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The symplectic egg in classical and quantum mechanics

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Symplectic geometry is the language of Classical Mechanics in its Hamiltonian formulation, and it also plays a crucial role in Quantum Mechanics. Symplectic geometry seemed to be well understood until 1985, when the mathematician Gromov discovered a surprising and unexpected property of canonical transformations: the non-squeezing theorem. Gromov's result, nicknamed the "principle of the symplectic camel," seems at first sight to be an abstruse piece of pure mathematics. It turns out that it has fundamental—and unsuspected—consequences in the interpretations of both Classical and Quantum Mechanics, because it is essentially a classical form of the uncertainty principle. We invite the reader to a journey taking us from Gromov's non-squeezing theorem and its dynamical interpretation to the quantum uncertainty principle, opening the way to new insights.
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I. PROLOGUE

Take an egg—preferably a hard boiled one—and cut it in half along its middle using a very sharp knife. The surface of section will be roughly circular and have area πr^2 (see Fig. 1). Next, take a new egg with the same size and cut it this time along a line joining the egg's tops, again as shown in Fig. 1. This time we get an elliptic surface of section with area πR^2 that is larger than that of the disk we got previously. So far, so good. But if you now take two *symplectic* eggs and do the same thing, then both sections will have exactly same area. What's more, it doesn't matter which plane passing through the center of the egg you cut, you will always get sections having the same area! This is admittedly a very strange property, which you probably have never experienced (at least in a direct way) in everyday life.

But what is a symplectic egg? The eggs we are cutting are metaphors for ellipsoids; an ellipsoid is a round ball that has been deformed by a linear transformation of space, i.e., a transformation preserving the alignment of three, or more, points. In mathematics such transformations are represented by matrices. Thus, the datum of an ellipsoid is the same thing as the data of a ball and a matrix. What we call a symplectic egg is an ellipsoid corresponding to the case where the matrix is symplectic (we'll define the concept in a moment). The reason for which the only symplectic egg you have seen on your breakfast table is flat (a fried egg!) is because the number of rows and columns of a symplectic matrix must always be even. Since we are unable to visualize things in more than three dimensions, the only symplectic eggs that are accessible to our perception are two dimensional.

But what is a symplectic matrix? In the case of the smallest (even) dimension, two, a (square) matrix

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

is symplectic if it has determinant one:

$$ad - bc = 1. \tag{2}$$

In higher dimensions (4, 6, 8, etc.) there are many more conditions—10 if the dimension is 4, 21 if it is 6, and $n(2n+1)$ if it is $2n$. We will write these conditions explicitly in Sec. III A.

So far, so good. But where do symplectic eggs come from, and what are they good for? Let me first tell you where symplectic matrices come from. They initially come from the study of the motions of celestial bodies, which is quite rich in mathematical concepts, some of which go back to the observations of Tycho Brahe and the work of Galileo Galilei and Johannes Kepler (some of the "Giants" on the shoulder's of which Isaac Newton stood). But the notion of a symplectic matrix, or more generally that of a symplectic transformation, did really have a long time to wait until it appeared explicitly and was recognized as a fundamental concept. Although implicit in the work of Hamilton and Lagrange on classical and celestial mechanics, the word "symplectic" was first coined by the mathematician Hermann Weyl in his book *The Classical Groups: their invariants and representations* (Princeton, 1939), just before World War II. Even still, as Ian Stewart reminds us in his Nature article *The Symplectic Camel*,[1] such transformations were a rather baffling oddity which presumably existed for some purpose—but which? It was only later agreed that the purpose of symplectic transformations is dynamics; that is, the study of *motion*.

Let me explain this in a little more detail. If we have a physical system consisting of "particles" (grains of sand, planets, spacecraft, or quarks), it is economical from both a notational and computational point of view to describe their motion (i.e., their location and velocity) by specifying a phase space vector, which is a matrix consisting of only one column. For instance, if we are dealing with a single particle with coordinates (x, y, z) and momentum (p_x, p_y, p_z), the phase space vector will be the column vector, whose entries are x, y, z, p_x, p_y, p_z. If we have a large number N of particles

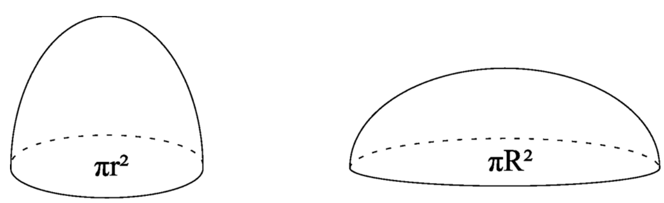

Fig. 1. An illustration of a hard-boiled egg that has been cut through its center by two different planes; the cross-sectional areas exposed by the cuts have different areas. For a symplectic egg, the cross-sectional areas will be the same no matter which plane cuts the egg.

with coordinates (x_i, y_i, z_i) and momenta $(p_{x_i}, p_{y_i}, p_{z_i})$, the phase space vector is obtained by first writing all the position coordinates and thereafter the momentum coordinates in corresponding order. The set of all such vectors form the phase space of our system of particles. It turns out that the knowledge of a certain function—the Hamiltonian (or energy) function—allows us to both predict and retrodict the motion of our particles; this is done by solving the Hamilton equations of motion, which, in the case $n=1$, are given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x}, \quad \frac{dp_x}{dt} = -\frac{\partial H}{\partial x}, \tag{3}$$

with similar relations for the other coordinates. Mathematically, these equations are just a fancy way to write Newton's second law $F=ma$. Thus, knowing (exactly) the positions and momenta at some initial time, we can determine them at any future (or past) time. The surprising, and for us very welcome, fact is that the transformation that takes the initial configuration to the final configuration is always a symplectic transformation! These transformations act on the phase vectors, and once this action is known we can determine the future of the entire system of particles (mathematicians would say we are in presence of a "phase space flow"). The relation between symplectic transformations and symplectic matrices is that we can associate a symplectic matrix to every symplectic transformation—it is just the Jacobian matrix of that transformation.

The symplectic egg is a special case of a deep mathematical theorem discovered in 1985 by the mathematician Gromov,[2] who won the Abel Prize in 2010 for his discovery (the Abel Prize is the equivalent of the Nobel Prize in mathematics). Gromov's theorem is nicknamed the "principle of the symplectic camel,"[1,3–5] and it tells us that it impossible to squeeze a symplectic egg through a hole in a plane of "conjugate coordinates" if its radius is larger than that of the hole. That one can do this with an ordinary (this time uncooked) egg is easy to demonstrate in your kitchen: simply place the egg in a cup of vinegar (Coca Cola will do as well) for 24 h; you will then be able to squeeze that egg through the neck of a bottle without any effort.

The marvelous thing about the symplectic egg is that it contains quantum mechanics in a nutshell, or perhaps more accurately, in an eggshell! Choose as radius $\sqrt{\hbar}$, where $\hbar = h/2\pi$ with h being Planck's constant. Then each surface of section will have area $\pi\hbar = h/2$. In Refs. 4 and 6, I have called such a tiny symplectic egg a *quantum blob*. It is possible—and in fact quite easy if you know the rules of the game—to show that this is equivalent to the uncertainty principle of quantum mechanics. The thing to remember here is that a classical property (i.e., a property involving usual motions, such as those of planets), here symbolized by the symplectic egg, contains an imprint of quantum mechanics (or is it the other way around?). In fact, the analogy between "classical" and "quantum" can actually be pushed much further, as I have shown with Basil Hiley.[7] But this, together with the notion of emergence, is another story.

Some of the ideas presented here are found in my *Physics Reports* paper[8] with Luef; they are developed and presented here in a way more accessible to a general audience.

II. NOTATION AND TERMINOLOGY

Position and moment vectors will be written as column vectors

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix},$$

and the corresponding phase vector is thus

$$\begin{pmatrix} x \\ p \end{pmatrix} = (x,p)^T = (x_1,\ldots,x_n;p_1,\ldots,p_n)^T,$$

where $(\cdots)^T$ indicates transposition. The integer n is unspecified; we will call it the number of degrees of freedom. If the vector $(x,p)^T$ denotes the phase vector of a system of N particles, then $n=3N$ and the numbers x_1, x_2, x_3 (p_1, p_2, p_3) can be identified with the positions x, y, z (momenta p_x, p_y, p_z) of the first particle, x_4, x_5, x_6 (p_4, p_5, p_6) with those of the second particle, and so on. This is not the only possible convention, but our choice has the advantage of making formulas involving symplectic matrices particularly simple and tractable. For instance, the "standard symplectic matrix" J is given by

$$J = \begin{pmatrix} 0 & I_{\mathrm{d}} \\ -I_{\mathrm{d}} & 0 \end{pmatrix}, \tag{4}$$

where I_{d} is the $n\times n$ identity matrix and 0 the $n\times n$ zero matrix. We note that

$$J^2 = -I_{\mathrm{d}}, \quad J^T = J^{-1} = -J. \tag{5}$$

III. THE SYMPLECTIC EGG

"*One should not increase, beyond what is necessary, the number of entities required to explain anything*" *(William of Ockham, alias Doctor Invincibilis)*

A. Symplectic matrices

Let S be a (real) square matrix of size $2n$. We say that S is a symplectic matrix if it satisfies the condition

$$S^T J S = J. \tag{6}$$

The standard symplectic matrix J is itself symplectic since we have $J^T J J = -J^2 J = J$, in view of Eq. (5).

The definition above is, admittedly, somewhat abrupt. Where does it come from? To answer this question, let us make a little geometric digression. The usual way to measure relative positions in our everyday world consists of using a *metric*. For instance, in Euclidean geometry, the metric is associated with the inner product: if $u=(u_1,\ldots,u_m)$, $v=(v_1,\ldots,v_m)$ are two vectors in an m-dimensional space, the inner product is $u\cdot v = u_1 v_1 + \cdots + u_m v_m$; the length (or norm) of the vector u is then $|u| = \sqrt{u\cdot u} = u_1^2 + \cdots + u_m^2$. When studying Euclidean geometry one is interested in linear transformations preserving length or, equivalently, preserving the inner product. Representing such a transformation by its matrix, say M, the condition $Mu\cdot Mv = u\cdot v$ is equivalent to $u^T M^T M v = u^T v$, which tells us that $M^T M = I$ (the identity matrix). Thus, linear transformations preserving the Euclidean metric are the well-known orthogonal transformations studied in elementary textbooks.

In symplectic geometry, one is not interested in calculating lengths, but rather one focuses on the notion of *area*.

Instead of an inner product, one defines a *symplectic (or skew) product*. This product can only be defined on even-dimensional linear spaces, e.g., the phase space of classical mechanics. In this case, it is customary to define the symplectic product of two vectors $z=(x,p)$ and $z'=(x',p')$ by

$$z \wedge z' = (z')^T Jz. \tag{7}$$

Notice that it does not make sense to define the "symplectic length" of a vector by the formula $|z| = \sqrt{z \wedge z}$, because we always have $z \wedge z = 0$. However, the number $z \wedge z'$ has a simple geometric interpretation: in position and momentum coordinates, we have

$$z \wedge z' = p \cdot x' - p' \cdot x = \sum_{j=1}^{n} p_j x'_j - p'_j x_j \tag{8}$$

which we can rewrite as

$$z \wedge z' = -\sum_{j=1}^{n} \begin{vmatrix} x_j & x'_j \\ p_j & p'_j \end{vmatrix}. \tag{9}$$

Thus, up to the sign, the symplectic product $z \wedge z'$ is the sum of the algebraic areas of the parallelograms spanned by the projections of the vectors z, z' on the planes x_j, p_j of conjugate coordinates.

Now, as Euclidean geometry is the study of linear transformations preserving the inner product, symplectic geometry is the study of linear transformations preserving the symplectic product. For a transformation described by a matrix S, this condition reads $Sz \wedge Sz' = z \wedge z'$, and taking the definition above into account, this gives $(Sz')^T JSz = (z')^T Jz$. But since $(Sz')^T JSz = (z')^T S^T JSz$, we see that $(z')^T S^T JSz = (z')^T Jz$ for all vectors z, z'; that is, $S^T JS = J$, which is precisely condition (6) defining a symplectic matrix. One often writes

$$\sigma(z, z') = z \wedge z' \tag{10}$$

and calls the function σ a *symplectic form*. Note the formal similarity between the definitions of orthogonal and symplectic transformations:

$$M \text{ is orthogonal} \iff M^T IM = I$$
$$S \text{ is symplectic} \iff S^T JS = J.$$

One passes from the first to the second by replacing the identity I with the standard symplectic matrix J.

Assume now that we write the matrix S in block form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{11}$$

where A, B, C, D are matrices of size n. It is a simple exercise in matrix algebra to show that condition (6) is equivalent to the following constraints on the blocks A, B, C, D:

$$A^T C = C^T A,\ B^T D = D^T B,\ \text{and}\ A^T D - C^T B = I_d. \tag{12}$$

Notice that the two first conditions imply that both products $A^T C$ and $B^T D$ *are symmetric*. These conditions collapse to the identity $ad - bc = 1$ [Eq. (2)] when $n = 1$. In this case A, B, C, D are the numbers a, b, c, d so that $A^T C = ac$ and $B^T D = bd$; the condition $A^T D - C^T B = I_d$ then reduces to $ad - bc = 1$.

The product of two symplectic matrices is a symplectic matrix, for if S and S' satisfy Eq. (6), then $(SS')^T JSS' = S'^T (S^T JS) S' = S'^T JS' = J$. Also, symplectic matrices are invertible and their inverses are symplectic as well. To see this, first take the determinant of both sides of $S^T JS = J$ to get $\det(S^T JS) = \det J$. But since $\det J = 1$ this becomes $(\det S)^2 = 1$; hence S is indeed invertible. Knowing this, we rewrite $S^T JS = J$ as $JS = (S^{-1})^T J$, from which it follows that $(S^{-1})^T JS^{-1} = JSS^{-1} = J$, and hence S^{-1} is symplectic. The symplectic matrices of equal size thus form a group called the symplectic group and denoted by $\mathrm{Sp}(2n)$.

An interesting property is that the symplectic group is closed under transposition—if S is a symplectic matrix, then so is S^T. To see this, take the inverse of the equality $(S^{-1})^T JS^{-1} = J$, which yields $SJ[(S^{-1})^T]^{-1} = SJS^T = J$ (noting that $J^{-1} = -J$ and $[(S^{-1})^T]^{-1} = S^T$). But this means that a matrix is symplectic if and only if its transpose is: inserting S^T in Eq. (6) and noting that $(S^T)^T = S$ gives the condition

$$SJS^T = J. \tag{13}$$

Replacing $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $S^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$ the conditions (12) is thus equivalent to the set of conditions:

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_d. \tag{14}$$

One can obtain other equivalent sets of conditions by using the fact that S^{-1} and $(S^{-1})^T$ are symplectic.

It is very interesting to note that the inverse of a symplectic matrix is

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \tag{15}$$

This is interesting because this formula is very similar to that giving the inverse $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant one. The inversion formula (15) suggests that in a sense symplectic matrices try very hard to mimic the behavior of 2×2 matrices—symplectic geometry is in essence a geometry of *areas*, as already noted in the discussion following Eq. (9). A major manifestation of this property will be discussed below, when we study Gromov's non-squeezing theorem. We will see that this is actually the essence of symplectic geometry, and at the origin of the symplectic egg property!

One final property of symplectic matrices. Recall that when we wanted to show that a symplectic matrix is always invertible, we established the identity $(\det S)^2 = 1$. From this it follows that the determinant of a symplectic matrix is *a priori* either $+1$ or -1. It turns out—though there is no elementary proof of this—that we always have $\det S = 1$ (see, for instance, Section 2.1.1 in Ref. 4; Mackey and Mackey's online paper[9] gives a nice discussion of several distinct methods for proving that symplectic matrices have determinant one).

Conversely, it is not true that any $2n \times 2n$ matrix with determinant one is symplectic when $n > 1$. Consider for instance

$$M = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1/\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1/\lambda \end{pmatrix} \tag{16}$$

with $\lambda \neq 0$. This matrix has determinant one but the condition $AD^T - BC^T = I_d$ in Eq. (14) is violated unless $\lambda = \pm 1$. Another simple example is provided by

$$M = \begin{pmatrix} R(\alpha) & 0 \\ 0 & R(\beta) \end{pmatrix}, \tag{17}$$

where $R(\alpha)$ and $R(\beta)$ are rotation matrices with angles $\alpha \neq \beta$ (this counterexample generalizes to an arbitrary number $2n$ of phase-space dimensions replacing $R(\alpha)$ and $R(\beta)$ by arbitrary but distinct rotations in the x and p spaces, respectively).

For a detailed exposition of symplectic matrices, with complete proofs, see Chapter 2 in Ref. 4.

B. The first Poincaré invariant

Consider $\gamma(t)$, with $0 \leq t \leq 2\pi$, as a loop in phase space. That is, we have $\gamma(t) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$, where $x(0) = x(2\pi)$ and $p(0) = p(2\pi)$; the functions $x(t)$ and $p(t)$ are assumed to be continuously differentiable. By definition, the first Poincaré invariant associated to $\gamma(t)$ is the integral

$$I(\gamma) = \oint_\gamma p\,dx = \int_0^{2\pi} p(t)^T \dot{x}(t)\,dt. \tag{18}$$

The fundamental property from which almost everything else in this paper stems is that $I(\gamma)$ is a symplectic invariant. By this we mean that if we replace the loop $\gamma(t)$ by a new loop $S\gamma(t)$, where S is a symplectic matrix, the first Poincaré invariant remains unchanged:

$$I(\gamma) = \oint_\gamma p\,dx = I(S\gamma) = \oint_{S\gamma} p\,dx. \tag{19}$$

The proof is not very difficult if we carefully use the relations characterizing symplectic matrices (see p.239 of Ref. 10 for a shorter but more abstract proof).

We will first need a differentiation rule for vector-valued functions, generalizing the product formula from elementary calculus. Suppose that

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \tag{20}$$

are vectors depending on the variable t and such that each component $u_j(t)$, $v_j(t)$ is differentiable. Let M be a symmetric matrix of size n and consider the real-valued function $u(t)^T M v(t)$. Its derivative is given by the formula

$$\frac{d}{dt}[u(t)^T M v(t)] = \dot{u}(t)^T M v(t) + u(t)^T M \dot{v}(t), \tag{21}$$

where we are writing $\dot{u} = du/dt$ and $\dot{v} = dv/dt$ as is customary in mechanics.

Let us now go back to the proof of the symplectic invariance of the first Poincaré invariant. Writing the symplectic matrix S in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the loop $S\gamma(t)$ is parametrized by

$$S\gamma(t) = \begin{pmatrix} Ax(t) + Bp(t) \\ Cx(t) + Dp(t) \end{pmatrix}, \quad 0 \leq t \leq 2\pi. \tag{22}$$

We thus have, by definition of the Poincaré invariant,

$$I(S\gamma) = \int_0^{2\pi} [Cx(t) + Dp(t)]^T [A\dot{x}(t) + B\dot{p}(t)]\,dt. \tag{23}$$

Expanding the product in the integrand we have $I(S\gamma) = I_1 + I_2$, where

$$I_1 = \int_0^{2\pi} x(t)^T C^T A \dot{x}(t)\,dt + \int_0^{2\pi} p(t)^T D^T B \dot{p}(t)\,dt \tag{24}$$

$$I_2 = \int_0^{2\pi} x(t)^T C^T B \dot{p}(t)\,dt + \int_0^{2\pi} p(t)^T D^T A \dot{x}(t)\,dt. \tag{25}$$

We claim that $I_1 = 0$. Recall that $C^T A$ and $D^T B$ are symmetric in view of the two first equalities in Eq. (12); applying the differentiation formula (21) with $u = v = x$ we have

$$\int_0^{2\pi} x(t)^T C^T A \dot{x}(t) dt = \frac{1}{2} \int_0^{2\pi} \frac{d}{dt}[x(t)^T C^T A x(t)]\,dt = 0, \tag{26}$$

because $x(0) = x(2\pi)$. Likewise, using Eq. (21) with $u = v = p$ gives

$$\int_0^{2\pi} p(t) D^T B \dot{p}(t)\,dt = 0, \tag{27}$$

hence $I_1 = 0$ as claimed. We next consider the term I_2. Rewriting the integrand of the second integral as

$$x(t)^T C^T B \dot{p}(t) = \dot{p}(t)^T B^T C x(t)^T \tag{28}$$

(because it is a number, and hence equal to its own transpose), we have

$$I_2 = \int_0^{2\pi} \dot{p}(t)^T B^T C x(t)^T\,dt + \int_0^{2\pi} p(t)^T D^T A \dot{x}(t)\,dt; \tag{29}$$

that is, because $D^T A = I_d + B^T C$ by transposition of the third equality in Eq. (12),

$$I_2 = \int_0^{2\pi} p(t)^T \dot{x}(t)\,dt + \int_0^{2\pi} [p(t)^T B^T C A \dot{x}(t) + \dot{p}(t)^T B^T C A x(t)]\,dt. \tag{30}$$

Again using the Eq. (21) and noting that the first integral is precisely $I(\gamma)$ we get, $D^T A$ being symmetric,

$$I_2 = I(\gamma) + \int_0^{2\pi} \frac{d}{dt}[p(t)^T B^T C A x(t)]\,dt. \tag{31}$$

The equality $I(S\gamma) = I(\gamma)$ then follows by noting that the integral on the right-hand-side is

$$p(2\pi)^T B^T C A x(2\pi) - p(0)^T B^T C A x(0) = 0, \tag{32}$$

since $x(2\pi) = x(0)$ and $p(2\pi) = p(0)$.

The observant reader will have observed that we needed all of the properties of a symplectic matrix contained in Eq. (12), showing that the symplectic invariance of the first Poincaré invariant is a characteristic property of symplectic matrices.

C. Proof of the symplectic egg property

Let us denote by B_R the phase space ball centered at the origin and having radius R. It is the set of all points $z = (x, p)$ such that $|z|^2 = |x|^2 + |p|^2 \leq R^2$. What we call a "symplectic egg" is the image $S(B_R)$ of B_R by a symplectic matrix S. It is thus an ellipsoid in phase space consisting of all points z' such that $z = S^{-1}z'$ is in the ball B_R; that is, $|S^{-1}z'|^2 \leq R^2$. Let us now cut $S(B_R)$ by a plane Π_j of conjugate coordinates x_j, p_j. We get an elliptic surface Γ_j, whose boundary is an ellipse denoted by γ_j. Because this ellipse lies in the plane Π_j we can parametrize it by only specifying coordinates $x_j(t)$, $p_j(t)$, all others being identically zero. Relabeling (if necessary) the coordinates we may as well assume that $j = 1$ so that the curve γ_j can be parametrized as

$$\gamma_j(t) = (x_1(t), 0, \ldots, 0; p_1(t), 0, \ldots, 0)^T \tag{33}$$

for $0 \leq t \leq 2\pi$ with $x_1(0) = x_1(2\pi)$ and $p_1(0) = p_1(2\pi)$. Since $x_k(t) = 0$ and $p_k(t) = 0$ for $k > 1$ the area of the ellipse is given by the formula

$$\text{Area}(\Gamma_1) = \int_0^{2\pi} p_1(t)\dot{x}_1(t)\, dt \tag{34}$$

$$= \sum_{k=1}^{n} \int_0^{2\pi} p_k(t)\dot{x}_k(t)\, dt \tag{35}$$

$$= \oint_{\gamma_1} p\, dx, \tag{36}$$

and hence $\text{Area}(\Gamma_1) = I(\gamma_1)$. Because the inverse matrix S^{-1} is also symplectic, we have $I(\gamma_1) = I(S^{-1}\gamma_1)$. But the loop $S^{-1}\gamma_1$ bounds a section of the ball B_R by a plane (the plane $S^{-1}\Pi_j$) passing through its center. This loop is thus a great circle of B_R and the area of the surface $S^{-1}\Gamma_1$ is exactly πR^2, which was to be proven.

We urge the reader to notice that the assumption that we are cutting $S(B_R)$ with a plane of *conjugate* coordinates is essential, because it is this assumption that allowed us to identify the area of the section with action. Here is a counter-example that shows that this property does not hold for arbitrary sections of $S(B_R)$. Take, for instance,

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1/\lambda_1 & 0 \\ 0 & 0 & 0 & 1/\lambda_2 \end{pmatrix}, \quad \text{with } \lambda_1 > 0,\ \lambda_2 > 0, \text{ and } \lambda_1 \neq \lambda_2, \tag{37}$$

so that $S(B_R)$ is defined by the inequality

$$\frac{1}{\lambda_1}x_1^2 + \frac{1}{\lambda_2}x_2^2 + \lambda_1 p_1^2 + \lambda_2 p_2^2 \leq R^2. \tag{38}$$

The section of $S(B_R)$ contained in the x_1, p_1 plane is the ellipse obtained by setting $x_2 = 0$ and $p_2 = 0$, giving

$$\frac{1}{\lambda_1}x_1^2 + \lambda_1 p_1^2 \leq R^2. \tag{39}$$

This elliptic section has area $\pi R^2\sqrt{\lambda_1}\sqrt{1/\lambda_1} = \pi R^2$ as predicted. If we instead intersect $S(B_R)$ with the x_2, p_1 plane (which is not a plane of conjugate variables), we get the ellipse

$$\frac{1}{\lambda_1}x_1^2 + \lambda_2 p_2^2 \leq R^2, \tag{40}$$

which has an area $\pi R^2\sqrt{\lambda_1/\lambda_2}$ different from πR^2, since $\lambda_1 \neq \lambda_2$.

The assumption that S is symplectic is also essential. Assume that we scramble the diagonal entries of the matrix S above, getting the new matrix

$$S' = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 0 & 1/\lambda_1 \end{pmatrix}. \tag{41}$$

The matrix S' still has determinant one, but it is not symplectic [*cf.* the matrix (16)]. The section $S'(B_R)$ contained in the x_2, p_2 plane is the ellipse

$$\frac{1}{\lambda_1}x_1^2 + \lambda_2 p_1^2 \leq R^2 \tag{42}$$

with area $\pi R^2\sqrt{\lambda_1/\lambda_2} \neq \pi R^2$.

IV. THE SYMPLECTIC CAMEL

The property of the symplectic camel is a generalization of the property of the symplectic egg to arbitrary canonical transformations; it reduces to the latter in the linear case.

A. Gromov's non-squeezing theorem: static formulation

As we mentioned in the Prologue, the property of the symplectic egg is related to a deep topological result, the "non-squeezing theorem" of Gromov[2] published in 1985. To understand this result fully, we have to introduce the notion of a canonical transformation.[4,10–12] A canonical transformation is an invertible, infinitely differentiable mapping

$$f : \begin{pmatrix} x \\ p \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ p' \end{pmatrix} \tag{43}$$

of phase space onto itself whose inverse f^{-1} is also infinitely differentiable and such that its Jacobian matrix

$$f'(x, p) = \frac{\partial(x', p')}{\partial(x, p)} \tag{44}$$

is symplectic at every point (x, p). A symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ automatically generates a linear canonical transformation by letting it act on phase space vectors $\begin{pmatrix} x \\ p \end{pmatrix} \rightarrow S\begin{pmatrix} x \\ p \end{pmatrix}$—it is an invertible transformation (because symplectic matrices are invertible), trivially infinitely differentiable, and its Jacobian matrix is S itself. Phase space translations, mappings of the form

$$\begin{pmatrix} x \\ p \end{pmatrix} \rightarrow \begin{pmatrix} x + x_0 \\ p + p_0 \end{pmatrix}, \tag{45}$$

are also canonical—their Jacobian matrix is just the identity $\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix}$, which is trivially symplectic. By composing linear canonical transformations with translations one obtains the class of all affine canonical transformations.

Here is an example of a nonlinear canonical transformation. Assume that $n = 1$ and denote the phase space variables by r and φ instead of x and p. The transformation defined by $(r, \varphi) \rightarrow (x, p)$ with

$$x = \sqrt{2r}\cos\varphi,\ p = \sqrt{2r}\sin\varphi,\ 0 \leq \varphi < 2\pi, \tag{46}$$

has Jacobian matrix

$$f'(r,\varphi) = \begin{pmatrix} \frac{1}{\sqrt{2r}}\cos\varphi & \frac{1}{\sqrt{2r}}\sin\varphi \\ -\sqrt{2r}\sin\varphi & \sqrt{2r}\cos\varphi \end{pmatrix}, \tag{47}$$

which has determinant one for every choice of r and φ. The transformation f is thus canonical and can be extended without difficulty to the multi-dimensional case by associating a similar transformation to each pair (x_j, p_j). It is in fact a symplectic version of the usual passage to polar coordinates (the reader can verify that the latter is not canonical by calculating its Jacobian matrix); it can also be viewed as the simplest example of action-angle variables.[10–12]

We will see in a moment why canonical transformations play such an important role in physics (and especially in classical mechanics), but let us first state Gromov's theorem.

Gromov's theorem:
No canonical transformation can squeeze a ball B_R through a circular hole in a plane Π_j of conjugate coordinates x_j, p_j with radius $r < R$.

This statement is surprisingly simple, and one can wonder why it took such a long time to discover it. There are many possible answers. The most obvious is that all known proofs of Gromov's theorem are extremely difficult and make use of highly non-trivial techniques from various parts of pure mathematics, so the result cannot be easily derived from elementary principles. Another reason is that it seems, as we will discuss below, to contradict the common conception of Liouville's theorem and was therefore unsuspected.

So what is the relation of Gromov's theorem with our symplectic eggs, and where does its nickname "principle of the symplectic camel" come from? The denomination apparently appeared for the first time in Arnol'd's paper.[3] Recalling that it is stated in the Scriptures

> *...Then Jesus said to his disciples, 'Amen, I say to you, it will be hard for one who is rich to enter the kingdom of heaven. Again I say to you, it is easier for a camel to pass through the eye of a needle than for one who is rich to enter the kingdom of God.'*

The biblical camel is here the ball B_R, and the eye of the needle is the hole in the x_j, p_j plane! (For various interpretations of the word "camel" see the comments following E. Samuel Reich's New Scientist paper[13] about our "symplectic camel" paper.[5])

Let us next show that the section property of the symplectic egg is indeed a linear (or affine) version of Gromov's theorem. It is equivalent to prove that no symplectic egg $S(B_R)$ with radius R larger than that (r) of the hole in the x_j, p_j plane can be threaded through that hole. Passing $S(B_R)$ through the hole means that the section of the symplectic egg contained in the x_j, p_j plane, which has area πR^2, is smaller than the area πr^2 of the hole; hence we must have $R \leq r$.

B. Dynamical interpretation

The reason that canonical transformations play an essential role in physics comes from the fact that Hamiltonian phase flows consist precisely of canonical transformations. Consider a particle of mass m moving along the x-axis under the action of a scalar potential $v(x)$; the particle is subject to a force $F = -\,dV/dx$. Because $F = m\,dv/dt = dp/dt$, the equations of motion can be written

$$\frac{dx}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -\frac{dV}{dx}. \tag{48}$$

Introducing the Hamilton function

$$H(x,p) = \frac{p^2}{2m} + v(x), \tag{49}$$

the system of differential equations (48) is equivalent to Hamilton's equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial p}. \tag{50}$$

More generally, we will consider the n-dimensional version of Eq. (50) that reads

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}, \quad 1 \leq j \leq n. \tag{51}$$

(In mathematical treatments of Hamilton's equations[4,10–12] the function H can be of a very general type, and even depend on time t.) In either case, these equations determine—as does any system of differential equations—a *flow*. By definition, the Hamiltonian flow is the infinite set of mappings ϕ_t^H defined as follows. Suppose we solve the system (51) subject to initial conditions $x_1(0), \ldots, x_n(0)$ and $p_1(0), \ldots, p_n(0)$. Denote the initial vector thus defined $\begin{pmatrix} x(0) \\ p(0) \end{pmatrix}$. Assuming that the solution to Hamilton's equations at time t exists (and is unique), we denote it by $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$. By definition, ϕ_t^H is just the mapping that takes the initial vector to the final vector:

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \phi_t^H \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}. \tag{52}$$

As time varies, the initial point describes a curve in phase space (often called a "flow curve" or "Hamiltonian trajectory").

The essential fact to remember is that each mapping ϕ_t^H is a canonical transformation. Hamiltonian flows are therefore, in particular, volume preserving; this is Liouville's theorem.[10–12] This property follows from the fact that symplectic matrices have determinant one. Since it is not true that every matrix with determinant one is symplectic, as soon as $n > 1$ volume preservation also holds for other transformations, and is therefore not a characteristic property of Hamiltonian flows (see Arnold,[10] Ch. 3, Section 16 for a discussion of this fact). The thing to observe is that volume preservation does not imply conservation of shape, and one could therefore imagine that under the action of a Hamiltonian flow a subset of phase space can be stretched in all directions, and eventually get very thinly spread out over huge regions of phase space, so that the projections on any plane could *a priori* become arbitrary small after some time t. In addition, one may very well envisage that the larger the number n of degrees of freedom, the more that spreading will occur since there are more directions in which the ball is likely to spread! This possibility, which is ruled out by the symplectic camel as we will explain below, has led to many philosophical speculations about Hamiltonian systems. For instance, in his 1989 book Roger Penrose (Ref. 14, pp. 174–184) comes to the conclusion that phase space spreading suggests that "*classical mechanics cannot actually be true of our world*" (p. 183).

Our discussion of Gromov's theorem shows that Hamiltonian evolution is much less disorderly than Penrose thought. To see this, consider again our phase space ball B_R. Its orthogonal projection (or "shadow") on any two-dimensional subspace Π of phase space is a circular surface with area πR^2. Suppose now that we move the ball B_R using a Hamiltonian flow ϕ_t^H and choose for Π the plane Π_j of conjugate coordinates x_j, p_j. The ball will slowly get deformed while maintaining the same volume. But, as a consequence of the principle of the symplectic camel, the area of its "shadow" on any plane Π_j will never decrease below its original value πR^2 (as illustrated in Fig. 2). Why is this so? First, it is clear that if the area of the projection of $f(B_R)$ on a plane x_j, p_j (f a canonical transformation) will never be smaller than πR^2, then we cannot expect that $f(B_R)$ lies inside a cylinder $(p_j - a_j)^2 + (x_j - b_j)^2 = r^2$ if $r < R$. So is the "principle of the symplectic camel" stronger than Gromov's theorem? Not at all, it is equivalent to it! Here is a simple proof. We assume as in Sec. III C that $j = 1$, which does not restrict the generality of the argument. Let γ_1 be the boundary of the projection of $f(B_R)$ on the x_1, p_1 plane; it is a loop encircling a surface Γ_1 with area at least πR^2. The surface Γ_1 can be deformed into a circle with the same area using an area-preserving mapping of the x_1, p_1 plane; call that mapping f_1 and define a global phase-space transformation f by the formula

$$f(x_1, p_1, x_2, p_2, \ldots, x_n, p_n) = (f_1(x_1, p_1), x_2, p_2, \ldots, x_n, p_n) \tag{53}$$

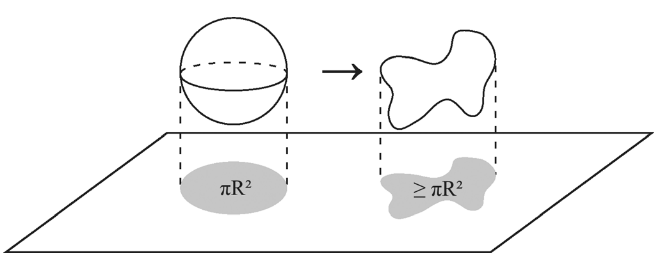

Fig. 2. An illustration demonstrating that a phase-space ball evolving according to Hamiltonian dynamics will always cast a shadow on a plane of conjugate coordinates with an area no less than its starting value.

(We are using in this formula, for obvious reasons of readability, an ordering of the position and momentum variables different from the standard one.) Calculating the Jacobian matrix it is easy to check that the matrix f is a canonical transformation, hence our claim. For a more detailed discussion of this and related topics, see Refs. 5 and 8.

C. The symplectic camel and Newton's second law

Recall that we derived Hamilton's equations for a particle moving in a force field $F = -dV/dx$ by writing down the equations of motion in the form

$$m\frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\frac{dV}{dx}. \tag{54}$$

The observant reader will have noticed that these two equations are just one way to express Newton's second law. More generally for a system of N point-like particles moving in three-dimensional physical space, Newton's second law would be

$$m\frac{dx_j}{dt} = p_j, \quad \frac{dp_j}{dt} = -\frac{dV}{dx_j}, \tag{55}$$

or, equivalently, Hamilton's equations (51) with $1 \le j \le n = 3N$. Thus, for Hamiltonian systems, Gromov's non-squeezing theorem just expresses a very deep and invisible property of Newton's second law!

V. QUANTUM BLOBS

What's in a name? That which we call a rose by any other name would smell as sweet.
Romeo and Juliet, Act 2, Scene 2 (W. Shakespeare)

By definition, a quantum blob is a symplectic egg with radius $R = \sqrt{\hbar}$. The section of quantum blob intersected by a plane of conjugate coordinates will thus have area $\pi\hbar = \frac{1}{2}h$. We will see that quantum blobs qualify as the smallest units of phase space allowed by the uncertainty principle of quantum mechanics. We begin with a very simple example illustrating the basic idea, namely, that a closed (phase space) trajectory cannot be carried by an energy shell smaller (in a sense to be made precise) than a quantum blob. As simple as this example is, it allows us to recover the ground energy of the anisotropic quantum harmonic oscillator.

A. The harmonic oscillator

The fact that the ground-state energy level of a one-dimensional harmonic oscillator

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{56}$$

is different from zero is heuristically justified in the physical literature by the following observation. Because Heisenberg's uncertainty relation $\Delta p_x \Delta x \ge \hbar/2$ prevent us from assigning simultaneously a precise value to both position and momentum, the oscillator cannot be at rest. To show that the lowest energy has the value $\hbar\omega/2$ predicted by quantum mechanics one can then argue as follows. Because we cannot distinguish the origin ($x = 0$, $p = 0$) of phase space from a phase plane trajectory lying inside the double hyperbola $p_x x < \hbar/2$, we must require that the points (x, p) of that trajectory are such that $|p_x x| \geq \hbar/2$; multiplying both sides of the inequality

$$\frac{p_x^2}{m\omega} + m\omega x^2 \geq 2|p_x x| \geq \hbar \tag{57}$$

by $\omega/2$ we then get

$$E = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \geq \frac{1}{2}\hbar\omega, \tag{58}$$

which gives the correct lower bound for the quantum energy. This argument can be reversed—since the lowest energy of an oscillator with frequency ω and mass m is $\hbar\omega/2$, the minimal phase space trajectory will be the ellipse

$$\frac{p_x^2}{m\hbar\omega} + \frac{x^2}{(\hbar/m\omega)} = 1, \tag{59}$$

which encloses a surface with area $h/2$. Everything in this discussion immediately extends to the generalized anisotropic n-dimensional oscillator

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2 x^2 \tag{60}$$

and one concludes that the smallest possible trajectories in x_j, p_j space are the ellipses

$$\frac{p_j^2}{m_j\hbar\omega_j} + \frac{x_j^2}{(\hbar/m_j\omega_j)} = 1. \tag{61}$$

By the same argument as above, using each of the Heisenberg uncertainty relations

$$\Delta p_j \Delta x_j \geq \frac{1}{2}\hbar \tag{62}$$

we recover the correct ground energy level

$$E = \frac{1}{2}\hbar\omega_1 + \frac{1}{2}\hbar\omega_2 + \cdots + \frac{1}{2}\hbar\omega_n \tag{63}$$

as predicted by standard quantum theory.[15] In addition, one finds that the projection of the motion on any plane of conjugate variables x_j, p_j will always enclose a surface having an area at least equal to $h/2$. In other words, the motions corresponding to the lowest possible energy must lie on a quantum blob!

These considerations suggest a strong relationship between quantum blobs and the uncertainty principle.

B. Quantum blobs and uncertainty

The Heisenberg inequalities (62) are a weak form of the quantum uncertainty principle; they are a particular case of the more accurate Robertson-Schrödinger[16,17] inequalities

$$(\Delta p_j)^2(\Delta x_j)^2 \geq \Delta(x_j, p_j)^2 + \frac{1}{4}\hbar^2 \tag{64}$$

(see Messiah[15] for a simple derivation). Here, in addition to the standard deviations Δx_j, Δp_j we have the covariances $\Delta(x_j, p_j)$, which are a measurement of how much the two variables x_j, p_j change together. (We take the opportunity to note that the interpretation of quantum uncertainty in terms of standard deviations goes back to Kennard;[18] Heisenberg's[19] initial formulation was much more heuristic). Contrary to what is often believed the Heisenberg inequalities (62) and the Robertson-Schrödinger inequalities (64) are not statements about the accuracy of our measurements; their derivation assumes perfect instruments (see the discussion in Peres,[20] p. 93). Their meaning is that if the same preparation procedure is repeated a large number of times on an ensemble of systems and is followed by either a measurement of x_j or a measurement of p_j, then the results obtained will have standard deviations Δx_j and Δp_j. In addition, these measurements need not be uncorrelated; this is expressed by the statistical covariances $\Delta(x_j, p_j)$ appearing in the inequalities (64).

It turns out that quantum blobs can be used to give a purely geometric and intuitive idea of quantum uncertainty. Let us first consider the case $n = 1$ and define the covariance matrix by

$$\Sigma = \begin{pmatrix} \Delta x^2 & \Delta(x,p) \\ \Delta(p,x) & \Delta p^2 \end{pmatrix}. \tag{65}$$

Its determinant is $\det\Sigma = (\Delta p)^2(\Delta x)^2 - \Delta(x,p)^2$, so in this case the Robertson-Schrödinger inequality is the same thing as $\det\Sigma \geq \hbar^2/4$. Now to the geometric interpretation. In statistics it is customary to associate with Σ the so-called *covariance ellipse* Ω_Σ defined by

$$\frac{1}{2}(x,p)\Sigma^{-1}\begin{pmatrix} x \\ p \end{pmatrix} \leq 1. \tag{66}$$

The area of this ellipse is $2\pi\sqrt{\det\Sigma}$, which, by virtue of Eq. (65), gives

$$\text{Area}(\Omega_\Sigma) = 2\pi\left[(\Delta p)^2(\Delta x)^2 - \Delta(x,p)^2\right]^{1/2}. \tag{67}$$

The inequality $\det\Sigma \geq \hbar^2/4$ is therefore equivalent to $\text{Area}(\Omega_\Sigma) \geq \pi\hbar = h/2$. We have thus succeeded in expressing the rather complicated Robertson-Schrödinger inequality (64) in terms of the area of a certain ellipse.

In higher dimensions, the same argument applies, but contrary to what common intuition suggests, the Robertson-Schrödinger inequalities will not be expressed in terms of volume (which is the generalization of area to higher dimensions), but again in terms of areas—namely, those of the intersections of the conjugate planes x_j, p_j with the covariance ellipsoid

$$\Sigma = \begin{pmatrix} \Delta(x,x) & \Delta(x,p) \\ \Delta(p,x) & \Delta(p,p) \end{pmatrix}. \tag{68}$$

Here $\Delta(x,x), \Delta(x,p)$, etc. are the $n \times n$ block-matrices $(\Delta(x_i,x_j))_{1\leq i,j\leq n}$, $(\Delta(x_i,p_j))_{1\leq i,j\leq n}$, etc. Notice that the diagonal terms of Σ are just the variances $\Delta x_1^2, \ldots, \Delta x_n^2; \Delta p_1^2, \ldots, \Delta p_n^2$ so that Eq. (68) reduces to Eq. (65) for $n = 1$. Defining the covariance ellipsoid Ω_Σ as above, one then proves that the inequalities (64) are equivalent to the property that the intersection of Ω_Σ with the planes x_j, p_j is at least $h/2$. These inequalities are saturated (i.e., they become equalities) if and only if these intersections have exactly area

$h/2$; that is, if and only if Ω_Σ is a quantum blob! The proof goes as follows (for a detailed argument see Refs. 5 and 8). One first remarks, using a simple algebraic argument, that if Σ is non-singular the Robertson-Schrödinger inequalities are equivalent to the following condition of the covariance matrix, due to Narcowich[21] and often used in quantum optics (see Refs. 22–24 and references therein):

The eigenvalues of the Hermitian matrix $\Sigma + i\hbar J/2$ are non-negative (which we write for short: $\Sigma + i\hbar J/2 \geq 0$).

One then shows that this condition implies that the covariance matrix Σ is positive definite, and hence invertible. The next step consists in noting that in view of Sylvester's theorem from linear algebra the leading principal minors of the matrix

$$\Sigma + \frac{i\hbar}{2}J = \begin{pmatrix} \Delta(x,x) & \Delta(x,p) + i\hbar I/2 \\ \Delta(p,x) - i\hbar I/2 & \Delta(p,p) \end{pmatrix} \tag{69}$$

are non-negative. This applies in particular to the minors of order 2 so that we must have

$$\begin{vmatrix} \Delta x_j^2 & \Delta(x_j,p_j) + i\hbar/2 \\ \Delta(p_j,x_j) - i\hbar/2 & \Delta p_j^2 \end{vmatrix} \geq 0; \tag{70}$$

expanding the determinant on the left side, this condition is precisely the Robertson-Schrödinger inequality (64).

As we have seen, the fact that the covariance ellipsoid is cut by the conjugate coordinate planes along ellipsoids with areas $\geq h/2$ implies the Robertson-Schrödinger inequalities. This is thus a geometric restatement of the quantum uncertainty principle; we can rephrase it as follows:

Every quantum covariance ellipsoid contains a quantum blob, i.e. a symplectic egg with radius $\sqrt{\hbar}$. When this ellipsoid is itself a quantum blob, the Robertson-Schrödinger inequalities are saturated.

This statement can be extended in various ways. In a very recent paper,[25] we have applied this geometric approach to the quantum uncertainty principle to the study of partial saturation of the Robertson-Schrödinger inequalities for mixed quantum states. We show, in particular, that partial saturation corresponds to the case where some (but not all) planes of conjugate coordinates cut the covariance ellipsoid along an ellipse with exactly area $h/2$; this allows us to characterize those states for which this occurs (they are generalized Gaussians, more precisely the "squeezed states" familiar from quantum optics).

Another important thing we will unfortunately not be able to discuss in detail because of length limitations, is that everything we have said above still holds true if we replace the phrase "planes of conjugate coordinates x_j, p_j" with "symplectic planes." A symplectic plane is a two-dimensional subspace of phase space which has the property that if we restrict the symplectic form (10) to pairs of vectors in the symplectic plane, then we again obtain a symplectic form. For instance, it is easy to check that the x_j, p_j are symplectic planes (but those of coordinates x_j, p_k, $j \neq k$, or x_j, x_k, or p_j, p_k are not). One proves[4,10] that every symplectic plane can be obtained from any of the x_j, p_j planes using a symplectic transformation, and that such transformations take symplectic planes to symplectic planes. This implies, in particular, that the Robertson-Schrödinger inequalities (64) are covariant under symplectic transformations—if one defines new coordinates x', p' by $(x', p')^T = S(x,p)^T$ (S a symplectic matrix), then if

$$(\Delta p_j)^2(\Delta x_j)^2 \geq \Delta(x_j,p_j)^2 + \frac{1}{4}\hbar^2 \tag{71}$$

we also have

$$(\Delta p'_j)^2(\Delta x'_j)^2 \geq \Delta(x'_j,p'_j)^2 + \frac{1}{4}\hbar^2. \tag{72}$$

On the other hand, it is moderately difficult (but we will not do it here) to show that the Robertson-Schrödinger inequalities do not retain their form under changes of coordinates that are not symplectic, so that linear symplectic transforms are the only linear transforms which preserves the uncertainty principle.

We mention that there are possible non-trivial generalizations of the uncertainty principle using new results in symplectic topology. For instance, in Ref. 26, Gromov's theorem (in the linear case) is extended to projections on symplectic subspaces with dimension greater than 2. The authors find that the volume of the projections is conserved during linear Hamiltonian motions. These results certainly deserve to be investigated further, since they lead to important "quantum universal invariants" which have not yet been studied.

C. Coarse-graining by quantum blobs

There is a very interesting and deep relation between the geometric notion of quantum blob and the Wigner formalism (for detailed studies of the latter see Refs. 4 and 27 and the numerous references therein).

It is customary in quantum statistics to "coarse-grain" phase space in "quantum cells" which are cubes with volume $(\sqrt{h})^{2n} = h^n$ (see the seminal paper[28]). These cells do not have any symmetry under general symplectic transformations: while such a transformation preserves volume, a cube will in general be distorted into a multidimensional polyhedron. What is more striking is the comparison of volumes. Since a quantum blob is obtained from a ball with radius $\sqrt{\hbar}$ by a symplectic (and hence volume-preserving) transformation its volume is $h^n/n!2^n$, which is $n!2^n$ smaller than that of a quantum cell. This is a huge number. For instance, in the case of the physical three-dimensional configuration space, this leads to a factor of 48. In the case of a macroscopic system with $n = 10^{23}$, this factor becomes unimaginably large. This is in strong contrast with the fact that the orthogonal projection of a quantum blob on any plane x_j, p_j of conjugate coordinates (or, more generally, on any symplectic plane) is an ellipse with area equal to $\pi\hbar = h/2$. The coarse graining of phase space by quantum blobs has several advantages, which are discussed in Ref. 6. Here is one: a quantum blob $S(B_{\sqrt{\hbar}})$ is the set of all points $z = (x, p)$ such that $|S^{-1}z| \leq \sqrt{\hbar}$, or equivalently $[(S^{-1})^T S^{-1} z]^T z \leq \hbar$. Set $G = (S^{-1})^T S^{-1}$ and consider the phase space Gaussian

$$\Psi(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-z^T G z/\hbar}. \tag{73}$$

One shows[4,27] that $\Psi(z)$ is the Wigner transform of the generalized coherent state

$$\psi(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-x^T M x/2\hbar}, \tag{74}$$

where M is a symmetric complex matrix of the type $M = X + iY$ whose real part X is positive definite. The matrices

X and Y can be determined in terms of the matrix G by solving the equation

$$\begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} = G \tag{75}$$

in X and Y. This shows that there is a one-to-one correspondence between quantum blobs and coherent states. For instance, if S is the identity, in which case the quantum blob is just the ball $B_{\sqrt{\hbar}}$, formula (75) yields $X = I$ and $Y = 0$ so that $\psi_0(x) = (\pi\hbar)^{-n}e^{-|x|^2/2\hbar}$, which is the fiducial coherent state initially introduced by Schrödinger[29,30] in 1926. We now make the following important remark: the Gaussians states (74) all saturate the Robertson-Schrödinger inequalities they satisfy. For instance, the fiducial coherent state ψ_0 satisfies the Heisenberg inequalities $\Delta p_j \Delta x_j = \hbar/2$; they are thus minimum uncertainty states. Now, quantum blobs are the smallest symplectic balls it makes sense to speak about. Our quantum blobs can thus be viewed as the phase space pictures of minimum uncertainty states.

VI. CONCLUSION

In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain (H. Weyl, 1939)[30]

This quotation from the mathematician Hermann Weyl goes straight to the point, and applies to physics as well. While algebra (in the large) has dominated the scene of quantum mechanics for a very long time (in fact, from its beginning: think about Heisenberg's "matrix mechanics"), we are witnessing a slow but steady emergence of geometric ideas, and to a "symplectization of science." In this paper, we had mainly in mind the applications of the "principle of the symplectic camel" to the correspondence between Classical and Quantum Mechanics. But there are other applications as well. In a joint paper with Scheeres and Maruskin, we have shown that this principle applies successfully as well to the study of orbit uncertainty of satellites. One might conclude by saying that not only do these geometric ideas add clarity to many concepts but they also lead to new insights. This is what we had in mind while writing the present paper.

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