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The symplectic Lazard ring

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§ 0. Introduction

In [16], D. Quillen determined the complex cobordism ring MU_* using the formal group theory. This method seems to be very powerful, but is not appliciable directly for the symplectic case.

However there are some works along this line.

Buhštaber-Novikov [7] studied two-valued formal groups and gave some applications to the symplectic cobordism ring MSp_*.

Gozman [9] and Shimakawa [23] defined the rings $\tilde{\Lambda}_{MU}$ and $\tilde{\Lambda}_{MSp}$ using the total symplectic Pontrjagin class of a certain symplectic vector bundle.

On the other hand, using the Adams spectral sequence, some important results were obtained.

In particular, Ōkita [14] has shown that the Hurewicz map induces an isomorphism

$$Q(MSp_*/\text{Torsion}) \cong Q(PKO_*(MSp)/\text{Torsion})$$

where $Q(\ )$ is the rational indecomposable functor (see § 5).

In this paper, we construct a ring $LMSp$ and a ring homomorphism

$$\theta : LMSp \longrightarrow MSp_*/\text{Torsion}.$$

Our $LMSp$ is defined by the several formal power series and the relations like as the Lazard ring and we can calculate the image of the compositions of θ and some generalized Hurewicz maps. Then the following theorem holds.

Theorem (see § 5, Theorem 5.7). *θ induces an isomorphism*

$$Q(LMSp/\text{Torsion}) \cong Q(MSp_*/\text{Torsion}).$$

(The corresponding result is not true for $\tilde{\Lambda}_{MSp}$, that is,

$$Q(\tilde{\Lambda}_{MSp}/\text{Torsion}) \ncong Q(MSp_*/\text{Torsion}).)$$

We can prove also that the image of the composition

$$LMSp \xrightarrow{\theta} MSp_*/\text{Torsion} \xrightarrow{\mu_{KO}} KO_*/\text{Torsion} \qquad \text{is equal to } \sum_{n\geq 0} KO_{4n}.$$

This paper is constructed as follows:

In § 1, we recall some notations, especially for the oriented theory.

In § 2, we construct some maps between the projective and the quasiprojective spaces. We note that we use the Becker-Gottlieb transfer and the Becker-Segal theorem. We determine also the homomorphisms induced by these maps on the ordinary cohomology theory.

In § 3, we recall some results in Adams [1] for the oriented theories.

In § 4, we define the symplectic formal system and the symplectic Lazard ring $LSMp$. We construct also the homomorphism $\theta: LMSp \to MSp_*/\mathrm{Torsion}$ and obtain the relation between $LMSp$ and $\tilde{\Lambda}_{MSp}$ in the process constructing θ.

In § 5, we obtain some basic relations between the generators of $LMSp$ and prove Ōkita's type theorem using these results.

§ 1. Notations

Let $\boldsymbol{C}$ (resp. $\boldsymbol{H}$) be the field of complex (resp. quaternionic) numbers. In this paper, a vector space over $\boldsymbol{H}$ has the right scaler multiplication.

Let $\boldsymbol{C}P^n$ (resp. $\boldsymbol{H}P^n$) be the n-th complex (resp. symplectic) projective space.

Let X_+ be the disjoint sum of a space X and a point $\{\infty\}$.

For a stable map from X to Y, we use the notation such as $f: X \underset{(s)}{\longrightarrow} Y$.

We use the similar notations in Adams [1], Switzer [26] and Conner-Floyd [8] for the oriented theories.

Let E be a complex (resp. symplectic) oriented theory and $\mathcal{T}_E(\xi) \in \tilde{E}^{2n}(M(\xi))$ (resp. $t_E(\xi) \in \tilde{E}^{4n}(M(\xi))$) a Thom class where ξ is an n-dim complex (resp. symplectic) vector bundle and $M(\xi)$ is its Thom space.

We may assume that $\mathcal{T}_E(\xi)$ (resp. $t_E(\xi)$) is natural for bundle maps, multiplicative and unitary i. e. $\mathcal{T}_E(n\text{-dim trivial bundle}) = \sigma^{-2n}1 \in \tilde{E}^{2n}(S^{2n})$ (resp. $t_E(n\text{-dim trivial bundle}) = \sigma^{-4n}1 \in \tilde{E}^{4n}(S^{4n})$) where $\sigma: \tilde{E}^{n+1}(\Sigma X) \xrightarrow{\cong} \tilde{E}^n(X)$ is a suspension isomorphism.

Let $\xi_n^{\boldsymbol{C}}$ (resp. $\xi_n^{\boldsymbol{H}}$) be the canonical line bundle over $\boldsymbol{C}P^n$ (resp. $\boldsymbol{H}P^n$). Recall that $M(\xi_n^{\boldsymbol{C}}) = \boldsymbol{C}P^{n+1}$ (resp. $M(\xi_n^{\boldsymbol{H}}) = \boldsymbol{H}P^{n+1}$).

Let $i_n: \boldsymbol{C}P^n \to (\boldsymbol{C}P^{n+1}, \infty)$ (resp. $i_n: \boldsymbol{H}P^n \to (\boldsymbol{H}P^{n+1}, \infty)$) be the inclusion and $i_n^*: \tilde{E}^*(\boldsymbol{C}P^{n+1}) \to E^*(\boldsymbol{C}P^n)$ (resp. $i_n^*: \tilde{E}^*(\boldsymbol{H}P^{n+1}) \to E^*(\boldsymbol{H}P^n)$) the induced homomorphism. We define the euler class $x^E \in E^2(\boldsymbol{C}P^\infty)$ (resp. $y^E \in E^4(\boldsymbol{H}P^\infty)$) for a complex (resp. symplectic) oriented theory E as $i_\infty^* \mathcal{T}_E(\xi_\infty^{\boldsymbol{C}})$ (resp. $i_\infty^* t_E(\xi_\infty^{\boldsymbol{H}})$).

Let $k: S^2 = \boldsymbol{C}P^1 \hookrightarrow \boldsymbol{C}P^\infty$ (resp. $S^4 = \boldsymbol{H}P^1 \hookrightarrow \boldsymbol{H}P^\infty$) be the inclusion. Then we can easily show that our euler classes satisfy $k^* x^E = \sigma^{-2}1$ (resp. $k^* y^E = \sigma^{-4}1$).

So in the case $E = H$, x^H and y^H are uniquely determined.

For the definition of the Thom classes in K, KO, MU and MSp theories, we use the same ones in Conner-Floyd [8]. We note that some euler classes in this paper are different from the usual ones in Adams [1] or Switzer [26].

For example, $x^K = t^{-1} \cdot (1 - \zeta^C)$ where ζ^C is the complex Hopf line bundle over $\boldsymbol{C}P^\infty$ and $t \in \pi_2(K)$ be a generator. We have also $y^{KO} = 1 - \zeta^H \in KSp^0(\boldsymbol{H}P^\infty) = KO^4(\boldsymbol{H}P^\infty)$ where $\zeta^{\boldsymbol{H}}$ is the symplectic Hopf line bundle. (We identify $KSp^0(\ )$

and $KO^4(\ )$ by the Bott periodicity.) On the other hand, Switzer [26] uses ζ^C-1 as the euler class of K-theory and ζ^H-1 as that of KO-theory.

One can easily show that the Conner-Floyd's definition of x^{MU} and y^{MSp} agrees with that by Adams [1] or Switzer [26].

Let $j: \boldsymbol{C}P^\infty \to BU$ (resp. $j: \boldsymbol{H}P^\infty \to BSp$) be the natural inclusion.

Let $\beta_n^E \in E_{2n}(\boldsymbol{C}P^\infty)$ (resp. $\eta_n^E \in E_{4n}(\boldsymbol{H}P^\infty)$) be the dual element of $(x^F)^n$ (resp. $(y^E)^n$) and we write $j_*\beta_n^E \in E_{2n}(BU)$ (resp. $j_*\eta_n^E \in E_{4n}(BSp)$) by β_n^E (resp. η_n^E).

Let $i: \boldsymbol{C}P^\infty \cong MU(1) \to \Sigma^2 MU$ (resp. $i: \boldsymbol{H}P^\infty \cong MSp(1) \to \Sigma^4 MSp$) be the canonical inclusion. We put $b_n^E = \sigma^{-2} i_* \beta_{n+1}^E \in E_{2n}(MU)$ (resp. $h_n^E = \sigma^{-4} i_* \eta_{n+1}^E \in E_{4n}(MSp)$).

For brevity, we will often abbreviate E in the case of $E=H$.

Throughout the paper the ring of integers is denoted by $\boldsymbol{Z}$ and the rational numbers by $\boldsymbol{Q}$.

If R is a ring with unit, then the formal power series ring over R is denoted by $R[[x]]$. If $f(x) = \sum_i f_i x^i \in R[[x]]$ where $f_i \in R$, then the coefficient of x^n in $f(x)$ is denoted by $[f(x)]_n$.

Then the binomial coefficient $\binom{n}{m}$ is equal to $[(1+x)^n]_m$.

§ 2. Stable maps

There is a symplectification map $q: \boldsymbol{C}P^\infty \to \boldsymbol{H}P^\infty$.

Since q is a fibre bundle whose fibre is S^2, there is a Becker-Gottlieb transfer $t: \boldsymbol{H}P_+^\infty \xrightarrow[(s)]{} \boldsymbol{C}P_+^\infty$. (See Becker-Gottlieb [5].) Then the next proposition is clear. (See Shimakawa [23], Lemma 1.)

Proposition 2.1. *Let x^H and y^H be the euler classes as in* § 1. *Then* $q^* y^H = -(x^H)^2$, $t^*(x^H)^{2i-1} = 0$ *and* $t^*(x^H)^{2i} = 2(-y^H)^i$ *for* $i>0$.

Next we recall the definition of the quasiprojective spaces. (See James [11], Yokota [27].)

Let $\boldsymbol{F}$ be $\boldsymbol{C}$ or $\boldsymbol{H}$ and S_F^n the unit sphere in $\boldsymbol{F}^n$.

Let $G_n(\boldsymbol{C}) = U(n)$ and $G_n(\boldsymbol{H}) = Sp(n)$. The quasiprojective space $Q_n(\boldsymbol{F})$ is defined to be the space of generalized reflections, that is, the image of $\phi: S_F^n \times S_F^1 \to G_n(\boldsymbol{F})$ where $\phi(u, q)$ is the automorphism which leaves v fixed if $\langle u, v\rangle = 0$ and sends u to uq.

We may define $Q_n(\boldsymbol{F})$ as the space obtained from $S_F^n \times S_F^1$ by imposing the equivalence relation $(u, q) \sim (ug, g^{-1}qg)$ $(g \in S_F^1)$ and collapsing $S_F^n \times 1$ to a point.

By the second definition, we can easily show that $Q_n(\boldsymbol{C}) = \Sigma(\boldsymbol{C}P_+^{n-1})$.

Put $\widetilde{\boldsymbol{C}P}^n = Q_n(\boldsymbol{C})$ and $\widetilde{\boldsymbol{H}P}^n = Q_n(\boldsymbol{H})$. Clearly, we have a symplectification map $\tilde{q}: \widetilde{\boldsymbol{C}P}^\infty \to \widetilde{\boldsymbol{H}P}^\infty$.

We define $k_n: \Sigma^2(\boldsymbol{C}P_+^n) \to BU$ as the composition

$$\Sigma^2(\boldsymbol{C}P_+^n) = \Sigma\widetilde{\boldsymbol{C}P}^{n+1} \xrightarrow{\Sigma\tilde{j}} \Sigma U(n+1) \xrightarrow{\Sigma i_{n+1}} \Sigma U \xrightarrow{\iota} BU$$

where $\tilde{j}$, i_{n+1} are the natural inclusions and ι is the adjoint map of the equivalence $U \simeq \Omega BU$.

Define $i_{n,+}:CP_+^n\to BU\times Z$ by $i_{n,+}|CP^n:CP^n\to CP^\infty\to BU\times\{1\}$ and $i_{n,+}|\{\infty\}:\{\infty\}\to BU\times\{0\}$ where all maps are the canonical inclusions.

Let $B':BU\times Z\simeq\Omega^2BU$ be the Bott periodicity map.

Lemma 2.2. *$k_n:\Sigma^2(CP_+^n)\to BU$ is homotopic to the adjoint map of the composition $CP_+^n\xrightarrow{i_{n,+}}BU\times Z\xrightarrow{B'}\Omega^2BU$.*

Proof. We define $\tilde{k}_n:CP_+^n\to\Omega U(n+1)$ by

$$\tilde{k}_n([u])(t)(v)=(u,\ e^{2i\pi t})(v)\quad\text{and}\quad\tilde{k}_n(\infty)(t)(v)=v\,.$$

Clearly k_n is an adjoint map of the composition

$$CP_+^n\xrightarrow{\tilde{k}_n}\Omega U(n+1)\longrightarrow\Omega U\,.$$

We define $b_{n,m}:\dfrac{U(n+m)}{U(n)\times U(m)}\to\Omega SU(n+m)$ by

$$b_{n,m}([A])(t)=\begin{pmatrix}e^{i\pi t}I_n & \\ & e^{-i\pi t}I_n\end{pmatrix}A\begin{pmatrix}e^{-i\pi t}I_n & \\ & e^{i\pi t}I_n\end{pmatrix}{}^t\bar{A}\qquad(A\in U(n+m))\,.$$

Notice that

$$\begin{array}{ccc}CP^n=\dfrac{U(n+1)}{U(n)\times U(1)} & \xrightarrow{\ b_{n,1}\ } & \Omega SU(n+1)\\ \Big\downarrow i & & \Big\downarrow \Omega i\\ \dfrac{U(2n)}{U(n)\times U(n)} & \xrightarrow{\ b_{n,n}\ } & \Omega SU(2n)\end{array}\qquad\text{commutes,}$$

$\varinjlim_n\dfrac{U(2n)}{U(n)\times U(n)}=BU$ and the Bott map B' is the composition

$$BU\times Z\xrightarrow{\varinjlim_n b_{n,n}\times id}\Omega SU\times Z=\Omega U\cong\Omega^2BU\,.$$

So we have to show that $\tilde{k}_n\cong b_{n,1}$.

Let $x=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_{n+1}\end{pmatrix}$ be the last vector of $A\in U(n+1)$ and $y=\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_{n+1}\end{pmatrix}\in C^{n+1}$.

Put

$$H([A],\ s)(t)(y)=\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\\ e^{-2i\pi ts}\cdot y_{n+1}\end{pmatrix}\qquad\text{if}\quad\langle y,\ x\rangle=0$$

and

$$H([A],\ s)(t)(x)=\begin{pmatrix} e^{2i\pi t}\cdot x_1 \\ e^{2i\pi t}\cdot x_2 \\ \vdots \\ e^{2i\pi t}\cdot x_n \\ e^{2i\pi t(1-s)}\cdot x_{n+1} \end{pmatrix}.$$

Since $H([A],\ 1)(t)(y)=b_{n,1}([A])(t)(y)$, this gives a homotopy $\tilde{k}_n \cong b_{n,1}$. Thus (2.2) holds. □

By (2.2), we have the following commutative diagram

(2.3)
$$\begin{array}{ccccc} \tilde{H}_{*+2}(\Sigma\widetilde{CP^\infty}) & & \xrightarrow{\quad k_{\infty *}\quad} & & \tilde{H}_{*+2}(BU) \\ \uparrow \sigma^2 & & & & \uparrow B_* \\ \tilde{H}_*(CP^\infty_+) & \xrightarrow{i_*} & \tilde{H}_*(BU) & \xrightarrow{\sigma^2} & \tilde{H}_{*+2}(\Sigma^2 BU) \end{array}$$

where B is the adjoint map of the Bott map B'.

As in Switzer [26] (16-23), $B_*\sigma^2(\beta^H_{m-1})=m\cdot\beta^H_m$ mod decomposable elements. So we obtain

Proposition 2.4. *$k_{\infty *}\sigma^2\beta^H_{m-1}=m\cdot\beta^H_m$ mod decomposable elements.*

Now we construct a map from $\widetilde{HP}^n$ to $\widetilde{CP}^{2n}$.

Let $z\in \boldsymbol{H}^n$ and $z=x+jy$ where $x,\ y\in \boldsymbol{C}^n$. We denote the complexification $c:\boldsymbol{H}^n\to \boldsymbol{C}^{2n}$ by setting $c(z)=x\oplus y\in \boldsymbol{C}^{2n}$.

Let $q=a+jb\in H$ where $a,\ b\in \boldsymbol{C}$. Since S^1_C is a maximal torus of S^1_H, there is a $g\in S^1_H$ such that $g^{-1}qg\in S^1_C$. If $g^{-1}qg=e^{i\pi t}$ where $-1<t<0$, then $(gj)^{-1}qgj=e^{-i\pi t}$. Thus there is a $g\in S^1_H$ such that $g^{-1}qg=e^{i\pi t}$ where $0\leqq t\leqq 1$.

So a representative element of $\widetilde{HP}^n$ can be taken as $(x+jy,\ e^{i\pi t})$ where $x,\ y\in \boldsymbol{C}^n$ and $0\leqq t\leqq 1$.

We define $\tilde{i}_n:\widetilde{HP}^n\to\widetilde{CP}^{2n}$ by the equation

$$\tilde{i}_n[(x+jy,\ e^{i\pi t})]=[(x\oplus y,\ e^{2\pi t})].$$

Then the following proposition holds.

Proposition 2.5. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \widetilde{HP}^n & \xrightarrow{\ \tilde{i}_n\ } & \widetilde{CP}^{2n} \\ \downarrow \tilde{j} & & \downarrow \tilde{j} \\ Sp(n) & \xrightarrow{\ c\ } & U(2n) \end{array}.$$

Proof. Clearly, by the definitions, we obtain

$$\tilde{j}[(u,q)]v=\phi(u,q)v=v-u(q-1)\langle u,v\rangle \qquad \text{where } \langle u,v\rangle=\sum_{i=1}^{n}\bar{u}_i\cdot v_i\,.$$

So,

$$\tilde{j}\circ\tilde{\imath}_n[(x+jy,e^{i\pi t})](x'\oplus y')=x'\oplus y'+(x\oplus y)(e^{2i\pi t}-1)\langle x\oplus y,\ x'\oplus y'\rangle$$

where $x', y'\in \boldsymbol{C}^n$. We also have the following equation

$$c\circ\tilde{j}[(x+jy,e^{i\pi t})](x'+jy')$$
$$=x'\oplus y'+(x\oplus y)(e^{i\pi t}-1)\langle x\oplus y,\ x'\oplus y'\rangle+(-\bar{y}\oplus\bar{x})(e^{-i\pi t}-1)\langle -\bar{y}\oplus\bar{x},\ x'\oplus y'\rangle\,.$$

We define $f_\theta[(x+jy,e^{i\pi t})]$ by the equation

$$f_\theta[(x+jy,e^{i\pi t})](x'\oplus y')$$
$$=x'\oplus y'+(x\oplus y)(e^{i\pi t(2-\theta)}-1)\langle x\oplus y,\ x'\oplus y'\rangle$$
$$+(-\bar{y}\oplus\bar{x})(e^{-i\pi t\theta}-1)\langle -\bar{y}\oplus\bar{x},\ x'\oplus y'\rangle\,.$$

This gives a homotopy $c\circ\tilde{j}\simeq\tilde{j}\circ\tilde{\imath}_n$. □

Clearly the following diagram is commutative:

$$\begin{array}{ccc}
\widetilde{HP}^n & \xrightarrow{\ \tilde{\imath}_n\ } & \widetilde{CP}^{2n} \\
\downarrow & & \downarrow \\
\widetilde{HP}^{n+1} & \xrightarrow{\ \tilde{\imath}_{n+1}\ } & \widetilde{CP}^{2n+2}
\end{array}$$

where vertical inclusions are induced by

$$\boldsymbol{H}^n=\boldsymbol{H}^n\oplus 0\subset\!\!\!\longrightarrow \boldsymbol{H}^{n+1} \quad\text{and}\quad \boldsymbol{C}^{2n}=\boldsymbol{C}^{2n}\oplus 0\subset\!\!\!\longrightarrow \boldsymbol{C}^{n+2}\,.$$

We define $\tilde{\imath}:\widetilde{HP}^\infty\to\widetilde{CP}^\infty$ to be $\varinjlim_n \tilde{\imath}_n$.

Now we determine the homomorphisms $(\Sigma\tilde{q})^*, (\Sigma\tilde{\imath})^*$. Let M_H^n (resp. M_C^n) be the principal S_H^1-(resp. S_C^1-)bundle

$$S_H^1\longrightarrow S_H^n\longrightarrow HP^{n-1} \qquad (\text{resp. } S_C^1\longrightarrow S_C^n\longrightarrow CP^{n-1})\,.$$

We regard $\boldsymbol{H}$ (resp. $\boldsymbol{C}$) as the S_H^1-(resp. S_C^1-)module by the adjoint action, and define γ_H^n (resp. γ_C^n) to be an associated $\boldsymbol{H}$ (resp. $\boldsymbol{C}$) bundle of M_H^n (resp. M_C^n). Clearly $\Sigma\widetilde{HP}^n=M(\gamma_H^n)$ and $\Sigma\widetilde{CP}^n=M(\gamma_C^n)$ where $M(E)$ is the Thom space of a vector bundle E.

Let $\boldsymbol{Q}(\ )$ be the stabilize functor $\varinjlim_n \Omega^n S^n(\ )$ and $j':\boldsymbol{Q}(HP^\infty)\to BSp$ the induced map from $j:HP^\infty\to BSp$ using the infinite loop space structure of BSp. Then by the theorem of Becker-Segal (Becker [4], Segal [22]), j' induces an epimorphism of the cohomology theories corresponding to these infinite loop spaces. So we have a map $r:\Sigma\widetilde{HP}^\infty\to\boldsymbol{Q}(HP^\infty)$ which satisfies

$$\iota\circ\Sigma\tilde{j}\simeq j'\circ r\,. \tag{2.6}$$

We may regard r as a stable map $r:\Sigma\widetilde{HP}^{\infty}\underset{(s)}{\longrightarrow} HP^{\infty}$.

Let E be a symplectic oriented theory. For any $b\in HP^{\infty}$ the inclusion $i_b:\{b\}\to HP^{\infty}$ induces $M(i_b):S^4\to M(\gamma_H^{\infty})$. Using (2.6) and the fact that $\tilde{j}:\widetilde{HP}^{\infty}\to Sp$ gives the cell decomposition of Sp, we can easily show that $M(i_b)^*r^*y^E=\sigma^{-4}1$. So $r^*y^E\in\tilde{E}^4(M(\gamma_H^{\infty}))$ is a Thom class.

Put $\tau_H=r^*y^E\in\tilde{H}^4(\Sigma\widetilde{HP}^{\infty})=\tilde{H}^4(M(\gamma_H^{\infty}))$ and $\tau_C=\mathcal{T}(\gamma_C^{\infty})$.

Proposition 2.7. $(\Sigma\tilde{q})^*(\tau_H\cdot y^m)=(-1)^{m+1}\cdot 2\cdot\tau_C\cdot x^{2m+1}=(-1)^{m+1}\cdot 2\cdot\sigma^{-2}x^{2m+1}$.

Proof. Since our Thom classes are unitary, $\tau_C\cdot x^{2m+1}=\sigma^{-2}x^{2m+1}$.

Let $D(E)$ (rssp. $S(E)$) be the disk (resp. sphere) bundle of a vector bundle $\pi:E\to B$.

The $H^*(B)$-module structure of $\tilde{H}^*(M(E))=H^*(D(E),S(E))$ is defined by

$$(D(E),S(E))\xrightarrow{\Delta}(D(E),S(E))\times(D(E),S(E))\xrightarrow{\pi\times id}B\times(D(E),S(E))$$

where Δ is the diagonal.

Since $\Sigma\tilde{q}$ is given by $q':\gamma_C^{\infty}\to\gamma_H^{\infty}$ where q' is the bundle map over $CP^{\infty}\xrightarrow{q}HP^{\infty}$, the module structure is compatible, i.e.,

$$(\Sigma\tilde{q}^*\tau_H\cdot y^m)=(\Sigma\tilde{q}^*\tau_H)\cdot q^*y^m=(\Sigma\tilde{q}^*\tau_H)\cdot(-1)^m x^{2m}.$$

So we have to show that $\Sigma\tilde{q}^*\tau_H=-2\cdot\tau_C\cdot x$. We have a commutative diagram

$$\begin{array}{ccccc}
\tilde{H}_4(\Sigma\widetilde{CP}^{\infty}) & \xrightarrow{\Sigma\tilde{j}_*} & \tilde{H}_4(\Sigma U) & \xrightarrow{\iota_*} & \tilde{H}_4(BU)\\
\downarrow{\scriptstyle\Sigma\tilde{q}_*} & & \downarrow{\scriptstyle\Sigma q_*} & & \downarrow{\scriptstyle Bq_*}\\
\tilde{H}_4(\Sigma\widetilde{HP}^{\infty}) & \xrightarrow{\Sigma\tilde{j}_*} & \tilde{H}_4(\Sigma Sp) & \xrightarrow{\iota_*} & \tilde{H}_4(BSp)
\end{array}.$$

Let $(\tau_H)^*\in\tilde{H}_4(\Sigma\widetilde{HP}^{\infty})$ be the dual element of τ_H. By the duality, we may prove that $\Sigma\tilde{q}_*\sigma^2\beta_1=-2\cdot(\tau_H)^*$. Since $j_*r_*(\tau_H)^*=\eta_1$ by the definition of τ_H, we have only to prove that $j_*r_*\Sigma\tilde{q}_*\sigma^2\beta_1=-2\cdot\eta_1$. By the above diagram and (2.6), $j_*r_*\Sigma\tilde{q}_*=Bq_*\iota_*\Sigma\tilde{j}_*$. By (2.1) and (2.4), we have

$$Bq_*\iota_*\Sigma\tilde{j}_*\sigma^2\beta_1=Bq_*k_{\infty*}\sigma^2\beta_1=Bq_*(2\cdot\beta_2+\text{decomposable elements})=-2\cdot\eta_1. \quad \square$$

Proposition 2.8. $(\Sigma\tilde{q})^*\circ(\Sigma\tilde{i})^*=\cdot 2$. *So we have*

$$\Sigma\tilde{i}^*(\tau_C\cdot x^m)=\begin{cases}0 & m=2k\\ (-1)^{k+1}\cdot\tau_H\cdot y^k & m=2k+1.\end{cases}$$

Proof. If $(\Sigma\tilde{q})^*\circ(\Sigma\tilde{i})^*=\cdot 2$, then the second result follows from (2.7). So we have to show that $(\Sigma\tilde{q})^*\circ(\Sigma\tilde{i})^*=\cdot 2$. There is a commutative diagram

$$\begin{array}{ccccc}
\tilde{H}_{4n}(\Sigma\widetilde{CP}^{\infty}) & \xrightarrow{\Sigma\tilde{q}_*} & \tilde{H}_{4n}(\Sigma\widetilde{HP}^{\infty}) & \xrightarrow{\Sigma\tilde{t}_*} & \tilde{H}_{4n}(\Sigma\widetilde{CP}^{\infty}) \\
\downarrow{\scriptstyle k_{\infty *}} & & & & \downarrow{\scriptstyle k_{\infty *}} \\
\tilde{H}_{4n}(BU) & \xrightarrow{Bq_*} & \tilde{H}_{4n}(BSp) & \xrightarrow{Bc_*} & \tilde{H}_{4n}(BU)\ .
\end{array}$$

As is well-known $Bc_*\circ Bq_*=\cdot 2$ modulo decomposable elements. Since $k_{\infty *}$ is monic, (2.8) is proved. □

Proposition 2.9. $r^*y^m=m\tau_H\cdot y^{m-1}$ $(m\geqq 1)$.

Proof. Let $z^*\in H_*(X)$ be the dual element of $z\in H^*(X)$.

Then $\beta_{2m}=j_*(x^{2m})^*$ and $\eta_m=j_*(y^m)^*$. So $Bq_*\beta_{2m}=(-1)^m\cdot\eta_m$ by (2.1). We have also $\Sigma\tilde{q}_*(\tau_C\cdot x^{2m-1})^*=(-1)^m\cdot 2\cdot(\tau_H\cdot y^{m-1})^*$. Then by (2.6), we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\tilde{H}_*(\Sigma\widetilde{CP}^{\infty}) & \xrightarrow{\Sigma\tilde{q}_*} & \tilde{H}_*(\Sigma\widetilde{HP}^{\infty}) & \xrightarrow{r_*} & \tilde{H}_*(HP^{\infty}) & \xrightarrow{j_*} & \tilde{H}_*(BSp) \\
\downarrow{\scriptstyle k_{\infty *}} & & \downarrow{\scriptstyle (\iota\circ\Sigma\tilde{j})_*} & & & & \downarrow{\scriptstyle \iota_{4*}} \\
\tilde{H}_*(BU) & \xrightarrow{Bq_*} & \tilde{H}_*(BSp) & & \xrightarrow{\iota_{4*}} & & H_*(\Sigma^4KO)\ ,
\end{array}$$

where KO is BO-spectrum and $\iota_4: BSp\rightarrow\Sigma^4KO$ the canonical inclusion. Since $k_{\infty *}(\tau_C x^{2m-1})^*=2m\cdot\beta_{2m}+$decomposable elements, we have

$$Bq_*k_{\infty *}(\tau_C x^{2m-1})^*=(-1)^m\cdot 2m\cdot\eta_m+\text{decomposable elements.}$$

If $r_*(\tau_H y^{m-1})^*=\alpha\cdot y_m$, then $j_*r_*\Sigma\tilde{q}_*(\tau_C x^{2m-1})^*=(-1)^m\cdot 2\alpha\cdot\eta_m$.

Since ι_{4*} kills the decomposable elements and since $\iota_{4*}\eta_m\neq 0$ (See Switzer [26].), $\alpha=m$. Thus (2.9) is proved. □

We put $\overline{HP}^{\infty}=\Sigma^{-1}\widetilde{HP}^{\infty}$, $\bar{q}=\Sigma^{-1}\tilde{q}$ and $\bar{t}=\Sigma^{-1}\tilde{t}$. Then we have the following stable maps:

$$CP_+^{\infty}\xrightarrow{q}HP_+^{\infty}\xrightarrow{t}CP_+^{\infty},\quad CP_+^{\infty}\xrightarrow{\bar{q}}\overline{HP}_+^{\infty}\xrightarrow{\bar{t}}CP_+^{\infty}\quad\text{and}\quad \Sigma^2\overline{HP}^{\infty}\xrightarrow{r}HP^{\infty}.$$

Let E be a symplectic oriented theory. Then we can regard $\tilde{E}^*(\overline{HP}^{\infty})$ as the $E^*(HP^{\infty})$-module by the suspension isomorphism $\tilde{E}^*(\Sigma\widetilde{HP}^{\infty})=\tilde{E}^*(\Sigma^2\overline{HP}^{\infty})\xrightarrow{\sigma^2}\tilde{E}^{*-2}(\overline{HP}^{\infty})$.

Since $r^*: H^4(HP^{\infty})\rightarrow H^4(\Sigma\widetilde{HP}^{\infty})$ is an isomorphism,

$$\sigma^2\circ r^*: \tilde{E}^4(HP^{\infty})\longrightarrow\tilde{E}^4(\Sigma\widetilde{HP}^{\infty})\longrightarrow\tilde{E}^2(\overline{HP}^{\infty})$$

is so.

We denote $\bar{y}^E \in \tilde{E}^2(\overline{HP^\infty})$ to be $\sigma^2 r^* y^E$. Then $\bar{y}^E \cdot (y^E)^m = \sigma^2(r^* y^E \cdot (y^E)^m)$ (for $m \geqq 0$) form a free $E_*(pt)$-base of $\tilde{E}^*(\overline{HP^\infty})$.

§ 3. Hurewicz homomorphism

Let E and F be the spectra of symplectic oriented theories. Then we have two symplectic classes in $\widetilde{E \wedge MSp}^*(HP^\infty)$:

$$y_L : HP^\infty \xrightarrow{y^E} \Sigma^4 E \xrightarrow{\sim} \Sigma^4 \wedge E \wedge \Sigma_0 \xrightarrow{id \wedge id \wedge \iota_{MSp}} \Sigma^4 \wedge E \wedge MSp$$

and

$$y_R : HP^\infty \xrightarrow{y^{MSp}} \Sigma^4 MSp \xrightarrow{\sim} \Sigma^4 \wedge \Sigma^0 \wedge MSp \xrightarrow{id \wedge \iota_E \wedge id} \Sigma^4 \wedge E \wedge MSp$$

where $\iota_{MSp} : \Sigma^0 \rightarrow MSp$ and $\iota_E : \Sigma^0 \rightarrow E$ are the unit maps.

We write y^E, y^{MSp} for y_L, y_R. We can compare y^E, y^{MSp} by the following lemma. (See Adams [1].) Put $h^E(y^E) = \sum_{i \geq 0} h_i^E (y^E)^{i+1}$.

Lemma 3.1. (Adams formula) $y^{MSp} = h^E(y^E)$.

By the universality of MSp for symplectic oriented theories, there is

$$u_F : MSp \longrightarrow F \quad \text{such that} \quad u_{F*}(y^{MSp}) = y^F .$$

Put $u_{F*} h^E(y) = \sum_{i \geq 0} u_{F*} h_i^E y^{i+1} \in E_*(F)[[y]]$. By (3.1), we have

Lemma 3.2. $y^F = u_{F*} h^E(y^E)$.

First, we consider the case of $E = H$. Let $\bar{y}^{MSp} = hur^H(\bar{y}^{MSp}) \in \widetilde{H \wedge MSp}^2(\overline{HP^\infty})$. We can easily show the following propositions by (3.1), (2.7), (2.8) and (2.9).

Proposition 3.3. *In $H \wedge MSp$-theory, we have*

$$q^*(y^{MSp})^m = (h(-x^2))^m \quad \text{and} \quad t^*(h(-x^2))^m = 2(y^{MSp})^m .$$

Proposition 3.4 *In $H \wedge MSp$-theory, we have*

$$\bar{q}^*(\bar{y}^{MSp} \cdot (y^{MSp})^m) = \frac{d}{dx} h(-x^2) \cdot (h(-x^2))^m$$

and

$$\bar{t}^*\Big(\frac{d}{dx} h(-x^2) \cdot (h(-x^2))^m\Big) = 2\bar{y}^{MSp} \cdot (y^{MSp})^m .$$

Next, we consider the case of $E = H \wedge KO$. In $H \wedge KO \wedge MSp$-theory, we have three euler classes y^H, y^{KO} and y^{MSp}.

By (3.1) and (3.2), we obtain the equation $y^{MSp} = h^{KO}(u_{KO*} h^H(y^H))$.

We can regard $H_*(KO)$ as the subring of $H_*(K) = \boldsymbol{Q}[t, t^{-1}]$, where $t \in H_2(K)$ is the generator in Adams [1] and Switzer [26]. In fact we have $c_*(H_*(KO)) = \boldsymbol{Q}[t^4, 2t^2, t^{-4}]$ where c_* is the monomorphism induced from complexification map $c: KO \to K$.

Then $u_{KO} \cdot h_i^H = 2(-1)^i \cdot t^{2i}/(2i+2)!$. (See Ōkita [14], Lemma 2.3., and recall that our y^H is different in sign from his one.)

Lemma 3.5. *In $H \wedge KO \wedge MSp$-theory,* $y^{MSp} = h^{KO}(-t^{-2} \cdot (2 \cdot \cosh(t\sqrt{-y}) - 2))$.

Put $f(x) = h^{KO}(-t^{-2} \cdot (\cdot 2 \cdot \cosh(tx) - 2))$ and $\bar{f}(x) = \frac{1}{2} f'(x)$. Put also $\bar{y}^{MSp} = hur^{H \wedge KO}(\bar{y}^{MSp}) \in \widetilde{H \wedge KO \wedge MSp}^2(\boldsymbol{HP}^\infty)$. The proofs of the following two propositions are similar to those of (3.4) and (3.5).

Proposition 3.3′. *In $H \wedge KO \wedge MSp$-theory,*

$$q^*(y^{MSp})^m = (f(x))^m \quad \textit{and} \quad t^*(f(x))^m = 2(y^{MSp})^m .$$

Proposition 3.4′. *In $H \wedge KO \wedge MSp$-theory,*

$$\bar{q}^*(\bar{y}^{MSp} \cdot (y^{MSp}))^m = 2\bar{f}(x) \cdot (f(x))^m$$

and

$$\bar{t}^*(\bar{f}(x) \cdot (f(x)))^m = \bar{y}^{MSp} \cdot (y^{MSp})^m .$$

We denote $hur^E : \pi_*(\ ) \to E_*(\ )$ to be the generalized Hurewicz homomorphism. Since hur^E is induced from the unit map $\iota_E : \Sigma^0 \to E$, (3.2)~(3.4)′ give the informations for hur^E.

These results will be used in the following sections.

§ 4. Symplectic formal system and symplectic Lazard ring

Let R be a commutative ring with unit and $R[[X, \bar{X}, Y, \bar{Y}]]$ the formal power series ring with four variables X, $\bar{X}$, Y and $\bar{Y}$.

Definition 4.1. A symplectic formal system consists of a formal power series

$$E(X) = \sum_{i \geq 1} a_i \cdot X^i \in R[[X]] ,$$

and formal power series in $R[[X, \bar{X}, Y, \bar{Y}]]/(E(X) - \bar{X}^2, E(Y) - \bar{Y}^2)$,

$$F_k(X, \bar{X}, Y, \bar{Y}) = \sum_{i,j \geq 0} b_{i,j}^{(k)} \cdot X^i \cdot Y^j + \sum_{i,j \geq 1} c_{i,j}^{(k)} \cdot \bar{X} \cdot X^{i-1} \cdot \bar{Y} \cdot Y^{j-1} ,$$

$$G_k(X, \bar{X}, Y, \bar{Y}) = \sum_{i \geq 1, j \geq 0} d_{i,j}^{(k)} \cdot (\bar{X} \cdot X^{i-1} \cdot Y^j + \bar{Y} \cdot Y^{i-1} \cdot X^j) \qquad \text{for} \quad k \geqq 1$$

which satisfy

(i) (unitary relation) $b_{1,0}^{(1)} = d_{1,0}^{(1)} = 1$, $b_{n,0}^{(1)} = d_{n,0}^{(1)} = 0$ for $n \neq 1$,

(ii) (associative relation)

$$D(F_1(X, \bar{X}, Y, \bar{Y}), G_1(X, \bar{X}, Y, \bar{Y}), Z, \bar{Z}) = D(X, \bar{X}, F_1(Y, \bar{Y}, Z, \bar{Z}), G_1(Y, \bar{Y}, Z, \bar{Z}))$$

for $D = F_1$ or G_1,

(iii) (commutative relation) $b_{i,j}^{(1)} = b_{j,i}^{(1)}$, $c_{i,j}^{(1)} = c_{j,i}^{(1)}$,

(iv) (differential relation) $c_{1,1}^{(1)}=-2$, $c_{1,n}^{(1)}=c_{n,1}^{(1)}=0$ for $n\neq 1$,

(v) (power relation) $F_k(X, \bar{X}, Y, \bar{Y})=(F_1(X, \bar{X}, Y, \bar{Y}))^k$,

$$G_k(X, \bar{X}, Y, \bar{Y})=G_1(X, \bar{X}, Y, \bar{Y})\cdot F_{k-1}(X, \bar{X}, Y, \bar{Y})$$

and

(vi) (square relation) $(G_1(X, \bar{X}, Y, \bar{Y}))^2=E(F_1(X, \bar{X}, Y, \bar{Y}))$.

Definition 4.2. Let $\Gamma=\{E, F_k, G_k\}$ be a symplectic formal system over R. The associated symplectic ring R_Γ is the subring of R which is generated by the elements $8a_i$, $4b_{i,j}^{(2k-1)}$, $2b_{i,j}^{(2k)}$, $c_{i,j}^{(k)}$, $4d_{i,j}^{(k)}$ and 1.

Now we can define the symplectic Lazard ring $LMSp$ as follows. Let S be $\boldsymbol{Z}[a_i, b_{i,j}^{(k)}, c_{i,j}^{(k)}, d_{i,j}^{(k)}]$ where a_i, $b_{i,j}^{(k)}$, $c_{i,j}^{(k)}$ and $d_{i,j}^{(k)}$ are variables, and I the ideal of relations that appear in (i)~(vi) of (4.1).

Then we get a universal symplectic formal system over S/I. We denote Γ_{univ} as this system over S/I and do $LMSp$ as $(S/I)_{\Gamma_{univ}}$.

Then clearly, we have

Proposition 4.3. *Γ_{univ} and $LMSp$ are universal for symplectic formal systems and their associated symplectic rings.*

We can make $LMSp$ into a graded ring as follows.

Let assign the degree -2 to $\bar{X}$, $\bar{Y}$ and the degree -4 to X, Y. Let assign also the degree -4 to $E(X)$, the degree $-4k$ to $F_k(X, \bar{X}, Y, \bar{Y})$ and the degree $-4k+2$ to $G_k(X, \bar{X}, Y, \bar{Y})$. Then all the relations (i)~(vi) match these gradings. So the ideal I is graded and $LMSp$ is a graded ring.

We note that a_i, $b_{i,j}^{(k)}$, $c_{i,j}^{(k)}$ and $d_{i,j}^{(k)}$ have degrees $4(i-1)$, $4(i+j-k)$, $4(i+j-k-1)$ and $4(i+j-k)$, respectively. If a symplectic formal system over a positively graded ring R satisfies such conditions, then we say that Γ is graded.

Example. An easy computation shows $LMSp_0=\boldsymbol{Z}$ generated by 1, $LMSp_4=\boldsymbol{Z}$ generated by $4b_{1,1}^{(1)}$ and $LMSp_8=\boldsymbol{Z}\oplus\boldsymbol{Z}$ generated by $c_{3,3}^{(1)}$ and $2b_{2,2}^{(2)}$.

Next we want to construct a symplectic formal system over $H_*(MSp)$. Put $f(x)=h(-x^2)$ and $\bar{f}(x)=\frac{1}{2}\frac{d}{dx}h(-x^2)$ where $h(x)=\sum_{i\geq 0} h_i^H\cdot x^{i+1}$ as in § 3. Clearly, $f(x)$ and $\bar{f}(x)\in H_*(MSp)[[x]]$.

We denote the symplectic formal system Γ_H by setting,

$$E^H(f(x))=(\bar{f}(x))^2, \quad F_k^H(f(x), \bar{f}(x), f(y), \bar{f}(y))=(f(x+y))^k$$

and

$$G_k^H(f(x), \bar{f}(x), f(y), \bar{f}(y))=\bar{f}(x+y)\cdot(f(x+y))^{k-1} \quad \text{for } k\geqq 1.$$

Then the all the properties except (iv) are almost trivial.

Proposition 4.4. *In Γ_H, the differential relation holds.*

Proof. Put

$$F_1^H(f(x), \bar{f}(x), f(y), \bar{f}(y))=f(x+y)$$
$$=\sum_{i,j\geq 0} b_{i,j}\cdot(f(x))^i\cdot(f(y))^j+\sum_{i,j\geq 1} c_{i,j}\cdot\bar{f}(x)\cdot(f(x))^{i-1}\cdot\bar{f}(y)\cdot(f(y))^{j-1}$$

where $b_{i,j}$, $c_{i,j}\in H_*(MSp)$. Put $y^2=0$. Since $\bar{f}(x)=-x+$higher terms and $f(x)=-x^2+$higher terms and since the unitary relation holds, the above equation becomes

$$f(x+y)=f(x)+\sum_{i\geq 1} c_{i,1}\cdot\bar{f}(x)\cdot(f(x))^{i-1}\cdot(-y)\,.$$

Since $y^2=0$, this means

$$-2\bar{f}(x)=-y^{-1}\cdot(f(x+y)-f(x))=\sum_{i\geq 1} c_{i,1}\cdot\bar{f}(x)\cdot(f(x))^{i-1}\,.$$

Since $\bar{f}(x)\cdot(f(x))^{i-1}=(-1)^i x^{2i-1}+$higher terms, we have $c_{1,1}=-2$ and $c_{n,1}=0$ for $n\neq 1$ inductively. By the commutative relation, $c_{1,n}=0$ for $n\neq 1$. Thus (4.4) is proved. □

Then by (4.3), we have a ring homomorphism $\theta': LMSp\to H_*(MSp)$ such that $\theta'_*\Gamma_{univ}=\Gamma_H$ where θ'_* is defined by mapping each corresponding coefficients of $E(X)$, $F_k(X, \bar{X}, Y, \bar{Y})$ and $G_k(X, \bar{X}, Y, \bar{Y})$.

Proposition 4.5. *$\theta'(8a_i)$, $\theta'(4b_{i,j}^{(k)})$, $\theta'(c_{i,j}^{(k)})$ and $\theta'(4d_{i,j}^{(k)})$ are in* Im$(hur^H: MSp_*\to H_*(MSp))$ *for all $k\geq 1$.*

Proof. Since $t^*((\bar{q}^*\bar{y}^{MSp})^2)\in\widetilde{MSp}^*(HP_+^\infty)$, there is $\alpha_i\in MSp_*$ such that $\sum_{i\geq 0}\alpha_i\cdot(y^{MSp})^i=t^*((\bar{q}^*\bar{y}^{MSp})^2)$. If we map this equation into $(\widetilde{H\wedge MSp})^*(HP_+^\infty)$, then we have

$$\sum_{i\geq 0} hur^H(\alpha_i)\cdot(y^{MSp})^i=t^*((\bar{q}^*\bar{y}^{MSp})^2)=t^*((2\bar{f}(x))^2)=t^*(4\cdot E(f(x)))$$
$$=\sum_{i\geq 0}\theta'(8a_i)\cdot(y^{MSp})^i \qquad \text{by (3.3) and (3.4).}$$

Let $m: CP_+^\infty\wedge CP_+^\infty\to CP_+^\infty$ be the classifying map of the tensor product of canonical line bundle. Then

$$(t\wedge t)^*m^*q^*((y^{MSp})^k)\in\widetilde{MSp}^*(HP_+^\infty\wedge HP_+^\infty)\approx\widetilde{MSp}^*(HP_+^\infty)\otimes_{MSp_*}\widetilde{MSp}^*(HP_+^\infty)\,.$$

Similary we have the following equations:

$$(\bar{t}\wedge\bar{t})^*m^*q^*((y^{MSp})^k)\in\widetilde{MSp}^*(\overline{HP}^\infty\wedge\overline{HP}^\infty)\approx\widetilde{MSp}^*(\overline{HP}^\infty)\otimes_{MSp_*}\widetilde{MSp}^*(\overline{HP}^\infty)\,,$$

$$(\bar{t}\wedge t)^*m^*\bar{q}^*(\bar{y}^{MSp}\cdot(y^{MSp})^{k-1})\in\widetilde{MSp}^*(\overline{HP}^\infty\wedge HP_+^\infty)\approx\widetilde{MSp}^*(\overline{HP}^\infty)\otimes_{MSp_*}\widetilde{MSp}^*(HP_+^\infty)$$

and

$$(t\wedge\bar{t})^*m^*\bar{q}^*(\bar{y}^{MSp}\cdot(y^{MSp})^{k-1})\in\widetilde{MSp}^*(HP_+^\infty\wedge\overline{HP}^\infty)\approx\widetilde{MSp}^*(HP_+^\infty)\otimes_{MSp_*}\widetilde{MSp}^*(\overline{HP}^\infty)\,.$$

Then there are $\beta_{i,j}^{(k)}$, $\gamma_{i,j}^{(k)}$ and $\delta_{i,j}^{(k)}\in MSp_*$ which satisfy

$$\sum_{i,j\geq 0}\beta_{i,j}^{(k)}\cdot(y^{MSp})^i\otimes(y^{MSp})^j=(t\wedge t)^*m^*q^*((y^{MSp})^k)\,,$$

$$\sum_{i,j\geq 1}\gamma_{i,j}^{(k)}\cdot(\bar{y}^{MSp}\cdot(y^{MSp})^{i-1})\otimes(\bar{y}^{MSp}\cdot(y^{MSp})^{j-1})=(\bar{t}\wedge\bar{t})^*m^*q^*((y^{MSp})^k)$$

and

$$\sum_{i\geq 1,j\geq 0}\delta_{i,j}^{(k)}\cdot(\bar{y}^{MSp}\cdot(y^{MSp})^{i-1})\otimes(y^{MSp})^j=(\bar{t}\wedge t)^*m^*\bar{q}^*(\bar{y}^{MSp}\cdot(y^{MSp})^{k-1}).$$

And clearly

$$\sum_{i\geq 1,j\geq 0}\delta_{i,j}^{(k)}\cdot(y^{MSp})^j\otimes(\bar{y}^{MSp}\cdot(y^{MSp})^{i-1})=(t\wedge\bar{t})^*m^*\bar{q}^*(\bar{y}^{MSp}\cdot(y^{MSp})^{k-1}).$$

We can easily prove $hur^H(\beta_{i,j}^{(k)})=\theta'(4b_{i,j}^{(k)})$, $hur^H(\gamma_{i,j}^{(k)}=\theta'(c_{i,j}^{(k)})$ and $hur^H(\delta_{i,j}^{(k)})=\theta'(4d_{i,j}^{(k)})$ by the similar method used to prove $hur^H(\alpha_i)=\theta'(8a_i)$, using (3.3) and (3.4). Thus (4.5) is proved. □

To show $\theta'(2b_{i,j}^{(2k)})\in\mathrm{Im}(hur^H)$, we need some preparations.

Let $c:\boldsymbol{HP}_+^\infty\to BU(2)_+$ be the classifying map of the complexification $Sp(1)\to U(2)$ and $q:BU(n)_+\to BSp(n)_+$ that of the quaterniozation $U(n)\to Sp(n)$.

Let $m:BU(2)_+\wedge BU(2)_+\to BU(4)_+$ be the classifying map of the tensor product.

We abbreviate $X_+\wedge X_+\wedge\cdots\wedge X_+$ as X_+^n. Then we denote $m_4:(\boldsymbol{CP}^\infty)\to(\boldsymbol{CP}^\infty)_+^4$ as the classifying map of the endomorphism μ_4 of $U(1)\times U(1)\times U(1)\times U(1)$ defined by $\mu_4(a,b,c,d)=(ac,ad,bc,bd)$.

We denote $i_n:\underbrace{U(1)\times U(1)\times\cdots\times U(1)}_{(n\text{-times})}\to U(n)$ (resp. $i:\underbrace{Sp(1)\times Sp(1)\times\cdots\times Sp(1)}_{(n\text{-times})}\to Sp(n)$ as the canonical inclusion.

Then the diagram

$$\begin{array}{ccc}
\boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty & \xrightarrow{m_4} & \boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty \\
\Big\downarrow{\scriptstyle Bi_{2+}\wedge Bi_{2+}} & & \Big\downarrow{\scriptstyle Bi_{4+}} \\
BU(2)_+\wedge BU(2)_+ & \xrightarrow{m} & BU(4)_+
\end{array}\qquad\text{commutes.}$$

We denote also conj: $\boldsymbol{CP}_+^\infty\to\boldsymbol{CP}_+^\infty$ as the classifying map of the complex conjugation. Then the diagram

$$\begin{array}{ccc}
\boldsymbol{CP}_+^\infty & \xrightarrow{(id\wedge\mathrm{conj})\circ\Delta} & \boldsymbol{CP}_+^\infty\wedge\boldsymbol{CP}_+^\infty \\
\Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle Bi_{2+}} \\
\boldsymbol{HP}_+^\infty & \xrightarrow{c} & BU(2)_+
\end{array}\qquad\text{commutes.}$$

If we apply the functor $\widetilde{MSp}^*(\ )$, then we obtain a commutative diagram

$$
(4.6)\qquad
\begin{array}{ccccccc}
\widetilde{MSp}^*(BSp(4)_+) & \xrightarrow{q^*} & \widetilde{MSp}^*(BU(4)_+) & \xrightarrow{m^*} & \widetilde{MSp}^*(BU(2)_+\wedge BU(2)_+) & \xrightarrow{(c\wedge c)^*} & \widetilde{MSp}^*(HP^\infty_+\wedge HP^\infty_+) \\
\downarrow{\scriptstyle (Bi_{4+})^*} & & \downarrow{\scriptstyle (Bi_{4+})^*} & & \downarrow{\scriptstyle (Bi_{2+}\wedge Bi_{2+})^*} & & \downarrow{\scriptstyle (q\wedge q)^*} \\
\widetilde{MSp}^*((HP^\infty)^4_+) & \xrightarrow{q^*} & \widetilde{MSp}^*((CP^\infty)^4_+) & \xrightarrow{m^*} & \widetilde{MSp}^*((CP^\infty)^4_+) & \xrightarrow{(\Delta_c\wedge\Delta_c)^*} & \widetilde{MSp}^*(CP^\infty_+\wedge CP^\infty_+)
\end{array}
$$

where $\Delta_c=(id\wedge \mathrm{conj})\circ\Delta$.

Put $y_i^{MSp}=\pi_i^* y^{MSp}$. Then there is an isomorphism

$$\widetilde{MSp}^*((HP^\infty)^4_+)=MSp_*[[y_1^{MSp},\ y_2^{MSp},\ y_3^{MSp},\ y_4^{MSp}]]\,.$$

As is well-known, there are the symplectic Pontrjagin classes P_1, P_2, P_3 and P_4 such that $\widetilde{MSp}^*(BSp(4)_+)=MSp_*[[P_1,\ P_2,\ P_3,\ P_4]]$ and $(Bi_{4+})^*P_i$ is the i-th elementary symmetric function on y_1^{MSp}, y_2^{MSp}, y_3^{MSp} and y_4^{MSp}. (See Switzer [26].)

Put $r_i=hur^H(c\wedge c)^*m^*q^*P_i$ $(i=1, 2, 3, 4)$. We denote $B_{i,j}^{(k)}$ and $C_{i,j}^{(k)}$ as the elements of $H_*(MSp)$ which satisfy

$$F_k^H(X,\ \bar{X},\ Y,\ \bar{Y})=\sum_{i,j\geq 0} B_{i,j}^{(k)}\cdot X^i\cdot Y^j+\sum_{i,j\geq 1} C_{i,j}^{(k)}\cdot \bar{X}\cdot X^{i-1}\cdot \bar{Y}\cdot Y^{j-1}\,.$$

Let denote $x_i\in(\widetilde{H\wedge MSp})^2((CP^\infty)^n_+)$ for $1\leqq i\leqq n$ as π_i^*x where $x\in(\widetilde{H\wedge MSp})^2(CP^\infty_+)$ as in §2. Now we can calculate $(q\wedge q)^*r_i$.

Lemma 4.7.

(i) $(q\wedge q)^*r_1=\sum 4B_{i,j}^{(1)}\cdot(f(x_1))^i\cdot(f(x_2))^j$,

(ii) $(q\wedge q)^*r_2=\sum 6B_{i,j}^{(1)}\cdot B_{k,s}^{(1)}\cdot(f(x_1))^{i+k}\cdot(f(x_2))^{j+s}$

$$-\sum 2C_{i,j}^{(1)}\cdot C_{k,s}^{(1)}\cdot E^H(f(x_1))\cdot E^H(f(x_2))\cdot(f(x_1))^{i+k-2}\cdot(f(x_2))^{j+s-2}$$

and

(iii) $(q\wedge q)^*r_4=\sum B_{i,j}^{(1)}\cdot B_{k,s}^{(1)}\cdot B_{n,m}^{(1)}\cdot B_{p,q}^{(1)}\cdot(f(x_1))^{i+k+n+p}\cdot(f(x_2))^{j+s+m+q}$

$$-\sum 2B_{i,j}^{(1)}\cdot B_{k,s}^{(1)}\cdot C_{n,m}^{(1)}\cdot C_{p,q}^{(1)}\cdot E^H(f(x_1))\cdot E^H(f(x_2))$$
$$\cdot(f(x_1))^{i+k+n+p-2}\cdot(f(x_2))^{j+s+m+q-2}$$
$$+\sum C_{i,j}^{(1)}\cdot C_{k,s}^{(1)}\cdot C_{n,m}^{(1)}\cdot C_{p,q}^{(1)}\cdot(E^H(f(x_1)))^2\cdot(E^H(f(x_2)))^2$$
$$\cdot(f(x_1))^{i+k+n+p-4}\cdot(f(x_2))^{j+s+m+q-4}\,.$$

Proof. Put

$$S_i=(i\text{-th elementary symmetric function on } y_1^{MSp},\ y_2^{MSp},\ y_3^{MSp} \text{ and } y_4^{MSp})\,.$$

Then we obtain the equation

$$(q\wedge q)^*r_i=hur^H\circ(\Delta_c\wedge\Delta_c)^*\circ m_1^*\circ q^*\circ(Bi_{4+})^*P_i=(\Delta_c\wedge\Delta_c)^*\circ m_4^*\circ q^*\circ hur^H(S_i)$$

by (4.6).

Then the results of (4.7) follow from an easy calculation. Since the case (i)~(iii) are quite similar, we show the case (i) in detail and omit others.

$$(q\wedge q)^*r_1=(\varDelta_c\wedge\varDelta_c)^*m_4^*(f(x_1)+f(x_2)+f(x_3)+f(x_4)) \quad \text{(by (3.3).)}$$
$$=(\varDelta_c\wedge\varDelta_c)^*(f(x_1+x_3)+f(x_1+x_4)+f(x_2+x_3)+f(x_2+x_4))$$
$$\text{(by the definition of } m_4.)$$

Since $(id\wedge\text{conj}\wedge id\wedge\text{conj})^*x_i=(-1)^{i+1}x_i$, this equation becomes

$$(q\wedge q)^*r_1=(\varDelta\wedge\varDelta)^*(f(x_1+x_3)+f(x_1-x_4)+f(-x_2+x_3)+f(x+x_4))$$
$$=f(x_1+x_2)+f(x_1-x_2)+f(-x_1+x_2)+f(-x_1-x_2)\,.$$

Since $f(x)=h(-x^2)=f(-x)$ and $\bar{f}(x)=\frac{1}{2}\frac{d}{dx}h(-x^2)=-\bar{f}(-x)$, we obtain

$$(q\wedge q)^*r_1=2(f(x_1+x_2)+f(x_1-x_2))$$
$$=2(F_1^H(f(x_1),\ \bar{f}(x_1),\ f(x_2),\ \bar{f}(x_2))+F_1^H(f(x_1),\ \bar{f}(x_1),\ f(x_2),\ -\bar{f}(x_2)))$$
$$=\sum 4B_{i,j}^{(1)}\cdot(f(x_1))^i\cdot(f(x_2))^j\,. \quad \square$$

We have another commutative diagram

$$\begin{array}{ccccccc}
\widetilde{MSp}^*(BSp(2)_+) & \xrightarrow{q^*} & \widetilde{MSp}^*(BU(2)_+) & \xrightarrow{m^*} & \widetilde{MSp}^*(BU(2)_+\wedge CP_+^\infty) & \xrightarrow{(c\wedge t)^*} & \widetilde{MSp}^*(HP_+^\infty\wedge HP_+^\infty) \\
\downarrow{\scriptstyle (Bi_{2+})^*} & & \downarrow{\scriptstyle (Bi_{2+})^*} & & \downarrow{\scriptstyle (Bi_{2+}\wedge id)^*} & & \downarrow{\scriptstyle (q\wedge q)^*} \\
\widetilde{MSp}^*((HP^\infty)_+^2) & \xrightarrow{q^*} & \widetilde{MSp}^*((CP^\infty)_+^2) & \xrightarrow{m^*} & \widetilde{MSp}^*((CP^\infty)_+^2\wedge CP_+^\infty) & \xrightarrow{(\varDelta_c\wedge(tq))^*} & \widetilde{MSp}^*(CP_+^\infty\wedge CP_+^\infty)
\end{array}$$

where $m: BU(2)_+\wedge CP_+^\infty\to BU(2)_+$ is the classifying map of the tensor product $U(2)\times U(1)\to U(2)$ and $m_2:(CP^\infty)_+^3\to(CP^\infty)_+^2$ is that of the homomorphism $\mu_2: U(1)\times U(1)\times U(1)\to U(1)\times U(1)$ defined by $\mu_2(a,\ b,\ c)=(ac,\ bc)$.

Under the similar notations in (4.7), we obtain

Lemma 4.9.

$$(q\wedge q)^*\circ hur^H\circ(c\wedge t)^*\circ m^*\circ q^*P_2=\sum 2B_{i,j}^{(1)}\cdot B_{k,s}^{(1)}\cdot(f(x_1))^{i+k}\cdot(f(x_2))^{j+s}$$
$$-\sum 2C_{i,j}^{(1)}\cdot C_{k,s}^{(1)}\cdot E^H(f(x_1))\cdot E^H(f(x_2))\cdot(f(x_1))^{i+k-2}\cdot(f(x_2))^{j+s-2}\,.$$

Since the proof of (4.9) is quite similar to (4.7), we omit this.

We put $s_2=hur^H\circ(c\wedge t)^*\circ m^*\circ q^*P_2$. Then $r_2-2s_2\in\text{Im}\,(hur^H)$ and

$$(q\wedge q)^*(r_2-2s_2)=\sum 2B_{i,j}^{(1)}\cdot B_{k,s}^{(1)}\cdot(f(x_1))^{i+k}\cdot(f(x_2))^{j+s}$$
$$+\sum 2C_{i,j}^{(1)}\cdot C_{k,s}^{(1)}\cdot E^H(f(x_1))\cdot E^H(f(x_2))\cdot(f(x_1))^{i+k-2}\cdot(f(x_2))^{j+s-2}\,.$$

Since the right side of the above equation is $\sum 2B_{i,j}^{(2)}\cdot(f(x_1))^i\cdot(f(x_2))^j$ by the multiplicative relation and since there are elements $\beta_{i,j}^{(2)}\in MSp_*$ satisfying

$$r_2-2s_2=\sum hur^H(\beta_{i,j}^{(2)})\cdot(y_1^{MSp})^i\cdot(y_2^{MSp})^j\,, \qquad 2B_{i,j}^{(2)}=hur^H(\beta_{i,j}^{(2)})\in\text{Im}\,(hur^H)\,.$$

Since $2B_{i,j}^{(2)}=\theta'(2b_{i,j}^{(2)})$, we have already proved the following proposition in the case $k=1$.

Proposition 4.10. *$\theta'(2b_{i,j}^{(2k)})\in \mathrm{Im}(hur^H : MSp_* \to H_*(MSp))$ for $k\geqq 1$.*

Proof. Put $X=\sum B_{i,j}^{(1)}\cdot x_1^i\cdot x_2^j$, $Y=\sum C_{i,i}^{(1)}\cdot \bar{x}_1\cdot x_1^{i-1}\cdot \bar{x}_2\cdot x_2^{j-1}$ and $\bar{x}_i^2=E^H(x_i)$ for $i=1$ or 2. Then the coefficients of $(X+Y)^2+(X-Y)^2$ and those of $(X^2-Y^2)^2$ at $x_1^i\cdot x_2^j$ are in $\mathrm{Im}(hur^H)$ by the multiplicative relation, $2B_{i,j}^{(2)}\in \mathrm{Im}(hur^H)$ and (iii) of (4.7).

Notice that $(X+Y)^{2k}+(X-Y)^{2k}=\sum 2B_{i,j}^{(2k)}\cdot x_1^i\cdot x_2^j$ by the multiplicative relation. So, if the following lemma holds, then (4.10) can be proved by induction on k, easily.

Lemma 4.11.

$$(X+Y)^{2n}+(X-Y)^{2n}\in(\textit{ideal generated by } (X+Y)^{2m}+(X-Y)^{2m}\ (m<n) \textit{ and by } (X^2-Y^2)^2).$$

Proof. Put $A=(X+Y)^2$ and $B=(X-Y)^2$. Then we have only to prove that $A^n+B^n\in I_n=$(ideal generated by A^m+B^m $(m<n)$, AB). Since $A^n+B^n=(A+B)(A^{n-1}+B^{n-1})-AB(A^{n-2}+B^{n-2})\in I_n$, this is clear. □

Since $MSp_*\otimes \boldsymbol{Q}\xrightarrow{hur^H\otimes id} H_*(MSp)\otimes \boldsymbol{Q}$ is a monomorphism where $\boldsymbol{Q}$ is the field of rational numbers, and since $H_*(MSp)$ is torsion-free, $MSp_*/\mathrm{Torsion}\xrightarrow{hur^H} H_*(MSp)$ can be induced and is monic.

So $MSp_*/\mathrm{Torsion}\cong \mathrm{Im}(hur^H : MSp_*\to H_*(MSp))$. By (4.5) and (4.10), $\theta'(LMSp)\subset \mathrm{Im}(hur^H)$. Now the proof of the next theorem is clear.

Theorem 4.12. *There is a ring homomorphism $\theta : LMSp\to MSp_*/\mathrm{Torsion}$ such that $\theta'=hur^H\circ\theta$.*

We have some remarks.

(1) K. Shimakawa defined $\tilde{\Lambda}_{MSp}\subset MSp_*$ as the subring generated by the coefficients of $(c\wedge c)^*\circ m^*\circ q^*P_i\in MSp_*[[y_1^{MSp}, y_2^{MSp}]]$ (for $i=1\sim 4$). (See Shimakawa [23].) His approach was based on N. Ja. Gozman's method. (See Gozman [9].) These are closely related to the theory of 2-valued formal group studied by V. M. Buhštaber, S. P. Novikov and others. They introduced two functions $\Theta_1(x, y)$, $\Theta_2(x, y)\in(MSp_*\otimes\boldsymbol{Q})[[x, y]]$ such that

$$1+\sum_{i=1}^{4}(c\wedge c)^*\circ m^*\circ q^*P_i=(1+\Theta_1(y_1^{MSp}, y_2^{MSp})+\Theta_2(y_1^{MSp}, y_2^{MSp}))^2.$$

So the coefficients of $2\Theta_1$, $\Theta_1^2+2\Theta_2$, $2\Theta_1\Theta_2$ and Θ_2^2 are included in MSp_*. (See also Buhštaber [6].) Using our (4.5) and (4.10), one can easily proved that the coefficients of $2\Theta_2$ are in MSp_*.

(2) If we substitute MSp by KO, we have another example of symplectic formal system:

$$E(X)=-X+\frac{t^2}{4}\cdot X^2,$$

$$F_k(X, \bar{X}, Y, \bar{Y})=\left(X+Y-\frac{t^2}{2}\cdot X\cdot Y-2\cdot\bar{X}\cdot\bar{Y}\right)^k$$

and

$$G_k(X, \bar{X}, Y, \bar{Y})=\left(\bar{X}+\bar{Y}-\frac{t^2}{2}\cdot(\bar{X}\cdot Y+\bar{Y}\cdot X)\right)\cdot\left(X+Y-\frac{t^2}{2}\cdot X\cdot Y-2\cdot\bar{X}\cdot\bar{Y}\right)^{k-1}$$

for $k\geqq 1$.

If we denote LKO as the associated symplectic ring, then there is a ring homomorphism $\theta: LKO\rightarrow KO_*/\mathrm{Torsion}$. One can easily show that

$$\theta: LKO\cong\sum_{j\geq 0}KO_{4j}.$$

§5. Calculation in *LMSp*

First, we prove the following theorem.

Theorem 5.1. *$\theta'\otimes id: LMSp\otimes \boldsymbol{Q}\rightarrow(H_*(MSp))_{\Gamma_H}\otimes\boldsymbol{Q}$ is an isomorphism. So, $LMSp/\mathrm{Torsion}\xrightarrow{\theta'}(H_*(MSp))_{\Gamma_H}$ is also an isomorphism.*

There are some propositions.

Let $\Gamma=\{E, F_k, G_k\}$ be a symplectic formal system over R.

Proposition 5.2. *In $R\otimes\boldsymbol{Q}$, $\sum_{i\geq 0}d_{1,i}^{(1)}\cdot X^i=-\sum_{i\geq 1}i\cdot a_i\cdot X^{i-1}=-\frac{d}{dX}E(X)$.*

Proof. By square relation, we obtain the following equation

$$(G_1(X, \bar{X}, Y, \bar{Y}))^2=\sum_{i\geq 1}a_i\cdot(F_1(X, \bar{X}, Y, \bar{Y}))^i.$$

If we put $Y=0$, then $\bar{Y}^2=\sum_{i\geq 1}a_i\cdot Y^i=0$. Then

$$(G_1(X, \bar{X}, Y, \bar{Y}))^2=(\bar{X}+\sum_{i\geq 0}d_{1,i}^{(1)}\cdot X^i\cdot\bar{Y})^2=\bar{X}^2+2\sum_{i\geq 0}d_{1,i}^{(1)}\cdot\bar{X}\cdot X^i\cdot\bar{Y}$$

$$=E(X)+(2\sum_{i\geq 0}d_{1,i}^{(1)}\cdot X^i)\cdot\bar{X}\cdot\bar{Y}.$$

On the other hand, if $\bar{Y}^2=Y=0$, then we have the following equation:

$$\sum_{i\geq 1}a_i\cdot(F_1(X, \bar{X}, Y, \bar{Y}))^i=\sum_{i\geq 1}a_i\cdot(X-2\cdot\bar{X}\cdot\bar{Y})^i$$

$$=E(X)-(2\sum_{i\geq 1}i\cdot a_i\cdot X^{i-1})\cdot\bar{X}\cdot\bar{Y}.$$

Thus (5.2) holds. □

Proposition 5.3. *In $R\otimes\boldsymbol{Q}$, $2G_1(X, \bar{X}, Y, \bar{Y})=\bar{Y}\cdot\left(\frac{\partial}{\partial Y}F_1(X, \bar{X}, Y, \bar{Y})\right)$.*

Proof. If we put $Z=0$ on the associative relation

$$F_1(F_1(X,\bar{X},Y,\bar{Y}),G_1(X,\bar{X},Y,\bar{Y}),Z,\bar{Z})=F_1(X,\bar{X},F_1(Y,\bar{Y},Z,\bar{Z}),G_1(Y,\bar{Y},Z,\bar{Z}))$$

and compare the coefficient at $\bar{Z}$, then the similar calculations to those in the proof of (5.2) deduce the following equation

$$\begin{aligned}
&c_{1,1}^{(1)}\cdot G_1(X,\bar{X},Y,\bar{Y})\\
&=-(\sum b_{i,j}^{(1)}\cdot X^i(2j)\bar{Y}\cdot Y^{j-1}+\sum c_{i,j}^{(1)}\cdot\bar{X}\cdot X^{i-1}((\sum d_{1,s}^{(1)}\cdot Y^s)Y^{j-1}+2(j-1)\bar{Y}^2Y^{j-2}))\\
&=-\left(\sum b_{i,j}^{(1)}\cdot X^i(2j)\bar{Y}\cdot Y^{j-1}+\sum c_{i,j}^{(1)}\cdot\bar{X}\cdot X^{i-1}(\frac{d}{dY}E(Y)Y^{j-1}+2(j-1)E(Y)Y^{j-2})\right).
\end{aligned}$$

Since $\bar{Y}^2=E(Y)$, $\frac{\partial}{\partial\bar{Y}}(\bar{Y}^2)=2\cdot\bar{Y}\cdot\frac{\bar{Y}}{Y}=\frac{d}{dY}E(Y)$. So we have

$$\begin{aligned}
&\sum b_{i,j}^{(1)}\cdot X^i\cdot 2\bar{Y}\cdot j\cdot Y^{j-1}+\sum c_{i,j}^{(1)}\cdot\bar{X}\cdot X^{i-1}\left(\frac{d}{dY}E(Y)\cdot Y^{j-1}+2(j-1)E(Y)\cdot Y^{j-2}\right)\\
&=2\bar{Y}\cdot\left(\sum b_{i,j}^{(1)}\cdot X^i\cdot\frac{d}{dY}(Y^j)+\sum c_{i,j}^{(1)}\cdot\bar{X}\cdot X^{i-1}\cdot\frac{d}{dY}(\bar{Y}\cdot Y^{j-1})\right)\\
&=2\bar{Y}\cdot\frac{\partial}{\partial Y}F_1(X,\bar{X},Y,\bar{Y}).
\end{aligned}$$

Thus (5.3) is proved. □

If $\Gamma=\{E,F_k,G_k\}$ is a symplectic formal system over a commutative ring R, then the R-algebra $R[[X,\bar{X},Y,\bar{Y}]]/(\bar{X}^2-E(X),\bar{Y}^2-E(Y))$ has a free R-module base $\bar{X}^\varepsilon\cdot X^n\cdot\bar{Y}^{\varepsilon'}Y^m$, $\varepsilon=0$ or 1, $\varepsilon'=0$ or 1 and $n, m\geqq 0$.

So, (5.2) and (5.3) can be interpreted as

(5.3)′ $$d_{i,j}^{(1)}=j\cdot b_{i,j}^{(1)}=i^{-1}\cdot j\cdot d_{j,i}^{(1)},$$

$$2d_{i,j}^{(1)}=\sum_{j=n+m-2}(n+2m-2)\cdot a_n\cdot c_{i,m}^{(1)}\qquad\text{for}\quad j\geqq 1$$

and

(5.2)′ $$d_{1,i}^{(1)}=-(i+1)\cdot a_{i+1},\qquad d_{i,1}^{(1)}=-i^{-1}\cdot(i+1)\cdot a_{i+1}\qquad\text{for}\quad i\geqq 1.$$

Let R be a commutative ring which is graded and is connected and Γ be a graded symplectic formal system over R. Let P be the augumentation ideal of R and J be the intersection

$$P\cap(\text{the subring generated by } a_i,\ b_{i,j}^{(1)},\ c_{i,j}^{(1)},\ d_{i,j}^{(1)}).$$

Proposition 5.4. *In* $R\otimes\boldsymbol{Q}$,

$$c_{n,m}^{(1)}\equiv-2(d_{n,m-1}^{(1)}+d_{m,n-1}^{(1)})+N(n,m)a_{n+m-1}\pmod{J^2}$$

for $n, m\geqq 1$ *and* $(n,m)\neq(1,1)$ *where* $N(n,m)\in\boldsymbol{Z}$.

Proof. We consider the square relation $(G_1(X,\bar{X},Y,\bar{Y}))^2=E(F_1(X,\bar{X},Y,\bar{Y}))$. We denote the coefficient at $\bar{X}^\varepsilon\cdot X^n\cdot\bar{Y}^{\varepsilon'}\cdot Y^m$ as $[\ ]_{(\varepsilon,n,\varepsilon',m)}$. If we compare the coefficients at $\bar{X}\cdot X^{n-1}\cdot\bar{Y}\cdot Y^{m-1}$ modulo J^2, then we obtain the following equation

$$2(d^{(1)}_{n,m-1}+d^{(1)}_{m,n-1})\equiv[(\bar{X}+\bar{Y}+\sum_{i,j\geq 1} d^{(1)}_{i,j}\cdot(\bar{X}\cdot X^{i-1}Y^j+\bar{Y}\cdot Y^{i-1}X^j))^2]_{(1,n-1,1,m-1)}$$

$$=[\sum_{i\geq 1} a_i\cdot(F_1(X,\bar{X},Y,\bar{Y}))^i]_{(1,n-1,1,m-1)}$$

$$\equiv[a_1\cdot F_1(X,\bar{X},Y,\bar{Y})+a_{n+m-1}\cdot(F_1(X,\bar{X},Y,\bar{Y}))^{n+m-1}]_{(1,n-1,1,m-1)}$$

$$\equiv a_1\cdot c^{(1)}_{n,m}+N(n,m)\cdot a_{n+m-1}\quad (\mathrm{mod}\, J^2)\qquad \text{for } n,m\geqq 1 \text{ and } (n,m)\neq(1,1)$$

where $N(n,m)=[(F_1(X,\bar{X},Y,\bar{Y}))^{n+m-1}]_{(1,n-1,1,m-1)}$. If we compare the coefficients at $\bar{X}\cdot\bar{Y}$, then we have $2=c^{(1)}_{1,1}\cdot a_1$. Then $a_1=-1$ and (5.4) follows from the above equations. □

Let A be the subring of R generated by a_i $(i\geqq 1)$. Then under the same hypothesis as in (5.4), we have

Proposition 5.5. *$J\otimes\boldsymbol{Q}\subset A\otimes\boldsymbol{Q}$. So $R_\Gamma\otimes\boldsymbol{Q}=A\otimes\boldsymbol{Q}$.*

Proof. First, we will prove $J\otimes\boldsymbol{Q}\subset(A+J^2)\otimes\boldsymbol{Q}$. If we can prove this, then by an easy induction on degree, we can prove (5.5).

By using the second equation of (5.3)′, we have

$$2d^{(1)}_{i,j}\equiv a_1\cdot c^{(1)}_{i,j+1}\cdot(2j+1)\equiv-(2j+1)\cdot c^{(1)}_{i,j+1}\quad(\mathrm{mod}\, J^2)\qquad \text{for } j\geqq 1\,.$$

So we have only to prove that $d^{(1)}_{i,j}\in(A+J^2)\otimes\boldsymbol{Q}$ by (5.3)′.

If $j=1$, then (5.2)′ says that $d^{(1)}_{i,j}\in A\otimes\boldsymbol{Q}$ for all $i\geqq 1$. So, we assume that $d^{(1)}_{i,k-1}\in(A+J^2)\otimes\boldsymbol{Q}$ for some $k\geqq 2$ and all $i\geqq 1$.

Since $2d^{(1)}_{i,k-1}\equiv-(2k-1)c^{(1)}_{i,k}$ (mod J^2), $c^{(1)}_{i,k}\in(A+J^2)\otimes\boldsymbol{Q}$ for all $i\geqq 1$. On the other hand, $c^{(1)}_{i,k}\equiv-2(d^{(1)}_{i,k-1}+d^{(1)}_{k,i-1})$ (mod $A+J^2$) by (5.4). So $d^{(1)}_{k,i-1}\in(A+J^2)\otimes\boldsymbol{Q}$ for all $i\geqq 2$. And we have $d^{(1)}_{k,i-1}=(i-1)\cdot b^{(1)}_{k,i-1}=(i-1)\cdot k^{-1}\cdot k\cdot b^{(1)}_{i-1,k}=(i-1)\cdot k^{-1}\cdot d^{(1)}_{i-1,k}$ for all $i\geqq 2$ by the first equation of (5.3)′.

Thus by induction on k, we have $d^{(1)}_{i,j}\in(A+J^2)\otimes\boldsymbol{Q}$. □

Now we can prove (5.1). Let $T=\boldsymbol{Q}[t_2,t_3,\cdots,t_k,\cdots]$ and $\alpha: T\to LMSp\otimes\boldsymbol{Q}$ the homomorphism defined by $\alpha(t_i)=a_i$ for $i\geqq 2$. Put $t_1=-1$. We assign the degree $4(i-1)$ to t_i. Then α is graded and is an epimorphism by (5.5).

We consider the following composition

$$T\xrightarrow{\alpha} LMSp\otimes\boldsymbol{Q}\xrightarrow{\theta'}(H_*(MSp))_{\Gamma_H}\otimes\boldsymbol{Q}\xhookrightarrow{\kappa} H_*(MSp)\otimes\boldsymbol{Q}\,.$$

By the definition of Γ_H, we have a square relation $(\bar{f}(x))^2=\sum_{i\geqq 1}\theta\circ\alpha(t_i)\cdot(f(x))^i$ where $f(x)$ and $\bar{f}(x)$ are as in § 4. So, we obtain the following equation

$$(\sum_{i\geqq 1}(-1)^i\cdot i\cdot h_{i-1}\cdot x^{2i-1})^2=\sum_{i\geqq 1}\theta'\circ\alpha(t_i)\cdot(\sum_{i\geqq 1}(-1)^j h_{j-1}\cdot x^{2j})^i\,.$$

Let $D=$(the ideal generated by $\{h_i\}$ $(i\geqq 1))^2$. If we compare the coefficients at x^{2i} modulo D, then we obtain easily $\theta'\circ\alpha(t_i)\equiv-(2i-1)h_{i-1}$ modulo D for $i\geqq 2$. Thus $\kappa\circ\theta'\circ\alpha: T\to H_*(MSp)\otimes\boldsymbol{Q}=\boldsymbol{Q}[h_1,h_2,\cdots,h_k,\cdots]$ is an isomorphism.

Since α and θ' are surjective, we can easily conclude that $\theta': LMSp\otimes\boldsymbol{Q}\to$

$(H_*(MSp))_{\Gamma_H}\otimes \boldsymbol{Q}$ is an isomorphism. □

Let L_*, M_* be graded rings which are commutative, unitary and free as modules. Then we denote the rational indecomposable module $Q(L_*)$ as the quotient $L_*/L_*\cap D_*$ where D_* is the ideal of all decomposable elements in $L_*\otimes \boldsymbol{Q}$.

If $f: L_*\to M_*$ is a ring homomorphism, then it gives the induced homomorphism $Q(f): Q(L_*)\to Q(M_*)$.

Ōkita [14] has studied $Q(MSp_*/\text{Torsion})$ in detail. He determined completely the image of $Q(MSp_*/\text{Torsion})$ in $Q(H_*(MSp))$ by $Q(hur^H)$.

We use the same notation $h_i\in Q(H_*(MSp))$ for the quotient image of $h_i\in H_{4i}(MSp)$. Clearly $Q(H_*(MSp))$ is generated freely by h_i $(0\leqq i)$.

Then Ōkita [14] has proved the following theorem. (See Ōkita [14], Theorem 1.1, Propositions 4.1, 4.2 and 4.3.)

Theorem 5.6. (Ōkita) *$\operatorname{Im} Q(hur^H)$ is generated freely by $2^{s_i}\cdot t_i\cdot h_i$ for $i\geqq 0$ where s_i and t_i are integers defined as follows:*

$$s_i=\begin{cases} 2 & \text{if } i\equiv 0 \pmod 2,\ i\neq 2^j \text{ for any } j \\ 4 & \text{if } i=2^j \text{ for some } j \\ 4 & \text{if } i\equiv 1 \pmod 2,\ i\neq 2^j-1 \text{ for any } j \\ 8 & \text{if } i=2^j-1 \text{ for some } j, \end{cases}$$

$$t_i=\begin{cases} p & \text{if } 2i+1 \text{ is a power of an odd prime } p \\ 1 & \text{otherwise}. \end{cases}$$

We have a commutative diagram

$$\begin{array}{ccc} LMSp/\text{Torsion} & \xrightarrow{\ \theta'\ } & (H_*(MSp))_{\Gamma_H} \\ \Big\downarrow{\scriptstyle\theta} & & \cap\Big\downarrow \\ MSp_*/\text{Torsion} & \xrightarrow{\ hur^H\ } & H_*(MSp) \end{array} .$$

Now we can prove the following theorem.

Theorem 5.7. $\operatorname{Im} Q(hur^H)=Q((H_*(MSp))_{\Gamma_H})$. *So $Q(\theta): Q(LMSp/\text{Torsion})\to Q(MSp_*/\text{Torsion})$ is an isomorphism.*

Proof. Since $\theta': LMSp/\text{Torsion}\to (H_*(MSp))_{\Gamma_H}$ is an isomorphism, the first statement deduces the second one. So we have only to determine $Q((H_*(MSp))_{\Gamma_H})$.

Let $B_{i,j}$ and $C_{i,j}$ be the elements in $H_*(MSp)$ satisfying

$$f(x+y)=\sum_{i,j\geqq 0} B_{i,j}\cdot(f(x))^i\cdot(f(y))^j+\sum_{i,j\geqq 1} C_{i,j}\cdot\bar{f}(x)\cdot(f(x))^{i-1}\cdot\bar{f}(y)\cdot(f(y))^{j-1}$$

where $f, \bar{f}$ are as in § 4. If we compare the coefficients at $x^{2n}\cdot y^{2m}$, then we have

$$B_{n,m}=\binom{2n+2m}{2n}h_{n+m-1} \quad \text{in} \quad Q(H_*(MSp)). \tag{5.8}$$

Also, if we compare the coefficients at $x^{2n-1}\cdot y^{2m-1}$, then we have easily

$$C_{n,m}=-\binom{2n+2m-2}{2n-1}h_{n+m-2} \quad \text{in} \quad Q(H_*(MSp)). \tag{5.9}$$

So, if the following lemma can be proved, then (5.6) deduces the first statement of (5.7).

Let S be a set of integers. Then we denote the greatest common divisor of all elements in S by GCD(S).

Lemma 5.10.

(1) $\mathrm{GCD}\left(\binom{2N+2}{2n-1}\middle|1<n<N+1\right)$

$$\begin{cases} \equiv 2 \pmod 4 & \text{if } N\equiv 0 \pmod 2,\ N\neq 2^j \text{ for any } j \\ \equiv 4 \pmod 8 & \text{if } N=2^j \text{ for some } j, \end{cases}$$

(2) $4\cdot\mathrm{GCD}\left(\binom{2N+2}{2n}\middle|0<n<N+1\right)$

$$\begin{cases} \equiv 4 \pmod 8 & \text{if } N\equiv 1 \pmod 2,\ N\neq 2^j-1 \text{ for any } j \\ \equiv 8 \pmod{16} & \text{if } N=2^j-1 \text{ for some } j \text{ and} \end{cases}$$

(3) $\mathrm{GCD}\left(\binom{2N+2}{n}\middle|1<n<2N+1\right)$

$$\begin{cases} =2^s\cdot p & \text{for some } s \text{ if } 2N+1 \text{ is a power of an odd prime } p \\ =2^s & \text{for some } s \text{ otherwise.} \end{cases}$$

The proof of (5.10) is easy but tedious. So, we prove only the first statement of (1). The proofs of the rest are quite similar.

We may put $N=2^a\cdot(2b+1)$ where a, b are positive integers. Then we have the equation

$$\binom{2^{a+1}(2b+1)}{2n-1}=[(1+t)^{2^{a+1}(2b+1)+2}]_{2n-1}\equiv[(1+t^2)^{2^a(2b+1)+1}]_{2n-1}\equiv 0 \pmod 2$$

where t is a variable. On the other hand, we have

$$\binom{2^{a+1}(2b+1)+2}{2^{a+1}+1}=\binom{2^{a+1}(2b+1)}{2^{a+1}+1}+2\binom{2^{a+1}(2b+1)}{2^{a+1}}+\binom{2^{a+1}(2b+1)}{2^{a+1}-1}.$$

If q is an integer, then we have also

$$\binom{2^{a+1}(2b+1)}{2q+1}=[(1+t)^{2^{a+1}(2b+1)}]_{2q+1}\equiv[(1+2t^2+t^4)^{2^{a-1}(2b+1)}]_{2q+1}\equiv 0 \pmod 4.$$

So, $\binom{2^{a+1}(2b+1)+2}{2^{a+1}+1}\equiv 2\binom{2^{a+1}(2b+1)}{2^{a+1}} \pmod 4$. Since as is well-known $\binom{2^{a+1}(2b+1)}{2^{a+1}}\equiv 1 \pmod 2$, the result follows. □

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References

[1] J.F. Adams, Stable homotopy and generalized homology, Univ. of Chicago, Chicago (1970).
[2] J.F. Adams, A.S. Harris and R.M. Switzer, Hopf algebras of cooparations for real and complex K-theory, Proc. London Math. Soc. (3), **23** (1971), 385-408.
[3] S. Araki, Typical formal groups in complex cobordism and K-theory, Lectures in Math. 6, Kyoto Univ. (1973).
[4] J.C. Becker, Characteristic classes and K-theory, Lecture Notes in Math. 428 (1978), 132-143.
[5] J.C. Becker and D.H. Gottlieb, The transfer map and fibre bundles, Topology, **14** (1975), 1-13.
[6] V.M. Buhštaber, Topological applications of the theory of two-valued formal groups, Izv. Akad. Nauk SSSR Ser. Mat., **42** (1978), (Math. USSR Izv., **12** (1978), 125-177).
[7] V.M. Buhštaber and S.P. Novikov, Formal groups, power systems and Adams operators, Mat. Sb., **84** (1971), 81-118 (Math., USSR Sb., **13** (1971), 80-116).
[8] P.E. Conner and E.E. Floyd, The relation of cobordism to K-theory, Lecture Notes in Math. 28, Spriger Verlag.
[9] N.Ja. Gozman, On the image of self-conjugate cobordism ring in the complex and unoriented cobordism rings, Dokl. Akad. Nauk SSSR, **216** (1974) (Soviet Math. Dokl., **15** (1974), 953-956).
[10] A. Hattori, Integral characteristic numbers for weakly almost complex manifolds, Topology, **5** (1966), 259-280.
[11] I.M. James, The topology of Stiefel manifolds, London Math. Soc. Lecture note series **24** (1976).
[12] M. Lazard, Sur les groupes de Lie formels à un paramètre, Bull. Soc. Math. France, **83** (1955), 251-274.
[13] S.P. Novikov, The methods of algebraic topology from the viewpoint of cobordism theories, Izv. Akad. Nauk SSSR, Ser. Mat., **31** (1967), 855-951 (Math. USSR Izv., **1** (1967), 827-913).
[14] R. Ōkita, On the MSp Hattori-Stong problem, Osaka J. Math., **13** (1976), 547-566.
[15] D. Quillen, On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc., **75** (1969), 1293-1298.
[16] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod oparations, Advances in Math., **7** (1971), 29-56.
[17] N. Ray, A note on the symplectic bordism ring, Bull. London Math. Soc., **3** (1971), 159-162.
[18] N. Ray, Indecomposables in Tors MSp_*, Topology, **10** (1971), 261-270.
[19] N. Ray, Realizing symplectic bordism classes, Proc. Camb. Phil. Soc., **71** (1972), 301-305.

[20] D. M. Segal, Divisibility conditions on characteristic numbers of stably symplectic manifolds, Proc. Amer. Math. Soc., **27** (1971), 411-415.

[21] D. M. Segal, Halving the Milnor manifolds and some conjecture of Ray, Proc. Amer. Math. Soc., **39** (1973), 625-628.

[22] G. B. Segal, The stable homotopy of complex projective space, Quart. J. Math. Oxford Ser., **24** (1973), 1-5.

[23] K. Shimakawa, Remarks on the coefficient ring of quaternionic oriented cohomology theories, RIMS, Kyoto Univ., **12** (1976), 241-254.

[24] R. E. Stong, Relations among characteristic number I, Topology, **4** (1965), 267-281.

[25] R. E. Stong, Some remarks on symplectic cobordism, Ann. of Math., **86** (1967), 425-433.

[26] R. M. Switzer, Algebraic topology-homotopy and homology, Springer Verlag (1975).

[27] I. Yokota, On the cellular decompositions of unitary groups, J. Inst. Poly. Osaka City Univ., **7** (1956), 39-49.

[28] I. Yokota, On the homology of classical Lie groups, J. Inst. Poly. Osaka, City Univ., **8** (1957), 93-120.