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THE SUBCOMODULE STRUCTURE OF THE QUANTUM SYMMETRIC POWERS

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In this paper we study the symmetric powers of quantum GL_n. These are comodules and we give a complete description of the subcomodule structure.

In [3] Doty gives a complete description of the submodule structure for the symmetric powers of the natural module for GL_n. We want to do the same thing for quantum GL_n, that is, determine the complete subcomodule structure of the q-symmetric powers (defined in [2] by Dipper and Donkin).

Throughout the paper we shall look at quantum GL_n defined over some field k. From [2, Theorem 2.1.9] we get a k basis of the q-symmetric powers. Following Doty's idea [3] we shall determine the subcomodule generated by a basis element by relating it to the carry pattern which we shall define, and hence get a complete description of the subcomodule structure. As one would expect, the quantum parameter $q \in k$ will play an important role in this description.

1. PRELIMINARIES

We shall first recall the set-up from [2].

1.1 Let R be a commutative ring, $q \in R$ and $2 \leqslant n \in \mathbf{Z}$. Let $A_q(n)$ be defined as the R-algebra generated by $\{c_{i,j} \mid 1 \leqslant i,j \leqslant n\}$ with the following relations:

$$c_{i,k}c_{j,l} = qc_{j,l}c_{i,k} \quad \text{for } i > j \text{ and } k \leqslant l$$
$$c_{i,k}c_{j,l} = c_{j,l}c_{i,k} + (q-1)c_{j,k}c_{i,l} \quad \text{for } i > j \text{ and } k > l$$
$$c_{i,k}c_{i,l} = c_{i,l}c_{i,k} \quad \text{for all } i, k \text{ and } l.$$

1.2 For some positive integer r, we denote by $I(n,r)$ the set of $\mathbf{i} = (i_1, \ldots, i_r)$ where $i_j \in \{1, \ldots, n\}$ $(1 \leqslant j \leqslant r)$. For $\mathbf{i}, \mathbf{s} \in I(n,r)$ let $c_{\mathbf{i},\mathbf{s}} = c_{i_1,s_1}c_{i_2,s_2} \ldots c_{i_r,s_r} \in A_q(n)$. Let $A_q(n,r)$ denote the subspace of $A_q(n)$ spanned by $\{c_{\mathbf{i},\mathbf{s}} \mid \mathbf{i}, \mathbf{s} \in I(n,r)\}$.

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Then $A_q(n)$ is a graded R-algebra $A_q(n) = \oplus_{0\leqslant r} A_q(n,r)$ and it is free as an R-module. The graded part $A_q(n,r)$ has R-basis $\{c_{\mathbf{i},\mathbf{s}} \mid (\mathbf{i},\mathbf{s}) \in I^2(n,r)^+\}$. (See [2] for a definition of $I^2(n,r)^+$. It means that a basis element will be of the form $c_{1,1}^{a_{11}} c_{1,2}^{a_{12}} \dots c_{1,n}^{a_{1n}} c_{2,1}^{a_{21}} \dots . c_{n,n}^{a_{nn}}$ where $a_{ij} \geqslant 0$ for all i,j and $\sum_{i,j} a_{i,j} = r$.)

1.3 $A_q(n)$ is made into a bialgebra with comultiplication $\Delta(c_{i,s}) = \sum_{j=1}^{n} c_{i,j} \otimes c_{j,s}$ and counit $\varepsilon(c_{i,s}) = \delta_{i,s}$. For every $0 \leqslant r \in \mathbf{Z}$, $A_q(n,r)$ is a subcoalgebra of $A_q(n)$

1.4 Let E be the free R-module of dimension n with basis $\{e_1, \dots, e_n\}$. This means $E = Re_1 \oplus \dots \oplus Re_n$. Let $E^{\otimes r}$ denote the r-fold tensor product of E with itself. For $\mathbf{i} = (i_1, i_2, \dots, i_r) \in I(n,r)$ let $e_{\mathbf{i}} = e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_r} \in E^{\otimes r}$. Now $E^{\otimes r}$ becomes a right $A_q(n,r)$-comodule with structure map $\tau_r : E^{\otimes r} \to E^{\otimes r} \otimes A_q(n,r)$ defined by $\tau_r(e_{\mathbf{s}}) = \sum_{\mathbf{i} \in I(n,r)} e_{\mathbf{i}} \otimes c_{\mathbf{i},\mathbf{s}}$. Let $T(E)$ denote the tensor algebra $T(E) = \oplus_{0 \leqslant r} E^{\otimes r}$, which is also a right $A_q(n)$-comodule.

1.5 There is also a left comodule structue on $E^{\otimes r}$, but to distinguish this from the right comodule it is given the name $V^{\otimes r}$, that is, V is the free R-module with basis $\{v_1, \dots, v_n\}$ and $V^{\otimes r}$ becomes a left $A_q(n,r)$-comodule with structure map $\sigma_r : V^{\otimes r} \to A_q(n,r) \otimes V^{\otimes r}$ defined by $\sigma_r(v_{\mathbf{s}}) = \sum_{\mathbf{i} \in I(n,r)} c_{\mathbf{i},\mathbf{s}} \otimes v_{\mathbf{i}}$ (again $v_{\mathbf{i}} = v_{i_1} \otimes \dots \otimes v_{i_r} \in V^{\otimes r}$ for $\mathbf{i} = (i_1, \dots, i_r) \in I(n,r)$). The tensor algebra $T(V)$ is a left $A_q(n)$-comodule.

1.6 The ideal $\langle e_i \otimes e_j - e_j \otimes e_i \mid 1 \leqslant i,j \leqslant n \rangle$ is a subcomodule of $T(E)$. By $S_q(E)$ we denote the factor comodule. The ideal $\langle q(v_i \otimes v_j) - v_j \otimes v_i \mid 1 \leqslant i < j \leqslant n \rangle$ is a subcomodule of $T(V)$. By $S_q(V)$ we denote the factor comodule.

Theorem. [**2**, 2.1.9]

(i) *$S_q(E)$ is a graded algebra $S_q(E) = \oplus_{0 \leqslant r} S_q^r(E)$. The graded part $S_q^r(E)$ is R-free with basis $\{e_{i_1} \dots e_{i_r} \mid i_1 \leqslant i_2 \leqslant \dots \leqslant i_r\}$.*

(ii) *$S_q(V)$ is a graded algebra $S_q(V) = \oplus_{0 \leqslant r} S_q^r(V)$. The graded part $S_q^r(V)$ is R-free with basis $\{v_{i_1} \dots v_{i_r} \mid i_1 \leqslant i_2 \leqslant \dots \leqslant i_r\}$.*

(Here $e_{i_1} \dots e_{i_r}$ (respectively $v_{i_1} \dots v_{i_r}$) denotes the image of $e_{i_1} \otimes \dots \otimes e_{i_r}$ (respectively $v_{i_1} \otimes \dots \otimes v_{i_r}$) in $S_q^r(E)$ (respectively $S_q^r(V)$).

We shall call $S_q^r(E)$ and $S_q^r(V)$ the q-symmetric powers of E and V.

2. Calculations

We want to give a complete description of the subcomodule structure of $S_q^r(E)$ and $S_q^r(V)$. To do so we shall, following Doty [**3**], determine the subcomodule generated by some basis element $e_{i_1} \dots e_{i_r} \in S_q^r(E)$ (respectively $v_{i_1} \dots v_{i_r} \in S_q^r(V)$). From now on we shall assume that R is some field k.

2.1 For $s \in \mathbf{Z}$ let $[s] = (q^s - 1)/(q-1) \in k$. We have that $[s] = 1 + q + \dots q^{s-1}$. For $s, t \in \mathbf{N}$ let $\begin{bmatrix} s \\ t \end{bmatrix} = ([s][s-1]\dots[s-t+1])/([1][2]\dots[t])$ (called the Gaussian polynomial). Now we get:

LEMMA. *Let $s, t \in \mathbf{N}$.*

(i) *If q is a non root of unity then $\begin{bmatrix} s \\ t \end{bmatrix} \neq 0$ for all $t \leqslant s$.*

(ii) *If q is an l^{th} root of unity, then write $s = s_1 l + s_0$, $t = t_1 l + t_0$ where $0 \leqslant t_0, s_0 < l$. Then we have*

$$\begin{bmatrix} s \\ t \end{bmatrix} = \binom{s_1}{t_1} \begin{bmatrix} s_0 \\ t_0 \end{bmatrix}$$

where $\binom{s_1}{t_1}$ is the ordinary binomial coefficient. Furthermore we have that

$$\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} \neq 0 \Leftrightarrow s_0 \geqslant t_0.$$

PROOF:

(i) This is clear.

(ii) This is proved by induction on t using that

$$\begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ t-1 \end{bmatrix} \begin{bmatrix} s-t+1 \\ 1 \end{bmatrix}.$$

□

For more information about Gaussian polynomials see [**1**].

2.2 We shall first consider the right comodule. As $\tau_1(e_j) = \sum_{i=1}^{n} e_i \otimes c_{i,j}$ $(1 \leqslant j \leqslant n)$ we have that $e_1^{a_1} \dots e_n^{a_n} \in S_q^r(E)$ is sent to

(∗)

$$(e_1 \otimes c_{1,1} + \dots + e_n \otimes c_{n,1})^{a_1} (e_1 \otimes c_{1,2} + \dots + e_n \otimes c_{n,2})^{a_2} \dots (e_1 \otimes c_{1,n} + \dots + e_n \otimes c_{n,n})^{a_n}$$

by τ_r. So we shall determine the term in (∗) involving some arbitrary basis element $e_1^{b_1} \dots e_n^{b_n} \in S_q^r(E)$.

LEMMA 2.3. *Let $\sum_{i=1}^{n} b_i = a_j$ ($b_i \in \mathbf{N}$ for all i). The term involving $e_1^{b_1} e_2^{b_2} \dots e_n^{b_n}$ in $(e_1 \otimes c_{1,j} + e_2 \otimes c_{2,j} + \dots + e_n \otimes c_{n,j})^{a_j}$ is equal to*

$$e_1^{b_1} e_2^{b_2} \dots e_n^{b_n} \otimes \begin{bmatrix} a_j \\ b_1 \end{bmatrix} \begin{bmatrix} a_j - b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_j - b_1 - b_2 \\ b_3 \end{bmatrix} \dots \begin{bmatrix} a_j - b_1 - b_2 \dots - b_{n-1} \\ b_n \end{bmatrix} c_{1,j}^{b_1} c_{2,j}^{b_2} \dots c_{n,j}^{b_n}$$

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PROOF: Assume first $n = 2$. So we shall look at $(e_1 \otimes c_{1,j} + e_2 \otimes c_{2,j})^{a_j}$ for $1 \leqslant j \leqslant 2$. If $a_j = 1$ the lemma is true, so by induction we shall assume that the lemma is true for $a_j - 1$. Hence the term involving $e_1^{b_1} e_2^{b_2}$ is equal to

$$\begin{aligned}\left(e_1^{b_1} e_2^{b_2-1} \otimes \begin{bmatrix} a_j - 1 \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 - 1 \\ b_2 - 1 \end{bmatrix} c_{1,j}^{b_1} c_{2,j}^{b_2-1}\right)(e_2 \otimes c_{2,j}) \\ + \left(e_1^{b_1-1} e_2^{b_2} \otimes \begin{bmatrix} a_j - 1 \\ b_1 - 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} c_{1,j}^{b_1-1} c_{2,j}^{b_2}\right)(e_1 \otimes c_{1,j}) \\ = e_1^{b_1} e_2^{b_2} \otimes \left(\begin{bmatrix} a_j - 1 \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 - 1 \\ b_2 - 1 \end{bmatrix} + q^{b_2} \begin{bmatrix} a_j - 1 \\ b_1 - 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix}\right) c_{1,j}^{b_1} c_{2,j}^{b_2} \\ = e_1^{b_1} e_2^{b_2} \otimes \begin{bmatrix} a_j \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} c_{1,j}^{b_1} c_{2,j}^{b_2}\end{aligned}$$

as $b_2 = a_j - b_1$.

Now for the general case. Using the same argument as above we can first move $c_{1,j}$ to the left, that is, the term involving $e_1^{b_1} \dots e_n^{b_n}$ will be of the form

$$e_1^{b_1} \dots e_n^{b_n} \otimes \begin{bmatrix} a_j \\ b_1 \end{bmatrix} c_{1,j}^{b_1} (\dots)$$

where $(\dots)$ does not include $c_{1,j}$. Now using the same argument on $(\dots)$ we can move $c_{2,j}$ to the left in $(\dots)$ and we have to multiply by $\begin{bmatrix} a_j - b_1 \\ b_2 \end{bmatrix}$. By repeating this we get the lemma. ∎

COROLLARY 2.4. *Let $e^{\mathbf{a}} = e_1^{a_1} \dots e_n^{a_n}$, $e^{\mathbf{b}} = e_1^{b_1} \dots e_n^{b_n} \in S_q^r(E)$. The term involving $e^{\mathbf{b}}$ in*

$$(e_1 \otimes c_{1,1} + \dots + e_n \otimes c_{n,1})^{a_1} (e_1 \otimes c_{1,2} + \dots + e_n \otimes c_{n,2})^{a_2} \dots (e_1 \otimes c_{1,n} + \dots + e_n \otimes c_{n,n})^{a_n}$$

is a sum of elements of the form

$$e_1^{b_1} e_2^{b_2} \dots e_n^{b_n} \otimes d\, q^{s}\, c_{1,1}^{d_{1,1}} c_{1,2}^{d_{1,2}} \dots c_{1,n}^{d_{1,n}} c_{2,1}^{d_{2,1}} \dots c_{n,n}^{d_{n,n}}$$

where $\sum_i d_{i,j} = a_j$, $\sum_j d_{i,j} = b_i$, s is some integer and

$$d = \begin{bmatrix} a_1 \\ d_{1,1} \end{bmatrix} \begin{bmatrix} a_1 - d_{1,1} \\ d_{2,1} \end{bmatrix} \dots \begin{bmatrix} a_1 - d_{1,1} - \dots - d_{n-1,1} \\ d_{n,1} \end{bmatrix} \begin{bmatrix} a_2 \\ d_{1,2} \end{bmatrix} \dots \begin{bmatrix} a_n - d_{1,n} - \dots d_{n-1,n} \\ d_{n,n} \end{bmatrix}.$$

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PROOF: From the lemma it follows that we have a sum of elements of the form

$$e_1^{b_1} e_2^{b_2} \dots e_n^{b_n} \otimes d\, c_{1,1}^{d_{1,1}} c_{2,1}^{d_{2,1}} \dots c_{n,1}^{d_{n,1}} c_{1,2}^{d_{1,2}} \dots c_{n,n}^{d_{n,n}}.$$

And as $c_{i,j} c_{s,t} = q c_{s,t} c_{i,j}$ for $i > s$ and $j < t$ the corollary follows. ∎

2.5 Now consider the left comodule case. As $\sigma_1(v_j) = \sum_{i=1}^{n} c_{j,i} \otimes v_i$ $(1 \leqslant j \leqslant n)$ we have that $v_1^{a_1} \dots v_n^{a_n} \in S_q^r(V)$ is sent to

(∗)

$$(c_{1,1} \otimes v_1 + \dots + c_{1,n} \otimes v_n)^{a_1} (c_{2,1} \otimes v_1 + \dots + c_{2,n} \otimes v_n)^{a_2} \dots (c_{n,1} \otimes v_1 + \dots + c_{n,n} \otimes v_n)^{a_n}$$

by σ_r. So we shall describe the term in (∗) involving some arbitrary basis element $v_1^{b_1} \dots v_n^{b_n} \in S_q^r(V)$.

LEMMA 2.6. *Let $\sum_{i=1}^{n} b_i = a_j$ $(b_i \in \mathbb{N}$ for all $i)$. Then the term involving $v_1^{b_1} v_2^{b_2} \dots v_n^{b_n}$ in $(c_{j,1} \otimes v_1 + c_{j,2} \otimes v_2 + \dots + c_{j,n} \otimes v_n)^{a_j}$ is equal to*

$$c_{j,1}^{b_1} c_{j,2}^{b_2} \dots c_{j,n}^{b_n} \otimes \begin{bmatrix} a_j \\ b_1 \end{bmatrix} \begin{bmatrix} a_j - b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} a_j - b_1 - b_2 \\ b_3 \end{bmatrix} \dots \begin{bmatrix} a_j - b_1 - b_2 \dots - b_{n-1} \\ b_n \end{bmatrix} v_1^{b_1} v_2^{b_2} \dots v_n^{b_n}.$$

PROOF: Assume first $n = 2$. If $a_j = 1$ the lemma is true, so by induction we shall assume that the lemma is true for $a_j - 1$. Hence the term involving $v_1^{b_1} v_2^{b_2}$ is equal to

$$\begin{aligned}
&\left(c_{j,1}^{b_1 - 1} c_{j,2}^{b_2} \otimes \begin{bmatrix} a_j - 1 \\ b_1 - 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} v_1^{b_1 - 1} v_2^{b_2}\right)(c_{j,1} \otimes v_1) \\
&\quad + \left(c_{j,1}^{b_1} c_{j,2}^{b_2 - 1} \otimes \begin{bmatrix} a_j - 1 \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 - 1 \\ b_2 - 1 \end{bmatrix} v_1^{b_1} v_2^{b_2 - 1}\right)(c_{j,2} \otimes v_2) \\
&\quad = c_{j,1}^{b_1} c_{j,2}^{b_2} \otimes \left(q^{b_2} \begin{bmatrix} a_j - 1 \\ b_1 - 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_j - 1 \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 - 1 \\ b_2 - 1 \end{bmatrix}\right) v_1^{b_1} v_2^{b_2} \\
&\quad = c_{j,1}^{b_1} c_{j,2}^{b_2} \otimes \begin{bmatrix} a_j \\ b_1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} v_1^{b_1} v_2^{b_2}
\end{aligned}$$

as $b_2 = a_j - b_1$.

Now for the general case the lemma follows by the same argument as in 2.3. ∎

COROLLARY 2.7. *Let $v^{\mathbf{a}} = v_1^{a_1} \dots v_n^{a_n}, v^{\mathbf{b}} = v_1^{b_1} \dots v_n^{b_n} \in S_q^r(V)$. The term involving $v^{\mathbf{b}}$ in*

$$(c_{1,1} \otimes v_1 + \dots + c_{1,n} \otimes v_n)^{a_1} (c_{2,1} \otimes v_1 + \dots + c_{2,n} \otimes v_n)^{a_2} \dots (c_{n,1} \otimes v_1 + \dots + c_{n,n} \otimes v_n)^{a_n}$$

is a sum of elements of the form

$$c_{1,1}^{d_{1,1}} c_{1,2}^{d_{1,2}} \dots c_{1,n}^{d_{1,n}} c_{2,1}^{d_{2,1}} \dots c_{n,n}^{d_{n,n}} \otimes d\, q^s\, v_1^{b_1} v_2^{b_2} \dots v_n^{b_n}$$

where $\sum_j d_{i,j} = a_i$, $\sum_i d_{i,j} = b_j$, *s is some integer and*

$$d = \begin{bmatrix} a_1 \\ d_{1,1} \end{bmatrix} \begin{bmatrix} a_1 - d_{1,1} \\ d_{1,2} \end{bmatrix} \dots \begin{bmatrix} a_1 - d_{1,1} - \dots - d_{1,n-1} \\ d_{1,n} \end{bmatrix} \begin{bmatrix} a_2 \\ d_{2,1} \end{bmatrix} \dots \begin{bmatrix} a_n - d_{n,1} - \dots d_{n,n-1} \\ d_{n,n} \end{bmatrix}.$$

PROOF: From the lemma it follows that we have a sum of elements of the form

$$c_{1,1}^{d_{1,1}} c_{1,2}^{d_{1,2}} \dots c_{1,n}^{d_{1,n}} c_{2,1}^{d_{2,1}} \dots c_{n,n}^{d_{n,n}} \otimes d\, v_1^{d_{1,1}} v_2^{d_{1,2}} \dots v_n^{d_{1,n}} v_1^{d_{2,1}} \dots v_n^{d_{n,n}}.$$

And as $v_i v_j = q v_j v_i$ for $1 \leqslant j < i \leqslant n$ the corollary follows. ∎

2.8 Let $e^{\mathbf{a}}, e^{\mathbf{b}}$ (respectively $v^{\mathbf{a}}, v^{\mathbf{b}}$) be basis elements of $S_q^r(E)$ (respectively $S_q^r(V)$). From 2.4 and 2.7 it follows that

$$e^{\mathbf{b}} \in \langle e^{\mathbf{a}} \rangle \Leftrightarrow v^{\mathbf{b}} \in \langle v^{\mathbf{a}} \rangle$$

where $\langle e^{\mathbf{a}} \rangle$ (respectively $\langle v^{\mathbf{a}} \rangle$) is the subcomodule of $S_q^r(E)$ (respectively $S_q^r(V)$) generated by $e^{\mathbf{a}}$ (respectively $v^{\mathbf{a}}$). Hence the subcomodule structure of $S_q^r(E)$ and $S_q^r(V)$ is the same. From now on we shall only consider the right comodule case.

3. The main result

In this section we shall give a complete description of the subcomodule structure of the q-symmetric powers. In 2.4 we have described the subcomodule of $S_q^r(E)$ generated by some basis element. Now we want to know when the coefficient d in 2.4 is non zero. We shall do so by introducing the carry pattern. If the quantum parameter $q \in k$ is a non root of unity, then there is no work to be done (see 3.1). If, however, q is a root of unity, then we shall consider two cases: when k is of characteristic 0 (see 3.2—3.6) and when k is of prime characteristic (see 3.7—3.10).

THEOREM 3.1. *Let q be a non root of unity. Then $S_q^r(E)$ is irreducible for each $r \geqslant 0$.*

PROOF: By 2.1 and 2.4 it follows that the subcomodule generated by some arbitrary basis element $e^{\mathbf{a}} \in S_q^r(E)$ is equal to $S_q^r(E)$. ∎

From now on we shall assume that q is an l'th root of unity ($l \geqslant 1$).

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3.2 Assume first that k is of characteristic 0. Let $e^{\mathbf{a}} = e_1^{a_1} e_2^{a_2} \dots e_n^{a_n} \in S_q^r(E)$. For each $1 \leqslant i \leqslant n$ write $a_i = a_i^1 l + a_i^0$ where $0 \leqslant a_i^0 < l$. Futhermore let $r = r_1 l + r_0$ where $0 \leqslant r_0 < l$.

DEFINITION: $C(e^{\mathbf{a}})$ is defined by

$$\sum_{i=1}^{n} a_i^0 = C(e^{\mathbf{a}})l + r_0$$

Following [**3**], we call $C(e^{\mathbf{a}})$ the carry pattern of $e^{\mathbf{a}}$.

PROPOSITION 3.3. *Let $e^{\mathbf{a}} = e_1^{a_1} e_2^{a_2} \dots e_n^{a_n}$, $e^{\mathbf{b}} = e_1^{b_1} e_2^{b_2} \dots e_n^{b_n} \in S_q^r(E)$. Then $C(e^{\mathbf{a}}) \geqslant C(e^{\mathbf{b}})$ if and only if there exists $d_{i,j} \in \mathbf{N}$ $(1 \leqslant i,j \leqslant n)$ such that $\sum_{i=1}^{n} d_{i,j} = a_j$, $\sum_{j=1}^{n} d_{i,j} = b_i$ and*

$$\begin{bmatrix} a_1 \\ d_{1,1} \end{bmatrix} \begin{bmatrix} a_1 - d_{1,1} \\ d_{2,1} \end{bmatrix} \dots \begin{bmatrix} a_1 - d_{1,1} - \dots - d_{n-1,1} \\ d_{n,1} \end{bmatrix} \begin{bmatrix} a_2 \\ d_{1,2} \end{bmatrix} \dots \begin{bmatrix} a_n - d_{1,n} - \dots d_{n-1,n} \\ d_{n,n} \end{bmatrix} \neq 0.$$

PROOF: Assume $C(e^{\mathbf{a}}) \geqslant C(e^{\mathbf{b}})$. This means that $\sum_{i=1}^{n} a_i^0 \geqslant \sum_{i=1}^{n} b_i^0$. For $1 \leqslant i,j \leqslant n$ we shall define $d_{i,j}$. If $b_i \leqslant a_i$ set $d_{i,i} = b_i^1 l + b_i^0$ if $b_i^0 \leqslant a_i^0$ and set $d_{i,i} = b_i^1 l + a_i^0$ otherwise. If $b_i > a_i$ set $d_{i,i} = a_i^1 l + a_i^0$ if $b_i^0 > a_i^0$ and set $d_{i,i} = a_i^1 l + b_i^0$ otherwise. As $\sum_{i=1}^{n} a_i^0 \geqslant \sum_{i=1}^{n} b_i^0$ we see by 2.1 (ii) that we can choose $d_{i,j}$ $(i \neq j)$ such that we get the statement in the proposition.

Assume now that there exist $d_{i,j} \in \mathbf{N}$ $(1 \leqslant i,j \leqslant n)$ as required in the proposition. For each $d_{i,j}$ we write $d_{i,j} = d_{i,j}^1 l + d_{i,j}^0$ where $0 \leqslant d_{i,j}^0 < l$. For each $1 \leqslant j \leqslant n$

$$\begin{bmatrix} a_j \\ d_{1,j} \end{bmatrix} \begin{bmatrix} a_j - d_{1,j} \\ d_{2,j} \end{bmatrix} \dots \begin{bmatrix} a_j - d_{1,j} - \dots - d_{n-1,j} \\ d_{n,j} \end{bmatrix} \neq 0.$$

Hence by 2.1 (ii) we have $a_j^0 \geqslant d_{1,j}^0 + \dots + d_{n,j}^0$. But as $\sum_{i=1}^{n} d_{i,j} = a_j$ we must then have that

$$a_j^0 = d_{1,j}^0 + \dots + d_{n,j}^0$$

for each $1 \leqslant j \leqslant n$. Now as $\sum_{j=1}^{n} d_{i,j} = b_i$ we have (for $1 \leqslant i \leqslant n$) that $\sum_{j=1}^{n} d_{i,j}^0 = n_i l + b_i^0$ where $n_i \geqslant 0$. Hence

$$\sum_{j=1}^{n} a_j^0 = \sum_{j=1}^{n} \sum_{i=1}^{n} d_{i,j}^0 = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{i,j}^0 = \sum_{i=1}^{n} (n_i l + b_i^0) = \sum_{i=1}^{n} n_i l + \sum_{i=1}^{n} b_i^0,$$

that is,

$$C(e^{\mathbf{a}}) \geqslant C(e^{\mathbf{b}}).$$

$\square$

3.4 From 2.4 and 3.3 we immediately get

COROLLARY. *Let $e^{\mathbf{a}}, e^{\mathbf{b}}$ be basis elements in $S_q^r(E)$. Then*

$$e^{\mathbf{b}} \in \langle e^{\mathbf{a}} \rangle \Leftrightarrow C(e^{\mathbf{b}}) \leqslant C(e^{\mathbf{a}}).$$

3.5 Let $C(r)$ denote the set of numbers which occur as the carry pattern for some $e^{\mathbf{a}} \in S_q^r(E)$. For $c \in C(r)$ let $L(c)$ denote the span of all $e^{\mathbf{a}} \in S_q^r(E)$, where $C(e^{\mathbf{a}}) = c$. Let $T(c) = \sum_{c' \leqslant c} L(c')$. The complete description of the subcomodule structure of $S_q^r(E)$ is then an easy consequence of 3.4:

THEOREM. *We have a 1-1 correspondence*

$$c \leftrightarrow T(c)$$

between the set of carry patterns $C(r)$ and the set of subcomodules of $S_q^r(E)$.

In particular, the irreducible composition factors of $S_q^r(E)$ are in one-to-one correspondence with $L(c)$ for $c \in C(r)$. It follows that all composition factors must have multiplicity one. (In fact we know this already by [**2**], as all weight multiplicities are 0 or 1.)

3.6 It is easy to determine the carry patterns which occur: Let $r = r_1 l + r_0$ where $0 \leqslant r_0 < l$. Let $M = r_1$ if $r < (n-1)(l-1)$ and otherwise let

$$M = \begin{cases} n-1 & \text{if } r_0 + n - 1 < l \\ n-2 & \text{otherwise}. \end{cases}$$

Then we have that $C(r) = \{0, 1, \ldots, M\}$. By 3.5 the subcomodule structure of $S_q^r(E)$ can then be described by

$$\begin{array}{c} S_q^r(E) = T(M) \\ | \\ T(M-1) \\ | \\ \vdots \\ \vdots \\ | \\ T(0) \end{array}$$

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3.7 Let now k be of prime characteristic p. Let $e^{\mathbf{a}} = e^{a_1} \dots e^{a_n} \in S_q^r(E)$, and for each $1 \leqslant i \leqslant n$ write $a_i = a_i^1 l + a_i^0$ where $0 \leqslant a_i^0 < l$. Furthermore let $a_i^1 = \sum_j c_i^{1,j} p^j$ where $0 \leqslant a_i^{1,j} < p$ for all i, j. Let $r = r_1 l + r_0$, $0 \leqslant r_0 < l$ and $r_1 = \sum_j r_1^j p^j$, $0 \leqslant r_1^j < p$ for all j.

Let $C_0(e^{\mathbf{a}})$ be defined by

$$\sum_{i=1}^{n} a_i^0 = C_0(e^{\mathbf{a}}) l + r_0,$$

and for $t \geqslant 1$ define $C_t(e^{\mathbf{a}})$ by

$$C_{t-1}(e^{\mathbf{a}}) + \sum_{i=1}^{n} a_i^{1,t-1} = C_t(e^{\mathbf{a}}) p + r_1^{t-1}.$$

DEFINITION: The carry pattern of $e^{\mathbf{a}}$ is defined to be

$$C(e^{\mathbf{a}}) = (C_0(e^{\mathbf{a}}), C_1(e^{\mathbf{a}}), \dots, C_m(e^{\mathbf{a}}))$$

where m is the biggest j such that $r_1^j > 0$.

The carry pattern is defined as a mixture of the carry pattern in characteristic 0, and the carry pattern defined in [**3**].

EXAMPLE. Let $n = 3$, $r = 99$, $p = 3$ and $l = 5$. Let $e^{\mathbf{a}} = e_1^{29} e_2^{37} e_3^{33}$ and $e^{\mathbf{b}} = e_1^{26} e_2^{52} e_3^{21}$. Then we have that

$$C(e^{\mathbf{a}}) = (1,1,2) \text{ and } C(e^{\mathbf{b}}) = (0,1,1).$$

3.8 Let $C(r)$ be the set of carry patterns which occur as carry pattern for some $e^{\mathbf{a}} \in S_q^r(E)$. We let $C(r)$ be partially ordered by $c \leqslant c'$ if and only if $c_i \leqslant c_i'$ for all i (where $c = (c_0, c_1, \dots c_m)$ and $c' = (c_o', c_1', \dots, c_m')$).

PROPOSITION. *Let $e^{\mathbf{a}}, e^{\mathbf{b}}$ be basis elements of $S_q^r(E)$. Then*

$$e^{\mathbf{b}} \in \langle e^{\mathbf{a}} \rangle \Leftrightarrow C(e^{\mathbf{b}}) \leqslant C(e^{\mathbf{a}}).$$

PROOF: The proposition can be proved as in 3.3. If $s, t \in \mathbf{N}$ then by 2.1

$$\begin{bmatrix} s \\ t \end{bmatrix} = \binom{s_1}{t_1} \begin{bmatrix} s_0 \\ t_0 \end{bmatrix}$$

where $s = s_1 l + s_0$, $t = t_1 l + t_0$, $0 \leqslant s_0, t_0 < l$.

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$C_0(e^{\mathbf{a}})$ plays exactly the same role as the carry pattern does in the characteristic 0 case, and can be handled as in the proof of 3.3. But we have to be a bit more careful with the binomial coefficient. We have that

$$\binom{s_1}{t_1} \equiv \binom{s_1^m}{t_1^m}\binom{s_1^{m-1}}{t_1^{m-1}}\cdots\binom{s_1^0}{t_1^0} \mod p$$

where $s_1 = \sum_j s_1^j p^j$, $t_1 = \sum_j t_1^j p^j$, $0 \leqslant s_1^j, t_1^j < p$ for all j.

The proof of 3.3 uses the fact that $\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} \neq 0 \Leftrightarrow s_0 \geqslant t_0$. As $\binom{s_1^j}{t_1^j} \neq 0 \Leftrightarrow t_1^j \leqslant s_1^j$ the present proposition can be proved using the proof of 3.3 on each level of the p-adic expansion. ∎

3.9 For $c \in C(r)$ let $L(c)$ denote the span of all $e^{\mathbf{a}} \in S_q^r(E)$, where $C(e^{\mathbf{a}}) = c$. For a subset $B \subset C(r)$ let $T(B) = \sum_{c \in B} L(c)$. We say that the subset B is order closed if it contains the predecessors of all its elements under the given order relation. As a consequence of 3.8 we now have a complete description of the subcomodule structure of $S_q^r(E)$.

Theorem. *We have a 1-1 correspondence*

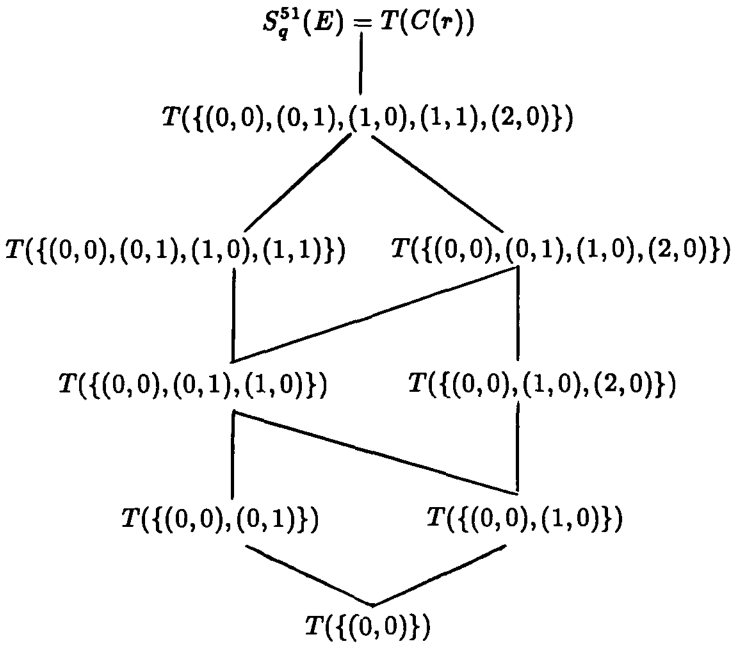
$$B \leftrightarrow T(B)$$

between the set of order closed subsets of $C(r)$ and the set of subcomodules of $S_q^r(E)$.

In particular, the irreducible composition factors of $S_q^r(E)$ are in one-to-one correspondence with $L(c)$ for $c \in C(r)$, and all composition factors have multiplicity one.

3.10 We can determine $C(r)$ and hence the order closed subsets of $C(r)$ by combining 3.6 with Lemma 3 in [**3**].

Example. Let $n = 3$, $l = 5$, $p = 7$ and $r = 51 = (1\,p + 3)l + 1$.
Then $C(r) = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)\}$. Hence by 3.9 the subcomodule structure of $S_q^{51}(E)$ can be described by



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