we make an ansatz of two superimposed plane waves

$$\left|\psi_1^\alpha\psi_2^\beta\right\rangle^{\mathrm{II}} = \sum_{k<l=1}^{K} \Psi_k(y_1)\Psi_l(y_2)\left|\phi_1^1\ldots\psi_k^\alpha\ldots\psi_l^\beta\ldots\phi_K^1\right\rangle^{\mathrm{I}}. \tag{4.12}$$

This solves the compatibility condition $\mathcal{S}_\pi^{\mathrm{I}}\left|\psi_1^\alpha\psi_2^\beta\right\rangle^{\mathrm{II}} = S_\pi^{\mathrm{I}}\left|\psi_1^\alpha\psi_2^\beta\right\rangle_\pi^{\mathrm{II}}$ except when the two excitations are neighbours. We should also consider a state with one excitation $\phi^2\mathcal{Z}^+$ which can undergo mixing with the previous state

$$\left|\phi_{12}^2\mathcal{Z}^+\right\rangle^{\mathrm{II}} = \sum_{k=1}^{K}\Psi_k(y_1)\Psi_k(y_2)f(y_1,y_2,x_k)\left|\phi_1^1\ldots\phi_k^2\mathcal{Z}^+\ldots\phi_K^1\right\rangle^{\mathrm{I}}. \tag{4.13}$$

Here, $f(y_1,y_2,x_k)$ represents a factor which occurs when two excitations reside on the same site. A generic two-excitation eigenstate must be of the form

$$\left|\Psi^{\mathrm{II}}\right\rangle = \left|\psi_1^\alpha \psi_2^\beta\right\rangle^{\mathrm{II}} + \varepsilon^{\alpha\beta}\left|\phi_{12}^2 \mathcal{Z}^+\right\rangle^{\mathrm{II}} + \mathcal{S}_{12}^{\mathrm{II}}\left|\psi_1^\alpha \psi_2^\beta\right\rangle^{\mathrm{II}} \tag{4.14}$$

with an $\mathfrak{su}(2)$ symmetric S-matrix

$$\mathcal{S}_{12}^{\mathrm{II}}\left|\psi_1^\alpha \psi_2^\beta\right\rangle^{\mathrm{II}} = M_{12}\left|\psi_2^\alpha \psi_1^\beta\right\rangle^{\mathrm{II}} + N_{12}\left|\psi_2^\beta \psi_1^\alpha\right\rangle^{\mathrm{II}}. \tag{4.15}$$

Again we impose the compatibility condition

$$\mathcal{S}_\pi^{\mathrm{I}}\left|\Psi^{\mathrm{II}}\right\rangle = S_\pi\left|\Psi^{\mathrm{II}}\right\rangle_\pi \tag{4.16}$$

which is trivially satisfied when the two excitations are not neighbours. To solve the relation exactly we need to consider only a two-site state

$$\begin{aligned}\left|\Psi^{\mathrm{II}}\right\rangle &= f(y_1,x_1)f(y_2,x_2)S^{\mathrm{II,I}}(y_2,x_1)\left|\psi_1^\alpha \psi_2^\beta\right\rangle^{\mathrm{I}} \\ &\quad + f(y_1,x_1)f(y_2,x_1)f(y_1,y_2,x_1)(x_2^-/x_2^+)\,\varepsilon^{\alpha\beta}\left|\phi_1^2\phi_2^1\mathcal{Z}^+\right\rangle^{\mathrm{I}} \\ &\quad + f(y_1,x_2)f(y_2,x_2)S^{\mathrm{II,I}}(y_1,x_1)S^{\mathrm{II,I}}(y_2,x_1)f(y_1,y_2,x_2)\,\varepsilon^{\alpha\beta}\left|\phi_1^1\phi_2^2\mathcal{Z}^+\right\rangle^{\mathrm{I}} \\ &\quad + M(y_1,y_2)f(y_2,x_1)f(y_1,x_2)S^{\mathrm{II,I}}(y_1,x_1)\left|\psi_1^\alpha \psi_2^\beta\right\rangle^{\mathrm{I}} \\ &\quad + N(y_1,y_2)f(y_2,x_1)f(y_1,x_2)S^{\mathrm{II,I}}(y_1,x_1)\left|\psi_1^\beta \psi_2^\alpha\right\rangle^{\mathrm{I}}\end{aligned} \tag{4.17}$$

and the state $\left|\Psi^{\mathrm{II}}\right\rangle_\pi$ where x_1 and x_2 are interchanged. We find the unique solution of (4.16)

$$\begin{aligned}M_{12} &= -\frac{i}{y_1 + g^2/2y_1 - y_2 - g^2/2y_2 + i} = -\frac{i}{v_1 - v_2 + i}\,, \\ N_{12} &= -\frac{y_1 + g^2/2y_1 - y_2 - g^2/2y_2}{y_1 + g^2/2y_1 - y_2 - g^2/2y_2 + i} = -\frac{v_1 - v_2}{v_1 - v_2 + i},\end{aligned} \tag{4.18}$$

where the new spectral parameter v_k is related to y_k as

$$v_k = y_k + \frac{g^2}{2y_k}. \tag{4.19}$$

The factor for two coincident excitations in (4.13) is

$$f(y_1,y_2,x_k) = \frac{\alpha}{\gamma_k^2}\,\frac{x_k^- - x_k^+}{x_k^+}\,\frac{y_1 y_2 - x_k^- x_k^+}{y_1 y_2}\,\frac{y_1 - y_2}{y_1 + g^2/2y_1 - y_2 - g^2/2y_2 + i}. \tag{4.20}$$

In Appendix C we will present an alternative notation for wave functions which is somewhat more transparent and should naturally generalize to more than two excitations.

4.4 Final level

The level-II S-matrix $\mathcal{S}^{\mathrm{II}}_{12}$ has the standard form of a $\mathfrak{su}(2)$ invariant S-matrix with spectral parameters $v_k = y_k + \frac{g^2}{2y_k}$. It is therefore clear that the remaining elements of the diagonalized S-matrix are[7]

$$\begin{aligned} S^{\mathrm{II,II}}(y_1, y_2) &= -M(y_1, y_2) - N(y_1, y_2) = 1, \\ S^{\mathrm{III,II}}(w_1, y_2) &= \frac{w_1 - y_2 - g^2/2y_2 - \frac{i}{2}}{w_1 - y_2 - g^2/2y_2 + \frac{i}{2}} = \frac{w_1 - v_2 - \frac{i}{2}}{w_1 - v_2 + \frac{i}{2}}, \\ S^{\mathrm{III,III}}(w_1, w_2) &= \frac{w_1 - w_2 + i}{w_1 - w_2 - i}. \end{aligned} \tag{4.21}$$

Eigenstates of the Hamiltonian are now determined through a set of main parameters $\{x_1, \ldots, x_{K^{\mathrm{I}}}\}$ as well as several auxiliary parameters $\{y_1, \ldots, y_{K^{\mathrm{II}}}\}$ and $\{w_1, \ldots, w_{K^{\mathrm{III}}}\}$. The spin chain picture has completely dissolved.

4.5 Bethe equations

Bethe equations are periodicity conditions for a state of the original spin chain. As the length fluctuates, we cannot define the period, but if we also impose cyclicity this is not a problem. The generic Bethe equations for a diagonalized S-matrix $S^{AB}(x_k^A, x_l^B)$ read

$$1 = \prod_{\substack{B=0 \\ (B,l)\neq(A,k)}}^{\mathrm{III}} \prod_{l=1}^{K^B} S^{BA}(x_l^B, x_k^A). \tag{4.22}$$

Here K^A is the number of excitations of type $A \in \{0, \mathrm{I}, \mathrm{II}, \mathrm{III}\}$. So far we have not introduced the quasi-excitations of type 0: These are sites of the original spin chain and they do not carry an individual momentum parameter for this homogeneous spin chain. They only scatter with excitations of type I defining the wave function of a homogeneous plane wave

$$S^{\mathrm{I},0}(x_k, \cdot) = \frac{x_k^+}{x_k^-} = e^{ip_k}. \tag{4.23}$$

Imposing a Bethe equation at level 0 implies that sites can be permuted around the chain without a net phase shift. This operation is a global shift and invariance is equivalent to the zero-momentum condition (2.23),

[7] Note that the excitation of type II is fermionic. For the diagonalized S-matrix we shall use the convention that scattering of two fermions introduces an additional factor of -1. Hence $S^{\mathrm{II,II}} = 1$.

i.e. the physicality constraint $\mathfrak{P}|\Psi\rangle = \mathfrak{K}|\Psi\rangle = 0$ in Section 2.5. Clearly, the S-matrix satisfies the involution condition

$$S^{A,B}\big(x_k^A, x_l^B\big) = \frac{1}{S^{B,A}\big(x_l^B, x_k^A\big)} \tag{4.24}$$

from which the remaining matrix elements can be read off. The asymptotic Bethe equations are summarized in table 2. Here $v_k = \frac{y_k + g^2}{2y_k}$ and $x_k^\pm$ are related by (2.21).

In order to understand the number of excitations K^A, we first of all convert all fields ϕ^2 into $\frac{\psi^1\psi^2}{\mathcal{Z}\phi^1}$. Then we follow through the earlier nested Bethe ansatz and find

$$\begin{aligned}
K^0 &= N(\mathcal{Z}) + N(\phi^1) + N(\psi^1) + N(\psi^2) = p + 2q + 2r - s = r_1 + 2r_2 - r_3,\\
K^{\mathrm{I}} &= N(\phi^1) + N(\psi^1) + N(\psi^2) + N(\phi^2) = q + 2r - s = r_2 + r_4,\\
K^{\mathrm{II}} &= N(\psi^1) + N(\psi^2) + 2N(\phi^2) = 2r - s = r_3 + 2r_4,\\
K^{\mathrm{III}} &= N(\psi^2) + N(\phi^2) = r - s = r_4.
\end{aligned} \tag{4.25}$$

Here, $\big[p, q; r + \frac{1}{2}\delta D; s\big]$ are the Dynkin labels of the state when the Dynkin diagram is O–O–X–O and $\big[r_1; r_2 + \frac{1}{2}\delta D; r_3; r_4 + \frac{1}{2}\delta D\big]$ are the Dynkin labels when the diagram is O–X–O–X. These are related by $p = r_1, q = r_2 - r_3 - r_4; s = r_3; r = r_3 + r_4$. Note that the highest-weight state in a multiplet is determined using the Dynkin diagram O–X–O–X.

The derived Bethe equations agree with the equations conjectured in [13]. To see this, we first eliminate the flavours $1, 2, 3$ to restrict to the $\mathfrak{su}(2|3)$

Table 2: Asymptotic Bethe equations for the dynamic $\mathfrak{su}(2|3)$ spin chain

$$1 = \prod_{l=1}^{K^{\mathrm{I}}} \frac{x_l^+}{x_l^-},$$

$$1 = \left(\frac{x_k^-}{x_k^+}\right)^{K^0} \prod_{\substack{l=1\\l\neq k}}^{K^{\mathrm{I}}} \left(S^0(x_l, x_k)\, \frac{x_k^+ - x_l^-}{x_k^- - x_l^+}\right) \prod_{l=1}^{K^{\mathrm{II}}} \frac{x_k^- - y_l}{x_k^+ - y_l},$$

$$1 = \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k - x_l^+}{y_k - x_l^-} \prod_{l=1}^{K^{\mathrm{III}}} \frac{v_k - w_l + \frac{i}{2}}{v_k - w_l - \frac{i}{2}},$$

$$1 = \prod_{l=1}^{K^{\mathrm{II}}} \frac{w_k - v_l + \frac{i}{2}}{w_k - v_l - \frac{i}{2}} \prod_{\substack{l=1\\l\neq k}}^{K^{\mathrm{III}}} \frac{w_k - w_l - i}{w_k - w_l + i}.$$

sector. Then we trade in all Bethe roots of type 7 for Bethe roots of type 5 by means of the duality transformation. Finally, we identify flavours I, II, III with $4, 5, 6$, respectively. In other words, the Bethe roots x, u, y, v, w correspond to x_4, u_4, x_5, u_5, u_6.

4.6 Symmetry enhancement

Superficially, the Bethe equations in table 2 look as though they originate from the Dynkin diagram O–X–O, i.e., a spin chain with $\mathfrak{su}(2|2)$ symmetry. However, the full symmetry algebra of the considered spin chain is $\mathfrak{su}(3|2)$ by construction. This means that some of the symmetry must be hidden.

Symmetries in the Bethe equations are represented by Bethe roots at special positions, conventionally at ∞. Indeed, one can add a Bethe root $x^{\pm} \to \infty$ (flavour I), $y, v \to \infty$ (flavour II) or $w \to \infty$ (flavour III) to any existing set of Bethe roots. If the original set satisfies the Bethe equations, the new set does so as well, because the scattering between these special excitations and any other excitation is trivial, $S = 1$.

Symmetry enhancement for the Bethe equations in table 2 works as follows: one adds a Bethe root $y = 0$ and removes a quasi-excitation of type 0 at the same time. In the Bethe equation for x_k, the effect of adding $y = 0$ and removing a quasi-excitation cancels. In the Bethe equation for a w_k, the scattering with $y = 0$ is trivial, because $v = \frac{y+g^2}{2y} = \infty$. The equation for some other y_k is not modified due to the absence of self-scattering terms for fermions. Finally, in the equation for y itself, the net scattering with all x_ls is equivalent to the zero-momentum condition (2.23). The latter is effectively the Bethe equation for the (removed) quasi-excitation.

In conclusion, the Bethe equations have a hidden $\mathfrak{su}(2|3)$ symmetry. This however requires that the physicality constraint holds. One can also derive the S-matrix and Bethe equations assuming that the residual symmetry at level I is $\mathfrak{su}(1|2)$. This avenue is considered in Appendix B.

4.7 Abelian phase

We have solved the asymptotic spectrum of the $\mathfrak{su}(2|3)$ dynamic spin chain [4] up to the overall function $S_0(x_k, x_l)$. The analysis of a similar class of long-range spin chains in [36] has produced a suggestive generic form for this function. Clearly, it does not necessarily have to apply to this particular spin chain, but it is worth contemplating the possibility. Here is summary of the results of [36]: the overall factor is

$$S_0(x_l, x_k) = \frac{1 - g^2/2x_k^+ x_l^-}{1 - g^2/2x_k^- x_l^+} \exp\Bigl(2i\theta_{kl}\Bigr) \tag{4.26}$$

with the dressing phase

$$\theta_{kl} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{rs}(g^2)\Bigl(q_{r,k}\, q_{s,l} - q_{s,k}\, q_{r,l}\Bigr). \tag{4.27}$$

and the r-th moment of the k-th excitation

$$q_{r,k} = \frac{1}{r-1}\left(\frac{i}{(x_k^+)^{r-1}} - \frac{i}{(x_k^-)^{r-1}}\right). \tag{4.28}$$

The coefficient functions $\beta_{rs}(g)$, $r > s$, can be chosen freely, but the structure of the algebra generators imposes some constraints: Compatibility with the range of the interactions requires $\beta_{rs}(g^2) = \mathcal{O}(g^{2s-2})$. Compatibility with gauge theory Feynman diagrams imposes the more restrictive constraint $\beta_{rs}(g^2) = \mathcal{O}(g^{2r+2s-4})$. Finally, the coefficients β_{rs} with odd $r+s$ violate parity. The author believes that all these coefficients can be realized by the underlying $\mathfrak{su}(2|3)$ spin chain. In the analysis of [4] only the first, $\beta_{23}(g^2)$ can be seen at $\mathcal{O}(g^4)$.

In [36] two further sequences of parameters related to propagation and mixing of charges were identified. Here, these degrees of freedom are fixed by the structure of the algebra, cf. (2.24), and the inclusion of the Hamiltonian in the algebra. Finally, we note that (2.24) is not the correct map for the Inozemtsev spin chain [9], cf. the appendix of [12]. This proves that the Inozemtsev spin chain cannot be an accurate description of the $\mathfrak{su}(2)$ sector of planar $\mathcal{N}=4$ SYM beyond three loops which remained as a logical possibility after [10]. Conversely, we cannot guarantee that the spin chain of [12] is the correct (asymptotic) description at starting from four loops; proper scaling in the thermodynamic limit may be violated in other ways or even integrability might break (although the latter does not seem likely).

5 Generalization to $\mathfrak{psu}(2,2|4)$ and $\mathcal{N}=4$ SYM

In $\mathcal{N}=4$ super Yang–Mills, there are (8|8) types of level-I excitations [15]. These transform under the residual algebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$ [5].[8] The generators of the bosonic subalgebra $\mathfrak{su}(2)^4$ are $\mathfrak{L}, \mathfrak{R}, \dot{\mathfrak{L}}, \dot{\mathfrak{R}}$, the fermionic generators

[8] A very similar algebra appeared in the study of mass-deformed M2 branes [45, 34]. It would be interesting to find out if there is a deeper connection. Also the residual algebra $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ for the $\mathfrak{su}(2|3)$ sector of $\mathcal{N}=4$ SYM appears to play a for M5 branes [34].

are $\mathfrak{Q}, \mathfrak{S}, \dot{\mathfrak{Q}}, \dot{\mathfrak{S}}$. The dotted algebra relations are the same as for the undotted ones (2.5,2.6) with the central charges shared among the two algebras $(\dot{\mathfrak{C}}, \dot{\mathfrak{K}}, \dot{\mathfrak{P}}) = (\mathfrak{C}, \mathfrak{K}, \mathfrak{P})$. The set of $(8|8) = (2|2) \times (2|2)$ excitations now transforms under each extended $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ subalgebra as $(2|2)$ in Section 2.4. The $(8|8)$ composite fields are of four types: $(\phi\dot{\phi})$ is a quartet of scalars, $(\phi\dot{\psi})$ and $(\psi\dot{\phi})$ are two quartets of fermions and $(\psi\dot{\psi})$ is a quartet of covariant derivatives.

We can now apply these results for the algebra, S-matrix and Bethe equations to $\mathcal{N} = 4$ SYM. Due to invariance under each factor of the residual symmetry, the S-matrix should be

$$\mathcal{S}^{\mathcal{N}=4}_{kl} = \frac{\mathcal{S}^{\mathrm{I}}_{kl}\dot{\mathcal{S}}^{\mathrm{I}}_{kl}}{A_{kl}} \tag{5.1}$$

with some overall undetermined phase $S_0(x_k, x_l)$, c.f. the remarks in Section 4.7. Similarly, the asymptotic Bethe equations can be composed from those in table 2. Here, the main Bethe roots $x^{\pm}_k$ are shared among the two sectors, but the auxiliary Bethe roots y_k, w_k are duplicated $\dot{y}_k, \dot{w}_k$. The complete Bethe equations are as in [13].

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Appendix A A contraction of $\mathfrak{d}(2, 1; \epsilon)$

The exceptional superalgebra $\mathfrak{d}(2, 1; \epsilon)$ consists of three triplets of $\mathfrak{su}(2)$ generators $\mathfrak{J}^a{}_b$, $\mathfrak{J}^\alpha{}_\beta$, $\mathfrak{J}^\mathfrak{a}{}_\mathfrak{b}$ and an octet of fermionic generators $\mathfrak{J}^{a\beta\mathfrak{c}}$. The $\mathfrak{su}(2)^3$ generators commute canonically

$$[\mathfrak{J}^a{}_b, \mathfrak{J}^c{}_d] = \delta^c_b \mathfrak{J}^a{}_d - \delta^a_d \mathfrak{J}^c{}_b,$$
$$[\mathfrak{J}^\alpha{}_\beta, \mathfrak{J}^\gamma{}_\delta] = \delta^\gamma_\beta \mathfrak{J}^\alpha{}_\delta - \delta^\alpha_\delta \mathfrak{J}^\gamma{}_\beta,$$

$$[\mathfrak{J}^{\mathfrak{a}}{}_{\mathfrak{b}}, \mathfrak{J}^{\mathfrak{c}}{}_{\mathfrak{d}}] = \delta^{\mathfrak{c}}_{\mathfrak{b}} \mathfrak{J}^{\mathfrak{a}}{}_{\mathfrak{d}} - \delta^{\mathfrak{a}}_{\mathfrak{d}} \mathfrak{J}^{\mathfrak{c}}{}_{\mathfrak{b}}. \tag{A.1}$$

The fermionic generators transform in the fundamental representation of each $\mathfrak{su}(2)$ factor

$$\begin{aligned}
\left[\mathfrak{J}^{a}{}_{b}, \mathfrak{J}^{c\delta\mathfrak{e}}\right] &= \delta^{c}_{b} \mathfrak{J}^{a\delta\mathfrak{e}} - \tfrac{1}{2}\delta^{a}_{b} \mathfrak{J}^{c\delta\mathfrak{e}},\\
\left[\mathfrak{J}^{\alpha}{}_{\beta}, \mathfrak{J}^{c\delta\mathfrak{e}}\right] &= \delta^{\delta}_{\beta} \mathfrak{J}^{c\alpha\mathfrak{e}} - \tfrac{1}{2}\delta^{\alpha}_{\beta} \mathfrak{J}^{c\delta\mathfrak{e}},\\
\left[\mathfrak{J}^{\mathfrak{a}}{}_{\mathfrak{b}}, \mathfrak{J}^{c\delta\mathfrak{e}}\right] &= \delta^{\mathfrak{e}}_{\mathfrak{b}} \mathfrak{J}^{c\delta\mathfrak{a}} - \tfrac{1}{2}\delta^{\mathfrak{a}}_{\mathfrak{b}} \mathfrak{J}^{c\delta\mathfrak{e}}.
\end{aligned} \tag{A.2}$$

Finally, the anticommutator of the fermionic generators is

$$\left\{\mathfrak{J}^{a\beta\mathfrak{c}}, \mathfrak{J}^{d\epsilon\mathfrak{f}}\right\} = \alpha\, \varepsilon^{ak} \varepsilon^{\beta\epsilon} \varepsilon^{\mathfrak{c}\mathfrak{f}} \mathfrak{J}^{d}{}_{k} + \beta\, \varepsilon^{ad} \varepsilon^{\beta\kappa} \varepsilon^{\mathfrak{c}\mathfrak{f}} \mathfrak{J}^{\epsilon}{}_{\kappa} + \gamma\, \varepsilon^{ad} \varepsilon^{\beta\epsilon} \varepsilon^{\mathfrak{c}\mathfrak{k}} \mathfrak{J}^{\mathfrak{f}}{}_{\mathfrak{k}}. \tag{A.3}$$

The Jacobi identity requires $\alpha + \beta + \gamma = 0$ and a rescaling of $\mathfrak{J}^{a\beta\mathfrak{c}}$ leads to a rescaling of (α, β, γ). The parameter of $\mathfrak{d}(2,1;\epsilon)$ is given by $\epsilon = \frac{\gamma}{\alpha}$ or any other of the six quotients made from two of the coefficients α, β, γ.

We now derive the algebra in Section 2.3 as a contraction of the algebra given before. First of all we identify two of the $\mathfrak{su}(2)$s

$$\mathfrak{J}^{a}{}_{b} = \mathfrak{R}^{a}{}_{b}, \quad \mathfrak{J}^{\alpha}{}_{\beta} = \mathfrak{L}^{\alpha}{}_{\beta}. \tag{A.4}$$

The third $\mathfrak{su}(2)$ will be contracted, we split up the generator $\mathfrak{J}^{\mathfrak{a}}{}_{\mathfrak{b}}$ as follows:

$$\mathfrak{J}^{1}{}_{2} = \epsilon^{-1}\, \mathfrak{P}, \quad \mathfrak{J}^{1}{}_{1} = -\mathfrak{J}^{2}{}_{2} = -\epsilon^{-1}\, \mathfrak{C}, \quad \mathfrak{J}^{2}{}_{1} = -\epsilon^{-1}\, \mathfrak{K}. \tag{A.5}$$

The fermionic generator yields the supersymmetry generators

$$\mathfrak{J}^{a\beta 1} = \varepsilon^{ac} \mathfrak{Q}^{\beta}{}_{c}, \quad \mathfrak{J}^{a\beta 2} = \varepsilon^{\beta\gamma} \mathfrak{S}^{a}{}_{\gamma}. \tag{A.6}$$

Finally, the three constants of the exceptional algebra are adjusted to $\mathfrak{d}(2,1;\epsilon)$

$$\alpha = -1 - \epsilon, \quad \beta = 1, \quad \gamma = \epsilon. \tag{A.7}$$

Sending $\epsilon \to 0$ leads to the commutation relations in Section 2.3.

Appendix B Alternative notation with $\mathfrak{su}(1|2)$ symmetry

The manifest symmetry of the Bethe equations is $\mathfrak{su}(1|2)$, i.e., the residual symmetry at level I appears to be $\mathfrak{su}(1|2)$ and not $\mathfrak{su}(2|2)$. In fact, we can work with $\mathfrak{su}(1|2)$ as the manifest symmetry of the S-matrix and thereby avoid the effects of a fluctuating length. Let us outline this picture here.

We first define the two bosonic excitations as $\phi := \phi^1$ and $\chi := \phi^2 \mathcal{Z}^+$. Then the multiplet $(\phi|\, \psi^1, \psi^2|\, \chi)$ transforms in a typical representation $(\mathbf{1}|\mathbf{2}|\mathbf{1})$ of $\mathfrak{su}(1|2)$. This representation is like the one discussed in Section 2.4 but the index a is restricted to the value 1. There is no complication from a fluctuating length as in (2.8) for $\mathfrak{Q}^{\alpha}$ transforms $\phi = \phi^1$ to ψ^{α} and ϕ^{β}

to $\varepsilon^{\alpha\beta}\phi^2\mathcal{Z}^+ = \varepsilon^{\alpha\beta}\chi$. Similarly, $\mathfrak{S}_\alpha$ transforms between $(\phi|\,\psi^1,\psi^2|\,\chi)$ in the opposite direction. The spin chain becomes static. Note that for an excitation with central charge $C = +\frac{1}{2}$, the representation splits in two parts $(\mathbf{1}|\mathbf{2}|\mathbf{0})$ and $(\mathbf{0}|\mathbf{0}|\mathbf{1})$, i.e., a fundamental and a trivial representation. This is the common breaking pattern for typical representations of $\mathfrak{su}(2|1)$.

To understand the possible degrees of freedom of an invariant S-matrix one should investigate the irreducible representations in the tensor product $(\mathbf{1}|\mathbf{2}|\mathbf{1})^2$ [46]. There are three irreps which could be described by the symbols $(\mathbf{1}|\mathbf{2}|\mathbf{1}|\mathbf{0}|\mathbf{0})$, $(\mathbf{0}|\mathbf{2}|\mathbf{4}|\mathbf{2}|\mathbf{0})$ and $(\mathbf{0}|\mathbf{0}|\mathbf{1}|\mathbf{2}|\mathbf{1})$. The S-matrix thus acts on selected representatives as

$$\begin{aligned} \mathcal{S}^{\mathrm{I}}_{12}\,|\phi_1\phi_2\rangle^{\mathrm{I}} &= S^1_{12}\,|\phi_2\phi_1\rangle^{\mathrm{I}},\\ \mathcal{S}^{\mathrm{I}}_{12}\,|\psi_1^{\{\alpha}\psi_2^{\beta\}}\rangle^{\mathrm{I}} &= -S^2_{12}\,|\psi_2^{\{\alpha}\psi_1^{\beta\}}\rangle^{\mathrm{I}},\\ \mathcal{S}^{\mathrm{I}}_{12}\,|\chi_1\chi_2\rangle^{\mathrm{I}} &= S^3_{12}\,|\chi_2\chi_1\rangle^{\mathrm{I}}, \end{aligned} \tag{B.1}$$

the action on the other states is determined through $\mathfrak{su}(1|2)$ invariance. From symmetry arguments alone the three factors S^k_{12} are independent. It is, however, very likely that they are interrelated by the Yang–Baxter equation (3.8).

In the main text we have used invariance under $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ to relate the coefficients and found

$$\begin{aligned} S^1_{12} &= S^0_{12}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+},\\ S^2_{12} &= S^0_{12},\\ S^3_{12} &= S^0_{12}\,\frac{g^2/2x_2^+ - g^2/2x_1^-}{g^2/2x_2^- - g^2/2x_1^+} = S^0_{12}\,\frac{x_2^-}{x_2^+}\,\frac{x_1^+}{x_1^-}\,\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}. \end{aligned} \tag{B.2}$$

It is straightforward to see that this S-matrix agrees with table 1. For the last line in (B.1, B.2) one should note that $\chi = \phi^2\mathcal{Z}^+$ requires the introduction of factors of $\exp(ip) = x^+/x^-$ due to shifts of $\mathcal{Z}^+$ (2.13).

Spin chains with the same symmetry group and the same type of representation have been investigated in [46,47]. The expressions (B.2) for the eigenvalues of the S-matrix however do not agree with the expressions in [46,47]. Also the Bethe equations for the same model in [48] are incompatible with our equations in table 2. The results in [46–48] are certainly correct and it seems that (B.2) is an exceptional solution of the YBE. The existence of such a solution might be attributed to the fact that the representation of the excitations is correlated to the momentum by (2.18), see also [25]. The distinction to the $\mathfrak{su}(1|1)$ case in [25] appears to be that we cannot use an arbitrary dispersion relation, but only (2.18) is valid. It would be useful to understand the derivation with manifest $\mathfrak{su}(2|1)$ symmetry better.

Appendix C Using generators to construct level-II states

In Section 4.1, 4.2, 4.3 we have determined the diagonalized wave functions of two level-II excitations. Here we will present an alternative notation which easily generalizes to more than two level-II excitations. This ansatz makes use of the supersymmetry generators $(\mathfrak{Q}^\alpha{}_1)_k$ to create an excitation ψ^α from the vacuum of ϕ^1s

$$\begin{aligned}(\mathfrak{Q}^\alpha{}_1)_k\,|0\rangle^{\mathrm{II}} &= a_k|\phi^1_1\ldots\psi^\alpha_k\ldots\phi^1_K\rangle^{\mathrm{I}},\\ (\mathfrak{Q}^\alpha{}_1)_k(\mathfrak{Q}^\beta{}_1)_l|0\rangle^{\mathrm{II}} &= a_k a_l|\phi^1_1\ldots\psi^\alpha_k\ldots\psi^\beta_l\ldots\phi^1_K\rangle^{\mathrm{I}},\\ (\mathfrak{Q}^\alpha{}_1)_k(\mathfrak{Q}^\beta{}_1)_k|0\rangle^{\mathrm{II}} &= a_k b_k\varepsilon^{\alpha\beta}|\phi^1_1\ldots\phi^2_k\mathcal{Z}^+\ldots\phi^1_K\rangle^{\mathrm{I}}.\end{aligned} \tag{C.1}$$

The advantage of this notation is that various factors from the algebra, such as a_k, b_k, will be absorbed into the application of the symmetry generators. The single-excitation state in (4.4) will now be written in a slightly different way

$$|\psi^\alpha\rangle^{\mathrm{II}} = \sum_{k=1}^{K}\frac{x_k^-\Psi_{k-1}(y)-x_k^+\Psi_k(y)}{x_k^- - x_k^+}\,(\mathfrak{Q}^\alpha{}_1)_k\,|0\rangle^{\mathrm{II}},\quad \Psi_k(y)=\prod_{l=1}^{k}S^{\mathrm{II,I}}(y,x_l). \tag{C.2}$$

Being somewhat sloppy about the terms at $k=0,K$ we can rewrite the one-excitation state as

$$|\psi^\alpha\rangle^{\mathrm{II}} = \sum_{k=0}^{K}\Psi_k(y)\Big((\mathfrak{Q}^\alpha{}_1)^-_k + (\mathfrak{Q}^\alpha{}_1)^+_{k+1}\Big)|0\rangle^{\mathrm{II}}. \tag{C.3}$$

Here we have introduced the dressed generators

$$(\mathfrak{Q}^\alpha{}_1)^\pm_k = \frac{x_k^\mp}{x_k^\mp - x_k^\pm}\,(\mathfrak{Q}^\alpha{}_1)_k. \tag{C.4}$$

The formula (C.3) can now be interpreted as follows: the level-II excitation y is permuted along the level-I chain using the scattering phase $S^{\mathrm{II,I}}$ until it is between x_k and x_{k+1}. At this point it can be joined with the vacuum either to the left by $(\mathfrak{Q}^\alpha{}_1)^-_k$ or to the right by $(\mathfrak{Q}^\alpha{}_1)^+_{k+1}$.

It becomes straightforward to write the two-excitation state as

$$\begin{aligned}\big|\psi_1^\alpha\psi_2^\beta\big\rangle^{\mathrm{II}} = \tfrac{1}{2}\sum_{k=0}^{K}\varPsi_k(y_1)\varPsi_k(y_2)\big[\big(\mathfrak{Q}^\alpha{}_1\big)^-_k(\mathfrak{Q}^\beta{}_1)^-_k + 2(\mathfrak{Q}^\alpha{}_1)^-_k(\mathfrak{Q}^\beta{}_1)^+_{k+1}\\ + (\mathfrak{Q}^\alpha{}_1)^+_{k+1}(\mathfrak{Q}^\beta{}_1)^+_{k+1}\big]\big|0\big\rangle^{\mathrm{II}} + \sum_{k<l=0}^{K}\varPsi_k(y_1)\varPsi_l(y_2)\Big((\mathfrak{Q}^\alpha{}_1)^-_k\\ + (\mathfrak{Q}^\alpha{}_1)^+_{k+1}\Big)\Big((\mathfrak{Q}^\beta{}_1)^-_l + (\mathfrak{Q}^\beta{}_1)^+_{l+1}\Big)\big|0\big\rangle^{\mathrm{II}}.\end{aligned} \tag{C.5}$$

Here, we have to make sure that the two excitations y_1, y_2 do not cross when they are joined with the vacuum. This leads to the slightly asymmetric form of the first term which should be understood as a chain-ordered version of the second term. Now the two asymptotic regions are joined by

$$\big|\varPsi_2^{\alpha\beta}\big\rangle = \big|\psi_1^\alpha\psi_2^\beta\big\rangle^{\mathrm{II}} + \mathcal{S}_{12}^{\mathrm{II}}\big|\psi_1^\alpha\psi_2^\beta\big\rangle^{\mathrm{II}}. \tag{C.6}$$

There is no term for two coincident excitations anymore, the correct factor in (4.20) has been distributed among the two asymptotic regions. The level-II S-matrix is

$$\mathcal{S}_{12}^{\mathrm{II}}\big|\psi_1^\alpha\psi_2^\beta\big\rangle^{\mathrm{II}} = M_{12}\big|\psi_2^\alpha\psi_1^\beta\big\rangle^{\mathrm{II}} + N_{12}\big|\psi_2^\beta\psi_1^\alpha\big\rangle^{\mathrm{II}}. \tag{C.7}$$

It should be clear how to generalize this framework to more than two excitations.

Appendix D A singlet state

In this appendix we construct and investigate a composite excitation which transforms as a singlet of the symmetry algebra. We have no direct use for it, but its existence appears exciting.

D.1 The state

Considerations of the manifest symmetries $\mathfrak{su}(3)$ and $\mathfrak{su}(2)$ suggest that the singlet must be composed from the two building blocks $\varepsilon_{ab}\big|\phi_1^a\phi_2^b\mathcal{Z}^+\mathcal{Z}^+\mathcal{Z}^+\big\rangle^{\mathrm{I}}$

and $\varepsilon_{\alpha\beta}\big|\psi_1^\alpha\psi_2^\beta\mathcal{Z}^+\mathcal{Z}^+\big\rangle^{\mathrm{I}}$. To obtain the relative coefficient we demand invariance under the fermionic generators in (2.8) and find

$$\big|\mathbf{1}_{12}\big\rangle^{\mathrm{I}} = \frac{\alpha}{\gamma_1\gamma_2}\left(\frac{x_1^+}{x_1^-}-1\right)\varepsilon_{ab}\big|\phi_1^a\phi_2^b\mathcal{Z}^+\mathcal{Z}^+\mathcal{Z}^+\big\rangle^{\mathrm{I}} + \varepsilon_{\alpha\beta}\big|\psi_1^\alpha\psi_2^\beta\mathcal{Z}^+\mathcal{Z}^+\big\rangle^{\mathrm{I}} \quad \text{(D.1)}$$

with $x_2^\pm = \frac{g^2}{2x_1^\mp}$. Also the central charges $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$ annihilate this state. It is clear that one of the excitations is not physical, it has a negative central charge which balances the positive central charge of the other excitation. This composite of the two excitations might be interpreted as a particle–antiparticle pair. One could also say that one of the components is a creation operator while the other is an annihilation operator.

For $\mathcal{N}=4$ SYM, the invariant combination essentially consists of two covariant derivatives. Their total anomalous dimension is -2 which cancels precisely their contribution to the classical dimension.

D.2 Scattering

We can scatter the compound with any other excitation $\mathcal{X}$. Remarkably, the compound stays intact and the scattering phase is independent of the type of excitation. We find

$$\begin{aligned}\mathcal{S}^{\mathrm{I}}_{13}\mathcal{S}^{\mathrm{I}}_{12}\big|\mathcal{X}_3\mathbf{1}_{12}\big\rangle^{\mathrm{I}} &= \frac{x_3^+x_3^+}{x_3^-x_3^-}\,\frac{x_3^+-x_1^+}{x_3^+-x_1^-}\,\frac{1-g^2/2x_3^-x_1^+}{1-g^2/2x_3^-x_1^-}\\ &\quad\times S^0(x_3,x_1)\,S^0\left(x_3,\frac{g^2}{2x_1}\right)\big|\mathbf{1}_{12}\mathcal{X}_3\big\rangle^{\mathrm{I}}. \qquad \text{(D.2)}\end{aligned}$$

Note that this is not symmetric under the map of $x_1^\pm \to \frac{g^2}{2x_1^\mp}$. This is okay as the compound $\mathbf{1}_{12}$ is not symmetric under the interchange of x_1 and x_2. In fact, trying to interchange the components of $\mathbf{1}_{12}$ is not well-defined due to divergencies in the S-matrix.

We can also represent the state by means of diagonalized excitations. Then it is composed from the excitations $(K^0, K^{\mathrm{I}}, K^{\mathrm{II}}, K^{\mathrm{III}}) = (2,2,2,1)$. We can obtain trivial scattering for all but the main excitations by setting $w=u$, $v_1 = u+\frac{i}{2}$, $v_2 = u-\frac{i}{2}$, $x_2^\pm = \frac{g^2}{2x_1^\mp}$. When we then set $y_1 = x_1^+$ and $y_2 = \frac{g^2}{2x_1^-}$ we obtain the same phase as in (D.2).

For the complete $\mathcal{N}=4$ SYM model we can construct a similar invariant state from the excitations $(K^0, \dot K^{\mathrm{III}}, \dot K^{\mathrm{II}}, K^{\mathrm{I}}, K^{\mathrm{II}}, K^{\mathrm{III}}) = (0,1,2,2,2,1)$.

Here, in addition we must set $\dot{w} = u$, $\dot{v}_1 = u + \frac{i}{2}$, $\dot{v}_2 = u - \frac{i}{2}$. For $\dot{y}_{1,2}$ we have to choose between $x_1^\pm$ and $\frac{g^2}{2x_1^\pm}$. Setting as above $\dot{y}_1 = \frac{g^2}{2x_1^+}$ and $\dot{y}_2 = x_1^-$ the overall phase is

$$\mathcal{S}_{13}^{\mathrm{I}}\mathcal{S}_{12}^{\mathrm{I}}|\mathcal{X}_3\mathbf{1}_{12}\rangle^{\mathrm{I}} = \left(\frac{x_3^+ - x_1^+}{x_3^+ - x_1^-}\,\frac{1 - g^2/2x_3^- x_1^+}{1 - g^2/2x_3^- x_1^-}\right)^2 \times S^0(x_3, x_1)\, S^0\left(x_3, \frac{g^2}{2x_1}\right)\, |\mathbf{1}_{12}\mathcal{X}_3\rangle^{\mathrm{I}}. \tag{D.3}$$

This particular choice is most likely the correct one because the first term in the scattering factor matches the function f_{13}^2 obtained in the context of crossing symmetry [49].

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