THE SWALLOWTAIL AND BUTTERFLY CUSPOIDS AND THEIR APPLICATION IN THE INITIAL POST-BUCKLING OF SINGLE-MODE STRUCTURAL SYSTEMS*

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Abstract. This paper investigates the swallowtail and butterfly catastrophes from the point of view which is applicable in the theory of elastic stability. Thus, the results are concerned with the various forms of these instabilities as well as the determination of the critical load surfaces which are of engineering significance. It is demonstrated that the results are applicable to an axially loaded beam resting on a nonlinear elastic foundation.

Introduction. Catastrophe theory [1] and the theory of elastic stability [2, 3, 4] are known to be similar in nature. For example, Thompson and Hunt [5] showed that imperfection sensitivity in the anticlinal point of bifurcation results in an elliptic umbilic catastrophe. Further, they showed that the stability analysis of a two-mode axially loaded stiffened flat plate could be classified as a hyperbolic umbilic catastrophe. In addition, the parabolic umbilic and its application to a two-mode pressurized spherical shell and to a two-mode uniformly compressed plate on a nonlinear elastic foundation were examined by Hui and Hansen [6, 7]. For single-mode systems, the catastrophes which have been considered are the fold, cusp, swallowtail and butterfly cuspoids. The fold model occurs when the cubic term of the potential energy is non-zero. It arises, for example, in the single-mode overall buckling of a wide integrally stiffened flat plate (where the stringers are closely spaced so that overall buckling occurs first) under axial compression [8] and in the Cox buckling problem [2]. The cusp model is found in both the single local mode and the single overall mode buckling of axially stiffened cylindrical shells under compressive loads [9, 10, 11], oval cylindrical shells under axial compression [12], compressed sandwich cylindrical panels [13] as well as in the initial post-buckling analysis of a spherical cap under a concentrated or an axisymmetric distributed load [14, 15]. The reason for the present investigation is that all of the above-mentioned elementary catastrophes have been investigated and applied to structural instability problems except the swallowtail and the butterfly cuspoids. Thus, it is of interest to investigate the implications of these remaining catastrophes and to determine the role they play in the theory of elastic stability.

In this paper, the swallowtail and butterfly cuspoids are unfolded and analyzed in terms of the theory of elastic stability. They are solved in the most general form so that

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the results are applicable to any system where the potential energy expression falls into their standard forms. Previous work [16, 17, 18] has been done on these models; however, it was not specifically toward the theory of elastic stability. In particular, the equilibrium paths of the present work illustrate important physical insight and as a result emphasize the distinction between primary and secondary critical surfaces. The swallowtail and butterfly cuspoids are in general applicable to any cusp catastrophe (including all the aforementioned examples) when the quartic term of the potential energy is either zero or sufficiently small. Therefore, in order to demonstrate the general results, a stability analysis of a beam resting on a nonlinear elastic foundation is presented in the final section. This analysis is based on a Koiter-style approach [2].

Classification of single-mode systems.   The most general form of the potential energy of a single-mode system expanded about the classical critical load of the perfect system is

$$
\begin{aligned}
PE = {} & \left[A_2'(\lambda - \lambda_{cl}) + \frac{1}{2!} A_2''(\lambda - \lambda_{cl})^2 + \cdots \right] \xi^2 + [A_3 + A_3'(\lambda - \lambda_{cl}) + \cdots] \xi^3 \\
& + [A_4 + A_4'(\lambda - \lambda_{cl}) + \cdots] \xi^4 + [A_5 + A_5'(\lambda - \lambda_{cl}) + \cdots] \xi^5 \\
& + [A_6 + A_6'(\lambda - \lambda_{cl}) + \cdots] \xi^6 + \cdots \\
& + [\varepsilon_1 B_1 + \varepsilon_1 B_1'(\lambda - \lambda_{cl}) + \cdots] \xi + [\varepsilon_2 B_2 + \varepsilon_2 B_2'(\lambda - \lambda_{cl}) + \cdots] \xi^2 \\
& + [\varepsilon_3 B_3 + \varepsilon_3 B_3'(\lambda - \lambda_{cl}) + \cdots] \xi^3 + [\varepsilon_4 B_4 + \varepsilon_4 B_4'(\lambda - \lambda_{cl}) + \cdots] \xi^4 + \cdots \quad (1)
\end{aligned}
$$

where A_1, A_1', A_1'', ... and A_2 vanish because of the condition that the first and second variations of the potential energy are zero in the critical state of equilibrium. In the above λ is the applied load parameter, λ_{cl} is the classical critical load, ξ is the amplitude of the buckling mode, A_2', A_2'', A_3, A_3', ..., B_1, B_1', B_2, ... are constants and ε_1, ε_2, ε_3 and ε_4 are the amplitudes of various imperfection quantities. That is, these imperfections represent deviations from the idealized model of the structure. Thus, they may appear as initial geometric deformation changes in material specification or any other similar quantity. It should be noted that, for an asymptotic analysis, it is usual to retain only the first non-vanishing term in each of the above square brackets.

In catastrophe theory, the form of a single mode instability is classified according to the first non-vanishing coefficient A_n, where $n = 3, 4, \ldots$. The standard form after division by A_n and a suitable scaling is

$$
PE = \xi^n + \alpha_{n-2} \xi^{n-2} + \alpha_{n-3} \xi^{n-3} + \cdots + \alpha_3 \xi^3 + \alpha_2 \xi^2 + \alpha_1 \xi
$$

$$
\text{with} \quad A_n \neq 0, \qquad A_{n-1} = \cdots A_2 = 0 \qquad (2)
$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-2}$ are control parameters and are associated with quantities such as applied load(s) and deviations of the real system from the idealized model. Furthermore, all these control parameters are assumed to be zero in the reference state and it is the stability of the perturbations of the reference state which is under consideration.

The fold cuspoid occurs whenever the cubic term of the potential energy is non-zero. The standard form is

$$
PE = \xi^3 + \alpha_1 \xi, \qquad A_3 \neq 0. \qquad (3)
$$

Also, the standard form of the cusp model is given by

$$PE = \pm\xi^4 + \alpha_2\xi^2 + \alpha_1\xi, \qquad A_3 = 0, \qquad A_4 \neq 0. \tag{4}$$

The swallowtail cuspoid instability occurs when both the cubic and quartic terms of the potential energy vanish (that is, when the quartic term of the cusp model vanishes) and the quintic term is non-zero. Its standard form is

$$PE = \xi^5 + \alpha_3\xi^3 + \alpha_2\xi^2 + \alpha_1\xi, \qquad A_3 = A_4 = 0, \qquad A_5 \neq 0. \tag{5}$$

Continuing, if the quintic term along with the cubic and quartic terms of the potential energy vanish, the next higher form of instability is called the butterfly cuspoid. The standard form of this model is

$$PE = \pm\xi^6 + \alpha_4\xi^4 + \alpha_3\xi^3 + \alpha_2\xi^2 + \alpha_1\xi$$

$$A_3 = A_4 = A_5 = 0, \qquad A_6 \neq 0 \tag{6}$$

where the plus or minus sign corresponds respectively to the butterfly catastrophe and to its dual. In the present, because of physical implications, the plus sign corresponds to what is defined as the stable butterfly cuspoid and the negative sign to the unstable butterfly cuspoid. The forms of the instability such as the fold and swallowtail cuspoids which begin with the odd powers of ξ are always unstable at the classical critical load. It is noted that one of the objectives of this paper is to emphasize the importance of the sign of the leading term of the even-ordered catastrophes such as the cusp and the butterfly models. Other higher forms of instability are also of possible interest; however, they are out of the scope of the elementary catastrophes. For example, the standard form of the Wigwam and Star catastrophes are [16], respectively,

$$PE = \xi^7 + \alpha_5\xi^5 + \alpha_4\xi^4 + \alpha_3\xi^3 + \alpha_2\xi^2 + \alpha_1\xi$$

$$A_3 = A_4 = A_5 = A_6 = 0, \qquad A_7 \neq 0, \tag{7}$$

$$PE = \pm\xi^8 + \alpha_6\xi^6 + \alpha_5\xi^5 + \alpha_4\xi^4 + \alpha_3\xi^3 + \alpha_2\xi^2 + \alpha_1\xi$$

$$A_3 = A_4 = A_5 = A_6 = A_7 = 0, \qquad A_8 \neq 0. \tag{8}$$

Swallowtail cuspoid. By setting the first and second variations of the appropriate potential energy expression to zero, the equilibrium and stability equations of the swallowtail cuspoid are

$$5\xi^4 + 3\alpha_3\xi^2 + 2\alpha_2\xi + \alpha_1 = 0, \qquad 10\xi^3 + 3\alpha_3\xi + \alpha_2 = 0. \tag{9}$$

The critical load surface is obtained by eliminating the behavior variable from the equilibrium and stability equations. The primary critical load surface is defined to be the first intersection of the equilibrium path with the stability boundary as the applied load is increased from zero. Any subsequent intersection with the stability boundary is termed the secondary surface. Throughout this paper, the primary surfaces are shown as solid lines while the secondary surfaces are shown as dotted lines. All stability boundaries are shown as dash-dot lines. Stable and unstable equilibrium paths are shown as solid and dotted lines respectively.

Fig. 1a shows the swallowtail cuspoid critical load surface of $1 - \alpha_2$ versus α_1 for various values of α_3. The corresponding equilibrium paths for three typical values of α_3 are shown in Figs. 1b–d. Fig. 1b shows that for a system with positive values of α_3 and
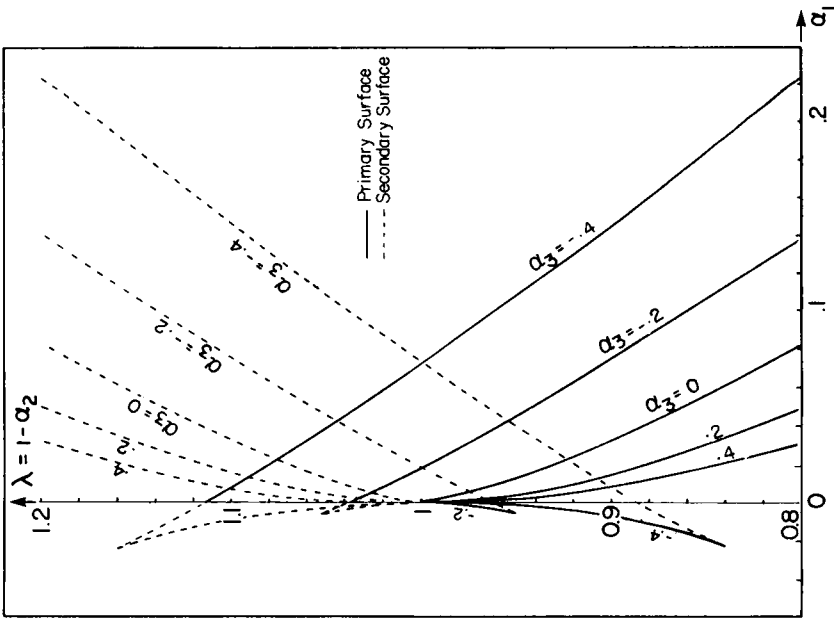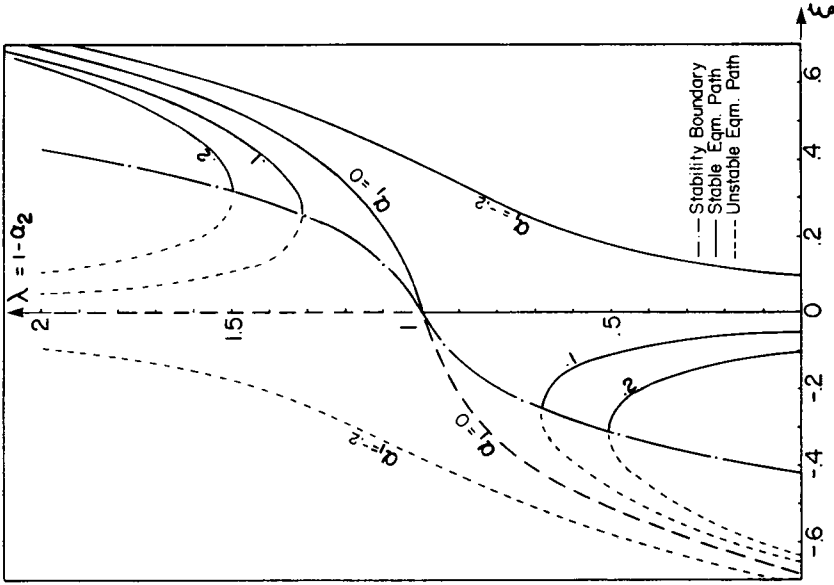
FIG. 1b. Swallowtail cuspoid equilibrium paths, $(\alpha_3 = 0.2) > 0$.
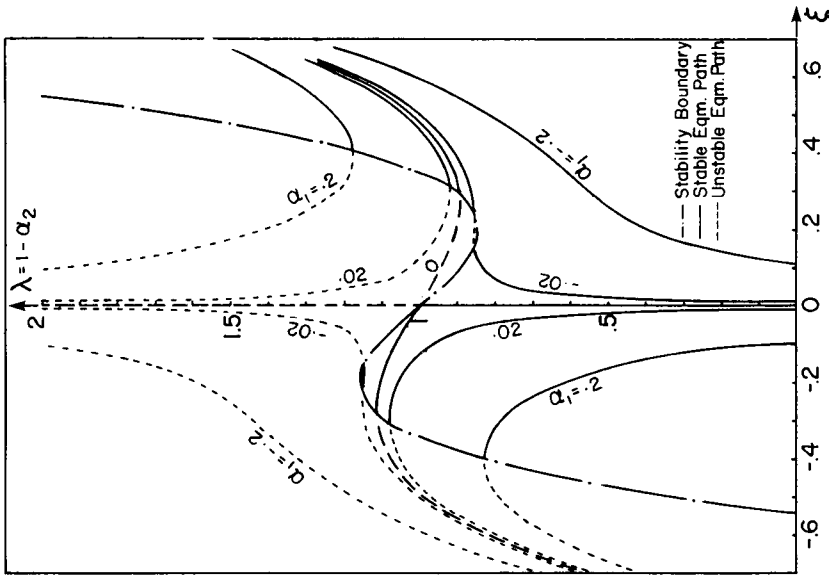

FIG. 1a. Swallowtail cuspoid critical set.

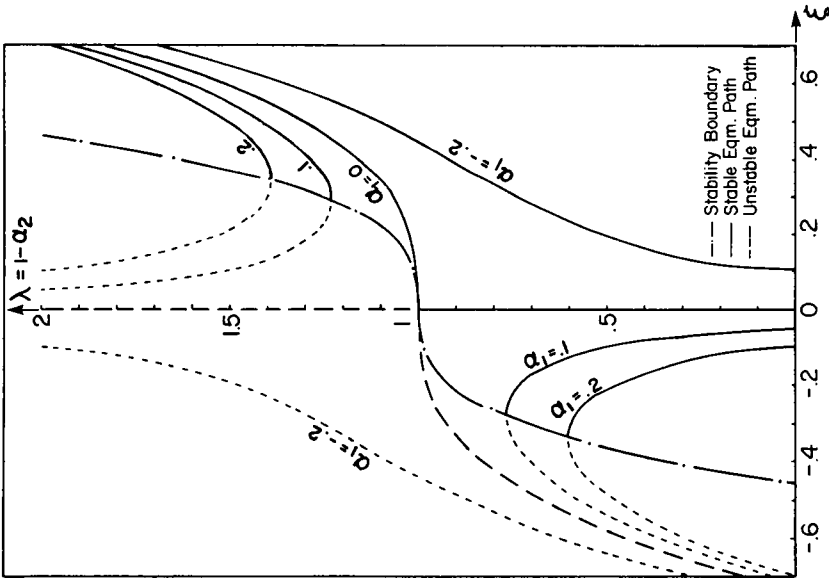FIG. 1d. Swallowtail cuspoid equilibrium paths, $(\alpha_3 = -0.4) < 0$.

FIG. 1c. Swallowtail cuspoid equilibrium paths, $\alpha_3 = 0$.

negative values of α_1, the initially stable equilibrium paths do not intersect the stability boundary. Hence, there is no critical load surface as shown in Fig. 1a. Fig. 1d shows that when α_3 is negative, the stability boundary has two peaks and therefore, for sufficiently small (in magnitude) negative values of α_3 the equilibrium paths intersect the stability boundary and the critical surface exists.

Butterfly cuspoid. The equilibrium and stability equations of the stable butterfly cuspoid are, respectively,

$$6\xi^5 + 4\alpha_4\xi^3 + 3\alpha_3\xi^2 + 2\alpha_2\xi + \alpha_1 = 0, \qquad 15\xi^4 + 6\alpha_4\xi^2 + 3\alpha_3\xi + \alpha_2 = 0. \quad (10)$$

Similarly, for the unstable butterfly cuspoid, these equations become

$$-6\xi^5 + 4\alpha_4\xi^3 + 3\alpha_3\xi^2 + 2\alpha_2\xi + \alpha_1 = 0, \qquad -15\xi^4 + 6\alpha_4\xi^2 + 3\alpha_3\xi + \alpha_2 = 0. \quad (11)$$

It may be noted that all four of the above equations remain invariant when the parameters α_1, α_3 and deflection ξ change sign simultaneously. Thus the four-dimensional $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ critical surface will have reflective symmetry in the sense that when α_3 is replaced by $-\alpha_3$, the surface is symmetric with respect to the α_1 axis. Further, it is obvious that by changing the signs of $\alpha_1, \alpha_2, \alpha_3$ and α_4 in the stable butterfly cuspoid, the result is an unstable butterfly cuspoid and vice versa. Similarly, by changing the signs of α_2, α_4 and ξ in the stable butterfly cuspoid, the result is also an unstable butterfly cuspoid and vice versa. Hence, the equilibrium curves and critical surfaces of the unstable butterfly cuspoid are obtained by simply inverting the corresponding graphs of the stable butterfly model (that is, by changing the signs of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$).

Fig. 2a shows the stable butterfly cuspoid critical load surface when the parameter α_4 is set to .05. Also, due to symmetry considerations, only negative values of α_3 are plotted. The critical load surface for positive values of α_3 can be obtained by reflecting the graph about the load axis. The corresponding equilibrium paths for typical values of α_3 being $-.4$ are shown in Fig. 2b. Here, assuming that the applied load is increased from zero, the system will buckle only for sufficiently small negative parameters α_1. Fig. 2c shows that when $\alpha_3 = 0$ the initially stable system will buckle only when $\alpha_1 = 0$. Again, from symmetry considerations, the equilibrium paths for positive values of α_3 are similar to Fig. 2b except α_1 and the deflection ξ change sign. This implies that the system will buckle only for sufficiently small positive values of α_1.

Fig. 3a shows the stable butterfly cuspoid critical load surface for a typical negative value of α_4 being $-.5$. Again, only negative values of α_3 are plotted due to symmetry. This figure differs from Fig. 2a in that the primary surfaces exist for sufficiently small positive and negative values of α_1. This is demonstrated in Figs. 3b,c where the stability boundaries have two minima instead of one. Fig. 3b shows that when $\alpha_3 = -1$, the relative major peak of the stability boundary corresponds to negative values of α_1 and thus the primary surface for negative values of α_1 is higher than the primary surface for positive values of α_1 as shown in Fig. 3a. The reverse is true for positive values of α_3 and are therefore not plotted.

Fig. 4a shows the unstable butterfly cuspoid critical load surfaces for a typical negative value of α_4 being -0.05. This graph is obtained by inverting Fig. 2a and then changing the signs of α_4 and the deflection ξ. Although the equilibrium and stability
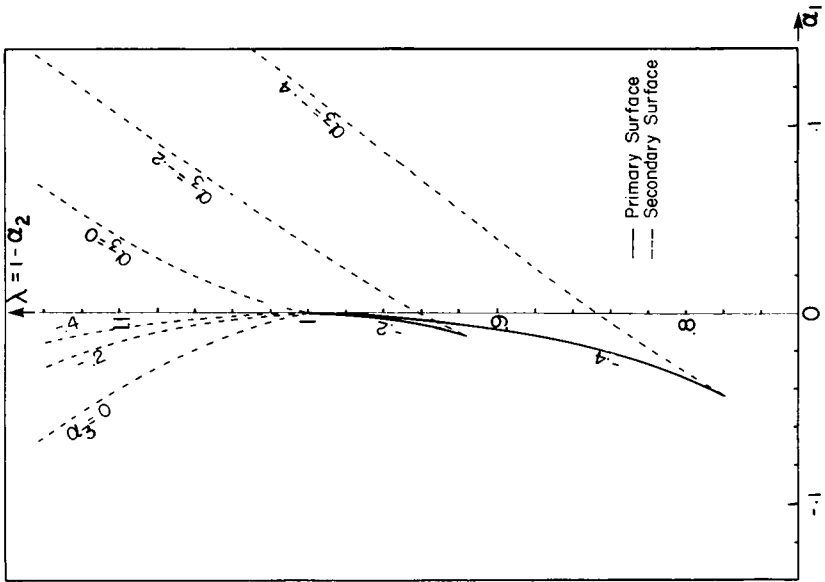
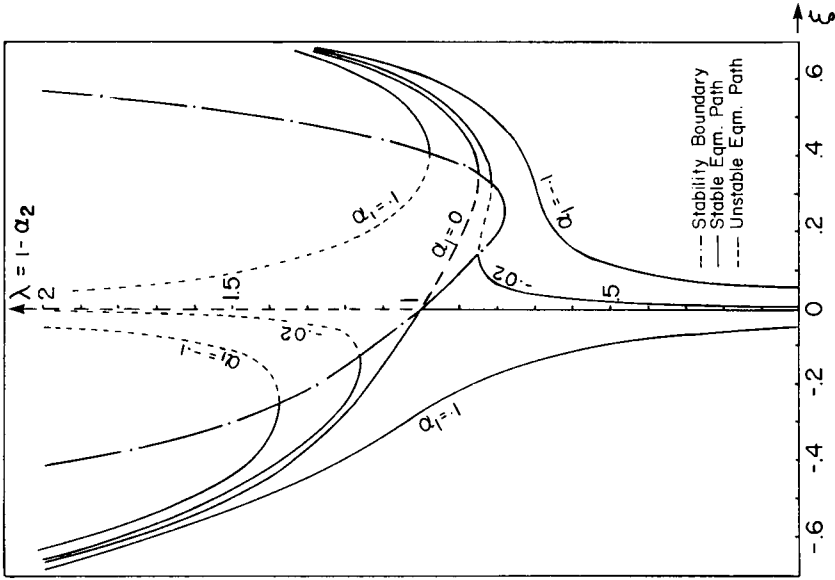FIG. 2b. Butterfly cuspoid type one equilibrium paths, $(\alpha_4 = .05) > 0$ and $(\alpha_3 = -.4) < 0$.

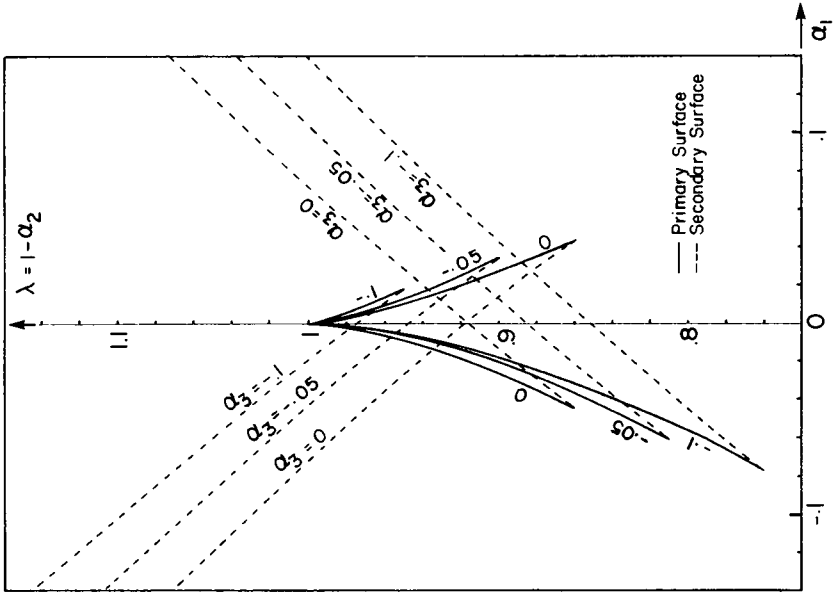FIG. 2a. Butterfly cuspoid type one critical set, $(\alpha_4 = .05) > 0$.

FIG. 3a. Butterfly cuspoids type one critical set, $(\alpha_4 = -.5) < 0$.
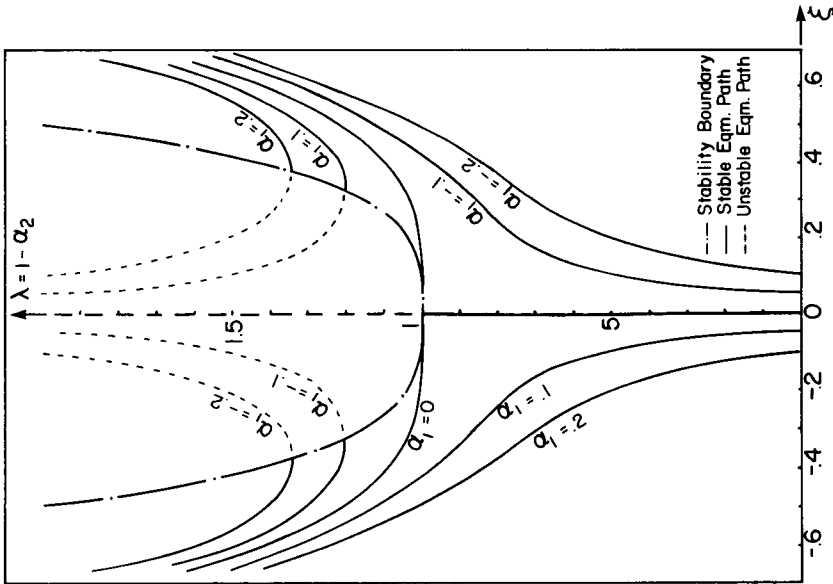

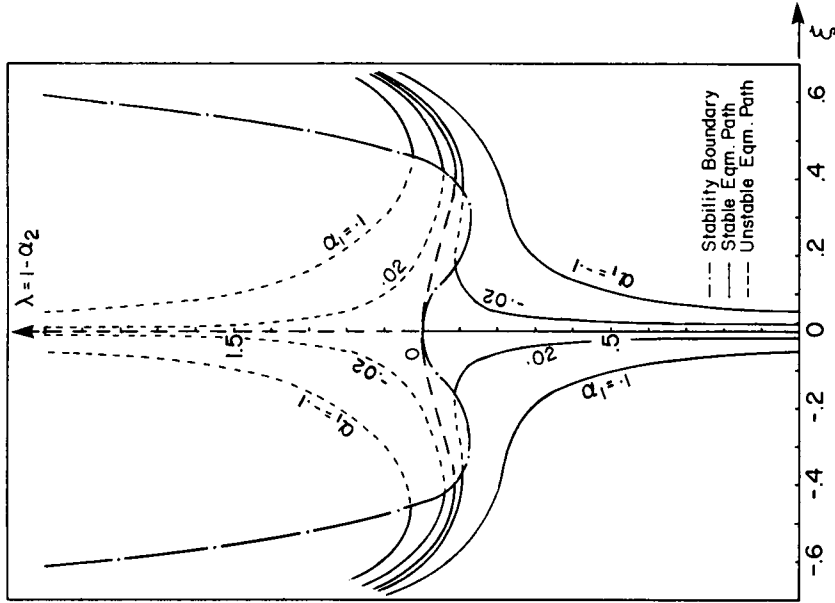FIG. 2c. Butterfly cuspoid type one equilibrium paths, $(\alpha_4 = .05) > 0$ and $\alpha_3 = 0$.

FIG. 3c. Butterfly cuspoid type one equilibrium paths, $(\alpha_4 = -0.5) < 0$ ank $\alpha_3 = 0$.

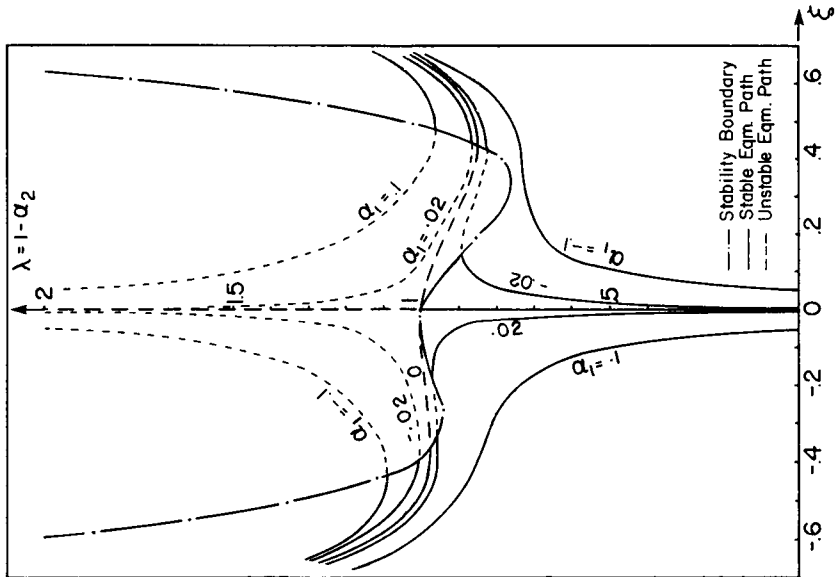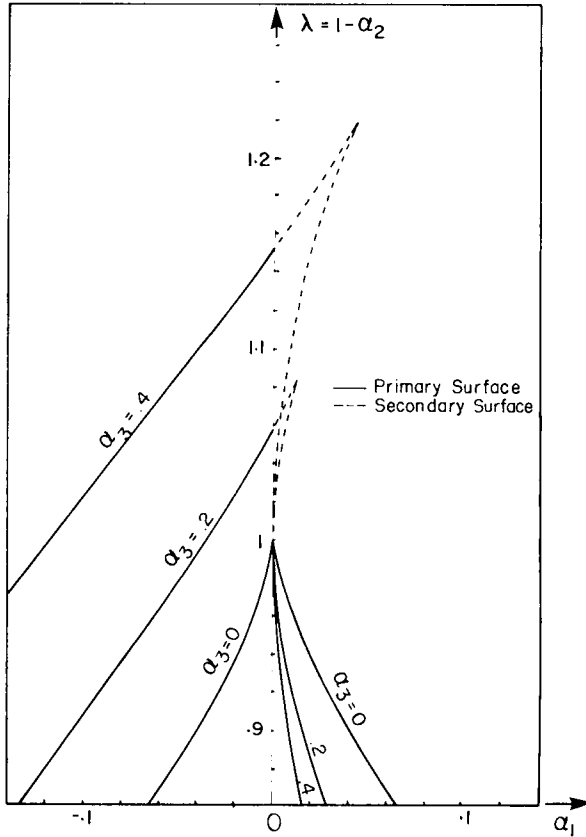FIG. 3b. Butterfly cuspoid type one equilibrium paths, $(\alpha_4 = -0.5) < 0$ and $(\alpha_3 = -0.1) < 0$.

FIG. 4a. Butterfly cuspoid type two critical set, $(\alpha_4 = -.05) < 0$.

equations of these two figures are the same, the primary surfaces which are of engineering significance are totally different. The system in Fig. 4a is seen to buckle for all values of the imperfection α_1. This can be explained from the corresponding equilibrium paths of Figs. 4b and 4c. Since the stability boundary is inverted to open downwards, there is a severe reduction of the critical load which indicates that the system is more sensitive to imperfections than the stable butterfly cuspoid.

Figs. 5a–c show the unstable butterfly cuspoid critical load surfaces and equilibrium paths when α_4 is set to 0.5. Again, this model is shown to be more imperfection-sensitive than the corresponding stable case. Comparison of the equilibrium paths of Fig. 4a and Fig. 5b where the value of α_3 is fixed at 0.4 shows that an initially stable system with positive values of α_1 buckled at a relatively higher value of the critical load in Fig. 5b than in Fig. 4b. This is because the stability boundary, Fig. 5b, has two peaks instead of one, thus causing a less severe reduction of the critical load.

Koiter's general theory of initial post-buckling: single-mode systems. In order to demonstrate the present results in an example it is appropriate to first present a summary of the pertinent aspects of Koiter's general theory of elastic buckling and initial post-buckling for single-mode systems. This section does so. The first approximation to the
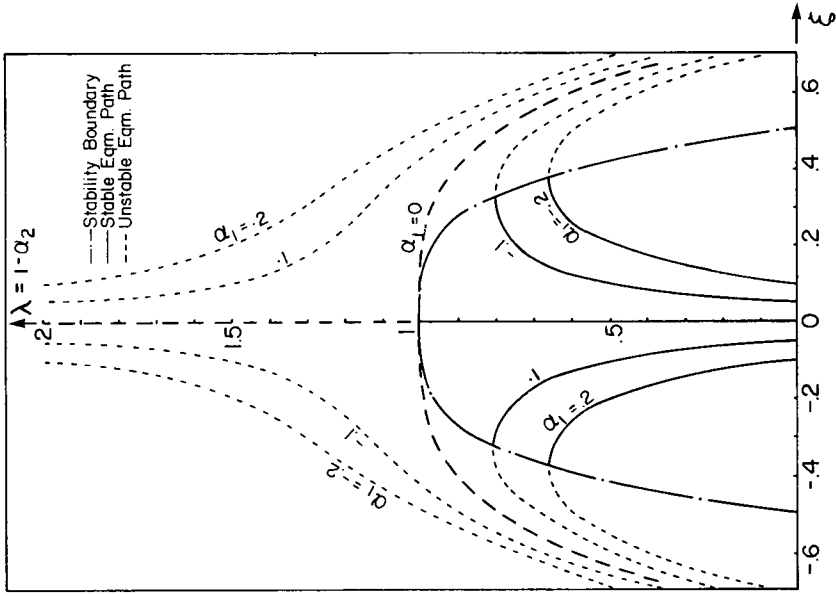
FIG. 4c. Butterfly cuspoid type two equilibrium paths, $(\alpha_4 = -.05) < 0$ and $\alpha_3 = 0$.

FIG. 4b. Butterfly cuspoid type two equilibrium paths, $(\alpha_4 = -.05) < 0$ and $(\alpha_3 = .4) > 0$.

DAVID HUI AND JORN S. HANSEN

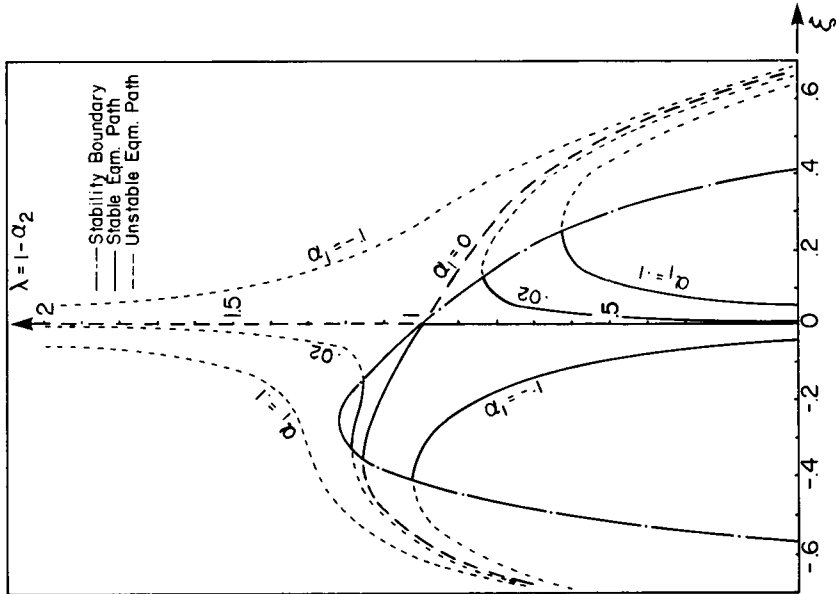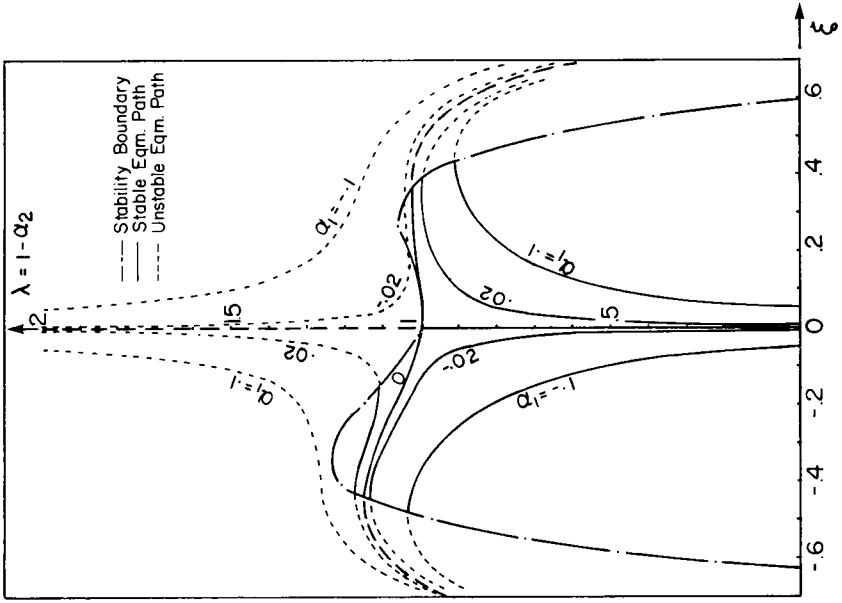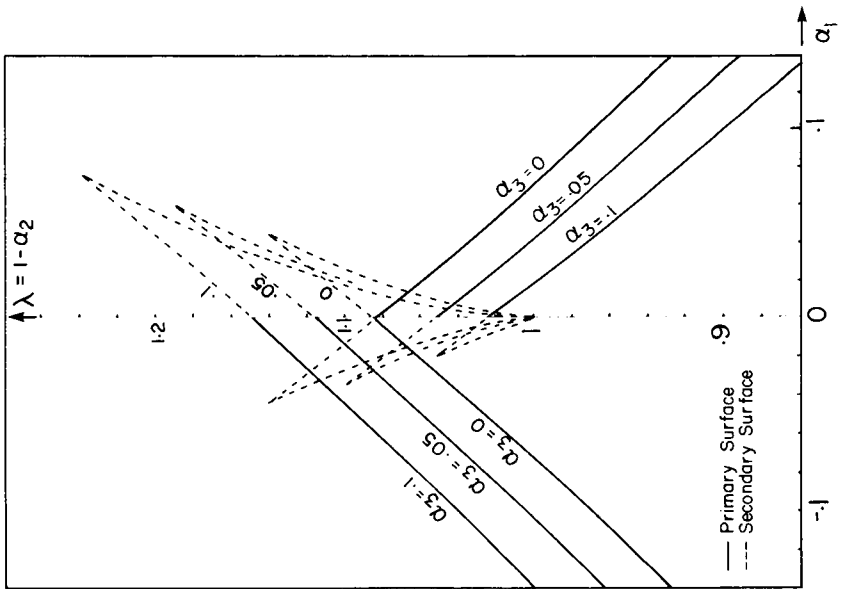FIG. 5b. Butterfly cuspoid type two equilibrium path, ($\alpha_4 = 0.5$) > 0 and ($\alpha_3 = 0.1$) > 0.

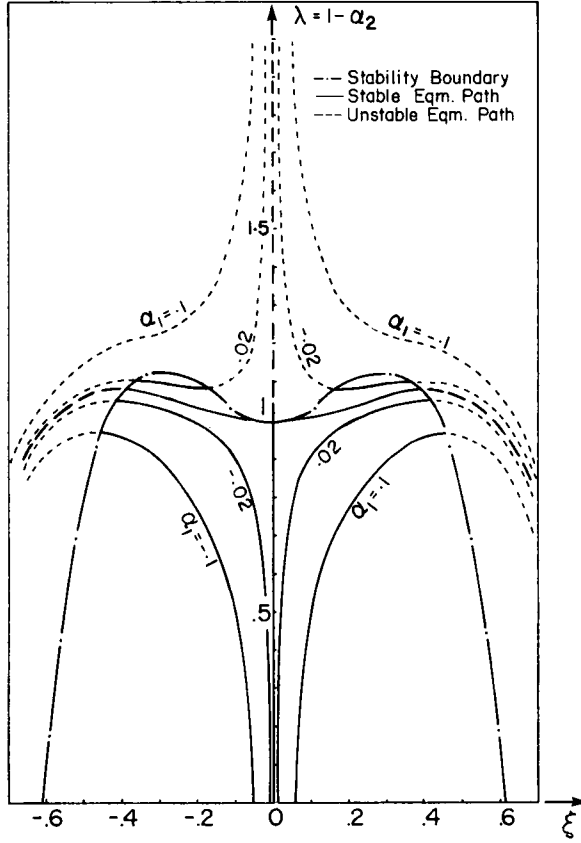FIG. 5a. Butterfly cuspoid type two critical set, ($\alpha_4 = .5$) > 0.

FIG. 5c. Butterfly cuspoid type two equilibrium paths, $(\alpha_4 = .5) > 0$ and $\alpha_3 = 0$.

potential energy expanded about the classical critical load of the perfect system is

$$PE = A'_2(\lambda - \lambda_{cl})\xi^2 + \cdots + A'_{n-1}(\lambda - \lambda_{cl})\xi^{n-1}$$
$$+ A_n \xi^n + \cdots + \varepsilon_1 B_1 \xi + \cdots + \varepsilon_{n-1} B_{n-1} \xi^{n-1}. \qquad (12)$$

Since only the most predominant terms are retained and the prebuckling state is assumed to be linear, Koiter's general theory is valid only for sufficiently small amplitudes of the imperfection parameters. (The swallowtail and butterfly cuspoids correspond to the case when $n = 5$ and $n = 6$ respectively.) Although many of the terms in the above potential energy expression are considered to be negligibly small in Koiter's approach, they are considered to be essential in order to unfold the particular form of instability properly. The coefficients of the potential energy are

$$A_2 = \frac{d}{d\lambda}(P_2[u_{cr}]),$$

$$A_3 = (P_3[u_{cr}])_{\lambda = \lambda_{cl}}, \qquad A'_3 = \frac{d}{d\lambda}(P_3[u_{cr}])|_{\lambda = \lambda_{cl}},$$

$$A_4 = (P_4[u_{cr}] - P_2[\phi_2])_{\lambda = \lambda_{cl}}, \qquad A'_4 = \frac{d}{d\lambda}(P_4[u_{cr}] - P_2[\phi_2])|_{\lambda = \lambda_{cl}},$$

$$A_5 = (P_5[u_{cr}] + P_{31}[u_{cr}, \phi_2] + P_{12}[u_{cr}, \phi_2])_{\lambda = \lambda_{cl}},$$

$$A_6 = (P_6[u_{cr}] + P_{41}[u_{cr}, \phi_2] + P_{22}[u_{cr}, \phi_2] + P_3[\phi_2] - P_2[\psi_3])_{\lambda = \lambda_{cl}},$$

$$B_1 = (Q_1[u_{cr}])_{\lambda = \lambda_{cl}}, \qquad B_2 = (Q_2[u_{cr}] - P_{11}[\bar{\phi}_0, \phi_2])_{\lambda = \lambda_{cl}},$$

$$B_3 = (Q_3[u_{cr}] - P_{11}[\bar{\phi}_0, \phi_3] - P_{11}[\bar{\phi}_1, \phi_2])_{\lambda = \lambda_{cl}},$$

$$B_4 = (Q_4[u_{cr}] - P_{11}[\bar{\phi}_1, \phi_3] - P_{11}[\bar{\phi}_2, \phi_2])_{\lambda = \lambda_{cl}}. \tag{13}$$

In the above $P_{mn}[u, v]$ represents terms in the potential energy of the perfect system which are independent of the imperfection and are of mth order in the deflection u and nth order in v, and $Q_n[u]$ represents terms in the potential energy which are linear both in the applied load and generalized imperfection and are of nth order in the deflection u. The functions ϕ_2, ϕ_3, ψ_3 of the perfect system and $\bar{\phi}_0$, $\bar{\phi}_1$, and $\bar{\phi}_2$ of the perfect system are obtained by solving the appropriate differential equations, that is, from [2],

$$\left[P_{11}[\phi_2, \delta u] + P_{21}[u_{cr}, \delta u] - \left(\frac{3P_3[u_{cr}]}{2T_2[u_{cr}]} \right) T_{11}[u_{cr}, \delta u] \right]_{\lambda = \lambda_{cl}} = 0,$$

$$\left[P_{11}[\phi_3, \delta u] + P_{31}[u_{cr}, \delta u] - \left(\frac{4P_4[u_{cr}]}{2T_2[u_{cr}]} \right) T_{11}[u_{cr}, \delta u] \right]_{\lambda = \lambda_{cl}} = 0,$$

$$\left[P_{11}[\psi_3, \delta u] + P_{31}[u_{cr}, \delta u] + P_{111}[u_{cr}, \phi_2, \delta u] \right.$$
$$\left. - \frac{4P_4[u_{cr}] + 2P_{21}[u_{cr}, \phi_2]}{2T_2[\phi_1]} T_{11}[\psi_1, \delta u] \right]_{\lambda = \lambda_{cl}} = 0,$$

$$\left[P_{11}[\bar{\phi}_0, \delta u] + Q_1[\delta u] - \left(\frac{Q_1[u_{cr}]}{2T_2[u_{cr}]} \right) T_{11}[u_{cr}, \delta u] \right]_{\lambda = \lambda_{cl}} = 0,$$

$$\left[P_{11}[\bar{\phi}_1, \delta u] + Q_{11}[u_{cr}, \delta u] - \left(\frac{Q_2[u_{cr}]}{T_2[u_{cr}]} \right) T_{11}[u_{cr}, \delta u] \right]_{\lambda = \lambda_{cl}} = 0,$$

$$\left[P_{11}[\bar{\phi}_2, \delta u] + Q_{21}[u_{cr}, \delta u] - \left(\frac{3Q_3[u_{cr}]}{2T_2[u_{cr}]} \right) T_{11}[u_{cr}, \delta u] \right]_{\lambda = \lambda_{cl}} = 0, \tag{14}$$

where δu is a variational quantity and the functions ϕ_2, ϕ_3, ψ_3, $\bar{\phi}_0$, $\bar{\phi}_1$ and $\bar{\phi}_2$ satisfy the orthogonality condition

$$T_{11}[u_{cr}, \phi_2] = T_{11}[u_{cr}, \phi_3] = T_{11}[u_{cr}, \psi_3] = T_{11}[u_{cr}, \bar{\phi}_0]$$
$$= T_{11}[u_{cr}, \bar{\phi}_1] = T_{11}[u_{cr}, \bar{\phi}_2] = 0 \tag{15}$$

and $T_{11}[u, v]$ can be taken as the inner product of the two vector quantities u and v.

Example: beam on a nonlinear elastic foundation. The single-mode buckling problems discussed will be considered within the context of the stability analysis of a beam resting on a nonlinear elastic foundation subjected to a concentrated axial compressive load P (positive for compression). The potential energy of the system consists of the sum of the strain energy of the beam due to bending and the strain energy absorbed by the

elastic foundation and the work term, that is,

$$PE = U_0 + U_F - \text{Work}$$

$$= \frac{1}{2} EI \int_0^L \left| \frac{W_{,XX}^2}{1 - W_{,X}^2} \right| dX$$

$$+ \int_0^L \int_0^W \{K_1 W + K_2 W^2 + K_3 W^3 + \cdots\} \, dW \, dX - P \int_0^L \{1 - \sqrt{1 - W_{,X}^2}\} \, dX. \quad (16)$$

In the above, E is Young's modulus, I is the moment of inertia of the beam, W is the vertical deflection, X is the horizontal coordinate and L is the length of the beam and K_1, K_2, K_3, ... are the spring constants of the elastic foundation. Introducing the following nondimensional quantities x, w, λ, k_1, k_2, k_3, ... defined by

$$x = X/L, \qquad w = W/L, \qquad \lambda = PL^2/EI, \qquad k_i = K_i L^{i+3}/EI, \qquad i = 1, 2, 3, \ldots, \quad (17)$$

the potential energy expression becomes

$$PE = \left(\frac{EI}{2L}\right) \left\{ \int_0^1 \left[\frac{w_{,xx}^2}{1 - w_{,x}^2} \right] dx \right.$$

$$+ 2 \int_0^1 \left[\tfrac{1}{2} k_1 w^2 + \tfrac{1}{3} k_2 w^3 + \tfrac{1}{4} k_3 w^4 + \cdots \right] dx + 2\lambda \int_0^1 \left[-1 + \sqrt{1 - w_{,x}^2} \right] dx \right\}. \quad (18)$$

By grouping the terms according to the order of the deflection, the potential energy can be expressed as

$$PE = P_2^\lambda[u] + P_3^\lambda[u] + P_4^\lambda[u] + P_5^\lambda[u] + P_6^\lambda[u] + \cdots \quad (19)$$

where

$$P_2^\lambda[u] = \frac{EI}{2L} \int_{x=0}^1 (w_{,xx}^2 + k_1 w^2) - \lambda w_{,x}^2 \, dx$$

$$P_3^\lambda[u] = \left(\frac{EI}{2L}\right) \left(\tfrac{2}{3} k_2 \int_{x=0}^1 w^3 \, dx \right)$$

$$P_4^\lambda[u] = \left(\frac{EI}{2L}\right) \int_{x=0}^1 (w_{,xx}^2 w_{,x}^2 + \tfrac{1}{2} k_3 w^4) - \tfrac{1}{4} \lambda w_{,x}^4 \, dx$$

$$P_5^\lambda[u] = \left(\frac{EI}{2L}\right) \left(\tfrac{2}{5} k_4 \int_{x=0}^1 w^5 \, dx \right)$$

$$P_6^\lambda[u] = \left(\frac{EI}{2L}\right) \int_{x=0}^1 (w_{,xx}^2 w_{,x}^4 + \tfrac{1}{3} k_5 w^6) - \tfrac{1}{8} \lambda w_{,x}^6 \, dx. \quad (20)$$

It is worth noting that the odd functionals $P_3^\lambda[u]$, $P_5^\lambda[u]$, $P_7^\lambda[u]$, ... are independent of the applied load and that λ is a positive quantity which is assumed to increase from zero. Furthermore, the prebuckling state admits the zero-displacement solution.

Upon setting the first variation of the quadratic terms $P_2^\lambda[u]$ to zero, the differential equation for classical buckling is

$$w_{,xxxx} + \lambda w_{,xx} + k_1 w = 0 \quad (21)$$

where the forced boundary conditions are $w = 0$ at $x = 0, 1$, and the natural boundary conditions are $w_{,xx} = 0$ at $x = 0, 1$. The solution to this eigenvalue problem is

$$w = \xi_m \sum_{n=1}^{\infty} \sin m\pi x \tag{22}$$

so that the eigenvalues are

$$\lambda_m = \frac{m^4\pi^4 + k_1}{m^2\pi^2} \qquad m = 1, 2, \ldots \tag{23}$$

where m is a positive integer. By minimizing λ_m with respect to m, it follows that the critical wave number $m_{cr} = j$ must lie somewhere near the value $(4\sqrt{k_1})/\pi$. In addition, for

$$\lambda_{cl} = \lambda_j = \frac{j^4\pi^4 + k_1}{j^2\pi^2}, \tag{24}$$

$\lambda_l > \lambda_j$, $l < j$ and $\lambda_n > \lambda_j$, $n > j$. Further, the least upper bounds may be shown to occur for $l = j - 1$ and $n = j + 1$ respectively. Thus it follows that the critical wave number j is obtained from the inequalities

$$(j - 1)^2 j^2 \pi^4 < k_1 < (j + 1)^2 j^2 \pi^4. \tag{25}$$

In order to investigate the stability of the critical state, it is necessary to consider higher-order terms in the potential energy. Thus, substituting $w = w_{cr}$ into $P_3^\lambda[u]$, the quantity A_3 is found to be $(m_{cr} = j)$

$$A_3\xi^3 = P_3^\lambda[u_{cr}] = \left(\frac{EI}{2L}\xi^3\right)\left(\tfrac{2}{3}k_2 \int_{x=0}^{1} (\sin j\pi x)^3 \, dx\right) = 0, \qquad j = \text{even}$$

$$= \left(\frac{EI}{2L}\xi^3\right)\left(\frac{8}{9j\pi}k_2\right), \qquad j = \text{odd.} \tag{26}$$

Thus, the swallowtail or butterfly catastrophes will only occur if $k_2 = 0$, or if the critical wave number j is even.

In order to compute A_4, it is necessary to solve the differential equation for ϕ_2, that is,

$$\delta P_2[\phi_2] = -\delta P_3[u_{cr}]. \tag{27}$$

If it is assumed that k_2 is zero, the right-hand side of the above equation is zero. Furthermore, since the displacement ϕ_2 must be orthogonal to the buckling mode w_{cr}, it follows that $\phi_2 = 0$. Thus the quantity A_4 becomes

$$A_4\xi^4 = P_4[u_{cr}]|_{\lambda = \lambda_{cl}} = \left(\frac{EI}{2L}\right)\int_0^1 w_{cr,xx}^2 w_{cr,x}^2 + \tfrac{1}{2}k_3 w_{cr}^4 - \tfrac{1}{4}\lambda_{cl}w_{cr,x}^4 \, dx$$

$$= \left(\frac{EI}{2L}\xi^4\right)[\tfrac{1}{32}j^6\pi^6 - \tfrac{3}{32}j^2\pi^2 k_1 + \tfrac{3}{16}k_3]. \tag{28}$$

It follows from the expression for A_4 that the only situation in which a higher-order catastrophe will arise is when $A_4 = 0$, or

$$\tfrac{1}{3}(j\pi)^6 + 2k_3 = (j\pi)^2 k_1. \tag{29}$$

Using our previous result that $\phi_2 = 0$, the quantity A_5 is found to be

$$A_5\,\xi^5 = P_5[u_{cr}]\big|_{\lambda = \lambda_{cl}} = \left(\frac{EI}{2L}\right)\left(\tfrac{2}{5}k_4 \int_{x=0}^{1} w_{cr}^5\, dx\right) = 0, \qquad j = \text{even}$$

$$= \left(\frac{EI}{2L}\,\xi^5\right)\left(\frac{32}{75j\pi}\,k_4\right), \qquad j = \text{odd}.$$

(30)

Thus with $k_2 = 0$ and by relating k_1 and k_3, one obtains $A_3 = A_4 = 0$. The swallowtail catastrophe occurs when k_4 is non-zero and j is an odd integer.

The butterfly catastrophe arises if $A_5 = 0$, which implies that the spring constant k_4 is zero or that j is odd. The quantity A_6 is, using our previous result that $\phi_2 = 0$,

$$A_6\,\xi^6 = P_6[u_{cr}]\big|_{\lambda = \lambda_{cl}} - P_2[\psi_3]\big|_{\lambda = \lambda_{cl}} \tag{31}$$

where the differential equation for the function ψ_3 is

$$\delta P_2[\psi_3]\big|_{\lambda = \lambda_{cl}} = -\delta P_4[u_{cr}]\big|_{\lambda = \lambda_{cl}}. \tag{32}$$

This equation, when expanded, becomes

$$2(\psi_{3,xxxx} + k_1\psi_3 + \lambda_{cl}\psi_{3,xx})(\delta\psi_3)$$
$$= [2(w_{cr,xx}^2\, w_{cr,x}),_x - 2(w_{cr,x}^2\, w_{cr,xx}),_{xx} - 2k_3 w_{cr}^3 - 3\lambda_{cl} w_{cr,x}^2\, w_{cr,xx}](\delta w) \tag{33}$$

with the forced and natural boundary conditions being those of the buckling mode. Substituting the buckling mode w_{cr} into the right-hand side of the above equation and assuming the solution for ψ_3 in the form

$$\psi_3 = \sum_{n=1}^{\infty} c_n \sin n\pi x, \tag{34}$$

then the result is obtained that

$$c_3 = \frac{\xi^3(j\pi)^2[3k_1 - 7(j\pi)^4]}{48[9(j\pi)^4 - k_1]}$$

and

$$c_1 = c_2 = c_4 = c_5 = \cdots = 0. \tag{35}$$

Substituting $u = \psi_3$ into $P_2[u]$ and carrying out the integration yields

$$P_2[\psi_3]\big|_{\lambda = \lambda_{cl}} = \frac{EI}{2L}\,(c_3)^2(36j^4\pi^4 - 4k_1). \tag{36}$$

Thus, the term $A_6\,\xi^6$ is found to be

$$A_6\,\xi^6 = P_6[u_{cr}]\big|_{\lambda = \lambda_{cl}} - P_2[\psi_3]\big|_{\lambda = \lambda_{cl}}$$
$$= \left(\xi^6\frac{EI}{2L}\right)\{(\tfrac{1}{16})(\tfrac{3}{8}j^8\pi^8 - \tfrac{5}{8}j^4\pi^4 k_1 + \tfrac{5}{3}k_5) - (c_3)^2(36j^4\pi^4 - 4k_1)\}, \tag{37}$$

where it should be noted that the sign of A_6 dictates whether the problem is classified as stable or unstable butterfly catastrophe.

In order to unfold a catastrophe where the leading term in the potential energy is of the form $A_n\,\xi^n$, it is necessary to include all those terms which involve $\xi, \ldots, \xi^{n-2}$ and to

perturb the coefficients of these terms about zero. Thus, in the context of the present example, the linear term in ξ, due to the influence of a geometric imperfection in the form of the buckling mode, is

$$\frac{1}{\lambda_{cl}} P'_{11}[u_{cr}, \bar{u}] = \frac{EI}{2L}\left(\frac{-\lambda}{\lambda_{cl}}\right)(\xi\bar{\xi})j^2\pi^2, \tag{38}$$

while the ξ^2 term is

$$\left(\frac{\lambda - \lambda_{cl}}{\lambda_{cl}}\right)\frac{\partial}{\partial\lambda}(P'_2[u_{cr}]) = \frac{EI}{2L}\left(1 - \frac{\lambda}{\lambda_{cl}}\right)(\tfrac{1}{2}\xi^2 j^2\pi^2). \tag{39}$$

Moreover, the remaining cubic and quartic unfolding terms are obtained by replacing k_1, k_2, k_3 and k_4 by $k_1 + \varepsilon_{k_1}$, $k_2 + \varepsilon_{k_2}$, $k_3 + \varepsilon_{k_3}$ and $k_4 + \varepsilon_{k_4}$ respectively, where ε_{k_1}, ε_{k_2}, ε_{k_3} and ε_{k_4} are small perturbations of the spring constants. Doing so yields

$$A'_3\xi^3 = \frac{EI}{2L}\xi^3\frac{8}{9j\pi}\varepsilon_{k_2} + B_3, \qquad j = \text{odd},$$

$$= B_3, \qquad j = \text{even}, \tag{40}$$

$$A'_4\xi^4 = \left(\frac{EI}{2L}\xi^4\right)[\tfrac{1}{32}j^6\pi^6 - \tfrac{3}{32}j^2\pi^2\varepsilon_{k_1} + \tfrac{3}{16}\varepsilon_{k_3}] + B_4, \tag{41}$$

and

$$A'_5\xi^5 = \left(\frac{EI}{2L}\xi^5\right)\left(\frac{32}{75j\pi}\varepsilon_{k_4}\right) + B_5, \qquad j = \text{odd},$$

$$= B_5, \qquad j = \text{even}. \tag{42}$$

In the above expressions, B_3 and B_5 are obtained by solving the appropriate differential equations (as shown in Eqs. (13)). These terms represent the influence of initial geometric imperfection on the cubic and quartic terms respectively.

It should be noted that ε_{k_1}, ε_{k_2}, ε_{k_3} and ε_{k_4} and $\lambda - \lambda_{cl}$ as well as the amplitude of the geometric imperfection constitute six independent control parameters. It is the sum of their influence, rather than the individual perturbation, which is important. For example, the individual influence of ε_{k_1} and ε_{k_3} is of little interest and it is the aggregate sum of these two perturbations which form the A'_4 coefficient which is of significance. Furthermore, the term ξ^{n-1} (i.e., A'_5 in the case of butterfly catastrophe and A'_4 in the case of swallowtail catastrophe) can be made to vanish by a straightforward coordinate transformation.

Finally, inspection of the expanded potential energy expression shows that the single-mode system is classified as the swallowtail catastrophe when $n = 5$ and as the butterfly catastrophe (stable or unstable depending on the sign of A_6) when $n = 6$. Consequently, the equilibrium paths and critical surfaces plotted provide useful information on the buckling and initial post-buckling analysis of a beam on a nonlinear elastic foundation.

Concluding remarks. Since the swallowtail and butterfly models in catastrophe theory are the only ones among the elementary catastrophes which have not been solved, the present paper completes the analysis of the seven elementary catastrophes from the point of view of the theory of elastic stability.

The three-dimensional critical load surface of the swallowtail cuspoid has been analyzed in its entirety. It corresponds to a stability problem where the first non-vanishing term of the potential energy at the singularity is quintic; thus, the structural configuration at the classical critical load is unstable. The four-dimensional butterfly cuspoid critical load surface has been analyzed by choosing typical values of one of the four control parameters. It arises in a stability problem where the first non-vanishing term of the potential energy at the singularity is ζ^6. The positive sign corresponds to the stable butterfly cuspoid and the negative sign corresponds to the unstable butterfly cuspoid. The latter is found to be more imperfection-sensitive because of the stability boundary opens downward rather than upward.

The equilibrium paths of the swallowtail model are antisymmetric with respect to the load axis. On the other hand, the equilibrium paths of the stable and unstable butterfly models are symmetric with respect to the load axis. Unlike the fold model, the slope of the load-deflection graph of the cusp, swallowtail, butterfly and any higher order cuspoids is always zero. Hence, the horizontal line $\lambda = \lambda_{cl}$ represents the first approximation of the post-buckling deflection path which is valid for vanishingly small control parameters.

Koiter's theory for single-mode structural system has been extended in a somewhat modified form. This is because catastrophe theory requires that in order to unfold the model, or present the form of the instability in its most general form, it is necessary to include the imperfection terms involving B_2, B_3, ..., B_{n-2} in addition to the B_1 term which involves the initial geometric deviation of the system. Practical application of the swallowtail and butterfly in single-mode structural system is shown in an axially loaded beam on a nonlinear elastic foundation. Finally, it should be emphasized that the results presented in this paper are asymptotic in a sense that they are valid for sufficiently small regions near the critical point.

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