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The Sub-Mean-Value Property of Subharmonic Functions and its Application to the Estimation of the Gaussian Curvature of the Span Metric

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Introduction

The main purpose of this paper is to prove the following theorem:

THEOREM. *Let W be a plane domain containing the origin 0. Let v be an integrable function on the complex plane $\boldsymbol{C}$ such that $v(z)\geqq c$ a.e. on W for a positive number c and $v(z)=0$ a.e. on the complement of W. If*

$$f'(0) = \frac{1}{\int v dm}\int_W f' v dm$$

for every analytic function f on W such that $\int_W |f'|^2 v dm < \infty$, then

$$\int_W s v dm \leqq c\int_{\Delta_r} s dm$$

for every subharmonic integrable function s on $\Delta_r=\{z\in\boldsymbol{C}\,|\,|z|<r\}$, where $r=\{\int v dm/(c\pi)\}^{1/2}$ and m denotes the two-dimensional Lebesgue measure.

The equality holds if and only if either s is harmonic on Δ_r or $v(z)=c$ a.e. on W and $W=\Delta_r-E$, where E denotes a relatively closed subset of Δ_r such that $E\cap K$ is removable with respect to analytic functions with finite Dirichlet integrals for every compact subset K of Δ_r.

In the above statement, one might wonder if s is defined on W and if $\int_W s v dm$ has a meaning. These follow from the following result:

PROPOSITION A ([7, Proposition 3.2]). *Under the same assumption as in the above theorem, it follows that $W\subset\Delta_r$ and the equality $\sup_{z\in W}|z|=r$ holds if and only if $v(z)=c$ a.e. on W and $W=\Delta_r-E$, where E denotes a relatively closed subset of Δ_r mentioned in the theorem.*

In fact, if $\sup_{z\in W}|z|=r$, then $\int_W s v dm = c\int_{\Delta_r} s dm$. Since every subharmonic function is locally bounded from above, if $\sup_{z\in W}|z|<r$, then $\max\{sv, 0\}$ is

integrable.

As an application of our theorem, we can estimate the Gaussian curvature $K(z)$ of the span metric, namely, the metric induced by the exact Bergman kernel differentials. The metric is equal to the Poincaré metric on an open disc, and so $K(z) \equiv -4$ on the disc.

In 1934, K. Zarankiewicz [10] showed that $K(z) < -4$ for annuli and, in 1967, S. Bergman and B. Chalmers [3] treated the case of triply-connected plane domains.

S. Bergman [2] obtained the following:

Let z_0 be a boundary point of a plane domain such that two circles can be constructed passing through z_0, one interior and one exterior to the domain. Then $\lim_{z \to z_0} K(z) = -4$ if z approaches z_0 so that $z - z_0$ makes with the interior normal an angle in absolute value less than $\pi/2$.

Therefore it is plausible that $K(z) \leqq -4$ for such a domain. N. Suita [9] conjectured that $K(z) \leqq -4$ for every Riemann surface $R \notin O_{AD}$, namely, for every Riemann surface on which there is a nonconstant analytic function with a finite Dirichlet integral, and that the equality holds if and only if R is conformally equivalent to $\Delta_1 - E$, where Δ_1 denotes the unit disc and E denotes a relatively closed subset of Δ_1 mentioned as in the above theorem.

Recently J. Burbea [5] proved that $K(z) \leqq -2$ for a plane domain which is not contained in the class O_{AD}.

We apply our theorem and show that the above conjecture is true.

This paper consists of eight sections. In Sections from 1 to 6 we make preparations for the proof of the theorem. On first reading this paper one could omit these sections except for the definition of the kernel functions $M_\nu(z; \zeta, t, R)$ in Section 5 and the statements of Propositions 2.1, 3.3, 4.2, 6.1 and Corollary 5.2.

The proof of the theorem is given in Section 7. The final section, Section 8, is devoted to its application to the estimation of the Gaussian curvature of the span metric.

§1. An areal inequality

Throughout this paper we denote by m the two-dimensional Lebesgue measure. In this section we shall show the following proposition:

PROPOSITION 1.1. *For every ε with $0 < \varepsilon < 1$, there is $\delta > 0$ such that if a measurable set E and open discs Δ_j, $j = 1, \ldots, n$, in the complex plane $\mathbf{C}$ satisfy*

$$m(E \cap \Delta_j) \leqq \delta m(\Delta_j)$$

for every j, then

$$m(E \cap (\bigcup_{j=1}^{n} \Delta_j)) \leqq \varepsilon m(\bigcup_{j=1}^{n} \Delta_j).$$

To prove Proposition 1.1 we prepare several lemmas.

LEMMA 1.2 ([7, Lemma 1.2]). *Let Δ_j, $j=1,\dots,n$, be open discs whose radii are not less than a positive number r. Then the length $\ell(\partial(\bigcup_{j=1}^{n}\Delta_j))$ of the boundary of $\bigcup_{j=1}^{n}\Delta_j$ is not greater than $(2/r)m(\bigcup_{j=1}^{n}\Delta_j)$.*

LEMMA 1.3. *Let J be a Jordan curve in $\boldsymbol{C}$ and set $J_\rho=\{z\in\boldsymbol{C} \mid d(z, J)<\rho\}$ for $\rho>0$, where $d(z, J)$ denotes the distance from z to J. Then*

$$m(J_\rho) \leqq 2\rho\ell(J) + \pi\rho^2.$$

LEMMA 1.4. *Let ω be a number with $0<\omega<1/10$. Set $\Delta=\{z\in\boldsymbol{C} \mid |z|<1\}$, $\Gamma=\{z\in\boldsymbol{C} \mid |z|>1+2\omega\}$ and $\Omega=\{z\in\boldsymbol{C} \mid -\omega/2\leqq\arg z<\omega/2\}$. Let Δ_j, $j=1,\dots,n$, be open discs such that the center c_j of each Δ_j belongs to Ω and each Δ_j satisfies $\Delta_j\cap\Delta\neq\emptyset$ and $\Delta_j\cap\Gamma\neq\emptyset$. Then $\bigcup_{j=1}^{n}\Delta_j$ is a domain starlike with respect to $1+\omega$. In particular, $\bigcup_{j=1}^{n}\Delta_j$ is a Jordan domain.*

PROOF. To prove the lemma, it is sufficient to show that $1+\omega\in\Delta_j$ for every j. Set $c_j=r_je^{i\theta_j}$ and denote by ρ_j the radius of Δ_j. Then

$$\begin{aligned} |(1+\omega)-c_j| &\leqq |(1+\omega)-r_je^{i\omega/2}| \\ &\leqq |(1+2\omega)e^{i\omega/2}-r_je^{i\omega/2}| \\ &= |(1+2\omega)e^{i\theta_j}-c_j| \\ &< \rho_j \end{aligned}$$

if $r_j\leqq 1+\omega$, and

$$|(1+\omega)-c_j| \leqq |e^{i\omega/2}-r_je^{i\omega/2}| < \rho_j$$

if $r_j>1+\omega$. These inequalities imply $1+\omega\in\Delta_j$.

LEMMA 1.5. *Let C be a convex set and let H_j, $j=1,\dots,n$, be closed half-planes in $\boldsymbol{C}$ such that $C\subset\bigcup_{j=1}^{n}H_j$. Then one can choose half-planes $H_{j_1},\dots, H_{j_l}$, $1\leqq l\leqq 3$, so that $C\subset\bigcup_{k=1}^{l}H_{j_k}$.*

LEMMA 1.6. *Let S be a closed square with sides of length d, let Δ be an open disc with radius r and let H be a closed half-plane such that the boundary ∂H of H is tangent to $\partial\Delta$ at a point belonging to S and H contains Δ. Then $m(S\cap(H-\Delta))\leqq 4d^3/r$.*

LEMMA 1.7. *For every* ε *with* $0<\varepsilon<1$, *there is* $\delta>0$ *such that if a measurable set* E *and open discs* Δ_j, $j=0, 1,\dots, n$, *in* $\boldsymbol{C}$ *satisfy* $\Delta_j\cap\Delta_0\neq\emptyset$ *and* $\rho_j\leqq\rho_0$, *for* $j=1,\dots, n$, *and* $m(E\cap\Delta_j)\leqq\delta m(\Delta_j)$ *for* $j=0, 1,\dots, n$, *where* ρ_j *denotes the radius of* Δ_j, *then*

$$m(E\cap(\bigcup_{j=0}^{n}\Delta_j))\leqq\varepsilon m(\Delta_0).$$

PROOF. We may assume $\Delta_0=\{z\in\boldsymbol{C}\,|\,|z|<1\}$. Then $\bigcup_{j=0}^{n}\Delta_j\subset\{z\in\boldsymbol{C}\,|\,|z|<3\}$. Set $\omega=\varepsilon/10$ and $\Gamma=\{z\in\boldsymbol{C}\,|\,1+2\omega<|z|<3\}$. We shall determine δ later. For a moment we assume $m(E\cap\Delta_j)\leqq\delta m(\Delta_j)$ for $j=0, 1,\dots, n$.

We first estimate the area of $E\cap(\bigcup_{j=1}^{n}\Delta_j)\cap\Gamma$. For d with $0<d<1/2\pi$, set $S_{jk}=\{x+iy\in\boldsymbol{C}\,|\,jd\leqq x\leqq(j+1)d$ and $kd\leqq y\leqq(k+1)d\}$, $A=\{S_{jk}\,|\,S_{jk}\cap\partial\{(\bigcup_{j=1}^{n}\Delta_j)\cap\Gamma\}\neq\emptyset\}$ and $B=\{S_{jk}\,|\,S_{jk}\subset(\bigcup_{j=1}^{n}\Delta_j)\cap\Gamma\}$. Let N be the smallest natural number not less than $2\pi/\omega$, set $\Omega_k=\{z\in\boldsymbol{C}\,|\,-\omega/2\leqq\arg z-2\pi k/N<\omega/2\}$, $k=0, 1,\dots, N-1$, and let D_k be the union of discs Δ_j such that $\Delta_j\cap\Gamma\neq\emptyset$ and the center of Δ_j belongs to Ω_k. Then, by Lemma 1.4, D_k are Jordan domains. Hence, by Lemmas 1.2 and 1.3, we have

$$\begin{aligned}m((\partial D_k)_{\sqrt{2}d})&\leqq 2\sqrt{2}d\,\ell(\partial D_k)+2\pi d^2\\&\leqq 2\sqrt{2}d\cdot\frac{2}{\omega}\cdot m(D_k)+2\pi d^2\\&\leqq(1600/\varepsilon+1)d.\end{aligned}$$

Since $m(\{z\in\boldsymbol{C}\,|\,|z|=1+2\omega\}_{\sqrt{2}d})\leqq 22d$ and $N\leqq 2\pi/\omega+1$, we have

$$\begin{aligned}m(\bigcup_{S_{jk}\in A}S_{jk})&\leqq N(1600/\varepsilon+1)d+22d\\&\leqq(100600/\varepsilon^2+1670/\varepsilon+23)d.\end{aligned}\tag{1.1}$$

To estimate the area $E\cap S$ for $S\in B$, we consider the discs Δ_j intersecting S. If $S\subset\Delta_j$ for some Δ_j, then

$$m(E\cap S)\leqq m(E\cap\Delta_j)\leqq\delta m(\Delta_j)\leqq\pi\delta.$$

If $S\cap\partial\Delta_j\neq\emptyset$ for every disc Δ_j intersecting S, take a closed half-plane H_j so that ∂H_j is tangent to $\partial\Delta_j$ at a point belonging to S and H_j contains Δ_j. Then $S\subset\bigcup_{\Delta_j\cap S\neq\emptyset}H_j$, and hence, by Lemma 1.5, there are closed half-planes $H_{j_1},\dots, H_{j_l}$, $1\leqq l\leqq 3$, such that $S\subset\bigcup_{k=1}^{l}H_{j_k}$. Therefore, by Lemma 1.6,

$$\begin{aligned}m(E\cap S)&\leqq\sum_{k=1}^{l}m(E\cap\Delta_{j_k})+\sum_{k=1}^{l}m(S\cap(H_{j_k}-\Delta_{j_k}))\\&\leqq 3(\pi\delta+4d^3/\omega)\end{aligned}$$

$$= 3\pi\delta + 120d^3/\varepsilon.$$

Set $\delta = \alpha d^2$. Then

$$m(E \cap S) \leqq (120d/\varepsilon + 3\pi\alpha)d^2$$

for every $S \in B$. Hence

$$(1.2) \qquad m(E \cap (\bigcup_{S_{jk}\in B} S_{jk})) \leqq \sum_{S_{jk}\in B} m(E \cap S_{jk})$$
$$\leqq \sum_{S_{jk}\in B} (120d/\varepsilon + 3\pi\alpha)m(S_{jk})$$
$$= (120d/\varepsilon + 3\pi\alpha)m(\bigcup_{S_{jk}\in B} S_{jk})$$
$$\leqq 8\pi(120d/\varepsilon + 3\pi\alpha).$$

From (1.1) and (1.2), we have

$$m(E \cap (\bigcup_{j=1}^{n} \Delta_j) \cap \Gamma) \leqq (100600/\varepsilon^2 + 5000/\varepsilon + 23)d + 240\alpha.$$

Since

$$m(E \cap \{z \in \boldsymbol{C} \mid |z| \leqq 1 + 2\omega\})$$
$$\leqq m(E \cap \Delta_0) + m(\{z \in \boldsymbol{C} \mid 1 \leqq |z| \leqq 1 + 2\omega\})$$
$$\leqq \pi\alpha d^2 + \varepsilon\pi/2,$$

we obtain

$$m(E \cap (\bigcup_{j=0}^{n} \Delta_j))$$
$$\leqq (100600/\varepsilon^2 + 5000/\varepsilon + 23)d + 240\alpha + \pi\alpha d^2 + \varepsilon\pi/2.$$

Take $d = 10^{-5}\varepsilon^3$ and $\delta = 10^{-13}\varepsilon^7$. Then $\alpha = 10^{-3}\varepsilon$, so that

$$m(E \cap (\bigcup_{j=0}^{n} \Delta_j)) \leqq \varepsilon\pi = \varepsilon m(\Delta_0).$$

This completes the proof.

Proof of Proposition 1.1. For ε with $0 < \varepsilon < 1$, let δ be a number considered in Lemma 1.7. We prove the proposition by mathematical induction on the number n of open discs. If $n = 1$, then our assertion is trivial. Assume that our assertion is true when the number of open discs is equal to $n - 1 \geqq 1$. Let E be a measurable set and Δ_j, $j = 1, \ldots, n$, be open discs satisfying $m(E \cap \Delta_j) \leqq \delta m(\Delta_j)$

for every j. Assume that Δ_k has the maximum radius ρ_k. Set $D=\cup_{\Delta_j\cap\Delta_k\neq\emptyset}\Delta_j$ and $D'=\cup_{\Delta_j\cap\Delta_k=\emptyset}\Delta_j$. Then $\cup_{j=1}^n\Delta_j=D\cup D'$. By the assumption,

$$m(E\cap D')\leqq\varepsilon m(D')$$

and, by Lemma 1.7, we have

$$m(E\cap D)\leqq\varepsilon m(\Delta_k).$$

Hence

$$\begin{aligned} m(E\cap(\bigcup_{j=1}^n\Delta_j)) &= m((E\cap D)\cup(E\cap D')) \\ &\leqq \varepsilon\{m(\Delta_k)+m(D')\} \\ &= \varepsilon m(\Delta_k\cup D') \\ &\leqq \varepsilon m(\bigcup_{j=1}^n\Delta_j). \end{aligned}$$

This completes the proof.

§2. Functional inequalities

In this section we deal with functional inequalities. Our aim is to prove the following proposition.

PROPOSITION 2.1. *Let α, β, γ, κ and σ be nonnegative numbers, let δ, λ and τ be positive numbers, and let ε be a number such that $0\leqq\varepsilon<1$. Let u and v be nonnegative integrable functions on $[0,\infty)$ with $\int_0^\infty u(x)dx=\alpha$ and $\int_0^\infty v(x)dx=\beta$. Set*

$$A(a,b)=\frac{\kappa}{b-a}\int_a^b\{u(x)+v(x)\}dx$$

for every pair of numbers a and b with $0\leqq a<b$. If $U(t)\equiv\int_t^\infty u(x)dx$ and $V(t)\equiv\int_t^\infty v(x)dx$ satisfy

$$U(t+\sigma\{(U+V)(t)\}^\tau)\leqq\varepsilon(U+V)(t) \tag{2.1}$$

and

$$V(b)\leqq\gamma A(a,b)^{1+\delta} \tag{2.2}$$

for every pair of numbers a and b such that $0\leqq a<b$, $A(a,b)\leqq 1$ and $2A(a,b)^\lambda\leqq$

$b-a$, then there is a nonnegative number M satisfying $U(M)=V(M)=0$ and depending only on α, β, γ, δ, ε, κ, λ, σ and τ.

PROOF. We may assume that $\alpha+\beta$, γ and κ are positive. Let q be a number satisfying $q\delta \geqq 6$, $q\lambda \geqq 3$ and $(2+q)\tau>1$, and let N be a natural number satisfying $N \geqq 2\kappa^{\lambda}$, $(N+1)^q \geqq \kappa$, $1+\delta/3 \geqq \log(N+2)/\log(N+1)$ and

$$1-\varepsilon\left(\frac{N+1}{N}\right)^2\left(\frac{N+2}{N+1}\right)^q \geqq \frac{\gamma\kappa^{1+\delta}}{(N+1)^{q\delta/3}}.$$

Set $\varepsilon_n = 1/\{(n+1)^2(n+2)^q\}$ for $n=N-1, N, N+1,\dots$ and $F(s)=\sigma s^{\tau}$.

First we show that there exists a positive number e depending only on α, β, γ, δ, ε, κ, λ, σ and τ such that $(U+V)(e) \leqq \varepsilon_{N-1}$. Set

$$c=\max\{(\alpha+\beta)\kappa,\ 2^{1/(1+\lambda)}\{(\alpha+\beta)\kappa\}^{\lambda/(1+\lambda)},\ (\alpha+\beta)\kappa\{(1-\varepsilon)\varepsilon_{N-1}/(2\gamma)\}^{-1/(1+\delta)}\}.$$

Since $A(0, c) \leqq (\alpha+\beta)\kappa/c \leqq 1$ and

$$\begin{aligned} 2A(0, c)^{\lambda} &\leqq 2\left\{\frac{(\alpha+\beta)\kappa}{2^{1/(1+\lambda)}\{(\alpha+\beta)\kappa\}^{\lambda/(1+\lambda)}}\right\}^{\lambda} \\ &= 2^{1/(1+\lambda)}\{(\alpha+\beta)\kappa\}^{\lambda/(1+\lambda)} \\ &\leqq c, \end{aligned}$$

by (2.2), we have

$$\begin{aligned} V(c) &\leqq \gamma A(0, c)^{1+\delta} \\ &\leqq \gamma\left\{\frac{(\alpha+\beta)\kappa}{(\alpha+\beta)\kappa\{(1-\varepsilon)\varepsilon_{N-1}/(2\gamma)\}^{-1/(1+\delta)}}\right\}^{1+\delta} \\ &= \frac{(1-\varepsilon)\varepsilon_{N-1}}{2}. \end{aligned}$$

Set $d=F((U+V)(0))=F(\alpha+\beta)$. Then, by (2.1),

$$U(c+d) \leqq U(c+F((U+V)(c))) \leqq \varepsilon(U+V)(c) \leqq \alpha\varepsilon + \varepsilon(1-\varepsilon)\varepsilon_{N-1}/2.$$

Since

$$\begin{aligned} U(c+nd) &\leqq U(c+(n-1)d+F((U+V)(c+(n-1)d))), \\ &\leqq \varepsilon(U+V)(c+(n-1)d), \end{aligned}$$

by mathematical induction, we have

$$(U+V)(c+nd) \leqq \alpha\varepsilon^n + (\textstyle\sum_{i=1}^{n}\varepsilon^i)(1-\varepsilon)\varepsilon_{N-1}/2 + V(c)$$

$$\leqq \alpha\varepsilon^n + \varepsilon_{N-1}/2$$

for every natural number n. Choose m so that $\alpha\varepsilon^m < \varepsilon_{N-1}/2$ and set $e = c + md$. Then e has the required property.

Consider $A(e, e+1/N^2)$. Since $A(e, e+1/N^2) \leqq \kappa N^2 \varepsilon_{N-1} = \kappa/(N+1)^q \leqq 1$ and

$$2A\left(e, e+\frac{1}{N^2}\right)^{\lambda} \leqq 2\left\{\frac{\kappa}{(N+1)^q}\right\}^{\lambda} \leqq \frac{N}{(N+1)^{q\lambda}} \leqq \frac{1}{N^2},$$

we have

$$\begin{aligned} V\left(e+\frac{1}{N^2}\right) &\leqq \gamma A\left(e, e+\frac{1}{N^2}\right)^{1+\delta} \\ &\leqq \gamma\left\{\frac{\kappa}{(N+1)^q}\right\}^{1+\delta} \\ &= \frac{1}{(N+1)^{q\delta/3}}\cdot\frac{1}{(N+1)^{q(1+\delta/3)}}\cdot\frac{\gamma\kappa^{1+\delta}}{(N+1)^{q\delta/3}} \\ &\leqq \frac{1}{(N+1)^2}\cdot\frac{1}{(N+2)^q}\cdot\left(1-\frac{\varepsilon\varepsilon_{N-1}}{\varepsilon_N}\right) = \varepsilon_N - \varepsilon\varepsilon_{N-1} \end{aligned}$$

by (2.2) and

$$U(e+F(\varepsilon_{N-1})) \leqq U(e+F((U+V)(e))) \leqq \varepsilon(U+V)(e) \leqq \varepsilon\varepsilon_{N-1}$$

by (2.1). Hence

$$(U+V)\left(e+\frac{1}{N^2}+F(\varepsilon_{N-1})\right) \leqq \varepsilon_N.$$

Repeating this process we have

$$(U+V)(e+\sum_{j=N}^{n} j^{-2} + \sum_{j=N}^{n} F(\varepsilon_{j-1})) \leqq \varepsilon_n.$$

Since

$$\sum_{j=N}^{n} F(\varepsilon_{j-1}) = \sigma\sum_{j=N}^{n}\left\{\frac{1}{j^2(j+1)^q}\right\}^{\tau} < \sigma\sum_{j=N}^{\infty} j^{-(2+q)\tau}$$

and $(2+q)\tau > 1$,

$$M = e + \sum_{j=N}^{\infty} j^{-2} + \sigma\sum_{j=N}^{\infty} j^{-(2+q)\tau}$$

has the required property.

§3. Modifications of bounded positive integrable functions on the plane

In [7, §1], we have dealt with modifications of positive measures. In this section we deal with bounded L^1 functions on $\boldsymbol{C}$ (i.e., bounded integrable functions on $\boldsymbol{C}$).

We begin with

LEMMA 3.1. *Let W be a plane domain and λ be a nonnegative bounded L^1 function on W. Then there is a nonnegative bounded lower semicontinuous L^1 function λ^* on W such that $\int_W s\lambda dm \leqq \int_W s\lambda^* dm$ for every $s \in SL^1(W)$, where $SL^1(W)$ denotes the class of subharmonic L^1 functions on W.*

PROOF. Exhaust W by an increasing sequence $\{W_j\}_{j=0}^{\infty}$ of relatively compact subdomains such that $\partial W_{j-1} \subset W_j$ for each $j \geqq 1$, and set $E_0 = W_0$ and $E_j = W_j - W_{j-1}$ for $j \geqq 1$. Set $\lambda_j = \lambda \chi_{E_j}$ for $j \geqq 0$, where χ_{E_j} denotes the characteristic function of E_j and set $d_j = d(E_j, \partial W_{j+1})$ for $j = 0, 1$ and $d_j = d(E_j, \partial W_{j-2} \cup \partial W_{j+1})$ for $j \geqq 2$. We write $D_j(z)$ for $\Delta_{d_j/2}(z)$, and consider

$$M_j(z) = \frac{\int_{D_j(z)} \lambda_j dm}{\pi(d_j/2)^2}$$

and

$$\lambda^* = \sum_{j=0}^{\infty} M_j.$$

It is easy to see that each M_j is continuous and that $\lambda^* \leqq 3 \sup_{z \in W} \lambda(z)$. Naturally, λ^* is lower semicontinuous. For every $s \in SL^1(W)$ we have

$$\begin{aligned}
\int_W s(\zeta)\lambda_j(\zeta)dm(\zeta) &\leqq \int_W \left\{\frac{1}{\pi(d_j/2)^2}\int_{D_j(\zeta)} s(z)dm(z)\right\} \lambda_j(\zeta)dm(\zeta) \\
&= \int_W \left\{\frac{1}{\pi(d_j/2)^2}\int_W s(z)\chi_{D_j(\zeta)}(z)dm(z)\right\} \lambda_j(\zeta)dm(\zeta) \\
&= \int_W \left\{\frac{1}{\pi(d_j/2)^2}\int_W \lambda_j(\zeta)\chi_{D_j(z)}(\zeta)dm(\zeta)\right\} s(z)dm(z) \\
&= \int_W s(z)M_j(z)dm(z).
\end{aligned}$$

Hence $\int_W s\lambda dm \leqq \int_W s\lambda^* dm$. By considering $s = 1$ we obtain $\int_W \lambda^* dm = \int_W \lambda dm < \infty$. Our lemma is now proved.

LEMMA 3.2. *Let λ be a positive lower semicontinuous function on a open set W. Then there is a sequence of functions $\lambda_n=\sum_{j=1}^{n}\beta_j\chi_j$ increasing to λ a.e. on W as $n\to\infty$, where each β_j is a positive constant and each χ_j denotes the characteristic function of an open disk in W.*

PROOF. We may assume that λ is bounded and $m(W)<\infty$. Set $M=\sup_{z\in W}\lambda(z)$ and denote the open set $\{z\in W|\lambda(z)>kM/2^n\}$ by $G_{n,k}$ for $n=1, 2,\dots, k=1, 2,\dots, 2^n-1$. Define f_n by $(M/2^n)\sum_{k=1}^{2^n-1}\chi_{G_{n,k}}$. Then $f_n\uparrow\lambda$ as $n\uparrow\infty$. In the following we let all $B_{n,k}$ be the union of finitely many mutually disjoint open discs. By making use of Vitali's covering theorem choose $B_{1,1}\subset G_{1,1}$ so that $m(G_{1,1}-B_{1,1})<1/2^2$. Set $\lambda_1=(M/2)\chi_{B_{1,1}}$ and $E_1=\{z\in W|f_1(z)\neq\lambda_1(z)\}$. Then $m(E_1)<1/2$. Suppose $B_{j,k}$, $j=1,\dots, n$, $k=1,\dots, 2^j-1$, are chosen so that $B_{j,k}\subset G_{j,k}$, $m(G_{j,k}-B_{j,k})<1/2^{2j}$, $\lambda_1\leqq\cdots\leqq\lambda_n$ and $m(E_j)<1/2^j$, where $\lambda_j=(M/2^j)\sum_{k=1}^{2^j-1}\chi_{B_{j,k}}$ and $E_j=\{z\in W|f_j(z)\neq\lambda_j(z)\}$. Choose $B_{n+1,k}$ so that $B_{n,k}\subset B_{n+1,2k}\subset G_{n+1,2k}$ and $m(G_{n+1,2k}-B_{n+1,2k})<1/2^{2(n+1)}$ for $k=1,\dots, 2^n-1$, and so that $B_{n+1,k}\subset G_{n+1,k}$ and $m(G_{n+1,k}-B_{n+1,k})<1/2^{2(n+1)}$ for other k. Set $\lambda_{n+1}=(M/2^{n+1})\sum_{k=1}^{2^{n+1}-1}\chi_{B_{n+1,k}}$ and $E_{n+1}=\{z\in W|f_{n+1}(z)\neq\lambda_{n+1}(z)\}$. Then $\lambda_n\leqq\lambda_{n+1}$ and $m(E_{n+1})<1/2^{n+1}$. The set $E=\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}E_j$ is of measure zero and $\lambda_n\uparrow\lambda$ on $W-E$. This proves our lemma.

Our main aim in this section is to prove the following proposition:

PROPOSITION 3.3. *Let W be a plane domain and let v be a bounded L^1 function on $\mathbf{C}$ such that $v(z)\geqq 1$ a.e. on W and $v(z)=0$ a.e. on the complement W^c of W. Then, for every $\varepsilon=(\varepsilon_1, \varepsilon_2)$ with $0<\varepsilon_1<1$ and $\varepsilon_2>0$, there are domains W_ε and $\tilde{W}_\varepsilon$, and a bounded L^1 function v_ε on $\mathbf{C}$ such that*

(1) $W\subset W_\varepsilon\subset\tilde{W}_\varepsilon$.

(2) $m(\tilde{W}_\varepsilon)\leqq\int vdm/(1-\varepsilon_1)$ *and* $m(\tilde{W}_\varepsilon-W_\varepsilon)\leqq\{\varepsilon_1/(1-\varepsilon_1)\}\int vdm$.

(3) v_ε *satisfies* $v_\varepsilon(z)\geqq 1$ *a.e. on* W_ε, $v_\varepsilon(z)=0$ *a.e. on* W_ε^c *and* $\int(v_\varepsilon-\chi_{W_\varepsilon})dm\leqq\varepsilon_2$.

(4) $\int_W svdm\leqq\int_{W_\varepsilon}sv_\varepsilon dm$ *for every* $s\in SL^1(\tilde{W}_\varepsilon)$.

(5) $U(r+2(1/\pi)^{1/2}\{(U+V)(r)\}^{1/2})\leqq\varepsilon_1(U+V)(r)$ *for every* $r\geqq 0$, *where*

$$u(r)=\int_0^{2\pi}\chi_{\tilde{W}_\varepsilon-W_\varepsilon}(re^{i\theta})rd\theta,\quad U(r)=\int_r^{\infty}u(t)dt,$$

$$v(r)=\int_0^{2\pi}v_\varepsilon(re^{i\theta})rd\theta\quad\text{and}\quad V(r)=\int_r^{\infty}v(t)dt.$$

REMARK. If h is harmonic, then h and $-h$ are both subharmonic. Hence it follows from (4) that $\int_W hvdm=\int_{W_\varepsilon}hv_\varepsilon dm$ for every harmonic L^1 function h on $\tilde{W}_\varepsilon$.

To prove the proposition we first give notation. Let Ω be a bounded domain whose boundary consists of a finite number of piecewise analytic curves or points, let c be a point of Ω, let α be a nonnegative number and let δ be a number with $0<\delta<1$. We shall define an open set $R(\Omega, c, \alpha, \delta)$ which is the union of a finite number of open rings, define a disc $\Delta(\Omega, c, \alpha, \delta)$ and define a domain $G(\Omega, c, \alpha, \delta)$ whose boundary consists of a finite number of piecewise analytic curves or points.

Set $\phi(r)=\int_0^{2\pi}\chi_\Omega(c+re^{i\theta})d\theta/(2\pi)$ for $r\geqq 0$, $F=\{r\geqq 0 \mid \phi(r)\leqq 1-\delta\}$ and $\Phi(r)=\int_0^r 2\pi t\chi_F(t)dt$, where χ_F denotes the characteristic function of F on $[0, \infty)$. Then ϕ is lower semicontinuous and analytic except, at most, at a finite number of points. Hence F is closed, $F\cap[0, r]$ consists of at most a finite number of closed intervals or points for every $r\geqq 0$ and $\Phi(r_1)=\Phi(r_2)$ implies $(F\cap[0, r_1])^\circ=(F\cap[0, r_2])^\circ$, where $(F\cap[0, r])^\circ$ denotes the interior of $F\cap[0, r]$ in $[0, \infty)$.

Since $\Phi(r)$ is continuous and increases from 0 to ∞ as r varies from 0 to ∞, there is a nonnegative number a such that $\Phi(a)=\alpha$. The number a may not be determined uniquely, but the open set $A=(F\cap[0, a])^\circ$ in $[0, \infty)$ is determined uniquely by α.

Set $R=R(\Omega, c, \alpha, \delta)=\{z\in\boldsymbol{C} \mid |z-c|\in A\}$, $\Delta=\Delta(\Omega, c, \alpha, \delta)=\{z\in\boldsymbol{C} \mid |z-c|<\sup_{r\in A} r\}$ and $G=G(\Omega, c, \alpha, \delta)=\Omega\cup R$. Then $m(R)=\Phi(a)=\alpha$, $m(\Omega\cap R)\leqq(1-\delta)\alpha$, $\int s\{\beta\chi_{\Delta_\rho(c)}\}dm\leqq\int s\chi_R\,dm$ for every $s\in SL^1(\Delta)$, where $\Delta_\rho(c)$ denotes the open disc with radius ρ and center at c such that $\Delta_\rho(c)\subset\Omega$ and β is a number with $\beta\pi\rho^2=\alpha$, and the domain G satisfies $m(\Delta-G)\leqq\delta m(\Delta)$.

Proof of Proposition 3.3. We may assume that W is a bounded domain whose boundary consists of a finite number of piecewise analytic curves, v is lower semicontinuous on W, $v(z)\geqq 1$ on W and $v(z)=0$ on W^c. In fact, for any W, v and $\varepsilon=(\varepsilon_1, \varepsilon_2)$ given in Proposition 3.3, let v^* be a bounded L^1 function on $\boldsymbol{C}$ such that v^* is lower semicontinuous on W, $v^*(z)\geqq 1$ on W, $v^*(z)=0$ on W^c and $\int_W svdm\leqq\int_W sv^*dm$ for every $s\in SL^1(W)$; its existence follows from Lemma 3.1. Choose a bounded subdomain Ω on W whose boundary consists of a finite number of piecewise analytic curves such that $\int_{W-\Omega}v^*dm\leqq\varepsilon_2/2$. If Proposition 3.3 is proved for Ω, $v'=v^*\chi_\Omega$ and $\varepsilon'=(\varepsilon_1, \varepsilon_2/2)$, then $W_\varepsilon=W\cup\Omega_{\varepsilon'}$, $\tilde{W}_\varepsilon=W\cup\tilde{\Omega}_{\varepsilon'}$ and $v_\varepsilon=v'_{\varepsilon'}+v^*\chi_{W-\Omega}$ satisfy conditions from (1) to (5). We shall show only that (5) is true. We set

$$U'(r)=\int_r^\infty\left\{\int_0^{2\pi}\chi_{\tilde{\Omega}_{\varepsilon'}-\Omega_{\varepsilon'}}(te^{i\theta})td\theta\right\}dt,$$

$$V'(r)=\int_r^\infty\left\{\int_0^{2\pi}v'_{\varepsilon'}(te^{i\theta})td\theta\right\}dt,$$

and define U and V corresponding to W_ε, $\tilde{W}_\varepsilon$ and v_ε. From the relation

$$\chi_{\tilde{\Omega}_{\varepsilon'}-\Omega_{\varepsilon'}} + v'_{\varepsilon'} \leqq \chi_{\tilde{W}_\varepsilon - W_\varepsilon} + v^*\chi_{W-\Omega} + v'_{\varepsilon'}$$

$$= \chi_{\tilde{W}_\varepsilon - W_\varepsilon} + v_\varepsilon,$$

we have $U'+V' \leqq U+V$. Since $U \leqq U'$ and $U'(r)$ is a decreasing function of r, we derive (5) for U and V from that for U' and V'.

Take a positive number δ so small that Proposition 1.1 is valid for ε_1 and fix it. Set $W_0 = W$ and $v_0 = v$.

Since v_0 is lower semicontinuous and satisfies $v_0(z) \geqq \chi_{W_0}(z)$ on $\boldsymbol{C}$, by Lemma 3.2 there is a sequence of functions $v_{0,n} = \chi_{W_0} + \sum_{j=1}^n \beta_{0,j}\chi_{0,j}$ such that $v_{0,n} \uparrow v_0$ a.e. as $n \uparrow \infty$, where each $\beta_{0,j}$ is a positive constant and each $\chi_{0,j}$ denotes the characteristic function of an open disc with radius $\rho_{0,j}$ and center at $c_{0,j}$.

Take n_1 so that

$$\int\Big(\sum_{j=n_1+1}^{\infty} \beta_{0,j}\chi_{0,j}\Big)dm \leqq \frac{\delta}{2}\int(v_0 - \chi_{W_0})dm$$

and set

$$R_n = R(W_{n-1},\ c_{0,n},\ \beta_{0,n}\pi\rho_{0,n}^2,\ \delta),$$

$$\Delta_n = \Delta(W_{n-1},\ c_{0,n},\ \beta_{0,n}\pi\rho_{0,n}^2,\ \delta),$$

$$W_n = G(W_{n-1},\ c_{0,n},\ \beta_{0,n}\pi\rho_{0,n}^2,\ \delta)$$

and

$$v_n = \chi_n + \sum_{j=1}^{n} \chi_{W_{j-1}\cap R_j} + \max\{v_0 - \chi_0 - \sum_{j=1}^{n}\beta_{0,j}\overline{\chi_{0,j}},\ 0\}$$

for $n = 1, \dots, n_1$, inductively, where χ_n means χ_{W_n} and $\overline{\chi_{0,j}}$ denotes the characteristic function of the closed disc with radius $\rho_{0,j}$ and center at $c_{0,j}$. Then

$$\int(v_{n_1} - \chi_{n_1})dm \leqq (1 - \delta/2)\int(v_0 - \chi_0)dm.$$

Since v_{n_1} is lower semicontinuous and satisfies $v_{n_1}(z) \geqq \chi_{n_1}(z)$ on $\boldsymbol{C}$, we can again find a sequence of functions $v_{1,n} = \chi_{n_1} + \sum_{j=n_1+1}^n \beta_{1,j}\chi_{1,j}$ such that $v_{1,n} \uparrow v_{n_1}$ a.e. as $n \uparrow \infty$, where each $\beta_{1,j}$ is a positive constant and each $\chi_{1,j}$ denotes the characteristic function of an open disc with radius $\rho_{1,j}$ and center at $c_{1,j}$. Take $n_2 > n_1$ so that

$$\int\Big(\sum_{j=n_2+1}^{\infty} \beta_{1,j}\chi_{1,j}\Big)dm \leqq \frac{\delta}{2}\int(v_{n_1} - \chi_{n_1})dm,$$

and set

$$R_n = R(W_{n-1},\ c_{1,n},\ \beta_{1,n}\pi\rho_{1,n}^2,\ \delta),$$

$$\Delta_n = \Delta(W_{n-1}, c_{1,n}, \beta_{1,n}\pi\rho_{1,n}^2, \delta),$$
$$W_n = G(W_{n-1}, c_{1,n}, \beta_{1,n}\pi\rho_{1,n}^2, \delta)$$

and

$$v_n = \chi_n + \sum_{j=n_1+1}^{n} \chi_{W_{j-1}\cap R_j} + \max\{v_{n_1} - \chi_{n_1} - \sum_{j=n_1+1}^{n} \beta_{1,j}\overline{\chi_{1,j}}, 0\}$$

for $n=n_1+1,\dots, n_2$. Then

$$\int(v_{n_2} - \chi_{n_2})dm \leqq (1-\delta/2)\int(v_{n_1} - \chi_{n_1})dm$$
$$\leqq (1-\delta/2)^2\int(v_0 - \chi_0)dm.$$

Repeating this process, choose $\{n_k\}$ and define v_n for $n_k < n \leqq n_{k+1}$ and $\beta_{k,j}$, $\chi_{k,j}$ for $j \geqq n_k + 1$ as above. Let N be a natural number such that $(1-\delta/2)^N\int(v_0 - \chi_0)dm \leqq \varepsilon_2$.

Set $W_\varepsilon = W_{n_N}$, $\tilde{W}_\varepsilon = W \cup \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{n_N}$ and $v_\varepsilon = v_{n_N}$. Then, by the definition, it is evident that these satisfy (1) and (3).

Set $n_0 = 0$. If $n_k + 1 < n \leqq n_{k+1}$, then

$$\int sv_n dm = \int s(\chi_n + \sum_{j=n_k+1}^{n} \chi_{W_{j-1}\cap R_j} + v_{n_k} - \chi_{n_k} - \sum_{j=n_k+1}^{n} \beta_{k,j}\chi_{k,j})dm$$

for every $s \in SL^1(W_n)$. Since $\chi_n + \chi_{W_{n-1}\cap R_n} = \chi_{W_{n-1}\cup R_n} + \chi_{W_{n-1}\cap R_n} = \chi_{n-1} + \chi_{R_n}$ and $\int s\chi_{R_n}dm \geqq \int s(\beta_{k,n}\chi_{k,n})dm$ for every $s \in SL^1(\Delta_n)$, we have

$$\int sv_n dm \geqq \int s(\chi_{n-1} + \sum_{j=n_k+1}^{n-1} \chi_{W_{j-1}\cap R_j} + v_{n_k} - \chi_{n_k} - \sum_{j=n_k+1}^{n-1} \beta_{k,j}\chi_{k,j})dm$$
$$= \int sv_{n-1}dm$$

for every $s \in SL^1(\tilde{W}_\varepsilon)$. Similarly $\int sv_{n_k+1}dm \geqq \int sv_{n_k}dm$. Hence (4) is satisfied.

Next we show (2). Set $E = \tilde{W}_\varepsilon - W_\varepsilon$. Then

$$m(E \cap \Delta_n) \leqq m(\Delta_n - W_n) \leqq \delta m(\Delta_n)$$

for $n=1, 2,\dots, n_N$. Hence, by Proposition 1.1,

$$m(E) = m(E \cap (\bigcup_{n=1}^{n_N} \Delta_n)) \leqq \varepsilon_1 m(\bigcup_{n=1}^{n_N} \Delta_n) \leqq \varepsilon_1 m(\tilde{W}_\varepsilon).$$

Therefore $m(\tilde{W}_\varepsilon) - m(W_\varepsilon) \leqq \varepsilon_1 m(\tilde{W}_\varepsilon)$, so that

$$m(\tilde{W}_\varepsilon) \leqq m(W_\varepsilon)/(1-\varepsilon_1) \leqq \int v_\varepsilon dm/(1-\varepsilon_1).$$

This implies $1 \in SL^1(\tilde{W}_\varepsilon)$, and so, by Remark to Proposition 3.3, $\int v_\varepsilon dm = \int v dm$. This completes the proof of (2).

Finally we show (5). Set $x(r) = \int_0^{2\pi} \chi_{\tilde{W}_\varepsilon}(re^{i\theta}) r d\theta$, $X(r) = \int_r^\infty x(t)dt$ and $A_r = \{\Delta_n \mid 1 \leqq n \leqq n_N,\ \Delta_n \cap \{z \in \boldsymbol{C} \mid |z| > r + 2(1/\pi)^{1/2} X(r)^{1/2}\} \neq \emptyset\}$ for $r \geqq 0$. If $\Delta_n \in A_r$, then $\Delta_n \cap \{z \in \boldsymbol{C} \mid |z| < r\} = \emptyset$. In fact, if $\Delta_n \cap \{z \in \boldsymbol{C} \mid |z| < r\} \neq \emptyset$, then

$$\begin{aligned} m(\Delta_n \cap \{z \in \boldsymbol{C} \mid r < |z| < r + 2(1/\pi)^{1/2} X(r)^{1/2}\}) &> \pi\{(1/\pi)^{1/2} X(r)^{1/2}\}^2 \\ &= X(r). \end{aligned}$$

Hence

$$\begin{aligned} X(r) &= m(\tilde{W}_\varepsilon \cap \{z \in \boldsymbol{C} \mid |z| > r\}) \\ &\geqq m(\Delta_n \cap \{z \in \boldsymbol{C} \mid r < |z| < r + 2(1/\pi)^{1/2} X(r)^{1/2}\}) \\ &> X(r). \end{aligned}$$

This contradiction implies $\Delta_n \cap \{z \in \boldsymbol{C} \mid |z| < r\} = \emptyset$. Therefore, by Proposition 1.1,

$$\begin{aligned} U(r + 2(1/\pi)^{1/2} X(r)^{1/2}) &= m(E \cap \{z \in \boldsymbol{C} \mid |z| > r + 2(1/\pi)^{1/2} X(r)^{1/2}\}) \\ &= m(E \cap (\bigcup_{\Delta_n \in A_r} \Delta_n)) \\ &\leqq \varepsilon_1 m(\bigcup_{\Delta_n \in A_r} \Delta_n) \\ &\leqq \varepsilon_1 X(r). \end{aligned}$$

Since $X(r) \leqq (U+V)(r)$, we obtain (5).

§4. Cauchy transforms

Let v be an L^1 function on $\boldsymbol{C}$. The Cauchy transform $\hat{v}$ of v is the function defined by

$$\hat{v}(z) = \int \frac{v(\zeta) dm(\zeta)}{\zeta - z}.$$

The integral on the right-hand side is absolutely convergent for almost all $z \in \boldsymbol{C}$ and analytic outside the support of v.

First we give a lemma without proof.

LEMMA 4.1. *Let* v *be an* L^1 *function satisfying* $0 \leqq v(z) \leqq N$ *a.e. on* $\boldsymbol{C}$. *Then*

$$\int \frac{v(\zeta)dm(\zeta)}{|\zeta - z|} \leqq 2\sqrt{N\pi \int v dm}.$$

The equality holds if and only if $v = N\chi_\Delta$ *a.e. on* $\boldsymbol{C}$, *where* $\Delta = \{\zeta \in \boldsymbol{C} \mid |\zeta - z| < \sqrt{\int v dm/(N\pi)}\}$.

The purpose of this section is to prove the following proposition:

PROPOSITION 4.2. *Let* W *be an open set, let* $\tilde{W}$ *be a domain such that* $W \subset \tilde{W}$ *and* $m(\tilde{W}) < \infty$, *and let* v *be an* L^1 *function on* $\boldsymbol{C}$ *such that* $v(z) \geqq 1$ *a.e. on* $W \cap \{z \in \boldsymbol{C} \mid |z| \geqq r_0\}$ *for some* $r_0 > 0$ *and* $v(z) = 0$ *a.e. on* W^c. *Set*

$$u(r) = \int_0^{2\pi} \chi_{\tilde{W}-W}(re^{i\theta}) r d\theta, \quad U(r) = \int_r^\infty u(t)dt,$$

$$\upsilon(r) = \int_0^{2\pi} v(re^{i\theta}) r d\theta \quad \text{and} \quad V(r) = \int_r^\infty \upsilon(t)dt.$$

If

$$\hat{v}(z) = -\frac{\int v dm}{z} \quad \text{on} \quad \tilde{W}^c \cap \{z \in \boldsymbol{C} \mid |z| \geqq r_0\}, \tag{4.1}$$

then

$$V(b) \leqq \gamma A(a, b)^{1+3/5}$$

for every pair of numbers a *and* b *such that* $r_0 \leqq a < b$, $A(a, b) \equiv \{\kappa/(b-a)\}\int_a^b \{u(t) + \upsilon(t)\}dt \leqq 1$ *and* $2A(a, b)^{1/5} \leqq b - a$, *where* $\gamma = 2\pi r_0(r_0^{1/2} + 4\max\{r_0, 1\}\int|v|dm)$ *and* $\kappa = 8/(\pi r_0)$.

PROOF. Set $T(re^{i\theta}) = -e^{i\theta}\hat{v}(re^{i\theta})$. Then

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} T(re^{i\theta}) r d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\left\{-e^{i\theta}\int_{\boldsymbol{C}} \frac{v(\zeta)dm(\zeta)}{\zeta - re^{i\theta}}\right\} r d\theta \\ &= \int_{\boldsymbol{C}}\left\{\frac{1}{2\pi i}\int_{|z|=r}\frac{dz}{z-\zeta}\right\} v(\zeta)dm(\zeta) \\ &= \int v dm - V(r) \end{aligned} \tag{4.2}$$

for almost all $r \in [0, \infty)$ and (4.1) implies

$$T(re^{i\theta}) = \frac{\int v dm}{r}$$

for every $re^{i\theta} \in \tilde{W}^c \cap \{z \in \boldsymbol{C} \mid |z| \geqq r_0\}$.

Let a and b be numbers such that $r_0 \leqq a < b$, $A = A(a, b) \leqq 1$ and $2A^{1/5} \leqq b - a$. We may assume $A > 0$. In fact, if $A = 0$, then $\int_a^b \int_0^{2\pi} \chi_{\tilde{W}}(re^{i\theta}) r d\theta dr \leqq \int_a^b (u+v) dr = 0$. Since $\tilde{W}$ is connected, we obtain the two cases $m(\tilde{W} \cap \{z \in \boldsymbol{C} \mid |z| > a\}) = 0$ or $m(\tilde{W} \cap \{z \in \boldsymbol{C} \mid |z| < b\}) = 0$. If $m(\tilde{W} \cap \{|z| > a\}) = 0$, then $V(b) \leqq V(a) = 0$. If $m(\tilde{W} \cap \{|z| < b\}) = 0$ and $\int v dm \neq 0$, then the function $\hat{v}$ analytic outside the support of v has a pole at the origin by (4.1). This is a contradiction. Hence if $m(\tilde{W} \cap \{|z| < b\}) = 0$, then $\int v dm = 0$, and so $v = 0$ a.e. on $\boldsymbol{C}$. Therefore we have $V(b) = 0$ in both cases.

Next we show that there is a number $c \in (a, b)$ such that

(1) $[c - A^{1/5}, c + A^{1/5}] \subset [a, b]$.

(2) $\int_{c-A^{1/5}}^{c+A^{1/5}} (u + v) dr \leqq \frac{\pi r_0}{2} A^{1+1/5}$.

(3) $\ell(\{r \in [c - (1/2)A^{1/5}, c + (1/2)A^{1/5}] \mid (u + v)(r) < \pi r_0 A\}) \geqq (1/2)A^{1/5}$,

where ℓ denotes the one-dimensional Lebesgue measure.

Let n be the largest natural number not greater than $(b-a)/(2A^{1/5})$ and set $c_j = a + (2j-1)A^{1/5}$, $j = 1, 2, \ldots, n$. Then $n \geqq (b-a)/(4A^{1/5})$ and

$$\sum_{j=1}^{n} \int_{c_j - A^{1/5}}^{c_j + A^{1/5}} (u+v) dr \leqq \int_a^b (u+v) dr = \frac{b-a}{\kappa} A = \frac{\pi r_0 (b-a)}{8} A.$$

Let $c = c_k$ be a number satisfying

$$\int_{c_k - A^{1/5}}^{c_k + A^{1/5}} (u + v) dr = \min_{1 \leqq j \leqq n} \int_{c_j - A^{1/5}}^{c_j + A^{1/5}} (u + v) dr.$$

Then (1) and (2) are satisfied. If $\ell(\{r \in [c - (1/2)A^{1/5}, c + (1/2)A^{1/5}] \mid (u+v)(r) \geqq \pi r_0 A\}) > (1/2)A^{1/5}$, then

$$\int_{c-A^{1/5}}^{c+A^{1/5}} (u + v) dr \geqq \int_{c-(1/2)A^{1/5}}^{c+(1/2)A^{1/5}} (u + v) dr > \pi r_0 A \cdot (1/2) A^{1/5}$$

$$= \frac{\pi r_0}{2} A^{1+1/5}.$$

This contradicts (2), so that (3) is satisfied.

We set $E = \{r \in [c - (1/2)A^{1/5}, c + (1/2)A^{1/5}] \mid (u+v)(r) < \pi r_0 A\}$ and $R = \{z \in \boldsymbol{C} \mid c - A^{1/5} < |z| < c + A^{1/5}\}$. In what follows, we consider only $r \in E$. If $r \in E$, then $r \geqq r_0$, $T(re^{i\theta}) = \int v dm / r$ for $re^{i\theta} \in \tilde{W}^c$ and $(u+v)(r) < \pi r_0 A$.

Next we define, for each $r \in E$, $\phi = \phi(\theta)$ on $[0, 2\pi)$ as follows: If $re^{i\theta} \in \tilde{W}^c$, then $\phi(\theta) = \theta$, and if $re^{i\theta} \in \tilde{W}$, then $\phi(\theta)$ is the minimum solution x satisfying $x \geqq \theta$, $re^{ix} \in \tilde{W}^c$ and

$$\int_\theta^x \chi_{\tilde{W}^c}(re^{it})dt = \int_0^\theta \chi_{\tilde{W}}(re^{it})dt.$$

In the latter case, since

$$r\int_0^{2\pi} \chi_{\tilde{W}}(re^{it})dt \leqq (u + v)(r) < \pi r_0 A \leqq \pi r$$

and $\tilde{W}^c$ is closed, $\phi(\theta)$ exists and satisfies $\theta \leqq \phi(\theta) < 4\pi$ and $r(\phi(\theta) - \theta) = r\int_0^{\phi(\theta)} \chi_{\tilde{W}}(re^{it})dt < 2\pi r_0 A$.

From (4.2) we have

$$V(r) = \frac{1}{2\pi}\int_0^{2\pi} \{T(re^{i\phi}) - T(re^{i\theta})\} r d\theta$$

$$= \frac{1}{2\pi}\int_{G_r} \{T(re^{i\phi}) - T(re^{i\theta})\} r d\theta$$

for almost all $r \in E$, where $G_r = \{\theta \in [0, 2\pi) \mid re^{i\theta} \in \tilde{W}\}$. Set

$$I(re^{i\theta}) = -e^{i\theta}\int_R \frac{\nu(\zeta) dm(\zeta)}{\zeta - re^{i\theta}}$$

and

$$O(re^{i\theta}) = -e^{i\theta}\int_{R^c} \frac{\nu(\zeta) dm(\zeta)}{\zeta - re^{i\theta}}.$$

Then $T(re^{i\theta}) = I(re^{i\theta}) + O(re^{i\theta})$ and

$$V(r) = \frac{1}{2\pi}\int_{G_r} \{I(re^{i\phi}) - I(re^{i\theta})\} r d\theta + \frac{1}{2\pi}\int_{G_r} \{O(re^{i\phi}) - O(re^{i\theta})\} r d\theta$$

for almost all $r \in E$. Hence, by integrating V on E, we have

$$(1/2)A^{1/5} V(b) \leqq \int_E V(r) dr$$

$$= \frac{1}{2\pi}\int_G \{I(re^{i\phi}) - I(re^{i\theta})\} r dr d\theta$$

$$+ \frac{1}{2\pi}\int_G \{O(re^{i\phi}) - O(re^{i\theta})\} r dr d\theta$$

by (3), where $G = \{(r, \theta) \mid r \in E \text{ and } \theta \in G_r\}$.

Since

$$\int_G r\,dr\,d\theta \leqq \int_E (u+v)\,dr \leqq \int_{c-A^{1/5}}^{c+A^{1/5}} (u+v)\,dr \leqq \frac{\pi r_0}{2} A^{1+1/5}$$

by (2), Lemma 4.1 gives

$$\int_G \frac{r\,dr\,d\theta}{|\zeta - re^{i\theta}|} \leqq \pi(2r_0)^{1/2}A^{3/5}.$$

Hence, by (2),

$$\begin{aligned}\int_G |I(re^{i\theta})|\,r\,dr\,d\theta &\leqq \int_G\left\{\int_R \frac{v(\zeta)\,dm(\zeta)}{|\zeta - re^{i\theta}|}\right\} r\,dr\,d\theta \\ &= \int_R\left\{\int_G \frac{r\,dr\,d\theta}{|\zeta - re^{i\theta}|}\right\} v(\zeta)\,dm(\zeta) \\ &\leqq \pi(2r_0)^{1/2}A^{3/5}\int_{c-A^{1/5}}^{c+A^{1/5}} v(r)\,dr \\ &\leqq \frac{\pi^2}{\sqrt{2}} r_0^{3/2}A^{1+4/5}.\end{aligned}$$

Consider the inverse function $\theta = (\phi\,|\,G_r)^{-1}$ of $\phi\,|\,G_r$. Then

$$\theta(\phi) = \int_0^{\phi} \chi_{W^c}(re^{it})\,dt$$

on $G'_r = \phi(G_r)$, and so the general derivative of θ is equal to $\chi_{W^c}(re^{i\phi})$ a.e. on G'_r. Therefore

$$\int_{G_r} \frac{d\theta}{|\zeta - re^{i\phi(\theta)}|} = \int_{G'_r} \frac{\chi_{W^c}(re^{i\phi})\,d\phi}{|\zeta - re^{i\phi}|} = \int_{G'_r} \frac{d\phi}{|\zeta - re^{i\phi}|}.$$

Since $G'_r \subset [0, 4\pi)$ and $\ell(G'_r) = \ell(G_r)$, by Lemma 4.1, we have

$$\begin{aligned}\int_G |I(re^{i\phi})|\,r\,dr\,d\theta &\leqq \int_R\left\{\int_G \frac{r\,dr\,d\theta}{|\zeta - re^{i\phi}|}\right\} v(\zeta)\,dm(\zeta) \\ &= \int_R\left\{\int_{G'} \frac{r\,dr\,d\phi}{|\zeta - re^{i\phi}|}\right\} v(\zeta)\,dm(\zeta) \\ &\leqq \pi^2 r_0^{3/2}A^{1+4/5},\end{aligned}$$

where $G' = \{(r, \phi)\,|\,r \in E \text{ and } \phi \in G'_r\}$.

Finally we consider the following:

$$|O(re^{i\phi}) - O(re^{i\theta})|$$

$$= \left| e^{i\phi}\int_{R^c} \frac{v(\zeta)dm(\zeta)}{\zeta - re^{i\phi}} - e^{i\theta}\int_{R^c} \frac{v(\zeta)dm(\zeta)}{\zeta - re^{i\theta}} \right|$$

$$\leqq \left(\int_{R^c \cap \Delta_{2r}} + \int_{R^c \cap \Delta_{2r}^c}\right) \frac{(|\zeta|/r)|re^{i\phi} - re^{i\theta}|}{|\zeta - re^{i\phi}||\zeta - re^{i\theta}|} |v(\zeta)| dm(\zeta),$$

where $\Delta_{2r} = \{z \in \boldsymbol{C} \mid |z| < 2r\}$. For $\zeta \in R^c$, we have $|\zeta - re^{i\phi}|$, $|\zeta - re^{i\theta}| \geqq (1/2)A^{1/5}$. If $|\zeta| \geqq 2r$, then

$$\frac{|\zeta|/r}{|\zeta - re^{i\theta}|} = \frac{1}{r} \frac{|\zeta|/r}{|\zeta/r - e^{i\theta}|} \leqq \frac{1}{r} \frac{|\zeta|/r}{|\zeta|/r - 1} \leqq \frac{2}{r}.$$

Since $|re^{i\phi} - re^{i\theta}| \leqq r(\phi - \theta) < 2\pi r_0 A$, we have

$$|O(re^{i\phi}) - O(re^{i\theta})|$$

$$\leqq \frac{4\pi r_0 A}{\left(\frac{1}{2}A^{1/5}\right)^2} \int_{R^c \cap \Delta_{2r}} |v(\zeta)| dm(\zeta) + \frac{4\pi r_0 A}{r \cdot \frac{1}{2} A^{1/5}} \int_{R^c \cap \Delta_{2r}^c} |v(\zeta)| dm(\zeta)$$

$$\leqq 16\pi \max\{r_0, 1\} \int |v| dm A^{3/5}.$$

Hence

$$\int_G |O(re^{i\phi}) - O(re^{i\theta})| r dr d\theta \leqq 8\pi^2 r_0 \max\{r_0, 1\} \int |v| dm\, A^{1+4/5}.$$

Combining the inequalities obtained above, we have

$$V(b) \leqq 2\pi r_0 (r_0^{1/2} + 4\max\{r_0, 1\} \int |v| dm) A^{1+3/5}.$$

This completes the proof.

§5. Convergence of kernel functions

Let R be a Riemann surface and let ζ be a point on R. Let v be a measurable function on R such that $v(z) \geqq c$ a.e. on R for a positive number c. We denote by $AD_v(R, \zeta)$ the complex linear space of analytic functions f on R such that $f(\zeta) = 0$ and $\int_R |f'(z)|^2 v(z) dx dy < \infty$, where $z = x + iy$.

An inner product on $AD_v(R, \zeta)$ is defined by

$$(f, g)_v = \frac{1}{\pi} \int_R f'(z)\overline{g'(z)} v(z) dx dy$$

for every pair of f and g in $AD_v(R, \zeta)$. With this inner product $AD_v(R, \zeta)$ becomes a Hilbert space.

Let t be a fixed local coordinate defined in a neighborhood of ζ. Since the functional $f\mapsto(df/dt)(\zeta)$ is bounded, there is a unique function $M(z)=M_v(z;\zeta, t, R)$ such that

$$\frac{df}{dt}(\zeta) = (f, M)_v$$

for every $f\in AD_v(R, \zeta)$. We call M the kernel function of $AD_v(R, \zeta)$.

The kernel function $M_v(z;\zeta, t, R)$ is identically equal to zero if and only if $(df/dt)(\zeta)=0$ for every $f\in AD_v(R, \zeta)$. If $M(z)=M_v(z;\zeta, t, R)\not\equiv 0$, then $(dM/dt)(\zeta)=(M, M)_v>0$. In this case we also consider the following normalized function:

$$M_v^*(z;\zeta, t, R) = \frac{M_v(z;\zeta, t, R)}{\frac{d}{dt}M_v(\zeta;\zeta, t, R)}.$$

This function is the unique function minimizing $(f, f)_v$ in the class of functions $f\in AD_v(R, \zeta)$ such that $(df/dt)(\zeta)=1$.

In this section we are concerned with the convergence of kernel functions and show the following proposition:

PROPOSITION 5.1. *Let R be a Riemann surface, let ζ be a point on R and let t be a fixed local coordinate defined in a neighborhood of ζ. Let $\{R_j\}_{j=1}^{\infty}$ be a sequence of subdomains of R such that $\cup R_j=R$, $R_j\subset R_{j+1}$ and $\zeta\in R_j$ for every j. Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of measurable functions on R such that $v_j(z)\geqq c$ a.e. on R_j for a fixed positive number c, $v_j(z)=0$ a.e. on $R-R_j$ and $v_j(z)\leqq v_{j+1}(z)$ a.e. on R for every j. We denote by $v(z)$ a measurable function on R which is equal to $\lim_{j\to\infty}v_j(z)$ a.e. on R. Set $M_j(z)=M_{v_j}(z;\zeta, t, R_j)$, $\mu_j(w)=\sum_{z\in M_j^{-1}(w)}v_j(z)$, $j=1, 2,\dots$, $M(z)=M_v(z;\zeta, t, R)$ and $\mu(w)=\sum_{z\in M^{-1}(w)}v(z)$. Then*

(1) $$\int_R |M_j'|^2 v_j dxdy = \int_C \mu_j dm < \infty, \qquad j = 1, 2,\dots$$

and

$$\int_R |M'|^2 v dxdy = \int_C \mu dm < \infty.$$

(2) $$\int_C \mu_j dm \downarrow \int_C \mu dm \quad as \quad j\uparrow\infty$$

and

$$\int_R |M_j' - M'|^2 v_j dxdy \leqq \int_C \mu_j dm - \int_C \mu dm.$$

In particular, M_j converges to M uniformly on every compact subset of R.

(3) $\int_R |M_j'^2 v_j - M'^2 v| dxdy \longrightarrow 0 \qquad as \quad j \longrightarrow \infty.$

(4) $\int_C |\mu_j - \mu| dm \longrightarrow 0 \qquad as \quad j \longrightarrow \infty.$

PROOF. Assertions (1) and (2) are easily verified. We write

$$\int_R |M_j'^2 v_j - M'^2 v| dxdy$$
$$\leqq \int_R |M'|^2 (v - v_j) dxdy + \int_R |M_j' - M'| (|M_j'| + |M'|) v_j dxdy.$$

Since $v_j \uparrow v$ and $\int_R |M'|^2 v dxdy < \infty$, $\int_R |M'|^2 |v - v_j| dxdy \to 0$ as $j \to \infty$. By applying Schwarz's inequality and (2), we see that $\int_R |M_j' - M'| (|M_j'| + |M'|) v_j dxdy \to 0$ as $j \to \infty$. Thus (3) is proved.

Next we prove (4). We may assume $M \not\equiv 0$. For every $\varepsilon > 0$, let Ω be a relatively compact subdomain of R such that its boundary $\partial\Omega$ consists of a finite number of smooth curves,

$$\int_{R-\Omega} |M'|^2 v dxdy < \varepsilon$$

and

$$\inf_{\bar{\Omega}} \frac{|M'|^2 dxdy}{\omega} > 0,$$

where ω denotes a second order differential on R with a positive continuous coefficient.

Set $G = M(\Omega)$, $G_{-\delta} = \{w \in G \mid d(w, M(\partial\Omega)) > \delta\}$ and $W = \Omega \cap M^{-1}(G_{-\delta})$, where δ is a positive number taken sufficiently small so that $\int_{\Omega - W} |M'|^2 v dxdy < \varepsilon$.

Take j sufficiently large so that $|M_j - M| \leqq \delta$ on $\bar{\Omega}$. Then every point $w \in G_{-\delta}$ is taken by $M_j|\Omega$ with the same number of times as taken by $M|\Omega$. Since $\inf_{\bar{\Omega}} |M'|^2 dxdy/\omega > 0$ and $z_j \to z$ as $j \to \infty$, where $z \in W$ and each z_j is an appropriate element of $M_j^{-1}(M(z))$, there is a 1-1 analytic mapping F_j of W into Ω such that $M_j \circ F_j = M$ for sufficiently large j. The sequence $\{F_j\}$ converges to the identity mapping of W. Take j so large that $\int_{\Omega - F_j(W)} |M'|^2 v dxdy < 2\varepsilon$ and set $W_j = F_j(W)$.

For a measurable set E in R, set $\mu_{j,E}(W) = \sum_{z \in M_j^{-1}(w) \cap E} v_j(z)$ for every j and $\mu_E(w) = \sum_{z \in M^{-1}(w) \cap E} v(z)$. Then

$$\int_C |\mu_j - \mu| dm$$

$$=\int_C|\mu_{j,\Omega}+\mu_{j,R-\Omega}-\mu_\Omega-\mu_{R-\Omega}|dm$$

$$\leqq\int_C|\mu_{j,\Omega}-\mu_\Omega|dm+\int_C\mu_{j,R-\Omega}dm+\int_C\mu_{R-\Omega}dm,$$

$$\int_C\mu_{R-\Omega}dm=\int_{R-\Omega}|M'|^2vdxdy$$

and

$$\int_C\mu_{j,R-\Omega}dm\leqq\int_{R-\Omega}|M'|^2vdxdy+\int_\Omega|M_j'^2v_j-M'^2v|dxdy$$
$$+\left(\int_C\mu_jdm-\int_C\mu dm\right).$$

We also have

$$\int_C|\mu_{j,\Omega}-\mu_\Omega|dm$$

$$=\int_C|\mu_{j,W_j}+\mu_{j,\Omega-W_j}-\mu_W-\mu_{\Omega-W}|dm$$

$$\leqq\int_{G_{-\delta}}|\mu_{j,W_j}-\mu_W|dm+\int_C\mu_{j,\Omega-W_j}dm+\int_C\mu_{\Omega-W}dm,$$

$$\int_C\mu_{\Omega-W}dm=\int_{\Omega-W}|M'|^2vdxdy$$

and

$$\int_C\mu_{j,\Omega-W_j}dm\leqq\int_{\Omega-W_j}|M'|^2vdxdy+\int_{\Omega-W_j}|M_j'^2v_j-M'^2v|dxdy.$$

Let $\phi=fdxdy$ be a second order differential on R with continuous coefficient such that the support of ϕ is compact and ϕ satisfies

$$\int_R|f-|M'|^2v|dxdy<\varepsilon.$$

Since

$$\int_{G_{-\delta}}|\mu_{j,W_j}-\mu_W|dm$$

$$=\int_{G_{-\delta}}|\textstyle\sum_{z\in M_j^{-1}(w)\cap W_j}v_j(z)-\sum_{z\in M^{-1}(w)\cap W}v(z)|dm$$

$$=\int_{G_{-\delta}}|\textstyle\sum_{z\in M^{-1}(w)\cap W}\{v_j(F_j(z))-v(z)\}|dm$$

$$\leqq \int_W |v_j(F_j(z)) - v(z)|\, |M'(z)|^2 dxdy,$$

we have

$$\int_{G_{-\delta}} |\mu_{j,W_j} - \mu_W| dm$$

$$\leqq \int_W \big|\, |M'(z)|^2 v_j(F_j(z)) - f(F_j(z))\big| dxdy$$

$$+ \int_W |f(F_j(z)) - f(z)| dxdy$$

$$+ \int_W |f(z) - |M'(z)|^2 v(z)| dxdy.$$

Set $z_j = x_j + iy_j = F_j(z)$ and $\phi = f_j dx_j dy_j$. Since

$$\int_W \big|\, |M'(z)|^2 v_j(F_j(z)) - f(F_j(z))\big| dxdy$$

$$= \int_{W_j} \big|\, |M'_j(z_j)|^2 v_j(z_j) - f_j(z_j)\big| dx_j dy_j$$

$$\leqq \int_R \big|\, |M'_j(z)|^2 v_j(z) - f(z)\big| dxdy$$

$$\leqq \int_R |M'^2_j v_j - M'^2 v| dxdy + \int_R \big|\, |M'|^2 v - f\big| dxdy$$

and

$$\int_W |f(F_j(z)) - f(z)| dxdy \longrightarrow 0 \qquad (j \longrightarrow \infty),$$

using all the above inequalities, we have

$$\limsup_{j\to\infty} \int_C |\mu_j - \mu| dm \leqq 7\varepsilon.$$

This completes the proof of (4).

COROLLARY 5.2. *In Proposition* 5.1, *assume further* $M \not\equiv 0$ *and set* $M^*_j(z) = M^*_{v_j}(z; \zeta, t, R_j)$, $\mu^*_j(w) = \sum_{z \in M^{*-1}_j(w)} v_j(z)$, $j = 1, 2, \ldots$, $M^*(z) = M^*_v(z; \zeta, t, R)$ *and* $\mu^*(w) = \sum_{z \in M^{*-1}(w)} v(z)$. *Then*

(1) $\int_C \mu^*_j dm \uparrow \int_C \mu^* dm$ *as* $j \uparrow \infty$.

(2) $\int_C |\mu^*_j - \mu^*| dm \longrightarrow 0$ *as* $j \longrightarrow \infty$.

§6. Estimates for the area

Let $M_\nu^*(z; \zeta, t, R)$ be the extremal function defined in §5. If R is a domain in the complex z-plane, then we choose z as a fixed local coordinate defined in a neighborhood of ζ. In this section, we deal with the case $\nu=\chi_R$ and abbreviate $M_{\chi_R}^*(z; \zeta, z, R)$ by $M^*(z; \zeta, R)$. We also consider a univalent function minimizing $(f, f)_{\chi_R}=\int_R |f'|^2 dxdy \equiv D_R[f]$ in the class of univalent functions f such that $(df/dz)(\zeta)=1$. This function may not be determined uniquely. We call it an extremal univalent function on R for ζ.

In this section we prove the following proposition:

PROPOSITION 6.1. *Let Ω be a plane domain such that $0 \in \Omega$ and $m(\Omega)<\infty$. Then*

$$m(\Omega - \Delta_r) \leqq 10\{m(\Omega)/\pi\}^{15/16} D_\Omega[z - M^*]^{1/16},$$

where $\Delta_r=\{z \in \boldsymbol{C} \mid |z|<r\}$, $r=\{m(\Omega)/\pi\}^{1/2}$ and $M^*(z)=M^*(z; 0, \Omega)$.

To prove Proposition 6.1 we prepare several lemmas.

LEMMA 6.2 (a length-area principle, cf. e.g. [1, p. 117]). *Let g be an analytic function on a ring domain $W=\{w \in \boldsymbol{C} \mid s<|w|<t\}$. Set $L(r)=\int_0^{2\pi} |g'(re^{i\theta})| r d\theta$ $(s<r<t)$. Then*

$$\int_s^t \frac{L(r)^2}{2\pi r} dr \leqq D_W[g].$$

PROOF. We may assume $D_W[g]<\infty$. Set $D(r)=\int_0^{2\pi}\int_s^r |g'(\rho e^{i\theta})|^2 \rho d\rho d\theta$. Then, by the Schwarz inequality, we have

$$L(r)^2 \leqq 2\pi r \int_0^{2\pi} |g'(re^{i\theta})|^2 r d\theta = 2\pi r D'(r).$$

Our assertion follows from this inequality.

LEMMA 6.3. *Let c and r be numbers with $0 \leqq c<r$. Then $F(z)=(r^2-c^2)z/(cz+r^2-c^2)$ is an extremal univalent function on $\Delta_r(c)=\{z \in \boldsymbol{C} \mid |z-c|<r\}$ for 0 and satisfies $F(\Delta_r(c))=\{w \in \boldsymbol{C} \mid |w|<r-c^2/r\}$.*

We omit the proof of this lemma. Next we show

LEMMA 6.4. *Let Ω be a finitely connected domain such that $0 \in \Omega$ and $m(\Omega)<\infty$. Let F be an extremal univalent function on Ω for 0. Then*

$$m(\Omega - \Delta_r) \leqq 9\{m(\Omega)/\pi\}^{15/16} D_\Omega[z - F]^{1/16},$$

where $\Delta_r = \{z \in \boldsymbol{C} \mid |z| < r\}$ *and* $r = \{m(\Omega)/\pi\}^{1/2}$.

PROOF. Without loss of generality we may assume $m(\Omega)=\pi$. Then $r=1$. Set $d=\{D_\Omega[z-F]/\pi\}^{1/2}$. If $d^{1/8} \geqq \pi^{-1/16}(\pi/9)$, then $m(\Omega-\Delta_1) \leqq m(\Omega)=\pi \leqq 9\pi^{1/16}d^{1/8}=9D_\Omega[z-F]^{1/16}$ and our lemma is proved. Hence we may assume $d^{1/8}<\pi^{-1/16}(\pi/9)=0.324\cdots$.

Since $\{D_\Omega[F]/\pi\}^{1/2} \geqq \{D_\Omega[z]/\pi\}^{1/2}-d=1-d$, the image $W=F(\Omega)$ is a circular slit disc of radius not less than $1-d$. Set $G=F^{-1}$, $r_1=1-2d$ and $V=\{w \in \boldsymbol{C} \mid r_1<|w|<r_1+d\}$. Then $\{D_W[w-G]/\pi\}^{1/2}=\{D_\Omega[F-z]/\pi\}^{1/2}=d$ and $D_{V \cap W}[G]^{1/2} \leqq m(V)^{1/2}+\sqrt{\pi}d$. Set $L(r)=\int_0^{2\pi}|G'(re^{i\theta})|rd\theta$ for r with $\{w \in \boldsymbol{C} \mid |w|=r\} \subset V \cap W$ and set $L=\inf\{L(r)\colon \{w \in \boldsymbol{C} \mid |w|=r\} \subset V \cap W\}$. Then, by Lemma 6.2, we have

$$\frac{L^2}{2\pi(r_1+d)} \cdot d \leqq [\{\pi(r_1+d)^2-\pi r_1^2\}^{1/2}+\sqrt{\pi}d]^2.$$

Hence

$$L^2 \leqq (2\pi r_1+2\pi d)\{2\pi r_1+(\sqrt{2}+\sqrt{d})2\pi\sqrt{d}\}$$
$$< (2\pi r_1+2\pi\sqrt{d})^2.$$

Therefore we can choose a number ρ so that $\{w \in \boldsymbol{C} \mid |w|=\rho\} \subset V \cap W$ and

$$L(\rho)-2\pi\rho < L(\rho)-2\pi r_1 < 2\pi\sqrt{d}.$$

Let U_ρ be the domain surrounded by the curve $G(\{w \in \boldsymbol{C} \mid |w|=\rho\})$ and R be the radius of the circumscribed circle of U_ρ; the circumscribed circle of U_ρ is the unique circle of the smallest radius that encloses U_ρ. Set $A(\rho)=m(U_\rho)$. Then

$$A(\rho) \geqq m(G(\{|w|<\rho\} \cap W)) \geqq \pi\rho^2,$$

and hence

$$L(\rho)^2-4\pi A(\rho) \leqq (2\pi\rho+2\pi\sqrt{d})^2-(2\pi\rho)^2.$$

By the Bonnesen inequality ([4], see [6]) we have

$$|L(\rho)-2\pi R| \leqq \{L(\rho)^2-4\pi A(\rho)\}^{1/2} \leqq 2\pi\sqrt{2+\sqrt{d}}\,d^{1/4}$$

so that

$$R-\rho \leqq (\sqrt{2+\sqrt{d}}+d^{1/4})d^{1/4} < 1.6\,d^{1/4}.$$

Let c be the center of the circumscribed circle of U_ρ. Then $G(\{|w|<\rho\} \cap W) \subset U_\rho \subset \Delta_R(c)$. Hence, by Lemma 6.3, we have

$$\rho \leqq R - |c|^2/R.$$

Therefore

$$|c|^2 = R(|c|^2/R) < (\rho + 1.6\,d^{1/4})1.6\,d^{1/4} < (1.4\,d^{1/8})^2.$$

Since $\Delta_R(c) \subset \Delta_{R+|c|/2}(c/2)$ and $\Delta_\rho(0) \subset \Delta_{R+|c|/2}(c/2)$, we obtain

$$\begin{aligned} m(\Delta_\rho(0) - \Omega) &\leqq m(\Delta_{R+|c|/2}(c/2) - G(\{|w| < \rho\} \cap W)) \\ &\leqq \pi(R + |c|/2)^2 - \pi\rho^2 \\ &< \pi\{(\rho + 1.6\,d^{1/4} + 0.7\,d^{1/8})^2 - \rho^2\} \\ &= \pi(1.6\,d^{1/8} + 0.7)(2\rho + 1.6\,d^{1/4} + 0.7\,d^{1/8})d^{1/8} \\ &< 9.2\,d^{1/8}. \end{aligned}$$

On the other hand, we have the following inequality:

$$m(\Delta_1 - \Delta_\rho(0)) = \pi(1 - \rho^2) \leqq \pi\{1 - (1 - 2d)^2\} < 4\pi d < 0.1\,d^{1/8}.$$

Therefore

$$\begin{aligned} m(\Omega - \Delta_1) &= m(\Delta_1 - \Omega) \\ &\leqq m(\Delta_1 - \Delta_\rho(0)) + m(\Delta_\rho(0) - \Omega) \\ &< 9.3\,d^{1/8} \\ &< 9\,D_\Omega[z - F]^{1/16}. \end{aligned}$$

LEMMA 6.5. *Let Ω be a plane domain and let F be an extremal univalent function on Ω for $\zeta \in \Omega$ and set $M^*(z) = M^*(z;\zeta,\Omega)$. Then*

$$D_\Omega[z - F] \leqq 4D_\Omega[z - M^*].$$

PROOF. Since $D_\Omega[F] = m(F(\Omega)) \leqq m(\Omega)$, we have

$$\begin{aligned} D_\Omega[z - F]^{1/2} &\leqq D_\Omega[z - M^*]^{1/2} + D_\Omega[F - M^*]^{1/2} \\ &= D_\Omega[z - M^*]^{1/2} + \{D_\Omega[F] - D_\Omega[M^*]\}^{1/2} \\ &\leqq D_\Omega[z - M^*]^{1/2} + \{m(\Omega) - D_\Omega[M^*]\}^{1/2} \\ &= 2\,D_\Omega[z - M^*]^{1/2}. \end{aligned}$$

PROOF OF PROPOSITION 6.1. Let $\{\Omega_n\}$ be an exhaustion of Ω such that each Ω_n is finitely connected and contains 0. Then, by Lemmas 6.4 and 6.5,

$$m(\Omega_n - \Delta_{r_n}) \leqq 10\{m(\Omega_n)/\pi\}^{15/16} D_{\Omega_n}[z - M_n^*]^{1/16},$$

where $r_n = \{m(\Omega_n)/\pi\}^{1/2}$ and $M_n^*(z) = M^*(z; 0, \Omega_n)$. Since $\lim_{n\to\infty} D_{\Omega_n}[z - M_n^*] = D_\Omega[z - M^*]$, by letting $n \to \infty$, we obtain the proposition.

§7. Proof of the theorem

In this section we prove our theorem and state a conjecture. At first we give two lemmas.

LEMMA 7.1. *Let $\Delta = \{z \in \boldsymbol{C} \mid |z| < r\}$, $R_1 = \{z \in \boldsymbol{C} \mid r_{11} < |z| < r_{12}\}$ and $R_2 = \{z \in \boldsymbol{C} \mid r_{21} < |z| < r_{22}\}$ with $0 \leqq r_{11} < r_{12} \leqq r_{21} < r_{22} \leqq r$. Let s be a subharmonic L^1 function on Δ. Then*

$$\frac{1}{m(R_1)} \int_{R_1} s\,dm \leqq \frac{1}{m(R_2)} \int_{R_2} s\,dm.$$

The equality holds if and only if s is harmonic on $\{z \in \boldsymbol{C} \mid |z| < r_{22}\}$.

LEMMA 7.2. *Let W be a domain and let v be an L^2 function on $\boldsymbol{C}$ such that $v(z) \geqq 1$ a.e. on W and $v(z) = 0$ a.e. on W^c. Then, for every $\varepsilon > 0$, there are an open set W_ε and a domain $\tilde{W}_\varepsilon$ satisfying the following conditions:*

(1) $W \subset W_\varepsilon \subset \tilde{W}_\varepsilon$.

(2) $m(\tilde{W}_\varepsilon) < \infty$ *and* $m(\tilde{W}_\varepsilon - W_\varepsilon) < \varepsilon$.

(3) $\int_W hv\,dm = \int_{W_\varepsilon} h\,dm$ *for every harmonic L^2 function h on $\tilde{W}_\varepsilon$.*

PROOF. To prove the lemma it is sufficient to construct the following W_n, $\tilde{W}_n$ and v_n, $n = 0, 1, 2, \ldots$:

(a) W_n is an open set and $\tilde{W}_n$ is a domain.

(b) $W_n \subset W_{n+1}$ and $\tilde{W}_n \subset \tilde{W}_{n+1}$.

(c) $W \subset W_n \subset \tilde{W}_n$.

(d) $m(\tilde{W}_n) < \infty$ for $n \geqq 0$, $m(\tilde{W}_0 - W_0) = 0$ and $m(\tilde{W}_n - W_1) < \varepsilon \sum_{m=1}^{n} 1/2^m$ for $n \geqq 1$.

(e) v_n is an L^2 function on $\boldsymbol{C}$ such that $v_n(z) \geqq 1$ a.e. on W_n and $v_n(z) = 0$ a.e. on W_n^c.

(f) $\|\mu_n\|_{1,W_n} < \varepsilon/2^{n+2}$ and $\|\mu_n\|_{2,W_n} < 1/2^n$ for $n \geqq 1$, where $\mu_n = v_n - \chi_{W_n}$ and $\|\mu_n\|_{p,W_n} = \left(\int_{W_n} |\mu_n|^p dm\right)^{1/p}$.

(g) $\int_W hv\,dm = \int_{W_n} hv_n\,dm$ for every harmonic L^2 function h on $\tilde{W}_n$.

We construct W_n, $\tilde{W}_n$ and v_n, $n = 0, 1, 2, \ldots$, by mathematical induction. Set $W_0 = \tilde{W}_0 = W$ and $v_0 = v$. Then W_0, $\tilde{W}_0$ and v_0 satisfy the above conditions for $n = 0$. Note that $m(W) < \infty$ because $v \in L^2(W)$ and $v \geqq 1$ a.e. on W.

Assume that W_n, $\tilde{W}_n$ and ν_n are constructed. Let Ω_n be a domain such that $\partial\Omega_n$ consists of a finite number of smooth curves, $\overline{\Omega_n}\subset\tilde{W}_n$, $\|\nu_n\|_{1,W_n-\Omega_n}<\varepsilon/2^{n+3}$ and $\|\nu_n\|_{2,W_n-\Omega_n}<1/2^{n+1}$. We apply Proposition 1.9 proved in [7] replacing W, $d\nu$ and ε by Ω_n, $(\nu_n\chi_{W_n\cap\Omega_n}+\chi_{\overline{\Omega_n}-W_n})dm$ and $\varepsilon/2^{n+2}$, respectively. Then there are a bounded open set $W_{\Omega,n}$ and a bounded domain $\tilde{W}_{\Omega,n}$ such that $\Omega_n\subset W_{\Omega,n}\subset\tilde{W}_{\Omega,n}$, $\overline{\Omega_n}\subset\tilde{W}_{\Omega,n}$, $m(\tilde{W}_{\Omega,n}-W_{\Omega,n})<\varepsilon/2^{n+2}$ and

$$\int_{W_n\cap\Omega_n}h\nu_n dm+\int_{\overline{\Omega_n}-W_n}hdm=\int_{W_{\Omega,n}}hdm$$

for every harmonic integrable function h on $\tilde{W}_{\Omega,n}$. Define $W_{n+1}=W_n\cup\{W_{\Omega,n}-(\overline{\Omega_n}-W_n)\}$, $\tilde{W}_{n+1}=\tilde{W}_n\cup\tilde{W}_{\Omega,n}$ and $\nu_{n+1}=\chi_{W_{n+1}}+\nu_n\chi_{W_n-\Omega_n}-\chi_{W_n-W_{\Omega,n}}=\chi_{W_{n+1}}+\mu_{n+1}$.

Substituting 1 in the above equality, we have

$$\int_{W_n\cap\Omega_n}\nu_n dm+\int_{\overline{\Omega_n}-W_n}dm=\int_{W_{\Omega,n}}dm.$$

Hence

$$\begin{aligned} m(W_{\Omega,n}-\Omega_n) &= \int_{W_n\cap\Omega_n}(\nu_n-1)dm \\ &\leqq \int_{W_n}\mu_n dm \\ &< \varepsilon/2^{n+2}. \end{aligned}$$

Therefore

$$\begin{aligned} m(\tilde{W}_{n+1}-W_1) &\leqq m((\tilde{W}_n\cup\tilde{W}_{\Omega,n})-(\tilde{W}_n\cup W_{\Omega,n})) \\ &\quad + m((\tilde{W}_n\cup W_{\Omega,n})-\tilde{W}_n)+m(\tilde{W}_n-W_1) \\ &\leqq m(\tilde{W}_{\Omega,n}-W_{\Omega,n})+m(W_{\Omega,n}-\Omega_n)+m(\tilde{W}_n-W_1) \\ &< \varepsilon\sum_{m=1}^{n+1}\frac{1}{2^m}. \end{aligned}$$

Thus $\tilde{W}_{n+1}$ satisfies (d). The other conditions are easily verified. The proof is complete.

We divide the proof of the theorem into three steps. We may assume that $\nu\geqq c$ everywhere on W and $\nu=0$ everywhere on W^c.

Step 1. Let ζ be a point on a Riemann surface R and let t be a local coordinate defined in a neighborhood of ζ. We abbreviate $M_{\chi_R}(z;\zeta,t,R)$ (resp. $M^*_{\chi_R}(z;\zeta,t,R)$) by $M(z)$ (resp. $M^*(z)$). Assume $M(z)\not\equiv 0$ and let f be an analytic function on $W=M^*(R)$ such that $\int_W |f'|^2 v dm < \infty$, where v denotes the valence function of M^*. By the reproducing property of M we have

$$(7.1) \qquad f'(0) = \frac{d(f\circ M^*)}{dt}(\zeta)$$

$$= \frac{1}{\pi}\int_R (f\circ M^*)'\overline{M}' dxdy$$

$$= \frac{1}{\pi}\frac{dM}{dt}(\zeta)\int_R (f\circ M^*)'\overline{M}^{*\prime} dxdy$$

$$= \frac{1}{\pi}\frac{dM}{dt}(\zeta)\int_W f' v dm$$

$$= \frac{1}{\int v dm}\int_W f' v dm.$$

In this step we show that if the valence function v of M^* satisfies $v(W) \geqq n$ on W for some natural number n, then

$$\int_W svdm \leqq n\int_\Delta sdm$$

for every function s which is subharmonic on $\boldsymbol{C}$ and bounded from below, where $\Delta = \{w \in \boldsymbol{C} \mid |w| < \{\int vdm/(n\pi)\}^{1/2}\}$.

Let $\{R_j\}$ be a regular exhaustion of R, namely, let $\{R_j\}$ be a sequence of subdomains of R such that $\cup R_j = R$, $\overline{R}_j \subset R_{j+1}$ for every j and each ∂R_j consists of a finite number of mutually disjoint analytic Jordan curves in R. We may assume that $\zeta \in R_j$ for every j. It is known that $M_j^*(z) = M_{\chi_j}^*(z; \zeta, t, R_j)$ can be extended analytically onto $\overline{R}_j$ (cf. [8; pp. 114–137]), where $\chi_j = \chi_{R_j}$. Hence each valence function v_j of M_j^* is bounded. Set $W_j = M_j^*(R_j)$ and $U_j = \{w \in W_j \mid v_j(w) \geqq n\}$.

For every $\delta > 0$, choose a compact set K containing 0 as its interior point so that $\int_{W-K} vdm < \delta$. Applying Rouché's theorem choose J_1 so that $K \subset U_j$ for any $j \geqq J_1$, and then choose $J \geqq J_1$ so that $\int_K (v - v_J)dm < \delta$ by Fatou's lemma. Set $\mu = \max\{v_J, n\chi_{W_J}\}$. Then μ is a bounded L^1 function on W_J such that $\mu(w) \geqq n$ on W_J and $\mu(w) = 0$ on W_J^c.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ be a pair of positive numbers ε_1 and ε_2 with $\varepsilon_1 < 1$, and apply Proposition 3.3 replacing W and v by W_J and μ/n, respectively. Then there are domains W_ε and $\widetilde{W}_\varepsilon$, and a bounded L^1 function v_ε on $\boldsymbol{C}$ such that

(1) $W_J \subset W_\varepsilon \subset \widetilde{W}_\varepsilon$.

(2) $m(\tilde{W}_\varepsilon) \leqq \frac{1}{n(1-\varepsilon_1)}\int \mu dm$ and $m(\tilde{W}_\varepsilon - W_\varepsilon) \leqq \frac{\varepsilon_1}{n(1-\varepsilon_1)}\int \mu dm.$

(3) v_ε satisfies $v_\varepsilon(z) \geqq 1$ a.e. on W_ε, $v_\varepsilon(z)=0$ a.e. on W_ε^c and $\int (v_\varepsilon - \chi_{W_\varepsilon})dm \leqq \varepsilon_2$.

(4) $\int_{W_J} s\mu dm \leqq n\int_{W_\varepsilon} sv_\varepsilon dm$ for every $s \in SL^1(\tilde{W}_\varepsilon)$.

(5) $U(r + 2(1/\pi)^{1/2}\{(U+V)(r)\}^{1/2}) \leqq \varepsilon_1(U+V)(r)$ for every $r \geqq 0$, where

$$u(r) = \int_0^{2\pi} \chi_{\tilde{W}_\varepsilon - W_\varepsilon}(re^{i\theta})rd\theta, \quad U(r) = \int_r^\infty u(t)dt,$$

$$v(r) = \int_0^{2\pi} v_\varepsilon(re^{i\theta})rd\theta \quad \text{and} \quad V(r) = \int_r^\infty v(t)dt.$$

By Remark to Proposition 3.3, we have

$$\int_{W_J} h\mu dm = n\int_{W_\varepsilon} hv_\varepsilon dm \tag{7.2}$$

for every $h \in HL^1(\tilde{W}_\varepsilon)$. Set $\lambda = v_\varepsilon + (\min\{v_J, n\chi_{W_J}\} - n\chi_{W_J})/n$ and $r_0 = (\int vdm/\pi)^{1/2}$. By (1) of Corollary 5.2, $\int v_J dm \leqq \int vdm$, so that by Proposition A, we have $\overline{W}_J \subset \{w \in \boldsymbol{C} \mid |w| < r_0\}$. Hence $\lambda(w) = v_\varepsilon(w) \geqq 1$ a.e. on $W_\varepsilon \cap \{w \in \boldsymbol{C} \mid |w| \geqq r_0\}$ and $\lambda(w) = 0$ a.e. on W_ε^c. Let us see that $\hat{\lambda}(w) = -\int \lambda dm/w$ on $\tilde{W}_\varepsilon^c \cap \{w \in \boldsymbol{C} \mid |w| \geqq r_0\}$. Let $w \in \tilde{W}_\varepsilon^c \cap \{w \in \boldsymbol{C} \mid |w| \geqq r_0\}$ and set $h(\zeta) = 1/(\zeta - w)$. Since both $\mathrm{Re}\, h$ and $\mathrm{Im}\, h$ belong to $HL^1(\tilde{W}_\varepsilon)$, we have by (7.2)

$$\begin{aligned}
\hat{\lambda}(w) &= \int_{W_\varepsilon} h\lambda dm \\
&= \int_{W_\varepsilon} hv_\varepsilon dm + \frac{1}{n}\int_{W_\varepsilon} h(\min\{v_J, n\chi_{W_J}\} - n\chi_{W_J})dm \\
&= \frac{1}{n}\int_{W_J} h(\max\{v_J, n\} + \min\{v_J, n\} - n)dm \\
&= \frac{1}{n}\int_{W_J} hv_J dm.
\end{aligned}$$

Let $f(\zeta)$ be a branch of $\log(\zeta - w)$ on $\{w \in \boldsymbol{C} \mid |w| < r_0\}$. Applying (7.1) to f we have

$$\int_{W_J} hv_J dm = h(0)\int_{W_J} v_J dm = -\frac{1}{w}\int_{W_J} v_J dm$$

and hence $\hat{\lambda}(w) = -(nw)^{-1}\int_{W_J} v_J dm$. If we carry the same computation with $h=1$, then we obtain $\int \lambda dm = n^{-1}\int_{W_J} v_J dm$. Hence $\hat{\lambda}(w) = -\int \lambda dm/w$.

By using Proposition 4.2, we have

$$V(b) \leqq \gamma A(a, b)^{1+3/5}$$

for every pair of numbers a and b such that $r_0 \leqq a < b$, $A(a, b) \equiv \{\kappa/(b-a)\}\int_a^b \{u(t)+v(t)\}dt \leqq 1$ and $2A(a, b)^{1/5} \leqq b-a$, where $\gamma = 2\pi r_0(r_0^{1/2} + 4\max\{r_0, 1\}\int|\lambda|dm)$ and $\kappa = 8/(\pi r_0)$. Hence, by Proposition 2.1, there is a number $M > \{\int v dm/(n\pi)\}^{1/2}$ satisfying $U(M) = V(M) = 0$. We can choose M so that it depends only on ε_1, r_0 and $\int|\lambda|dm$. Therefore $\tilde{W}_{\varepsilon'} \subset \{w \in \boldsymbol{C} \mid |w| < M\}$ for every $\varepsilon' = (\varepsilon_1', \varepsilon_2)$ with $0 < \varepsilon_1' \leqq \varepsilon_1$ and $\varepsilon_2 > 0$. Since every subharmonic function on $\boldsymbol{C}$ is locally bounded from above, s is bounded on $\{w \in \boldsymbol{C} \mid |w| < M\}$. Set $\|s\|_{\infty, M} = \sup_{|w|<M} |s(w)|$. Since

$$\int_{W_J - K} v_J dm = \int v_J dm - \int v dm + \int_{W-K} v dm + \int_K (v - v_J) dm < 2\delta,$$

we have

$$\begin{aligned}
\left| \int_{W_J} s v_J dm - \int_{W_J} s\mu dm \right| &= \left| \int_{W_J - K} s v_J dm - \int_{W_J - K} s\mu dm \right| \\
&\leqq \|s\|_{\infty, M} \int_{W_J - K} v_J dm + \|s\|_{\infty, M} \int_{W_J - K} n v_J dm \\
&\leqq \|s\|_{\infty, M}(n+1) \int_{W_J - K} v_J dm \\
&\leqq \|s\|_{\infty, M} 2(n+1)\delta.
\end{aligned}$$

From (3) and (4), we have

$$\begin{aligned}
\int_{W_J} s\mu dm &\leqq n \int_{W_\varepsilon} s v_\varepsilon dm \\
&= n\left\{\int_\Delta s dm + \int_{W_\varepsilon} s(v_\varepsilon - \chi_{W_\varepsilon}) dm + \int_{W_\varepsilon - \Delta} s dm - \int_{\Delta - W_\varepsilon} s dm\right\} \\
&\leqq n \int_\Delta s dm + n\|s\|_{\infty, M}\{\varepsilon_2 + m(W_\varepsilon - \Delta) + m(\Delta - W_\varepsilon)\}.
\end{aligned}$$

We next estimate $m(W_\varepsilon - \Delta)$ and $m(\Delta - W_\varepsilon)$. Apply Lemma 7.2 replacing W, v and ε by W_ε, v_ε and η, respectively. Then there are an open set W_η and a domain $\tilde{W}_\eta$ such that $W_\varepsilon \subset W_\eta \subset \tilde{W}_\eta$, $m(\tilde{W}_\eta) < \infty$, $m(\tilde{W}_\eta - W_\eta) < \eta$ and $\int_{W_\varepsilon} h v_\varepsilon dm = \int_{W_\eta} h dm$ for every $h \in HL^2(\tilde{W}_\eta)$. Hence, by (7.2),

$$\int_{W_J} f'\mu dm = n \int_{W_\eta} f' dm$$

for every analytic function f on $\Omega=\tilde{W}_\varepsilon \cup \tilde{W}_\eta$ with a finite Dirichlet integral $D_\Omega[f]$.

Set $M_\Omega(w)=M_{\chi\Omega}(w;\, 0,\, w,\, \Omega)$. Then

$$n\int_\Omega f'\left(\frac{1}{\int v_J dm}-\frac{\overline{M'_\Omega}}{n\pi}\right)dm=\frac{n}{\int v_J dm}\int_{\Omega-W_\eta} f'dm+\frac{1}{\int v_J dm}\int_{W_J} f'(\mu-v_J)dm$$
$$+\frac{1}{\int v_J dm}\int_{W_J} f'v_J dm-f'(0)$$

for every $f\in AD_{\chi\Omega}(\Omega,\, 0)$. Since

$$f'(0)=\frac{1}{\int v_J dm}\int_{W_J} f'v_J dm,$$

$\mu-v_J=0$ on U_J, $0\leqq\mu-v_J\leqq n-1$ on W_J-U_J and $m(\Omega-W_\eta)\leqq m(\tilde{W}_\varepsilon-W_\varepsilon)+m(\tilde{W}_\eta-W_\eta)\leqq\varepsilon_1(n(1-\varepsilon_1))^{-1}\int\mu dm+\eta\leqq\varepsilon_1(1-\varepsilon_1)^{-1}\int v_J dm+\eta\leqq\varepsilon_1(1-\varepsilon_1)^{-1}\int v dm+\eta$, we have

$$D_\Omega\left[\frac{w}{\int v_J dm}-\frac{M_\Omega}{n\pi}\right]^{1/2}\leqq\frac{1}{\int v_J dm}\{\sqrt{m(\Omega-W_\eta)}+\sqrt{m(W_J-U_J)}\}$$
$$<\frac{1}{\int v_J dm}\left(\sqrt{\frac{\varepsilon_1}{1-\varepsilon_1}\int v dm+\eta}+\sqrt{2\delta}\right).$$

Hence

$$D_\Omega\left[w-\frac{\int v_J dm}{nD_\Omega[M^*_\Omega]}M^*_\Omega\right]^{1/2}<\sqrt{\frac{\varepsilon_1}{1-\varepsilon_1}\int v dm+\eta}+\sqrt{2\delta},$$

where $M^*_\Omega(w)=M^*_{\chi\Omega}(w;\, 0,\, w,\, \Omega)$. Since $K\subset\Omega$ and K contains 0 as its interior point, $g(w)=w-M^*_\Omega\int v_J dm/(nD_\Omega[M^*_\Omega])$ tends to 0 uniformly on some neighborhood of 0 as δ, ε_1 and η tend to 0. Hence $g'(0)$ tends to 0 so that $\int v_J dm/(nD_\Omega[M^*_\Omega])$ tends to 1. It follows that $D_\Omega[w-M^*_\Omega]$ tends to 0.

Finally we show that $m(W_\varepsilon-\Delta)+m(\Delta-W_\varepsilon)$ tends to 0 as δ, ε_1, ε_2 and η tend to 0. We apply Proposition 6.1 and obtain $m(\Omega-\Delta_r)\to 0$, where $r=\{m(\Omega)/\pi\}^{1/2}$. From the relation $\int_{W_\varepsilon} v_\varepsilon dm=\int_{W_\eta} dm$ and (7.2) we obtain $m(W_\eta)=n^{-1}\int_{W_J}\mu dm$. Since $m(\Omega)-m(W_\eta)=m(\Omega-W_\eta)\leqq\varepsilon_1(1-\varepsilon_1)^{-1}\int v dm+\eta$ and

$$0\leqq\int_{W_J}\mu dm-\int_{W_J} v_J dm\leqq 2(n-1)\delta,$$

by (1) of Corollary 5.2, we have $m(\Omega)\to n^{-1}\int v dm$, and hence $r\to\{\int v dm/(n\pi)\}^{1/2}$. Therefore

$$m(W_\varepsilon-\Delta)\leqq m(\Omega-\Delta)\leqq m(\Omega-\Delta_r)+m(\Delta_r-\Delta)\longrightarrow 0,$$

and

$$|m(\Delta - W_\varepsilon) - m(W_\varepsilon - \Delta)| = |m(\Delta) - m(W_\varepsilon)| \longrightarrow 0.$$

Thus $m(\Delta - W_\varepsilon) \to 0$ too. By (2) of Corollary 5.2, we have

$$\int_W s\nu dm = \lim_{J\to\infty} \int_{W_J} s\nu_J dm \leqq n \int_\Delta sdm.$$

Step 2. Let W be a domain and let ν be an L^1 function on $\boldsymbol{C}$ given in the theorem. In this step we show

$$\int_W s\nu dm \leqq c \int_{\Delta_r} sdm \tag{7.3}$$

for every function s which is subharmonic on $\boldsymbol{C}$ and bounded from below.

Suppose first that ν is a lower semicontinuous function on $\boldsymbol{C}$ and $\nu(z)$ is a natural number not less than n for every $z \in W$. In this case, by using the same argument as in [7, Proposition 3.1], we can construct a Riemann surface R of infinite genus and $F \in AD_{\chi_R}(R, \zeta)$ for some $\zeta \in R$ such that

(1) The valence function ν_F of F is equal to ν a.e. on $\boldsymbol{C}$.

(2) $(dF/dt)(\zeta) \neq 0$ for some (and hence every) local coordinate t defined in a neighborhood of ζ.

(3) For every $g \in AD_{\chi_R}(R, \zeta)$, there is a function $f \in AD_\nu(W, 0)$ satisfying $g = f \circ F$.

By virtue of (2) we can choose F as a local coordinate defined in a neighborhood of ζ. As was shown in the proof of Proposition 3.1 of [7], $F(p) = M^*_{\chi_R}(p; \zeta, F, R)$. By Step 1, we have

$$\int_W s\nu dm = \int_W s\nu_F dm \leqq n \int_\Delta sdm$$

for every function s which is subharmonic on $\boldsymbol{C}$ and bounded from below, where $\Delta = \{z \in \boldsymbol{C} \mid |z| < \{\int \nu dm/(n\pi)\}^{1/2}\}$.

Suppose next that ν is lower semicontinuous on $\boldsymbol{C}$, that $\nu(z)/\varepsilon$ is a natural number for some fixed $\varepsilon > 0$ and for every $z \in W$ and that c/ε is a natural number. By the above argument, we also have (7.3) in this case.

Finally we show (7.3) for an arbitrary ν given in the theorem. We can construct functions ν_j on $\boldsymbol{C}$, $j = 1, 2, \dots$, such that

(1) $0 \leqq \nu_j \leqq \nu_{j+1}$ and $\lim \nu_j = \nu$ a.e. on $\boldsymbol{C}$.

(2) $\nu_j(z) \geqq c$ on W.

(3) ν_j is lower semicontinuous on $\boldsymbol{C}$.

(4) $(2^j/c)\nu_j(z)$ is a natural number for every $z \in W$.

Set $M^*_j(z) = M^*_{\nu_j}(z; 0, z, W)$, $\mu^*_j(w) = \sum_{z \in M^{*-1}_j(w)} \nu_j(z)$, $j = 1, 2, \dots$, $M^*(z) = M^*_\nu(z; 0, z, W)$ and $\mu^*(w) = \sum_{z \in M^{*-1}(w)} \nu(z)$. From the assumption in the theorem and the definition of the kernel function, we have $M^*(z) = z$ and so $\mu^* = \nu$. The function μ_j^* is lower semicontinuous on $\boldsymbol{C}$ and satisfies that $\mu_j^*(w) \geqq c$ on $M_j^*(W)$, $(2^j/c)\mu_j^*(w)=\mu_j^*(w)/(c/2^j)$ is a natural number for almost all $w \in M_j^*(W)$ and

$$f'(0) = \frac{1}{\int \mu_j^* dm} \int_{M_j^*(W)} f' \mu_j^* dm$$

for every $f \in AD_{\mu_j^*}(M_j^*(W), 0)$. Hence, by the above argument, we have

$$\int_{\boldsymbol{C}} s\mu_j^* dm \leqq c \int_{\Delta_j} s dm$$

for every function s which is subharmonic on $\boldsymbol{C}$ and bounded from below, where $\Delta_j = \{w \in \boldsymbol{C} \mid |w| < \{\int \mu_j^* dm/(c\pi)\}^{1/2}\}$. Therefore, by Corollary 5.2, we obtain

$$\int_W svdm = \lim \int_W s\mu_j^* dm \leqq c \int_{\Delta_r} sdm.$$

Step 3. Let W be a domain and let v be an L^1 function on $\boldsymbol{C}$ given in the theorem. Let s be a subharmonic function on $\boldsymbol{C}$ which is not necessarily bounded from below. Since $\max\{s, N\}$ is subharmonic for every number N, by Step 2, we have

$$\int_W \max\{s, N\} vdm \leqq c \int_{\Delta_r} \max\{s, N\} dm.$$

Letting $N \downarrow -\infty$, we obtain (7.3) for every subharmonic function on $\boldsymbol{C}$.

Now we prove (7.3) for every subharmonic L^1 function on Δ_r. By Proposition A, we may assume that there is a compact subset K of W such that $m(K)>0$ and $\inf_{z \in K} v(z) > c$. By using the same argument as in the proof of Proposition A ([7], Proposition 3.2), we can construct a measurable function v_0 on W such that

(1) $v_0(z) \geqq c + \alpha\chi_{\Delta_0}(z)$ on W and $v_0(z) = 0$ on W^c, where $\alpha > 0$ and Δ_0 denotes a disc centered at 0 with $\bar{\Delta}_0 \subset W$.

(2) $\int_W svdm \leqq \int_W sv_0 dm$ for every $s \in SL^1(W)$.

Set $\mu(z) = v_0(z) - \alpha\chi_{\Delta_0}(z)$. Then $\mu(z) \geqq c$ on W, $\mu(z)=0$ on W^c and

$$f'(0) = \frac{1}{\int \mu dm} \int_W f' \mu dm$$

for every $f \in AD_\mu(W, 0)$. Hence

$$\int_W s\mu dm \leqq c \int_{\Delta_\rho} sdm \tag{7.4}$$

for every subharmonic function s on $\boldsymbol{C}$, where $\Delta_\rho = \{z \in \boldsymbol{C} \mid |z| < \rho\}$ and $\rho = \{\int \mu dm/(c\pi)\}^{1/2}$.

For every $s \in SL^1(\Delta_r)$, there is a subharmonic function $\tilde{s}$ on $\boldsymbol{C}$ such that

$\tilde{s}\,|\,\bar{\Delta}_\rho = s\,|\,\bar{\Delta}_\rho$. Therefore (7.4) holds for every $s \in SL^1(\Delta_r)$.

By Lemma 7.1, we have

$$\int_W s(\alpha\chi_{\Delta_0})dm \leqq c\int_{\Delta_r - \Delta_\rho} sdm, \tag{7.5}$$

and so we obtain (7.3) for every $s \in SL^1(\Delta_r)$. The equality in (7.5) holds if and only if s is harmonic on Δ_r. This completes the proof of our theorem.

In our theorem we have treated upon the special case when v satisfies

$$f'(0) = \frac{1}{\int vdm}\int_W f'vdm$$

for every analytic function f on W such that $\int_W |f'|^2 vdm < \infty$. By our theorem and Lemma 7.2 we are led to the following conjecture:

CONJECTURE. *Let W be a domain and let v be an L^p $(1 \leqq p < \infty)$ function on $\mathbf{C}$ such that $v(z) \geqq 1$ a.e. on W and $v(z) = 0$ a.e. on W^c. Then there is a domain $\tilde{W}$ satisfying the following conditions:*

(1) $W \subset \tilde{W}$.

(2) $m(\tilde{W}) = \int vdm < \infty$.

(3) $\int_W svdm \leqq \int_{\tilde{W}} sdm$ *for every subharmonic L^q function s on $\tilde{W}$, where q satisfies $1/p + 1/q = 1$.*

If this conjecture were true, then it would be possible to simplify the proof of our theorem.

§8. An application to the estimation of the Gaussian curvature of the span metric

Let R be a Riemann surface and let v be a measurable function on R such that $v(z) \geqq c$ a.e. on R for a positive number c. For a natural number n, let $AD_v(R, \zeta^n)$ be the complex linear space of analytic functions f on R such that $\int_R |f'(z)|^2 v(z)dxdy < \infty$ and $f(\zeta) = (df/dt)(\zeta) = \cdots = (d^{n-1}f/dt^{n-1})(\zeta) = 0$ for a fixed local coordinate t defined in a neighborhood of ζ. We define an inner product by

$$(f, g)_v = \frac{1}{\pi}\int_R f'(z)\overline{g'(z)}v(z)dxdy$$

for every pair of f and g in $AD_v(R, \zeta^n)$. With this inner product $AD_v(R, \zeta^n)$ becomes a Hilbert space. Set $\|f\|_v = \sqrt{(f, f)_v}$.

Since the functional $f \mapsto (d^n f/dt^n)(\zeta)$ is bounded, there is a unique $M(z)=M_\nu(z; \zeta^n, t, R) \in AD_\nu(R, \zeta^n)$ such that

$$\frac{d^n f}{dt^n}(\zeta) = (f, M)_\nu$$

for every $f \in AD_\nu(R, \zeta^n)$. By Proposition A, we have $M(R) \subset \Delta = \{w \in \boldsymbol{C} \mid |w| < \|M\|_\nu/\sqrt{c}\}$; see the following proof. Applying our theorem we have

PROPOSITION 8.1. *If $M(z)=M_\nu(z; \zeta^n, t, R)$ is not identically zero, then*

$$D_R[f \circ M] \leqq D_\Delta[f] \tag{8.1}$$

for every analytic function f on $\Delta = \{w \in \boldsymbol{C} \mid |w| < \|M\|_\nu/\sqrt{c}\}$. The equality holds if and only if one of the following is satisfied:

(i) *f is constant.*

(ii) *$\nu(z)=c$ a.e. on R and f is a linear function, namely, $f(w)=aw+b$ for some constants a and b.*

(iii) *$n=1$, $\nu(z)=c$ a.e. on R and R is conformally equivalent to $\Delta - E$, where E denotes a relatively closed subset of Δ such that $E \cap K$ is removable with respect to analytic functions with finite Dirichlet integrals for every compact subset K of Δ.*

PROOF. Set $W=M(R)$ and $\mu(w)=\sum_{z \in M^{-1}(w)} \nu(z)$. Then $\int_{\boldsymbol{C}} \mu dm = \int_R |M'|^2 \nu dxdy < \infty$, $\mu(w) \geqq c$ a.e. on W, $\mu(w)=0$ on W^c and

$$\begin{aligned} f'(0) &= \frac{1}{(M, M)_\nu} f'(0) \cdot \frac{d^n M}{dt^n}(\zeta) \\ &= \frac{\pi}{\int \mu dm} \frac{d^n (f \circ M)}{dt^n}(\zeta) \\ &= \frac{1}{\int \mu dm} \int_R (f \circ M)' \overline{M'} \nu dxdy \\ &= \frac{1}{\int \mu dm} \int_W f' \mu dm \end{aligned}$$

for every $f \in AD_\mu(W, 0)$. To prove (8.1) we may assume $f \in AD_{\chi_\Delta}(\Delta, 0)$. Since $|f'|^2 = |f'^2| \in SL^1(\Delta)$, by our theorem, we have

$$cD_R[f \circ M] \leqq \int_R (f \circ M)' \overline{(f \circ M)'} \nu dxdy = \int_W |f'|^2 \mu dm \leqq c \int_\Delta |f'|^2 dm,$$

and so

$$D_R[f\circ M] \leqq D_\Delta[f].$$

Next we prove the equality assertion. It suffices to show the "only if" part. Assume $D_R[f\circ M]=D_\Delta[f]$ for a nonlinear analytic function f on Δ. If $D_\Delta[f]=\infty$, then $D_R[f\circ M]=\int_W |f'|^2\mu dm=\infty$. We know that $\int_C \mu dm<\infty$. Therefore $\overline{W}\not\subset\Delta$ so that $\sup_{w\in W}|w|$ is equal to the radius of Δ. By Proposition A, it follows that $\mu(w)=c$ a.e. on Δ, so that $\nu(z)=c$ a.e. on R and M is univalent. Hence $n=1$ and R is a Riemann surface mentioned in the proposition. If $D_\Delta[f]<\infty$, then the subharmonic L^1 function $|f'|^2$ is not harmonic on Δ, since f is nonlinear. Therefore, by our theorem, we have again $\mu(w)=c$ a.e. on Δ. It follows as above that (iii) is true. This completes the proof of our proposition.

COROLLARY 8.2. *If $M(z)=M_{\chi R}(z;\zeta,t,R)$ is not identically equal to zero, then*

$$D_R[M^2/2] \leqq D_R[M]^2/(2\pi).$$

The equality holds if and only if R is conformally equivalent to Δ_1-E, where Δ_1 denotes the unit disc and E denotes a relatively closed subset of Δ_1 mentioned in Proposition 8.1.

PROOF. Set $f(w)=w^2/2$. Then $D_\Delta[f]=\pi\|M\|_{\chi R}^4/2=D_R[M]^2/(2\pi)$, and so our assertion follows from Proposition 8.1.

REMARK. Set $\phi=M/\|M\|_{\chi R}$. Then $D_R[\phi]=\pi$. Hence, by Corollary 8.2, we have $D_R[\phi^2/2]\leqq D_R[\phi]^2/(2\pi)=\pi/2$. This has been conjectured by J. Burbea [5, Conjecture 2].

Now we deal with the "span metric". The span $S(\zeta)$ at $\zeta\in R$ is defined by

$$S(\zeta) = \frac{d}{dt} M_{\chi R}(\zeta;\zeta,t,R).$$

The span $S(\zeta)$ depends on the choice of the local coordinate t. But if the span vanishes for some local coordinate, then it vanishes for every local coordinate. We denote by N_R the set of points $\zeta\in R$ at which the spans vanish.

If $R\in O_{AD}$, namely, if there are no nonconstant analytic functions on R with finite Dirichlet integrals, then $N_R=R$. If $R\notin O_{AD}$, then N_R is a closed discrete subset of R. We note that $N_R=\emptyset$ if $R\notin O_{AD}$ and R is of finite genus.

The metric $\sqrt{S(\zeta)}\,|dt|$ defined on $R-N_R$ is called the span metric. Let $K(\zeta)$ be the Gaussian curvature of the span metric, namely,

$$K(\zeta) = -\frac{2}{S(\zeta)}\frac{\partial^2}{\partial t\partial\bar{t}}\log S(\zeta).$$

PROPOSITION 8.3. *It follows that*

$$K(\zeta) \leqq -4$$

for every ζ *in* $R-N_R$. *The following conditions are equivalent*:

(i) $K(\zeta)=-4$ *for some point* ζ *in* $R-N_R$.

(ii) $K(\zeta)=-4$ *for every point* ζ *in* $R-N_R$.

(iii) R *is conformally equivalent to* Δ_1-E, *where* Δ_1 *denotes the unit disc and* E *denotes a relatively closed subset of* Δ_1 *mentioned in Proposition* 8.1.

PROOF. Since $\zeta\in R-N_R$, $dM_{\chi_R}(\zeta;\zeta,t,R)/dt\neq 0$. In §5 we set $M^*_{\chi_R}(z;\zeta,t,R)=M_{\chi_R}(z;\zeta,t,R)/\{dM_{\chi_R}(\zeta;\zeta,t,R)/dt\}$. Denoting it by $F_1(z)$ here, we have $D_R[F_1^2/2]<\infty$ by Corollary 8.2. Hence $d^2M_{\chi_R}(\zeta;\zeta^2,t,R)/dt^2\neq 0$. Set $F_2(z)=M_{\chi_R}(z;\zeta^2,t,R)/\{d^2M_{\chi_R}(\zeta;\zeta^2,t,R)/dt^2\}$. Since

$$D_R[F_2]\leqq D_R[F_1^2/2],$$

by Corollary 8.2, we have

$$D_R[F_2]\leqq D_R[F_1]^2/(2\pi). \tag{8.2}$$

It is known that

$$K(\zeta)=-\frac{2}{\pi}\frac{D_R[F_1]^2}{D_R[F_2]}$$

(cf. e.g. [2, Chapter III]). By (8.2), we have

$$K(\zeta)\leqq -4$$

for every $\zeta\in R-N_R$.

Next we prove the second assertion. It is evident that (iii) implies (ii) and (ii) implies (i). If $K(\zeta)=-4$ for some ζ in $R-N_R$, then $D_R[F_1^2/2]=D_R[F_1]^2/(2\pi)$. Hence, by Corollary 8.2, R is a planar domain mentioned in (iii).

Finally we deal with the integral curvature of the span metric. It is the surface integral C of the Gaussian curvature $K(z)$, namely,

$$C=\int_{R-N_R}K(z)S(z)dxdy.$$

COROLLARY 8.4. *If* $R\notin O_{AD}$, *then* $C=-\infty$.

PROOF. By Proposition 8.3, we have $C\leqq -4\int_{R-N_R}S(z)dxdy=-4\int_R S(z)\cdot dxdy$. Let $\{\phi_n\}$ be a complete orthonormal system of $AD_{\chi_R}(R,\zeta)$. Then

$S(z)dxdy=\sum|\phi_n'(z)|^2dxdy$, and hence

$$\frac{1}{\pi}\int_R S(z)dxdy = \dim_C AD_{\chi_R}(R, \zeta).$$

Since $R\notin O_{AD}$, by [7, Corollary 2.5], $\dim_C AD_{\chi_R}(R, \zeta)=\infty$. Therefore $C=-\infty$.

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