THE SUBSET OF PIECEWISE-LINEAR MAPPINGS IS DENSE IN THE SPACE OF *K*-QUASICONFORMAL MAPPINGS OF THE PLANE

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1. Introduction

For each index n from the set $\mathbf{N}$ of natural numbers, let $\mathcal{N}_n$ denote the *regular net of equilateral triangles* in the complex plane $\mathbf{C}$, whose vertice set consists of the points $[p+(\frac{1}{2}+i\sqrt{3}/2)q]2^{-n}$ with integers p and q.

A mapping φ: $\mathbf{C}\to\mathbf{C}$ is called *linear*, if there are constants a, b, $c\in\mathbf{C}$ such that $\varphi(z)=az+bz^*+c$; the superscript star denotes complex conjugation. A mapping φ: $\mathbf{C}\to\mathbf{C}$ is said to be *piecewise-linear* with respect to the net $\mathcal{N}_n$, if its restrictions to the triangles of $\mathcal{N}_n$ are linear mappings. We define the *piecewise-linearized mapping* $\varphi^{\langle n\rangle}$: $\mathbf{C}\to\mathbf{C}$ for a mapping φ: $\mathbf{C}\to\mathbf{C}$ with respect to the net $\mathcal{N}_n$ as follows: $\varphi^{\langle n\rangle}$ is piecewise-linear with respect to $\mathcal{N}_n$, and it coincides with φ on the vertice set of $\mathcal{N}_n$.

The set of continuous mappings φ: $\mathbf{C}\to\mathbf{C}$ will be considered as a *topological space with the compact-open topology*; this induces *convergence* in the sense of uniform convergence on compact subsets. *Approximation* means convergence to a given mapping. Each continuous mapping φ: $\mathbf{C}\to\mathbf{C}$ is approximated by its piecewise-linearized mappings $\varphi^{\langle n\rangle}$.

In the subspace of quasiconformal mappings of the plane, there is the problem: can each φ be approximated by φ_n which are piecewise-linear with respect to $\mathcal{N}_n$?

METHOD OF BEURLING AND AHLFORS. *Let a quasiconformal mapping φ: $\mathbf{C}\to\mathbf{C}$ have maximal dilatation* $\mathrm{K}(\varphi)<\sqrt{3}$. *Then, φ is approximated by the piecewise-linearized mappings $\varphi^{\langle n\rangle}$; $\varphi^{\langle n\rangle}$ is quasiconformal* (*Ahlfors* [2], 768; [3], 298); *$\varphi^{\langle n\rangle}$ has maximal dilatation* $\mathrm{K}(\varphi^{\langle n\rangle})\leqslant\xi[\mathrm{K}(\varphi)]$, *where ξ is a certain function involving elliptic integrals* (*Agard* [1], 739); *for each index n, there are some φ such that* $\mathrm{K}(\varphi^{\langle n\rangle})=\xi[\mathrm{K}(\varphi)]$ *holds* (*Agard* [1], 739); *moreover, there are some φ such that* $\mathrm{K}(\varphi^{\langle n\rangle})=\xi[\mathrm{K}(\varphi)]$ *holds for all indices n* ([4], 49).

AGARD'S METHOD. *Let a quasiconformal mapping* φ: $\mathbf{C}\to\mathbf{C}$ *be composed as* $\varphi=\tilde{\varphi}_p\circ\ldots\circ\tilde{\varphi}_1$ *by quasiconformal mappings* $\tilde{\varphi}_q$: $\mathbf{C}\to\mathbf{C}$ *with maximal dilatations* $\mathrm{K}(\tilde{\varphi}_q)=[\mathrm{K}(\varphi)]^{1/p}<\sqrt{3}$. *Then* φ *is approximated by the mappings* $\varphi_n:=\tilde{\varphi}_p^{\langle n\rangle}\circ\ldots\circ\tilde{\varphi}_1^{\langle n\rangle}$; φ_n *is piecewise-linear, though not with respect to* $\mathcal{N}_n$; φ_n *is quasiconformal; for* $\varepsilon>0$ *and* p *sufficiently large,* φ_n *has maximal dilatation* $\mathrm{K}(\varphi_n)\leqslant[\mathrm{K}(\varphi)]^{3.243\ldots}+\varepsilon$ (*Agard* [1], 740); *further, there are some* φ *such that*

$$\mathrm{K}(\varphi_n)>[\mathrm{K}(\varphi)]^{3.243\ldots}$$

holds for all indices n and all admissible p ([4], 51).

THEOREM (abridged version). *Let a quasiconformal mapping* φ: $\mathbf{C}\to\mathbf{C}$ *have maximal dilatation* $\mathrm{K}(\varphi)>1$. *Then,* φ *can be approximated ("well" in the sense of Lehto–Virtanen* [5], 194) *by quasiconformal mappings* φ_n: $\mathbf{C}\to\mathbf{C}$ *with maximal dilatations* $\mathrm{K}(\varphi_n)<\mathrm{K}(\varphi)$, *which are piecewise-linear with respect to certain prescribed nets* $\mathcal{N}_n$ *of triangles.*

COROLLARY. *The subset of piecewise-linear mappings is dense in the space of K-quasiconformal mappings of the plane.*

With regard to maximal dilatations, Agard's result is weaker than our theorem. This is caused by the use of a sufficient condition, which is an unnecessary limitation: to approximate by piecewise linearizations of the given mapping or of mappings composing it. Now, consider the following necessary, but insufficient condition.

PROPOSITION. *Let a quasiconformal mapping* φ: $\mathbf{C}\to\mathbf{C}$ *be approximated by quasiconformal mappings* φ_n: $\mathbf{C}\to\mathbf{C}$, *which are piecewise-linear with respect to arbitrary nets* $\mathcal{N}_n$. *Then,* φ *can be approximated by quasiconformal mappings* ψ_n: $\mathbf{C}\to\mathbf{C}$ *such that* $\varphi_n=\psi_n^{\langle n\rangle}$.

Taking the φ_n as ψ_n proves it. To apply this proposition, we take certain sufficiently smooth ψ_n with maximal dilatations $\mathrm{K}(\psi_n)<\mathrm{K}(\varphi)$.

2. Linear mappings of a triangle

A *triangle* in $\mathbf{C}$ is given by its vertices, three non-collinear points z_1, z_2, $z_3\in\mathbf{C}$. We represent the vertices and thus the triangle by the *triple* $Z:=(z_1, z_2, z_3)$ in the Cartesian product space $\mathbf{C}^3$. The point set of the triangle is $\mathbf{T}:=\operatorname{conv} Z$.

Let a linear mapping φ: $\mathbf{C}\to\mathbf{C}$ be given by $\varphi(z):=az+bz^*+c$ with constants $a, b, c\in\mathbf{C}$. Then, $\varphi^3(z', z'', z'''):=(\varphi(z'), \varphi(z''), \varphi(z'''))$ defines the *Cartesian product mapping* φ^3: $\mathbf{C}^3\to\mathbf{C}^3$. Let us introduce the notation $E:=(1, 1, 1)$. φ is uniquely determined by its restriction to the vertice set of a triangle $\mathbf{T}=\operatorname{conv} Z$. From

$$W:=\varphi^3(Z)=aZ+bZ^*+cE,$$

Cramer's rule allows us to compute a, b, c for given triples Z (non-collinear) and W.

LEMMA 1. *Let* $Z:=(z_1, z_2, z_3)$ *denote the triple of vertices of a triangle in* $\mathbf{C}$, *and let* $W:=(w_1, w_2, w_3)$, $\tilde{W}:=(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ *be triples in* $\mathbf{C}^3$. *Then, there are uniquely determined linear mappings* $\varphi, \tilde{\varphi}$: $\mathbf{C}\to\mathbf{C}$ *such that* $\varphi^3(Z)=W$ *and* $\tilde{\varphi}^3(Z)=\tilde{W}$. φ *has the representation*

$$\varphi(z)=\frac{z\cdot\det(W, Z^*, E)+z^*\cdot\det(Z, W, E)+\det(Z, Z^*, W)}{\det(Z, Z^*, E)}.$$

φ *is quasiconformal if and only if* $\det(W, Z^*, E)>\det(Z, W, E)$; *then the complex dilatation is*

$$\varkappa=\frac{\det(Z, W, E)}{\det(W, Z^*, E)}.$$

If φ *and* $\tilde{\varphi}$ *are quasiconformal, then the difference of their complex dilatations is*

$$\tilde{\varkappa}-\varkappa=\frac{\det(W, \tilde{W}, E)\cdot\det(Z, Z^*, E)}{\det(W, Z^*, E)\cdot\det(\tilde{W}, Z^*, E)}.$$

Proof. It is trivial to check a, b, c and $\varkappa=b/a$. In order to show the last result, we work with known formulas for inner "$\cdot$" and outer "$\times$" multiplication of 3-vectors:

$$\begin{aligned}(\tilde{\varkappa}-\varkappa)&\cdot\det(W, Z^*, E)\cdot\det(\tilde{W}, Z^*, E)\\ &=\det(Z, \tilde{W}, E)\cdot\det(W, Z^*, E)-\det(Z, W, E)\cdot\det(\tilde{W}, Z^*, E)\\ &=[\det(Z, \tilde{W}, E)\cdot W-\det(Z, W, E)\cdot\tilde{W}]\cdot(Z^*\times E)\\ &=[(E\times Z)\times(W\times\tilde{W})]\cdot(Z^*\times E)\\ &=[\det(W, \tilde{W}, E)\cdot Z-\det(W, \tilde{W}, Z)\cdot E]\cdot(Z^*\times E)\\ &=\det(W, \tilde{W}, E)\cdot\det(Z, Z^*, E).\end{aligned}$$

3. Linearization of a nearly linear mapping

Let a triangle $\mathbf{T}=\operatorname{conv} Z$ in $\mathbf{C}$ be given, and consider a continuous mapping φ: $\mathbf{T}\to\mathbf{C}$. The *linearized mapping* $\hat{\varphi}$: $\mathbf{C}\to\mathbf{C}$ for φ with respect to $\mathbf{T}$ is defined as follows: $\hat{\varphi}$ is the uniquely determined linear mapping with $\hat{\varphi}^3(Z)=\varphi^3(Z)$.

If the mapping φ: $\mathbf{T}\to\mathbf{C}$ is quasiconformal, then its complex dilatation $\varkappa(z)$ equals

$$\frac{\partial\varphi}{\partial z^*}(z)\Big/\frac{\partial\varphi}{\partial z}(z)$$

almost everywhere in $\mathbf{T}$. We want to guarantee that $\hat{\varphi}$ is quasiconformal with complex dilatation $\hat{\varkappa}$ near $\varkappa(z)$. Since a condition on $\varkappa(z)$ alone would allow a subsequent conformal distortion of $\varphi(\mathbf{T})$ and thus of $\varphi^3(Z)$, let us try a limitation of both $\frac{\partial\varphi}{\partial z^*}(z)$ and $\frac{\partial\varphi}{\partial z}(z)$; this leads to the following lemma.

LEMMA 2. *Let a triangle* $\mathbf{T}$ *in* $\mathbf{C}$ *have angles* $\alpha_1, \alpha_2, \alpha_3$ *and put* $\beta := \max\{\alpha_1, \alpha_2, \alpha_3, \pi/2\}$. *Consider a continuously-differentiable quasiconformal mapping* $\psi: \mathbf{T} \to \mathbf{C}$, *and the linearized mapping* $\hat{\psi}: \mathbf{C} \to \mathbf{C}$ *for* ψ *with respect to* $\mathbf{T}$. *Let the differential of* ψ *satisfy an inequality*

$$|d\psi(z) - (adz + bdz^*)| \leq \varepsilon |dz|$$

with constants $a, b \in \mathbf{C}$ *and a positive constant* $\varepsilon < \frac{1}{2}(|a| - |b|)\sin\beta$. *Then* $\hat{\psi}$ *is quasiconformal, and its complex dilatation* $\hat{\varkappa}$ *satisfies the inequalities*

$$\left|\hat{\varkappa} - \frac{b}{a}\right| \leq \frac{|b/a| + 1}{|a/\varepsilon| \sin\beta - 1}, \quad |\hat{\varkappa}| \leq \frac{|b| \sin\beta + \varepsilon}{|a| \sin\beta - \varepsilon}.$$

Proof. We are going to assume the following special conditions. In case that $\beta > \pi/2$, the maximal angle is $\alpha_1 = \beta$, the corresponding vertex is $z_1 = 0$; further, $\psi(0) = 0$. In case that $\beta = \pi/2$, the orthocenter of $\mathbf{T}$ lies at the origin 0; further, $\psi(0) = 0$. This can be achieved by a renumbering of the vertices of $\mathbf{T}$, a translatory mapping of $\mathbf{T}$, and composition of ψ between two translatory mappings. Clearly, the values of the differentials and complex dilatations in Lemma 2 remain unchanged.

Now, the geometrical properties of $\mathbf{T}$ imply: $|(E \times Z) \cdot Z^*| = |E \times Z| \cdot |Z| \sin\beta$; absolute values are taken componentwise: $|Z| = |(z_1, z_2, z_3)| := (|z_1|, |z_2|, |z_3|)$.

Let us define linear mappings $\varphi, \tilde{\varphi}: \mathbf{C} \to \mathbf{C}$ by $\varphi(z) := az + bz^*$ and $\tilde{\varphi} := \hat{\psi}$. Integrating the differentials $d\psi(z)$ and $d\varphi(z)$ along the segments from 0 to the vertices of $\mathbf{T}$, we get

$$\tilde{\varphi}^3(Z) = \hat{\psi}^3(Z) = \psi^3(Z) = \varphi^3(Z) + D;$$

here, the triple D satisfies $|D| \leq \varepsilon |Z|$ componentwise. Thus, we can estimate as follows: $|(E \times Z) \cdot D| \leq |E \times Z| \cdot |D| \leq \varepsilon |E \times Z| \cdot |Z|$. Next, we apply Lemma 1; the above formula and estimate will be used some lines further.

$$\begin{aligned}
\hat{\varkappa} - \varkappa &= \frac{\det(aZ + bZ^*, aZ + bZ^* + D, E) \cdot \det(Z, Z^*, E)}{\det(aZ + bZ^*, Z^*, E) \cdot \det(aZ + bZ^* + D, Z^*, E)} \\
&= \frac{[a \cdot \det(Z, D, E) + b \cdot \det(Z^*, D, E)] \cdot \det(Z, Z^*, E)}{[a \cdot \det(Z, Z^*, E)]\,[a \cdot \det(Z, Z^*, E) + \det(D, Z^*, E)]} \\
&= \frac{(E \times Z) \cdot D + (b/a)\,(E \times Z^*) \cdot D}{a(E \times Z) \cdot Z^* + (Z^* \times E) \cdot D},
\end{aligned}$$

hence $$|\hat{\varkappa} - \varkappa| \leq \frac{\varepsilon |E \times Z| \cdot |Z| + \varepsilon |b/a| \cdot |E \times Z| \cdot |Z|}{|a| \cdot |E \times Z| \cdot |Z| \sin\beta - \varepsilon |E \times Z| \cdot |Z|} = \frac{1 + |b/a|}{|a/\varepsilon| \sin\beta - 1};$$

and finally $$|\hat{\varkappa}| \leq |\hat{\varkappa} - \varkappa| + |\varkappa| \leq \frac{\varepsilon + \varepsilon |b/a|}{|a| \sin\beta - \varepsilon} + \left|\frac{b}{a}\right| = \frac{|b| \sin\beta + \varepsilon}{|a| \sin\beta - \varepsilon}.$$

4. Nets of non-degenerating triangles

For each index $n \in \mathbf{N}$, let us consider a locally-finite covering $\mathcal{N}_n$ of $\mathbf{C}$ by triangles, such that any two different triangles of $\mathcal{N}_n$ intersect in a common side, in a common vertex, or not at all. The $\mathcal{N}_n$ will be called *general nets of triangles.*

The triangles of a general net can be very different in size; we want to describe the diameters of the triangles with the help of a continuous function δ_n: $\mathbf{C} \to \mathbf{R}$.

For any triangle $\mathbf{T} \in \mathcal{N}_n$ and any point $z \in \mathbf{C}$, put

$$\mathbf{T}_z := (\mathbf{T} - z) \cap (z - \mathbf{T}) := \{w \in \mathbf{C}\colon\ z \pm w \in \mathbf{T}\}.$$

In fact, $\mathbf{T}_z$ is the largest subset of $\mathbf{T} - z$ which is symmetrical with respect to the origin 0. $\mathbf{T}_z$ is void for $z \notin \mathbf{T} \in \mathcal{N}_n$. If $z \in \mathbf{T} \in \mathcal{N}_n$, then $\mathbf{T}_z$ is a non-void convex set which depends on $\mathcal{N}_n$, depends continuously on z, but does not depend on the particular $\mathbf{T}$ chosen. We introduce the *symmetrical diameter* $\delta_n(z)$ for the net $\mathcal{N}_n$ with respect to the point z as

$$\delta_n(z) := \max\{\operatorname{diam}[(\mathbf{T} - z) \cap (z - \mathbf{T})]\colon \mathbf{T} \in \mathcal{N}_n\}.$$

Let the triangle $\mathbf{T} = \operatorname{conv}\{z_1, z_2, z_3\}$ of the net $\mathcal{N}_n$ have angles α_1, α_2, α_3 and put $\beta := \max\{\alpha_1, \alpha_2, \alpha_3, \pi/2\}$. Elementary calculations lead to an upper bound and to a condition of Lipschitz type:

$$\max\{\delta_n(z)\colon z \in \mathbf{T}\} = \operatorname{diam} \mathbf{T} = \max\{|z_3 - z_2|,\ |z_1 - z_3|,\ |z_2 - z_1|\},$$

$$|\delta_n(z'') - \delta_n(z')| \leqslant |z'' - z'| \cdot 2 \tan\frac{\beta}{2} \quad \text{for } z', z'' \in \mathbf{T}.$$

Next, let us consider the shape of the triangles. We need good estimates for $\hat{\varkappa}$; yet, Lemma 2 is asymptotically sharp for $\beta \uparrow \pi$, because there are examples with

$$|\hat{\varkappa}| = \frac{|b| \sin\beta + \varepsilon \sin(\beta/2)}{|a| \sin\beta - \varepsilon \sin(\beta/2)}.$$

We can allow small angles, but we must prevent the triangles from being too obtuse. Therefore, we speak of a *non-degenerating sequence of nets* $\mathcal{N}_n$, if all angles of all triangles of all nets are bounded away from π.

This makes the functions δ_n: $\mathbf{C} \to \mathbf{R}$ uniformly continuous, even equicontinuous. Hence, a well-known theorem ([5], 74, Hilfssatz 5.1) implies the next lemma.

LEMMA 3. *Consider a non-degenerating sequence of nets $\mathcal{N}_n$ of triangles in* $\mathbf{C}$. *Let the symmetrical diameters $\delta_n(z)$ converge to* 0 *pointwise on a dense subset of* $\mathbf{C}$. *Then the δ_n converge to* 0 *in the compact-open topology.*

5. Approximation by smooth mappings with smaller dilatations

We are going to approximate a quasiconformal mapping φ: $\mathbf{C}\to\mathbf{C}$ by auxiliary mappings $\tilde{\psi}_n$: $\mathbf{C}\to\mathbf{C}$. We require certain smoothness properties which are to be used later.

LEMMA 4. *Let a quasiconformal mapping φ: $\mathbf{C}\to\mathbf{C}$ have complex dilatation $\varkappa(z)$. Then, there exists a sequence of continuously-differentiable quasiconformal mappings $\tilde{\psi}_n$: $\mathbf{C}\to\mathbf{C}$, converging to φ in the compact-open topology, with complex dilatations $\tilde{\varkappa}_n(z)$ converging to $\varkappa(z)$ pointwise almost everywhere in $\mathbf{C}$, and satisfying the following conditions. Each $\tilde{\psi}_n$ is conformal outside some compact set; $\tilde{\psi}_n(0)=\varphi(0)$ and $\tilde{\psi}_n(1)=\varphi(1)$ holds;*

$$\frac{K(\tilde{\psi}_n)-1}{K(\tilde{\psi}_n)+1}\leqslant\left(1-\frac{1}{n}\right)\frac{K(\varphi)-1}{K(\varphi)+1};$$

for each index n and for each $\varepsilon>0$ there are constants ϱ_n, σ_n with $0<\varrho_n, \sigma_n<\infty$ such that if two points $z', z''\in\mathbf{C}$ fulfil either $|z''-z'|\leqslant\varrho_n$ or $|z'|, |z''|\geqslant\sigma_n$, then $d\tilde{\psi}_n$ satisfies

$$|d\tilde{\psi}_n(z'')-d\tilde{\psi}_n(z')|\leqslant\varepsilon|dz|.$$

Proof. We use the standard method of defining a quasiconformal mapping implicitly by its complex dilatation. Let us start with a sequence of real-analytic functions $\varkappa_n$: $\mathbf{C}\to\mathbf{C}$, with $\varkappa_n(z)$ converging to $\varkappa(z)$ pointwise almost everywhere in $\mathbf{C}$, and such that

$$\sup\{|\varkappa_n(z)|: |z|\leqslant n+\pi/2\}\leqslant[K(\varphi)-1]/[K(\varphi)+1].$$

Further, let us put

$$\chi_n(\tau):=\begin{cases}1-\dfrac{1}{n} & \text{for } \tau\leqslant n\\ \left(1-\dfrac{1}{n}\right)[1-\sin^2(\tau-n)] & \text{for } n<\tau<n+\dfrac{\pi}{2}\\ 0 & \text{for } \tau\geqslant n+\dfrac{\pi}{2}\end{cases}$$

This defines a continuously differentiable function χ_n: $\mathbf{R}\to\mathbf{R}$. By $\tilde{\varkappa}_n(z):=\varkappa_n(z)\chi_n(|z|)$, we get a continuously differentiable function $\tilde{\varkappa}_n$: $\mathbf{C}\to\mathbf{C}$ with the required properties.

The generalized Riemann mapping theorem ([5], 204) guarantees a corresponding quasiconformal mapping $\tilde{\psi}_n$: $\mathbf{C}\to\mathbf{C}$ with $\tilde{\psi}_n(0)=\varphi(0)$ and $\tilde{\psi}_n(1)=\varphi(1)$. The condition for the differential $d\tilde{\psi}_n$ follows from the smoothness of $\tilde{\varkappa}_n$, because $\tilde{\psi}_n$ is regular ([5], 244) in $\mathbf{C}$ and conformal outside a compact subset of $\mathbf{C}$.

Finally, a well-known normality argument ([5], 218) combined with the fact that our convergence is derived from a topology, leads to the conclusion that the $\tilde{\psi}_n$ approximate φ.

6. Approximation by piecewise-linear mappings with smaller dilatations

THEOREM. *For a non-degenerating sequence of nets* $\mathcal{N}_n$ *of triangles in* $\mathbf{C}$, *let the symmetrical diameters* $\delta_n(z)$ *converge to* 0 *pointwise on a dense subset of* $\mathbf{C}$. *Further, let a quasiconformal mapping* φ: $\mathbf{C}\to\mathbf{C}$ *have complex dilatation* $\varkappa(z)$ *and maximal dilatation* $\mathrm{K}(\varphi)>1$. *Then there exist quasiconformal mappings* φ_n: $\mathbf{C}\to\mathbf{C}$, *converging to* φ *in the compact-open topology, with complex dilatations* $\varkappa_n(z)$ *converging to* $\varkappa(z)$ *pointwise almost everywhere in* $\mathbf{C}$, *such that* φ_n *has maximal dilatation* $\mathrm{K}(\varphi_n)<\mathrm{K}(\varphi)$, *and* φ_n *is piecewise-linear with respect to the net* $\mathcal{N}_n$.

Proof. I. In this part, we apply Lemmas 3 and 4. — Lemma 4 approximates the given mapping φ by certain mappings $\tilde{\psi}_n$. Since $\tilde{\psi}_n$ is a regular quasiconformal mapping, conformal outside of a certain compact subset of $\mathbf{C}$, the absolute value of $\partial\tilde{\psi}_n/\partial z$ has a positive lower bound γ_n. As the sequence of the nets $\mathcal{N}_n$ is non-degenerating, we can find a constant β_0 with $\pi/2\leqslant\beta_0<\pi$ which is an upper bound for the angles of the triangles of the nets $\mathcal{N}_n$. For each index n, we define the positive constant

$$\varepsilon_n:=\frac{\gamma_n}{3n}\left(1-\frac{1}{\mathrm{K}(\varphi)}\right)\sin\beta_0.$$

Putting $\varepsilon:=\varepsilon_n$ in Lemma 4, we get constants ϱ_n, σ_n such that $|z''-z'|\leqslant\varrho_n$ or $|z'|, |z''|\geqslant\sigma_n$ implies the inequality $|d\tilde{\psi}_n(z'')-d\tilde{\psi}_n(z')|\leqslant\varepsilon_n|dz|$.

Lemma 3 ensures us that the symmetrical diameters $\delta_n(z)$ converge to 0 uniformly on compact subsets of $\mathbf{C}$. For each index m, and for all sufficiently large indices n, any triangle $\mathbf{T}\in\mathcal{N}_n$ will satisfy either diam $\mathbf{T}\leqslant\varrho_m$ or $\mathbf{T}\subset\{z\colon |z|\geqslant\sigma_m\}$. We can choose an isotonic and surjective function λ: $\mathbf{N}\to\mathbf{N}$ such that the pairs (m, n) with $m\leqslant\lambda(n)$ are admissible for the above statement.

II. Choice of the mappings φ_n. — Since for each index m, the mapping $\tilde{\psi}_m$ is approximated by its piecewise-linearized mappings $\tilde{\psi}_m^{\langle n\rangle}$, it follows that we have $|\tilde{\psi}_m^{\langle n\rangle}(0)-\varphi(0)|=|\tilde{\psi}_m^{\langle n\rangle}(0)-\tilde{\psi}_m(0)|\leqslant 2^{-m}$ and $|\tilde{\psi}_m^{\langle n\rangle}(1)-\varphi(1)|=|\tilde{\psi}_m^{\langle n\rangle}(1)-\tilde{\psi}_m(1)|\leqslant 2^{-m}$ for all sufficiently large indices n. We can choose an isotonic and surjective function μ: $\mathbf{N}\to\mathbf{N}$ such that the pairs (m, n) with $m\leqslant\mu(n)$ are admissible above.

Taking $\nu:=\min\{\lambda,\mu\}$, we define another isotonic and surjective function ν: $\mathbf{N}\to\mathbf{N}$. We put $m:=\nu(n)$; this makes the pairs $(m, n)=(\nu(n), n)$ admissible in the sense of parts I and II. With the sequence of the mappings $\tilde{\psi}_m$, we associate the *diluted sequence* consisting of the mappings $\psi_n:=\tilde{\psi}_{\nu(n)}$. Finally, let us define $\varphi_n:=\psi_n^{\langle n\rangle}$; these mappings φ_n: $\mathbf{C}\to\mathbf{C}$ are piecewise-linear with respect to the nets $\mathcal{N}_n$.

III. Application of Lemma 2. — Let us take any one of the nets $\mathcal{N}_n$, any one of the triangles $\mathbf{T} \in \mathcal{N}_n$, and any one of the points $z \in \mathbf{T}$. We define ψ: $\mathbf{T} \to \mathbf{C}$ as the restriction of ψ_n to $\mathbf{T}$; its linearization $\hat{\psi}$: $\mathbf{C} \to \mathbf{C}$ with respect to $\mathbf{T}$ coincides with $\varphi_n = \psi_n^{\langle n \rangle}$ on $\mathbf{T}$. Finally, we put $a := \frac{\partial \psi_n}{\partial z}(z)$, $b := \frac{\partial \psi_n}{\partial z^*}(z)$, $\varepsilon := \varepsilon_{\nu(n)}$. Now we apply Lemma 2 and use the following estimates: $|a| \geqslant \gamma_{\nu(n)}$; $\sin \beta \geqslant \sin \beta_0$; $|b/a| \leqslant [1 - 1/\nu(n)][\mathrm{K}(\varphi) - 1]/[\mathrm{K}(\varphi) + 1]$.

$$|\varkappa_n(z) - \tilde{\varkappa}_{\nu(n)}| = \left|\hat{\varkappa} - \frac{b}{a}\right| \leqslant \frac{|b/a| + 1}{|a/\varepsilon| \sin \beta - 1} \leqslant \frac{[1 - 1/\nu(n)][\mathrm{K}(\varphi) - 1]/[\mathrm{K}(\varphi) + 1] + 1}{3\nu(n)\,\mathrm{K}(\varphi)/[\mathrm{K}(\varphi) - 1] - 1}$$

$$\leqslant \frac{2\nu(n)\,\mathrm{K}(\varphi) - [\mathrm{K}(\varphi) - 1]}{3\nu(n)\,\mathrm{K}(\varphi) - [\mathrm{K}(\varphi) - 1]} \cdot \frac{1}{\nu(n)} \cdot \frac{\mathrm{K}(\varphi) - 1}{\mathrm{K}(\varphi) + 1} \leqslant \frac{2}{3\nu(n)} \cdot \frac{\mathrm{K}(\varphi) - 1}{\mathrm{K}(\varphi) + 1},$$

hence

$$|\varkappa_n(z)| \leqslant \left|\hat{\varkappa} - \frac{b}{a}\right| + \left|\frac{b}{a}\right| \leqslant \left[1 - \frac{1}{3\nu(n)}\right] \cdot \frac{\mathrm{K}(\varphi) - 1}{\mathrm{K}(\varphi) + 1};$$

and finally

$$\frac{1 + |\varkappa_n(z)|}{1 - |\varkappa_n(z)|} \leqslant \frac{6\nu(n)\,\mathrm{K}(\varphi) - [\mathrm{K}(\varphi) - 1]}{6\nu(n) + [\mathrm{K}(\varphi) - 1]} < \mathrm{K}(\varphi).$$

IV. Properties of the mappings φ_n. — Part III implies the quasiconformality of φ_n in each triangle $\mathbf{T} \in \mathcal{N}_n$ and thus in $\mathbf{C}$. The estimates for $\varkappa_n(z)$ hold in $\bigcup \{\operatorname{int} \mathbf{T}\colon \mathbf{T} \in \mathcal{N}_n\}$, which is almost everywhere in $\mathbf{C}$; since we have upper bounds depending only on $\nu(n)$ and $\mathrm{K}(\varphi)$, we can deduce $\mathrm{K}(\varphi_n) < \mathrm{K}(\varphi)$. For $n \to \infty$, we find that $\varkappa_n(z) - \tilde{\varkappa}_{\nu(n)} \to 0$, and $\tilde{\varkappa}_{\nu(n)} \to \varkappa(z)$, hence $\varkappa_n(z) \to \varkappa(z)$ pointwise almost everywhere in $\mathbf{C}$. If we combine this with $\varphi_n(0) \to \varphi(0)$ and $\varphi_n(1) \to \varphi(1)$, we can conclude that the corresponding mappings φ_n: $\mathbf{C} \to \mathbf{C}$ approximate φ: $\mathbf{C} \to \mathbf{C}$ in the compact-open topology.

References

[1]. Agard, S., An extremal problem in quasiconformal mappings. *Duke Math. J.*, 33 (1966), 735–741.

[2]. Ahlfors, L. V., Extension of quasiconformal mappings from two to three dimensions. *Proc. Nat. Acad. Sci. USA*, 51 (1964), 768–771.

[3]. —— The modular function and geometric properties of quasiconformal mappings. *Proc. Conf. Compl. Anal.* (Minneapolis, 1964), Springer, Berlin, 1965, 296–300.

[4]. Jaenisch, S., Über die Approximierbarkeit quasikonformer Abbildungen der Ebene mittels stückweiser Linearisierung. *Mitt. Math. Sem. Giessen*, 79 (1968), 29–53.

[5]. Lehto, O. & Virtanen, K. I., Quasikonforme Abbildungen. Springer, Berlin, 1965.

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