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# THE SYMMETRIC CHOQUET INTEGRAL WITH RESPECT TO RIESZ-SPACE-VALUED CAPACITIES

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*Abstract.* A definition of "Šipoš integral" is given, similarly to [3], [5], [10], for real-valued functions and with respect to Dedekind complete Riesz-space-valued "capacities". A comparison of Choquet and Šipoš-type integrals is given, and some fundamental properties and some convergence theorems for the Šipoš integral are proved.

*Keywords*: Riesz spaces, capacities, integration, symmetric Choquet integral, monotone and dominated convergence theorems

*MSC 2000*: 28A70

## 1. Introduction

In [3] we introduced a "monotone-type" (that is, Choquet-type) integral for real-valued functions, with respect to finitely additive positive set functions, with values in a Dedekind complete Riesz space. A "Lebesgue-type" integral for such kind of functions was investigated in [7]. In [4] we gave some comparison results for these types of integrals.

In [10], a Choquet-type integral for real-valued functions with respect to Riesz-space-valued "capacities", that is, monotone set functions not necessarily finitely additive, is investigated. The study of these integrals is motivated by several branches of mathematics (for example, stochastic processes, see [16]) and has also some applications to probability theory and economics, for example for the study of the fundamental properties of the so-called "utility functions" (see for instance [14], [19], [21], [22]) and the study of "qualitative probabilities", that is, set functions which associate to an event not necessarily a real number (indeed, in reality it is often

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not so "natural" to represent the probability of a person about an event simply by means of an element of $[0,1]$, see for example [12]). For related topics, see also the bibliography [5] and [15].

In this paper we introduce a Šipoš-type, that is, "symmetric Choquet"-type integral, for real-valued functions with respect to Riesz-space-valued capacities, we investigate the fundamental properties and prove some convergence theorems (for real-valued capacities see [23] and [20], pp. 152–176).

## 2. Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$ and $\tilde{\mathbb{R}}$ be the sets of all natural, real, positive, negative and extended real numbers, respectively.

A Riesz space $R$ is said to be *Dedekind complete* if every nonempty subset of $R$, bounded from above, has supremum in $R$.

Throughout this paper we always suppose that $R$ is a Dedekind complete Riesz space. In some suitable cases we add to $R$ two extra element which we call $+\infty$ and $-\infty$, extending the ordering and operations. They have the same role as the usual $+\infty$ and $-\infty$ with the real numbers (see also [2], [13]). By the symbol $\overline{R}$ we denote the set $R \cup \{+\infty\} \cup \{-\infty\}$.

**Definition 2.1.** Given an element $r \in R$, we define $r^+ \equiv r \vee 0$, $r^- \equiv (-r) \vee 0$, $|r| \equiv r \vee (-r)$.

**Definition 2.2.** A sequence $(p_n)_n$ is called an $(o)$-*sequence* if $p_n \downarrow 0$, that is, if it is decreasing and $\inf\limits_n p_n = 0$. We say that a sequence $(r_n)_n$ is $(o)$-*convergent* (*order convergent*) to $r$ if there exists an $(o)$-sequence $(p_n)_n \in R$ such that $|r_n - r| \leqslant p_n$ $\forall\, n \in \mathbb{N}$, and in this case we write $(o) \lim\limits_n r_n = r$.

**Definition 2.3.** A directed net $(r_\alpha)_{\alpha\in\Xi}$ is called an $(o)$-*net* if $r_\alpha \downarrow 0$, that is, if it is decreasing and $\inf\limits_{\alpha\in\Xi} r_\alpha = 0$. We say that the directed net $(r_\alpha)_{\alpha\in\Xi}$ is $(o)$-*convergent* to $r$ if

$$(o) \limsup_\alpha r_\alpha \equiv \inf_\alpha [\sup_{\beta\geqslant\alpha} r_\beta] = (o) \liminf_\alpha r_\alpha \equiv \sup_\alpha [\inf_{\beta\geqslant\alpha} r_\beta] = r,$$

and in this case we write $(o) \lim\limits_{\alpha\in\Xi} r_\alpha = r$.

We begin with recalling the Choquet integral, introduced in [10], and we introduce and investigate the Šipoš (that is, the symmetric Choquet) integral for (extended) real-valued functions with respect to Riesz-space-valued capacities.

**Definition 3.1.** Let $X$ be any nonempty set, and let $\mathscr{A} \subset \mathscr{P}(X)$ be a $\sigma$-algebra (we suppose this for the sake of simplicity, though several results remain true if we consider more general structures). We say that a set function $P\colon \mathscr{A} \to R$ is a *capacity* if $P(\emptyset) = 0$ and $P(A) \leqslant P(B)$ whenever $A, B \in \mathscr{A}$, $A \subset B$; $P$ is said to be *submodular* if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) + P(A \cap B) \leqslant P(A) + P(B);$$

*supermodular*, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) + P(A \cap B) \geqslant P(A) + P(B);$$

*subadditive*, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) \leqslant P(A) + P(B);$$

*superadditive*, if

$$A, B \in \mathscr{A} \Longrightarrow P(A \cup B) \geqslant P(A) + P(B).$$

An $R$-valued capacity $P$ is said to be *continuous from below* if for every increasing sequence $(E_n)_n$ of elements of $\mathscr{A}$ we have

$$P\biggl(\bigcup_{n=1}^{\infty} E_n\biggr) = (o)\lim_n P(E_n) = \sup_n P(E_n);$$

*continuous from above*, if for every decreasing sequence $(E_n)_n$ of elements of $\mathscr{A}$ we have

$$P\biggl(\bigcap_{n=1}^{\infty} E_n\biggr) = (o)\lim_n P(E_n) = \inf_n P(E_n);$$

*continuous*, if it is continuous both from below and from above.

A function $P\colon \mathscr{A} \to R$ is called a *mean* (or a *finitely additive set function*) if $P(A) \geqslant 0\ \forall\, A \in \mathscr{A}$ and $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \emptyset$. It is easy to check that every mean is a capacity, but the converse is in general not true. We say that a set function $P$ is a *measure* or that $P$ is *$\sigma$-additive* if it is a continuous mean. Similarly to [3], given a function $f\colon X \to \tilde{\mathbb{R}}$ and a capacity $P$, for all $t \in \mathbb{R}$ set $\Sigma_t^f(\Sigma_t) \equiv \{x \in X\colon f(x) \geqslant t\}$; and, for every $t \in \mathbb{R}$, let $u_f(t) = u(t) \equiv P(\Sigma_t)$. We say that a function $f\colon X \to \tilde{\mathbb{R}}$ is *measurable* if $\Sigma_t^f \in \mathscr{A}$, $\forall\, t \in \mathbb{R}$.

We now recall a Riemann-type integral for functions defined in an interval of the real line and taking values in a Dedekind complete Riesz space (see also [3], [4]).

**Definition 3.2.** Given an interval $[a,b] \subset \mathbb{R}$, we call any finite set $\{x_0, x_1, \ldots, x_n\} \subset [a,b]$ where $x_0 = a$, $x_n = b$ and $x_i < x_{i+1}\ \forall\, i = 0, \ldots, n-1$ a *division of* $[a,b]$. We call the quantity $\delta(D) \equiv \max\limits_{i=0}^{n-1}(x_{i+1} - x_i)$ the *mesh* of $D$. We write $D_1 \geqslant D_2$ if $\delta(D_1) \leqslant \delta(D_2)$. A function $u\colon [a,b] \to R$ is said to be *Riemann integrable* if there exists an element $I \in R$ and an $(o)$-sequence $(p_j)_j$ such that

$$\sup_{\delta(D)\leqslant \frac{1}{j}} \left| \sum_{i=0}^{n-1} u(z_i)(x_{i+1} - x_i) - I \right| \leqslant p_j \quad \forall\, z_i \in [x_i, x_{i+1}]\ (i = 0, \ldots, n-1),$$

and we write $\int_a^b u(t)\,\mathrm{d}t \equiv I$.

The quantity $\sum\limits_{i=0}^{n-1} u(z_i)(x_{i+1} - x_i)$ is called the *Riemann sum* of $u$ associated with the division $\{x_0, x_1, \ldots, x_n\}$ with respect to the points $z_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n-1$.

We note that, similarly to the classical case, it is easy to check that every monotone function $u\colon [a,b] \to R$ is Riemann integrable.

We now introduce the Choquet integral for nonnegative functions with respect to Riesz space-valued capacities (see also [5], [10]).

**Definition 3.3.** A measurable nonnegative function $f \in \tilde{\mathbb{R}}^X$ is said to be *Choquet integrable* if the quantity

$$\int_0^{+\infty} u(t)\,\mathrm{d}t \equiv \sup_{a>0} \int_0^a u(t)\,\mathrm{d}t = (o) \lim_{a\to+\infty} \int_0^a u(t)\,\mathrm{d}t$$

exists in $R$ where $u(t) = P(\Sigma_t)$, $t \geqslant 0$. If $f$ is Choquet integrable, we denote its integral by the symbol $(C)\int_X f\,\mathrm{d}P$.

We now introduce the Šipoš integral, that is, the symmetric Choquet integral, for extended real-valued functions with respect to Riesz space-valued capacities. We begin with

**Definitions 3.4.** A measurable function $f\colon X \to \mathbb{R}$ is said to be *simple* if its range is finite.

Let $\mathscr{F}$ be the family of all finite subsets of $\mathbb{R}$ which contain zero. Given $F \in \mathscr{F}$ and $a \in \mathbb{R}$, set

$$F + a \equiv \{d \in \mathbb{R}\colon\ d = b + a,\ \text{with } b \in F\}$$

and

$$aF \equiv \{d \in \mathbb{R}\colon\ d = ab,\ \text{with } b \in F\}.$$

Let now $F \in \mathscr{F}$, $F = \{b_k, b_{k-1}, \ldots, b_1, b_0, a_0, a_1, \ldots, a_n\}$, where $b_k < b_{k-1} < \ldots < b_1 < b_0 = 0 = a_0 < a_1 < \ldots < a_n$, and let $f$ be a measurable function. As in [20], p. 153, set

$$f_F = \sum_{i=1}^{n} (a_i - a_{i-1}) \chi_{A_i} + \sum_{j=1}^{k} (b_j - b_{j-1}) \chi_{B_j},$$

where

$$\begin{aligned} A_i &= \{x \in X \colon\ f(x) \geqslant a_i\}, \quad i = 0, 1, \ldots, n; \\ B_j &= \{x \in X \colon\ f(x) \leqslant b_j\}, \quad j = 0, 1, \ldots, k. \end{aligned} \tag{1}$$

If $P$ is an $R$-valued capacity, we define the *integral sum* (with respect to $P$) associated with $f$ and $F$ as follows:

$$S_F(f) = \sum_{i=1}^{n} (a_i - a_{i-1}) P(A_i) + \sum_{j=1}^{k} (b_j - b_{j-1}) P(B_j)$$

(where the $A_i$'s and the $B_j$'s are as in (1)) if the right-hand side contains no expression of the type $+\infty - \infty$; moreover, we put *by convention* $S_{\{0\}}(f) = 0$. We note that the set $\mathscr{F}$ is directed. We say that $f \colon X \to \tilde{\mathbb{R}}$ (not necessarily positive) is *Šipoš integrable* (*(S)-integrable*) if the limit

$$(o) \lim_{F \in \mathscr{F}} S_F(f) \tag{2}$$

exists in $R$ and in this case we denote it by the symbol $(S)\int_X f \,\mathrm{d}P$. If the limit in (2) is $+\infty$ or $-\infty$, we write

$$(S)\int_X f \,\mathrm{d}P = +\infty$$

or

$$(S)\int_X f \,\mathrm{d}P = -\infty,$$

respectively, though $f$ is, of course, not $(S)$-integrable. Furthermore, given a set $A \subset X$, $A \in \mathscr{A}$, we say that $f$ is $(S)$-*integrable on* $A$ if $f\chi_A$ is $(S)$-integrable, and in this case we put, *by definition*,

$$(S)\int_A f \,\mathrm{d}P \equiv (S)\int_X f\chi_A \,\mathrm{d}P. \tag{3}$$

**Proposition 3.5.** *Let $f\colon X \to \tilde{\mathbb{R}}$ be a measurable function. The following assertions hold:*

a) *If $f \geqslant 0$ and $\mathscr{F} \ni F_1 \subset F_2 \in \mathscr{F}$, then $S_{F_1}(f) \leqslant S_{F_2}(f)$.*

b) *If $f \geqslant 0$, then $(S)\int_X f \,\mathrm{d}P$ exists in $R \cup \{+\infty\}$ and $(S)\int_X f \,\mathrm{d}P \geqslant 0$. Moreover, in this case we have*

$$(S)\int_X f \,\mathrm{d}P = \sup_{F \in \mathscr{F}} S_F(f).$$

c) *$(S)\int_X \cdot \,\mathrm{d}P$ is a monotone functional.*

d) *If $(S)\int_X f \,\mathrm{d}P$ exists in $R$, then for every $c \in \mathbb{R}$ we have*

$$(S)\int_X (cf) \,\mathrm{d}P = c \cdot (S)\int_X f \,\mathrm{d}P.$$

P r o o f. The proof is similar to the one of Lemma 7.3 (ii), p. 156, and Theorem 7.10 (i)–(iii), p. 155, of [20]. □

We now prove that, for measurable non negative extended real-valued functions, the Šipoš and Choquet integrals do coincide.

**Theorem 3.6.** *Let $f\colon X \to \tilde{\mathbb{R}}$ be a nonnegative measurable function. Then $f$ is Šipoš integrable if and only if it is Choquet integrable, and in this case*

$$(S)\int_X f \,\mathrm{d}P = (C)\int_X f \,\mathrm{d}P.$$

P r o o f. First of all we prove that every nonnegative bounded measurable function $f$ is Šipoš integrable.

Let $K \in \mathbb{N}$ be such that $f(x) \leqslant K$ for all $x \in X$. Then the function $u(t) \equiv P(\{x \in X\colon f(x) \geqslant t\})$ vanishes on $[K, +\infty[$, and therefore $u$ is Riemann integrable in $[0, K]$ and we get

$$R \ni \int_0^K u(t) \,\mathrm{d}t = \int_0^{+\infty} u(t) \,\mathrm{d}t = (C)\int_X f \,\mathrm{d}P. \tag{4}$$

Thus $f$ is Choquet integrable. Moreover, thanks also to Proposition 3.5 a), it is easy to check that $0 \leqslant S_F(f) \leqslant KP(X)$ for every $F \in \mathscr{F}$. From this and Proposition 3.5 b) it follows that $f$ is Šipoš integrable too and

$$(S)\int_X f \,\mathrm{d}P \leqslant KP(X).$$

Now, given $F \in \mathscr{F}$, let $\Sigma(u, F, K)$ be the Riemann sum of $u$ associated with that division of $[0, K]$ whose elements, which we denote by $\alpha_j$, $j = 1, \ldots, s$, are the points of $F$ belonging to $[0, K]$ and the points $0$ and $K$, with respect to the (right end-)points $\alpha_j$'s themselves, $j = 1, \ldots, s$. We have

$$\begin{aligned}
0 &\leqslant \left|(S)\int_X f \,\mathrm{d}P - (C)\int_X f \,\mathrm{d}P\right| \\
&= \left|(S)\int_X f \,\mathrm{d}P - \int_0^K u(t)\,\mathrm{d}t\right| \\
&= (o)\limsup_{F\in\mathscr{F}} \left|(S)\int_X f \,\mathrm{d}P - S_F(f) + S_F(f) - \int_0^K u(t)\,\mathrm{d}t\right| \\
&\leqslant (o)\limsup_{F\in\mathscr{F}} \left|(S)\int_X f \,\mathrm{d}P - S_F(f)\right| \\
&\quad + (o)\limsup_{F\in\mathscr{F}} \left|\Sigma(u, F, K) - \int_0^K u(t)\,\mathrm{d}t\right| \\
&= (o)\limsup_{F\in\mathscr{F}} \left|\Sigma(u, F, K) - \int_0^K u(t)\,\mathrm{d}t\right| = 0.
\end{aligned}$$

From this the assertion follows, at least when $f$ is bounded and measurable.

If $f$ is not bounded, then we have (finite or $+\infty$)

$$\begin{aligned}
(5)\qquad (S)\int_X f \,\mathrm{d}P &= \sup_{F\in\mathscr{F}} S_F(f) \\
&= \sup_{K\in\mathbb{N}} [\sup_{F\in\mathscr{F}} S_F(f \wedge K)] \\
&= \sup_{K\in\mathbb{N}} (S)\int_X (f\wedge K)\,\mathrm{d}P = \sup_{K\in\mathbb{N}} (C)\int_X (f\wedge K)\,\mathrm{d}P \\
&= \sup_{K\in\mathbb{N}} \int_0^K P(\{x\in X\colon\ f(x) \geqslant t\})\,\mathrm{d}t \\
&= \int_0^{+\infty} P(\{x\in X\colon\ f(x) \geqslant t\})\,\mathrm{d}t = (C)\int_X f\,\mathrm{d}P.
\end{aligned}$$

This concludes the proof. □

Proceeding in a way analogous to (5), it is possible to prove

**Proposition 3.7.** *A nonnegative measurable extended real-valued function $f$ is $(S)$-integrable if and only if one of the following three elements*

$$(o)\lim_{K\to+\infty}(S)\int_X (f\wedge K)\,\mathrm{d}P,$$
$$\sup_{K\in\mathbb{R},K\geqslant 0}(S)\int_X (f\wedge K)\,\mathrm{d}P,$$
$$\sup_{K\in\mathbb{N}}(S)\int_X (f\wedge K)\,\mathrm{d}P$$

*exists in $R$ and then these quantities coincide with $(S)\int_X f\,\mathrm{d}P$, not only if they belong to $R$, but also if they are equal to $+\infty$.*

We now prove

**Theorem 3.8.** *If $f$ is a measurable extended real-valued function (not necessarily nonnegative) and $a\in\mathbb{R}$, $a\geqslant 0$, then*

$$(S)\int_X f\,\mathrm{d}P=(S)\int_X (f\wedge a)\,\mathrm{d}P+(S)\int_X (f-f\wedge a)\,\mathrm{d}P \tag{6}$$

*(finite or $+\infty$), if one of the right-hand side expressions belongs to $R$.*

Proof. We prove the theorem in the case $(S)\int_X (f\wedge a)\,\mathrm{d}P\in R$: the proof in the other case is analogous.

By Proposition 3.5 b), the quantity $(S)\int_X (f-f\wedge a)\,\mathrm{d}P$ exists in $R\cup\{+\infty\}$. We consider first the case $(S)\int_X (f-f\wedge a)\,\mathrm{d}P\in R$. By definition of the Šipoš integral and the $(o)$-convergence of nets, there exist two $(o)$-nets $(p_F)_{F\in\mathscr{F}}$ and $(q_F)_{F\in\mathscr{F}}$ such that

$$\left|S_F(f\wedge a)-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|\leqslant p_F\quad\forall F\in\mathscr{F} \tag{7}$$

and

$$\left|S_F(f-f\wedge a)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P\right|\leqslant q_F\quad\forall F\in\mathscr{F}. \tag{8}$$

Fix arbitrarily $F_1,F_2\in\mathscr{F}$, let $F_0\equiv F_1\cup(F_2+a)$ and pick $F\supset F_0$. Since $F-a\supset F_2$, we get

$$\begin{aligned}\Bigl|S_F(f)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P-(S)\int_X (f\wedge a)\,\mathrm{d}P\Bigr|&\\ \leqslant\Bigl|S_{F-a}(f-f\wedge a)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P\Bigr|&\\ +\Bigl|S_F(f\wedge a)-(S)\int_X (f\wedge a)\,\mathrm{d}P\Bigr|\leqslant q_{F_1}+p_{F_2}.&\end{aligned}$$

From (9), thanks also to Proposition 3.5 a), it follows that

$$(10)\qquad (o)\limsup_{F\in\mathscr{F}}\left|S_F(f)-(S)\int_X (f-f\wedge a)\,\mathrm{d}P-(S)\int_X (f\wedge a)\,\mathrm{d}P\right|=0.$$

From (10) it follows that $(S)\int_X f\,\mathrm{d}P$ exists in $R$ and formula (6) holds true. In the case

$$(S)\int_X (f-f\wedge a)\,\mathrm{d}P=+\infty,$$

proceeding with the analogous notation as above, we get

$$S_F(f)=S_F(f\wedge a)+S_{F-a}(f-f\wedge a)\geqslant S_F(f\wedge a)+\int_X (f\wedge a)\,\mathrm{d}P-p_{\{0\}},$$

and taking the supremum in (11) we obtain that $(S)\int_X f\,\mathrm{d}P=+\infty$ if $(S)\int_X(f-f\wedge a)\,\mathrm{d}P=+\infty$, and hence (6) holds with the value $+\infty$. This concludes the proof. □

**Theorem 3.9.** *Let $f\colon X\to\tilde{\mathbb{R}}$ be a measurable function. If $(S)\int_X f^+\,\mathrm{d}P$ or $(S)\int_X f^-\,\mathrm{d}P$ belongs to $R$, then $(S)\int_X f\,\mathrm{d}P$ belongs to $R$ too and*

$$(12)\qquad (S)\int_X f\,\mathrm{d}P=(S)\int_X f^+\,\mathrm{d}P-(S)\int_X f^-\,\mathrm{d}P.$$

*Moreover, if $f$ is $(S)$-integrable, then (12) holds true, and $f^+$, $f^-$ are $(S)$-integrable too.*

Proof. The first part is an easy consequence of Theorem 3.8 (see also [20], p. 159).

We now turn to the second part. In order to prove it, it is sufficient to prove that $(S)\int_X f^+\,\mathrm{d}P$ and $(S)\int_X f^-\,\mathrm{d}P$ belong to $R$. We now report in detail only the proof of the first property. Since $f$ is $(S)$-integrable, there exists an $(o)$-net $(p_F)_{F\in\mathscr{F}}$ such that

$$(13)\qquad \left|S_F(f)-(S)\int_X f\,\mathrm{d}P\right|\leqslant p_F\quad\forall\,F\in\mathscr{F}.$$

Fix now arbitrarily $F_0\in\mathscr{F}$ and choose $F\in\mathscr{F}$ with $F\supset F_0$ and $F\cap\mathbb{R}^-=F_0\cap\mathbb{R}^-$. Proceeding analogously to [20], pp. 159–160, we get

$$(14)\qquad \begin{aligned}0&\leqslant S_{F\cap\mathbb{R}^+}(f^+)=S_F(f^+)=S_{-F}(f^-)+S_F(f)\\&=S_{-F_0}(f^-)+S_F(f)\leqslant S_{-F_0}(f^-)+(S)\int_X f\,\mathrm{d}P+p_{\{0\}}.\end{aligned}$$

We have

$$(15) \qquad (S)\int_X f^+ \,\mathrm{d}P = \sup_{F\in\mathscr{F}} S_F(f^+) = \sup_{F\in\mathscr{F}} S_{F\cap\mathbb{R}^+}(f^+).$$

By virtue of Proposition 3.5 a), the supremum in (15) is equal to the supremum with respect to those elements $F$ of $\mathscr{F}$ which contain $F_0$ and such that $F\cap\mathbb{R}^- = F_0\cap\mathbb{R}^-$. Since $F_0$ was *fixed, it follows* from (14) and (15) that $(S)\int_X f^+\,\mathrm{d}P \in R$. □

**Proposition 3.10.** *Let $X \supset A_1 \supset A_2 \supset \ldots \supset A_n \in \mathscr{A}$. Let $c_i$ be positive real numbers and $f_i \equiv c_i\chi_{A_i}$, $i = 1, 2, \ldots, n$. Then*

$$(S)\int_X \left(\sum_{i=1}^n f_i\right) \mathrm{d}P = \sum_{i=1}^n c_i P(A_i).$$

P r o o f. The proof is similar to the one of [8], Proposition 2.4, p. 65, and takes into account the equivalence between the Choquet and Šipoš integrals for nonnegative measurable functions. □

The proof of the following proposition is straightforward.

**Proposition 3.11.** *If $f$ is measurable and $|f|$ is $(S)$-integrable, then $f$ is $(S)$-integrable too. Moreover, if $f$ is measurable, $g$ is $(S)$-integrable and $|f| \leqslant g$, then $f$ is $(S)$-integrable too.*

From now on, given a nonnegative measurable function $f\colon X \to \tilde{\mathbb{R}}$, let $S_f$ be the set of all simple functions $g$ such that $0 \leqslant g(x) \leqslant f(x)\ \forall\, x \in X$.

**Proposition 3.12.** *If $f \geqslant 0$ is $(S)$-integrable, then*

$$(S)\int_X f\,\mathrm{d}P = \sup_{g\in S_f} (S)\int_X g\,\mathrm{d}P.$$

*Conversely, if $f \geqslant 0$ is measurable and such that the quantity $\sup\limits_{g\in S_f} (S)\int_X g\,\mathrm{d}P$ exists in $R$, then $f$ is $(S)$-integrable and*

$$(S)\int_X f\,\mathrm{d}P = \sup_{g\in S_f} (S)\int_X g\,\mathrm{d}P.$$

*Furthermore, if $f$ is nonnegative and $(S)$-integrable, then there exists a sequence of simple functions $(g_n)_n$ such that*

$$(S)\int_X f\,\mathrm{d}P = \sup_n (S)\int_X g_n\,\mathrm{d}P.$$

P r o o f. The proof of the first two parts is similar to the one of [8], Proposition 2.5, p. 65. The proof of the last part, in the case of a bounded $f$, is similar to the one of [3], Proposition 3.12, p. 798, and takes into account Proposition 3.10; the general case follows from the case of a bounded function and Proposition 3.7. □

**Proposition 3.13.** *If $f$ is $(S)$-integrable, then*

$$(o)\lim_{t\to+\infty} P(\{x\in X\colon\ |f(x)|\geqslant t\}) = 0 = P(\{x\in X\colon\ |f(x)| = +\infty\}).$$

P r o o f. The proof is similar to the one of [3], Proposition 3.10, p. 797, applied to $f^+$ and $f^-$, which are integrable by virtue of Theorem 3.9. □

We now show absolute continuity of the Šipoš integral. In order to do this, first we state a preliminary lemma (for the case $R=\mathbb{R}$, see [20], Lemma 7.5. (i), p. 163).

**Lemma 3.14.** *If $f$ is a nonnegative $(S)$-integrable function, then*

$$(o)\lim_{A\to+\infty}(S)\int_X (f - f\wedge A)\,\mathrm{d}P = 0.$$

P r o o f. Fix arbitrarily $F\in\mathscr{F}$, $F=\{b_k,b_{k-1},\dots,b_1,b_0=0=a_0,a_1,\dots,a_n\}$, where the elements of $F$ are ordered in the increasing order, and let

$$f_F = \sum_{i=1}^{n}(a_i-a_{i-1})\chi_{A_i}.$$

For $A\in\mathbb{R}^+$ large enough we get

$$f_F\leqslant f\wedge A\leqslant f. \tag{16}$$

Now, given $F\in\mathscr{F}$, let $A$ satisfy condition (16). From (16) and the monotonicity of the Šipoš integral we have

$$S_F(f) = (S)\int_X f_F\,\mathrm{d}P\leqslant (S)\int_X (f\wedge A)\,\mathrm{d}P\leqslant (S)\int_X f\,\mathrm{d}P. \tag{17}$$

Moreover, by virtue of Theorem 3.8 and (17), we get

$$(S)\int_X (f-f\wedge A)\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P - (S)\int_X (f\wedge A)\,\mathrm{d}P\leqslant (S)\int_X f\,\mathrm{d}P - S_F(f). \tag{18}$$

From (18) and the Šipoš integrability of $f$ it follows that

$$\begin{aligned} 0 &\leqslant (o)\limsup_{A\in\mathbb{R}^+}\left[(S)\int_X (f-f\wedge A)\,\mathrm{d}P\right] \\ &\leqslant (o)\limsup_{F\in\mathscr{F}}\left[(S)\int_X f\,\mathrm{d}P - S_F(f)\right] = 0. \end{aligned} \tag{19}$$

Thus the assertion follows. □

The next theorem is a consequence of Lemma 3.14

**Theorem 3.15.** *If $f\colon X \to \tilde{\mathbb{R}}$ is $(S)$-integrable, then the integral $(S)\int_{\cdot} f\,\mathrm{d}P$ is absolutely continuous, that is*

$$(o)\lim_n \int_{A_n} f dP = 0$$

*whenever $(A_n)_n$ is a sequence in $\mathscr{A}$ such that $(o)\lim\limits_n P(A_n) = 0$.*

Proof. The proof is similar to the one of [3], Proposition 3.17, p. 800, thanks to Lemma 3.14. □

## 4. Convergence theorems

In this section we prove some convergence theorems for the Šipoš integral with respect to Riesz space-valued capacities, not necessarily finitely additive.

Throughout this section, we always assume that $X$ is any nonempty set, $\mathscr{A} \subset \mathscr{P}(X)$ is a $\sigma$-algebra, $R$ is a Dedekind complete Riesz space, and $P\colon \mathscr{A} \to R$ is a *continuous* capacity.

We begin with the following theorem (for the real case, see [20], Theorem 7.13, pp. 162–163):

**Theorem 4.1.** *Let $c \in R$, $c \geqslant 0$ $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be an increasing sequence of nonnegative $(S)$-integrable functions with $(S)\int_X f_n dP \leqslant c$ for every $n \in \mathbb{N}$, and let $f \equiv \sup\limits_n f_n$ be the pointwise supremum.*

*Then $f$ is $(S)$-integrable, $(S)\int_X f\,\mathrm{d}P \leqslant c$ and*

$$(S)\int_X f\,\mathrm{d}P = \sup_n (S)\int_X f_n\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n\,\mathrm{d}P.$$

Proof. Fix arbitrarily $\varepsilon > 0$ and $F \in \mathscr{F}$, $F = \{a_0, a_1, \dots, a_n\}$, where $0 = a_0 < a_1 < \dots < a_n$. We choose $\delta$ such that

$$0 < 2\delta < \min\{(a_j - a_{j-1})\colon\ j = 1, 2, \dots, k\}$$

and

$$\frac{\delta}{a_1 - \delta} < \varepsilon$$

such a $\delta$ does exist. Proceeding analogously to the proof of Theorem 7.13 of [20], we get

$$\begin{aligned} S_F(f) &\leqslant (o)\lim_n (S)\int_X f_n \, \mathrm{d}P + \frac{\delta}{a_1-\delta}(o)\lim_n (S)\int_X f_n \, \mathrm{d}P \\ &\leqslant (o)\lim_n (S)\int_X f_n \, \mathrm{d}P + \varepsilon c, \end{aligned}$$

and hence

$$(S)\int_X f \, \mathrm{d}P = \sup_{F\in\mathscr{F}} S_F(f) \leqslant (o)\lim_n (S)\int_X f_n \, \mathrm{d}P + \varepsilon c.$$

Due to arbitrariness of $\varepsilon \in \mathbb{R}^+$, (20) yields

$$(S)\int_X f \, \mathrm{d}P \leqslant (o)\lim_n (S)\int_X f_n \, \mathrm{d}P.$$

The converse inequality follows easily from the monotonicity of the integral $(S)\int_X f \, \mathrm{d}P$. $\square$

We have the following consequences of Theorem 4.1:

**Corollary 4.2.** *If $(\alpha_n)_n$ is any decreasing sequence of positive real numbers with $\inf_n \alpha_n = 0$, then*

$$(o)\lim_{n\to+\infty} (S)\int_X (f \wedge \alpha_n) \, \mathrm{d}P = 0.$$

P r o o f. The proof is similar to the one of Lemma 7.5 (ii) of [20], p. 163. $\square$

**Corollary 4.3** (Fatou's Lemma)**.** *Let $c \in R$, $c \geqslant 0$ $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be any sequence of nonnegative $(S)$-integrable functions with $(S)\int_X f_n dP \leqslant c$ for every $n \in \mathbb{N}$, and $f \equiv \liminf_n f_n$.*

*Then*

$$(S)\int_X f \, \mathrm{d}P \leqslant (o)\liminf_n (S)\int_X f_n \, \mathrm{d}P.$$

P r o o f. First of all, we note that $f$ is $(S)$-integrable, thanks to Theorem 4.1. For each $n \in \mathbb{N}$, let $h_n = \inf_{i\geqslant n} f_i$. Then $0 \leqslant h_n \uparrow f$ and

$$(S)\int_X h_n \, \mathrm{d}P \leqslant (S)\int_X f \, \mathrm{d}P \quad \forall n \in \mathbb{N}.$$

Again by Theorem 4.1, we get

$$\begin{aligned}(S)\int_X f\,\mathrm{d}P &= (o)\lim_n (S)\int_X h_n\,\mathrm{d}P\\ &= (o)\liminf_n (S)\int_X h_n\,\mathrm{d}P \leqslant (o)\liminf_n (S)\int_X f_n\,\mathrm{d}P.\end{aligned}$$

This concludes the proof. □

We now recall the following fundamental representation theorem for Riesz spaces ([1], [18], [24]).

**Theorem 4.4.** *Given a Dedekind complete Riesz space $R$, there exists a compact Stonian topological space $\Omega$, unique up to homeomorphisms, such that $R$ can be embedded as a solid subspace of $\mathscr{C}_\infty(\Omega) = \{f \in \tilde{\mathbb{R}}^\Omega\colon f$ is continuous, and $\{\omega\colon |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$. Moreover, if $(a_\lambda)_{\lambda\in\Lambda}$ is any family such that $a_\lambda \in R\,\forall\lambda$ and $a = \inf_\lambda a_\lambda \in R$ (where the infimum is taken with respect to $R$), then $a = \inf_\lambda a_\lambda$ with respect to $\mathscr{C}_\infty(\Omega)$, and the set $\{\omega\in\Omega\colon (\inf_\lambda a_\lambda)(\omega) \neq \inf_\lambda a_\lambda(\omega)\}$ is meager in $\Omega$.*

We now turn to another version of the monotone convergence theorem. In order to prove it, we first establish

**Lemma 4.5.** *Let $[a,b]\subset\mathbb{R}$, $u_n\colon [a,b]\to R$ let $(n\in\mathbb{N}\cup\{0\})$ be monotone decreasing functions, such that*

$$u_n(t) = \inf_{s<t} u_n(s)\quad \forall\, t\in(a,b], \forall n\in\mathbb{N}\cup\{0\}; \tag{21}$$

$$u_n(t) \geqslant u_{n+1}(t)\quad \forall\, t\in[a,b], \forall\, n\geqslant 1; \tag{22}$$

$$\inf_n u_n(t) = u_0(t)\quad \forall\, t\in[a,b]. \tag{23}$$

*Then*

$$\int_a^b u_0(t)\,\mathrm{d}t = \inf_{n\geqslant 1}\int_a^b u_n(t)\,\mathrm{d}t = (o)\lim_{n\to+\infty}\int_a^b u_n(t)\,\mathrm{d}t.$$

P r o o f. First of all, we observe that for every $n\in\mathbb{N}\cup\{0\}$ the integral $\int_a^b u_n(t)\,\mathrm{d}t$ is the limit, for $l\to+\infty$, of the Riemann sums of the type

$$\sum_{i=1}^{2^l}(a_i^{(l)} - a_{i-1}^{(l)})u_n(a_i^{(l)}), \tag{24}$$

where the $a_i^{(l)}$'s, $l\in\mathbb{N}$, $i = 1,2,\ldots,2^l$, are taken in such a way that $a_0^{(l)} = a$, $a_{2^l}^{(l)} = b$, and the division generated by the $a_i^{(l)}$'s divides the interval $[a,b]$ in $2^l$ equal parts.

Denote by $\mathscr{Y}$ the set of points of all these divisions and let $\mathscr{Q}$ be the union of $\mathscr{Y}$ and the rational numbers contained in $[a,b]$; we note that $\mathscr{Q}$ is a countable dense subset of $[a,b]$.

Let now $\Omega$ be as in Theorem 4.4. We note that there exists a meager set $N^* \subset \Omega$ such that, for all $\omega \notin N^*$, we have

$$\left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega)\right] = \left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right]\right](\omega), \tag{25}$$

and for each $\omega \notin N^*$ and $s \in \mathscr{Q}$ we get

$$u_n(s)(\omega) \geqslant u_{n+1}(s)(\omega) \quad \forall\, n \geqslant 1,$$
$$\lim_{n\to+\infty} u_n(s)(\omega) = \inf_{n\geqslant 1} u_n(s)(\omega) = u_0(s)(\omega),$$

and all quantities involved are *real* numbers. Now, for all $s \in [a,b] \cap \mathscr{Q}$, $\forall \omega \notin N^*$ and $\forall\, n \in \mathbb{N} \cup \{0\}$, set

$$w_{n,\omega}(s) = u_n(s)(\omega).$$

For each $t \in [a,b]$, $\omega \notin N^*$ and $n \in \mathbb{N} \cup \{0\}$, put

$$w_{n,\omega}(t) = \inf_{s\leqslant t, s\in\mathscr{Q}} w_{n,\omega}(s). \tag{26}$$

By (26) and since the $w_{n,\omega}$'s are decreasing, their integrals can be evaluated analogously to in (24), and thus we get, $\forall\, n \in \mathbb{N} \cup \{0\}$ and $\forall\, \omega \notin N^*$,

$$\int_a^b w_{n,\omega}(t)\,\mathrm{d}t = \left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega). \tag{27}$$

We note that

$$w_{n,\omega}(s) \downarrow w_{0,\omega}(s) \quad \forall \omega \notin N^*, \quad \forall\, s \in [a,b] \cap \mathscr{Q},\ s \geqslant 0. \tag{28}$$

Furthermore, $\forall\, \omega \notin N^*$ and $t \geqslant 0$, $t \in [a,b]$, by “interchanging the infima involved” we get

$$\begin{aligned} \inf_n w_{n,\omega}(t) &= \inf_n \Big[\inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} w_{n,\omega}(s)\Big] = \inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} \big[\inf_n w_{n,\omega}(s)\big] \\ &= \inf_{s,t\in[a,b], s\leqslant t, s\in\mathscr{Q}} [w_{0,\omega}(s)] = w_{0,\omega}(t), \end{aligned} \tag{29}$$

and thus

$$w_{n,\omega}(t) \downarrow w_{0,\omega}(t) \quad \forall\, \omega \notin N^*, \quad \forall\, t \in [a,b]. \tag{30}$$

From (25), (27) and (30), and applying the classical (dominated) convergence theorem for real-valued functions, we get, $\forall\, \omega \notin N^*$:

$$\left[\int_a^b u_0(t)\,\mathrm{d}t\right](\omega) = \int_a^b w_{0,\omega}(t)\,\mathrm{d}t = \inf_n \left[\int_a^b w_{n,\omega}(t)\,\mathrm{d}t\right] \tag{31}$$
$$= \inf_n \left[\left[\int_a^b u_n(t)\,\mathrm{d}t\right](\omega)\right] = \left[\inf_n \left[\int_a^b u_n(t)\,\mathrm{d}t\right]\right](\omega).$$

From this, since $N^*$ is meager and the complement of every meager subset of $\Omega$ is dense in $\Omega$, it follows that

$$\int_a^b u_0(t)\,\mathrm{d}t = \inf_n \int_a^b u_n(t)\,\mathrm{d}t.$$

Thus we get the assertion. $\square$

We now are in position to prove

**Theorem 4.6.** *Let $(f_n\colon X \to \tilde{\mathbb{R}})_n$ be a decreasing sequence of nonnegative $(S)$-integrable functions and let $f = \inf_n f_n$ be the pointwise infimum. Then $f$ is $(S)$-integrable and*

$$(S)\int_X f\,\mathrm{d}P = \inf_n (S)\int_X f_n\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n\,\mathrm{d}P.$$

Proof. First of all, since $0 \leqslant f \leqslant f_1$, it follows from Proposition 3.11 that $f$ is integrable. Moreover, we observe that, proceeding similarly as in the first half of p. 164 of [20] and taking into account Lemma 3.14 we can suppose, without loss of generality, that the functions $f_n$ and $f$ are equibounded by a positive number $A$.

For each $t \geqslant 0$ and $n \in \mathbb{N}$, $n \geqslant 1$, let $u_n(t) = P(\{x \in X\colon\ f_n(x) \geqslant t\})$, and $\forall\, t \geqslant 0$ let $u_0(t) = P(\{x \in X\colon\ f(x) \geqslant t\})$.

Proceeding analogously to the proof of Theorem 3.6 we get

$$(S)\int_X f\,\mathrm{d}P = \int_0^A u_0(t)\,\mathrm{d}t \tag{32}$$

and

$$(S)\int_X f_n\,\mathrm{d}P = \int_0^A u_n(t)\,\mathrm{d}t \quad \forall\, n \geqslant 1. \tag{33}$$

Since $P$ is a continuous capacity and $f_n \downarrow f$, the functions $u_n$, $n \in \mathbb{N} \cup \{0\}$, satisfy conditions (21), (22) and (23). Applying Lemma 4.5 with $[a, b] = [0, A]$ and using (32) and (33), we conclude that

$$\begin{aligned}(S)\int_X f \,\mathrm{d}P &= \int_0^A u_0(t)\,\mathrm{d}t = \inf_{n\geqslant 1}\int_a^b u_n(t)\,\mathrm{d}t \\ &= (o)\lim_{n\to+\infty}\int_a^b u_n(t)\,\mathrm{d}t = \inf_n (S)\int_X f_n\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n\,\mathrm{d}P,\end{aligned}$$

which is the assertion. □

We now prove

**Theorem 4.7.** *Let $c \in R$, let $(f_n)_n$ be a sequence of $(S)$-integrable functions and $f$ a measurable function such that $f_n \downarrow f$ and*

$$\int_X f_n\,\mathrm{d}P \geqslant c \quad \forall\, n \in \mathbb{N}.$$

*Then $f$ is $(S)$-integrable and*

$$(S)\int_X f\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n\,\mathrm{d}P = \inf_n (S)\int_X f_n\,\mathrm{d}P.$$

Proof. Since $f_n \downarrow f$, we have $f_n^+ \downarrow f^+$ and $f_n^- \uparrow f^-$. Further,

$$\begin{aligned}0 \leqslant (S)\int_X f_n^-\,\mathrm{d}P &= (S)\int_X f_n^+\,\mathrm{d}P - (S)\int_X f_n\,\mathrm{d}P \\ &\leqslant (S)\int_X f_1^+\,\mathrm{d}P - (S)\int_X f_n\,\mathrm{d}P \leqslant (S)\int_X f_1^+\,\mathrm{d}P - c,\end{aligned}$$

and thus we get that the integrals $(S)\int_X f_n^-\,\mathrm{d}P$, $n \in \mathbb{N}$, are bounded from above by an element of $R$. By Theorem 4.1, $f$ is Šipoš-integrable and

(34) $$(S)\int_X f^-\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n^-\,\mathrm{d}P = \sup_n (S)\int_X f_n^-\,\mathrm{d}P.$$

Moreover, by Theorem 4.6, we get integrability of $f^+$ and

(35) $$(S)\int_X f^+\,\mathrm{d}P = (o)\lim_n (S)\int_X f_n^+\,\mathrm{d}P = \inf_n (S)\int_X f_n^+\,\mathrm{d}P.$$

Thus, from (34), (35) and Theorem 3.9 we obtain

$$\begin{aligned}(o)\lim_n (S)\int_X f_n\,\mathrm{d}P &= (o)\lim_n (S)\int_X f_n^+\,\mathrm{d}P - (o)\lim_n (S)\int_X f_n^-\,\mathrm{d}P \\ &= (S)\int_X f^+\,\mathrm{d}P - (S)\int_X f^-\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P,\end{aligned}$$

which is the assertion. □

The proof of the next theorem is similar to those of Theorem 4.7 and of Theorem 7.15, p. 166 of [20], if we take into account that $f_n \uparrow f$ implies $f_n^+ \uparrow f^+$ and $f_n^- \downarrow f^-$.

**Theorem 4.8.** *Let $c \in R$, $c \geqslant 0$, let $(f_n)_n$ be a sequence of $(S)$-integrable functions and $f$ a measurable function such that $f_n \uparrow f$ and*

$$\int_X f_n \,\mathrm{d}P \leqslant c \quad \forall\, n \in \mathbb{N}.$$

*Then $f$ is $(S)$-integrable and*

$$(S)\int_X f \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P = \sup_n (S)\int_X f_n \,\mathrm{d}P.$$

We now state a version of the Lebesgue convergence dominated theorem, which is a consequence of Theorems 4.7 and 4.8 and whose proof is similar to the one of Theorem 7.16 of [20]:

**Theorem 4.9.** *If $(f_n)_n$ is a sequence of measurable functions which converges pointwise to a measurable function $f$ and if $g$ is an $(S)$-integrable function with $|f_n| \leqslant g\ \forall\, n \in \mathbb{N}$, then $f$ is $(S)$-integrable and*

$$(S)\int_X f \,\mathrm{d}P = (o)\lim_n (S)\int_X f_n \,\mathrm{d}P.$$

## 5. The submodular theorems

In this section we prove some theorems for the Šipoš integral in the case when the involved capacities are submodular.

**Theorem 5.1.** *Let $P\colon \mathscr{A} \to R$ be a submodular capacity and let $f, g\colon X \to \tilde{\mathbb{R}}$ be two nonnegative measurable functions. Then*

$$(S)\int_X (f \wedge g) \,\mathrm{d}P + (S)\int_X (f \vee g) \,\mathrm{d}P \leqslant (S)\int_X f \,\mathrm{d}P + (S)\int_X g \,\mathrm{d}P$$

*(finite or $+\infty$). Moreover, if $f$ and $g$ are integrable, then $f \wedge g$ and $f \vee g$ are integrable too.*

Proof. If $(S)\int_X f\,\mathrm{d}P = +\infty$ or $(S)\int_X g\,\mathrm{d}P = +\infty$ the assertion is trivial. Let $f$ and $g$ be both $(S)$-integrable. $(S)$-integrability of $f \wedge g$ follows immediately from Proposition 3.11.

We now prove that $f \vee g$ is $(S)$-integrable. To this aim, pick arbitrarily $F \in \mathscr{F}$ with $F = \{a_0, a_1, \dots, a_n\}$, where $a_0 = 0 < a_1 < \dots < a_n$. Set

$$A_i = \{x\colon\ f(x) \geqslant a_i\}, \quad B_i = \{x\colon\ g(x) \geqslant a_i\}, \qquad i = 0, 1, \dots, n.$$

Proceeding analogously to the proof of Theorem 7.17 of [20], thanks to the submodularity of $P$ we get:

$$S_F(f \wedge g) + S_F(f \vee g) \leqslant S_F(f) + S_F(g) \leqslant \int_X f\,\mathrm{d}P + \int_X g\,\mathrm{d}P. \tag{36}$$

From (36), taking into account the Dedekind completeness of $R$, we have

$$\begin{aligned}(o) \lim_{F\in\mathscr{F}} &[S_F(f \wedge g) + S_F(f \vee g)] \\ &= (o) \lim_{F\in\mathscr{F}} [S_F(f \wedge g)] + (o) \lim_{F\in\mathscr{F}} [S_F(f \vee g)] \\ &= \sup_{F\in\mathscr{F}} [S_F(f \wedge g)] + \sup_{F\in\mathscr{F}} [S_F(f \vee g)] \in R.\end{aligned} \tag{37}$$

From the $(S)$-integrability of the function $f \wedge g$ and from (37) we get

$$(o) \lim_{F\in\mathscr{F}} [S_F(f \vee g)] = \sup_{F\in\mathscr{F}} [S_F(f \vee g)] \in R,$$

that is, the $(S)$-integrability of $f \vee g$. Taking the order limits for $F \in \mathscr{F}$, from (36) and (37) we obtain:

$$\begin{aligned}(S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P &\geqslant (o) \lim_{F\in\mathscr{F}} [S_F(f \wedge g) + S_F(f \vee g)] \\ &= (S)\int_X (f \wedge g)\,\mathrm{d}P + (S)\int_X (f \vee g)\,\mathrm{d}P,\end{aligned}$$

which is the assertion. □

Proceeding analogously to Theorem 5.1, it is possible to prove

**Proposition 5.2.** *If $f$ and $g$ are nonnegative measurable functions and $P$ is an $R$-valued subadditive capacity, then*

$$(S)\int_X (f \vee g)\,\mathrm{d}P \leqslant (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

(For the real case, see [20], Corollary 7.5, p. 168.)

We now state the submodular theorem (see also [5]).

**Proposition 5.3.** *Let $P\colon \mathscr{A} \to R$ be a submodular capacity and let $f, g \in \tilde{\mathbb{R}}^X$ be two nonnegative $(S)$-integrable functions. Then*

$$(S)\int_X (f+g)\,\mathrm{d}P \leqslant (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

*Moreover, if $f$ and $g$ are $(S)$-integrable, then $f+g$ is $(S)$-integrable too.*

Proof. If either $f$ or $g$ is not $(S)$-integrable, then the assertion is trivial. If both $f$ and $g$ are $(S)$-integrable, then, by virtue of the inequality $0 \leqslant f+g \leqslant 2(f \vee g)$ and Proposition 3.11, we get that $f+g$ is $(S)$-integrable. For the remaining part, see [5]. □

Proceeding analogously to Corollary 7.6 of [20], p. 173, it is possible to prove

**Theorem 5.4.** *Let $f$ be a measurable function and $P$ an $R$-valued submodular capacity. Then $f$ is $(S)$-integrable if and only if $|f|$ is $(S)$-integrable.*

**Remark 5.5.** We observe that, in general, the hypothesis of submodularity of $P$ cannot be dropped, not even in the case $R = \mathbb{R}$: indeed, if $P$ is a real-valued not submodular capacity, there exist some $(S)$-integrable functions $f$ (with respect to $P$) such that $|f|$ is not $(S)$-integrable (see [20], Example 3.16, p. 161).

Similarly to [20], Corollary 7.7, p. 174 and Corollary 7.8, p. 175, it is easy to prove the following two theorems:

**Theorem 5.6.** *If $P\colon \mathscr{A} \to R$ is a mean and $f, g$ are $(S)$-integrable, then*

$$(S)\int_X (f+g)\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

**Theorem 5.7.** *If $P\colon \mathscr{A} \to R$ is a capacity and $f$, $g$ are $(S)$-integrable and comonotonic, then*

$$(S)\int_X (f+g)\,\mathrm{d}P = (S)\int_X f\,\mathrm{d}P + (S)\int_X g\,\mathrm{d}P.$$

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