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The Synthesis of Noise Transfer Functions for Bandpass Delta-Sigma Modulators with Tunable Center Frequency

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***Abstract*—This paper presents a method to synthesize the noise transfer function (NTF) for tunable bandpass delta-sigma modulators, where the quantization noise stopband can be programmed over the whole Nyquist range. Instead of relying on traditional filter design theory, the proposed method allows to create NTFs of arbitrary order by directly placing the zeros and poles on the z-plane. The advantage is that the NTF can be re-calculated for each center frequency by using simple closed form expressions, thus avoiding the need of large lookup tables to store multiple pre-computed coefficient sets. Extensive system-level simulations show that our method yields equal performance as the Chebyshev-II design method. As an example, the synthesis of a binary 12th-order tunable bandpass delta-sigma modulator is demonstrated, and its stability is proven for any choice of the center frequency.**

I. Introduction

$\Delta\Sigma$ modulation is an established and effective technique to enhance the linearity of analog-to-digital and digital-to-analog data converters [1]–[4]. Tunable bandpass $\Delta\Sigma$ modulators are a special class of $\Delta\Sigma$ converters, where the center frequency of the signal band f_0 is programmable over a specific frequency range (Fig. 1). Common applications of tunable bandpass $\Delta\Sigma$ modulators include, for example, wireless receiver front-ends [5]–[7] and direct digital synthesizers [8]. Due to the increasing transmission bandwidth and flexibility needed in modern wireless communication systems, the tunability requirements on $\Delta\Sigma$ modulators are becoming increasingly challenging [9].

In discrete-time $\Delta\Sigma$ modulators, the most straightforward method to achieve tunability is to pre-compute the modulator's noise transfer function (NTF) for each supported center frequency, and to implement a loop filter where most or all of the coefficients are programmable [5], [6]. Changing f_0 is thus performed by re-programming the loop filter with the coefficients of the corresponding NTF. However, in a practical implementation, the pre-computed coefficients need to be stored in a lookup table, making this method unfeasible when the tunability range of f_0 is large.

Another method consists of using the discrete-time lowpass-to-bandpass transformation, given by

$$z^{-1} \rightarrow z^{-1} \frac{\cos\omega_0 - z^{-1}}{1 - z^{-1}\cos\omega_0}, \tag{1}$$

to shift a prototype lowpass NTF to the normalized angular frequency ω_0 [7], [8]. However, (1) does not offer the flexibility to control NTF parameters such as signal bandwidth and maximum gain, which is desirable in some modern applications.

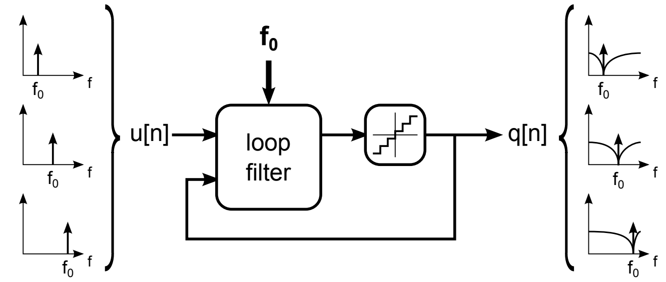


Fig. 1. General structure of a tunable bandpass $\Delta\Sigma$ modulator.

In our recent work [9], we proposed to use tunable bandpass $\Delta\Sigma$ modulation in digitally-intensive transceivers, to reduce the noise produced by the transmitter in the receive band. Due to the severe flexibility demands posed by the application, we showed that a highly tunable 4th-order NTF can be conveniently designed by directly placing the zeros and poles on the z-plane. Here, we generalize the method to arbitrary order, and we derive simple formulas to calculate the NTF coefficients starting from constraints on oversampling ratio, order, and maximum gain. In modern integrated circuits and system-on-chips, these formulas can be exploited by the already available processing unit to quickly reconfigure the $\Delta\Sigma$ modulator on a case-by-case basis, thus avoiding the need of large lookup tables to store multiple pre-computed coefficient sets. We demonstrate the validity of our results by simulating the signal-to-noise ratio (SNR) in a number of $\Delta\Sigma$ modulators synthesized with the proposed method, and comparing the results to modulators using classical Chebyshev-II NTFs. As an additional design example, a binary 12th-order tunable bandpass $\Delta\Sigma$ modulator is synthesized, and shown to be stable for any choice of f_0 over the Nyquist range.

The remainder of this paper is organized as follows. Section II describes the proposed NTF synthesis method from a mathematical point of view. Section III evaluates the achievable performance through system-level simulations, comparisons, and a design example. Finally, Section IV summarizes the formulas for NTF coefficient calculation, and draws the conclusions.

II. NTF Design Method

The feedback loop in Fig. 1 is physically realizable only if there is at least one delay between the input and the output of the loop filter. It can be demonstrated that this requirement translates into the important NTF realizability condition

$$H(z) = \frac{N(z)}{D(z)} = \frac{1 + \sum_{i=1}^{P} b_i z^{-i}}{1 + \sum_{i=1}^{P} a_i z^{-i}}, \tag{2}$$

where $H(z) = N(z)/D(z)$ is the NTF in the z-domain, P the modulator order, and $\{b_i, a_i\}$ the set of NTF coefficients [1].

The frequency response of $H(z)$ is given by evaluating the transfer function on the unit circle $z = e^{j\omega}$, where $\omega \in [-\pi, \pi)$ is the angular frequency normalized to the system sampling rate. Hence, $H(z)$ will provide the maximum possible attenuation at frequency ω_i if $H(e^{j\omega_i}) = 0$ holds. The simplest transfer function that satisfies this constraint is

$$C_{\omega_i}(z) = 1 - e^{j\omega_i} z^{-1}, \tag{3}$$

which is a 1st-order polynomial in z^{-1} with complex coefficients. In order to have real coefficients only, we need to add a second zero at the mirror frequency $-\omega_i$. This leads to

$$N_i(z) = C_{\omega_i}(z) \cdot C_{-\omega_i}(z) = 1 - (2\cos\omega_i)z^{-1} + z^{-2}, \tag{4}$$

which is the basic numerator factor of a digital single-notch filter. The position of the zero ω_i can be programmed over the whole Nyquist range by adjusting the coefficient of z^{-1}. It is clear that $N_i(z)$ fulfills the realizability condition given by (2).

In $\Delta\Sigma$ modulator design, especially with single-bit quantizer, we are not only interested in the stopband attenuation of $H(z)$, but also its maximum gain over all frequencies, given by the infinity-norm

$$||H||_\infty = \max_\omega |H(e^{j\omega})|. \tag{5}$$

Indeed, it has been shown that the stability of a binary modulator can be largely improved by lowering $||H||_\infty$. The most used empirical rule is known as *Lee criterion*, which states that a binary $\Delta\Sigma$ modulator is likely to be stable if $||H||_\infty < 1.5$ [1]. However, by studying the magnitude of (4) on the unit circle, it is easy to prove that the peak is always between 2 and 4, depending on ω_i.

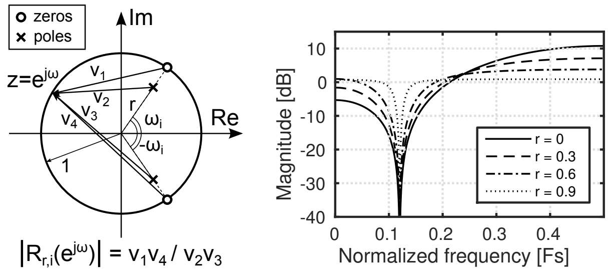
In order to reduce the maximum gain, we introduce factors similar to (3) to the denominator of the transfer function, to shift the poles away from $z = 0$. Because the poles must lie inside the unit circle (for stability), we modify (3) into

$$C_{r,\varphi_i}(z) = 1 - re^{j\varphi_i} z^{-1}, \tag{6}$$

where φ_i is the pole angle, and $r \in [0, 1)$ its distance from the origin. By placing a couple of complex-conjugate poles, we get the basic real-coefficient denominator factor

$$D_{r,i}(z) = C_{r,\varphi_i}(z) \cdot C_{r,-\varphi_i}(z) = 1 - (2r\cos\varphi_i)z^{-1} + r^2 z^{-2}. \tag{7}$$

For NTF design, the zero frequency ω_i must be located within the stopband, for maximum quantization noise attenuation. However, an obvious question concerns how to choose the pole angle φ_i. In this work, we assume $\varphi_i = \omega_i$, since it greatly simplifies the procedure to calculate the NTF coefficients. The validity of this choice will be proven through system-level simulations in Section III.



(a) (b)

Fig. 2. Single-notch digital filter. (a) Geometric interpretation of frequency response computation, from zero/pole placement in the z-plane. (b) Frequency response for $f_i = 0.12$ ($\omega_i = 2\pi f_i$) and different values of r.

Combining (4) and (7) yields the transfer function of a digital single-notch filter with parametrizable gain, given by

$$R_{r,i}(z) = \frac{N_i(z)}{D_{r,i}(z)} = \frac{1 - (2\cos\omega_i)z^{-1} + z^{-2}}{1 - (2r\cos\omega_i)z^{-1} + r^2 z^{-2}}. \tag{8}$$

Note that this expression also fulfills (2).

The reason why r controls the maximum gain of (8) can be intuitively understood by means of geometric interpretation [10]. Fig. 2(a) illustrates the zeros and poles of $R_{r,i}(z)$ in the z-plane. When evaluating the magnitude at $z = e^{j\omega}$, each zero (pole) brings a contribution to the numerator (denominator) equal to the distance between z and the zero (pole) itself. Therefore, we see that $\omega = \pm\omega_i$ will yield a magnitude of 0, since at least one of the zero vectors is null. On the other hand, as z moves away from the stopband, the distance from each zero and that from the corresponding pole become "similar", meaning that their ratio will be close to 1. The more r approaches 1, the more effective this zero/pole compensation will be. When $r = 0$, the poles coincide with the origin, thus their contribution to the denominator is always 1.

Fig. 2(b) verifies this geometric interpretation by plotting the frequency response of (8) for different values of r. Although the maximum gain is effectively reduced as r approaches 1, an increasing sharpness of the notch is also observed. The net effect is a degradation of the average stopband attenuation. Thus, r can be used to trade-off the SNR performance of the NTF with its maximum gain.

By studying the magnitude of (8) on the unit circle, it can be seen that the peak occurs always at $z = -1$ when $\cos\omega_i > 0$, and at $z = 1$ when $\cos\omega_i < 0$, independently from r. Therefore, we can write

$$||R_{r,i}||_\infty = 2\frac{1 + |\alpha_i|}{1 + 2r|\alpha_i| + r^2}, \tag{9}$$

where $\alpha_i = \cos\omega_i$. By inverting (9), we can derive the expression that relates r to a specified maximum gain constraint, which turns out to be

$$r = \sqrt{\alpha_i^2 + 2\frac{1 + |\alpha_i|}{||R_{r,i}||_\infty} - 1} - |\alpha_i|. \tag{10}$$

Furthermore, studying (9) as a function of $\alpha_i \in [-1, 1]$ reveals that $||R_{r,i}||_\infty$ is minimum for $\alpha_i = 0$ and maximum for $\alpha_i = \pm 1$. Hence, evaluating (10) for $|\alpha_i| = 1$ yields

$$r = \frac{2}{\sqrt{||R_{r,i}||_\infty}} - 1, \tag{11}$$

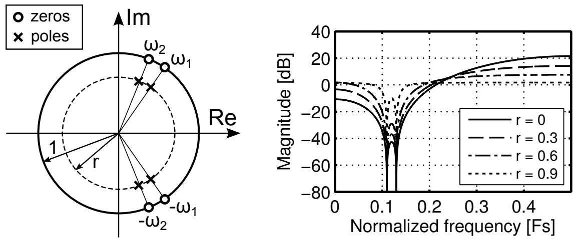


Fig. 3. Double-notch digital filter. (a) Zeros and poles in the z-plane. (b) Frequency response for $f_1 = 0.11$, $f_2 = 0.13$, and different values of r.

which is the minimum value of r that satisfies the maximum gain constraint for any $\omega_i \in [0, \pi]$.

The single-notch filter analyzed so far can be used as the basic building block for the NTF. In a bandpass $\Delta\Sigma$ modulator of order $P = 2M$, the overall NTF is defined as

$$H(z) = \prod_{i=1}^{M} R_{r,i}(z) = \prod_{i=1}^{M} \frac{1 - (2\cos\omega_i)z^{-1} + z^{-2}}{1 - (2r\cos\omega_i)z^{-1} + r^2 z^{-2}}, \tag{12}$$

where the zero frequencies are $\{\omega_1, \omega_2, \ldots, \omega_M\}$. Note that the above expression always fulfills (2). Having the same r for all M factors in (12) greatly simplifies the design method, and will be proven in Section III to be a valid design choice. Fig. 3 shows the zero-pole plot and frequency response of a NTF with $M = 2$. The trade-off between SNR and maximum gain described earlier can be still observed. Since all poles appear to be located on a circle of radius r, this parameter will be referred to as the *pole radius* in the following.

For a NTF of order greater than 2, an obvious question concerns the optimal zero location, that minimizes the total quantization noise power over the signal band. Under some reasonable assumptions, it was shown in [9] that, for the case $M = 2$, the optimal zero locations are given by

$$\omega_{1,2} = \omega_0 \pm \frac{\omega_S}{2\sqrt{3}}, \tag{13}$$

where ω_0 and ω_S are the center and width of the signal band respectively. An equivalent result for lowpass modulators is reported in [1]. Therefore, we can reuse the remaining results from [1] for $M > 2$, to obtain the optimal zero locations for higher orders. Table I lists the numeric values up to $M = 6$.

Deriving a relation between the pole radius and $||H||_\infty$ cannot be solved exactly for arbitrary M. However, an approximate solution can be easily found. First, we note that $||H||_\infty$ still occurs in $z = \pm 1$, depending on whether the signal band is on the left or right side of $\pi/2$. Second, we observe that all zero frequencies ω_i are very close to ω_0, especially for large oversampling ratios. In other words, the approximation $\alpha_i \approx \alpha_0 = \cos\omega_0$ holds when evaluating the $||R_{r,i}||_\infty$ factors. The above considerations lead to

$$||H||_\infty = \prod_{i=1}^{M} ||R_{r,i}||_\infty \approx \left(||R_{r,0}||_\infty\right)^M, \tag{14}$$

where

$$||R_{r,0}||_\infty = 2\frac{1 + |\alpha_0|}{1 + 2r|\alpha_0| + r^2}. \tag{15}$$

The pole radius is now found by inverting (14)–(15), yielding

$$r \approx \sqrt{\alpha_0^2 + 2\frac{1 + |\alpha_0|}{\sqrt[M]{||H||_\infty}} - 1} - |\alpha_0|. \tag{16}$$

Finally, in analogy with (11), the minimum r that satisfies the constraint on $||H||_\infty$ for any $\omega_0 \in [0, \pi]$ is

$$r \approx \frac{2}{\sqrt[2M]{||H||_\infty}} - 1. \tag{17}$$

For all practical purposes, the small error resulting from the approximations in (16) and (17) can be safely neglected.

III. Performance Evaluation

In order to evaluate fairly the performance of the proposed NTF synthesis method, we decided to compare it against the traditional Chebyshev-II filter design method, which has been widely used for NTF prototyping in $\Delta\Sigma$ modulators [2]–[4]. The comparison procedure works as follows.

1) We pick some values for NTF order, oversampling ratio (OSR), center frequency, and maximum gain.
2) The Chebyshev-II transfer function $H_c(z)$ that meets all chosen constraints is calculated, and properly scaled as to fulfill (2).
3) A second transfer function $H_p(z)$ is synthesized with the proposed method (using the expressions derived in Section II), starting from the same constraints.
4) The SNR vs. input amplitude curves are simulated, for two binary $\Delta\Sigma$ modulators implementing the NTFs calculated above. The simulations are performed with sinusoidal input signals.
5) $H_p(z)$ is re-synthesized with slightly different values of $||H_p||_\infty$, and the SNR simulations performed again, until the two $\Delta\Sigma$ modulators yield very similar SNR profiles.

Fig. 4 and 5 show the comparison results, for four different sets of design constraints. It can be seen that, in order to achieve the same SNR performance, $||H_p||_\infty$ typically needs to be increased by $1 \sim 1.5$ dB compared to $||H_c||_\infty$. Nevertheless, in practice this does not impair stability, as revealed by the simulated SNR vs. input amplitude curves.

The proposed NTF synthesis method can be used to design tunable bandpass $\Delta\Sigma$ modulators of very high order. For example, Fig. 6 demonstrates a binary 12th-order modulator. In this case, the same pole radius (calculated through (17)) was used for all center frequency settings. As the simulation results prove, the designed modulator is stable and operates correctly over the whole Nyquist range.

IV. Conclusion

In this paper, a flexible method to synthesize the NTF for tunable bandpass $\Delta\Sigma$ modulators is described. The method allows to create NTFs of arbitrary order, by directly placing the zeros and poles of the transfer function on the z-plane. Simple closed form expressions are used to calculate the NTF coefficients starting from constraints on OSR, order, and maximum gain. In modern applications with high flexibility demands, these formulas can be exploited to reconfigure the $\Delta\Sigma$

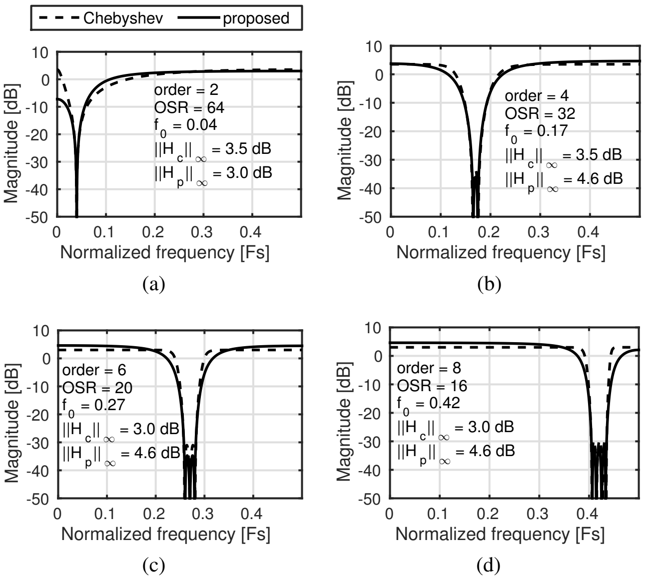


Fig. 4. Frequency response comparison between the Chebyshev-II design method and the proposed method. Each Chebyshev NTF $H_c(z)$ has been designed to fulfill some predefined constraints on order, OSR, f_0, and $||H_c||_\infty$. The corresponding $H_p(z)$ have been designed with the proposed method starting from the same constraints, but $||H_p||_\infty$ has been adjusted to achieve the same simulated SNR performance (shown in Fig. 5).

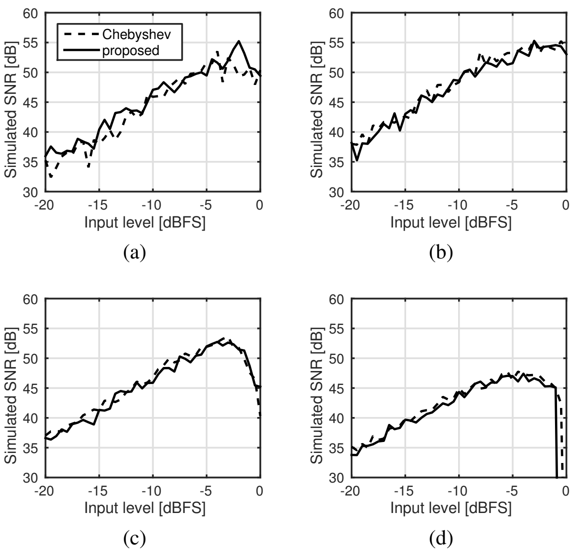


Fig. 5. Simulated SNR vs. input amplitude curves, for binary $\Delta\Sigma$ modulators using sinusoidal input signals and the NTFs of Fig. 4.

modulator on a case-by-case basis, thus avoiding the need of large lookup tables to store multiple pre-computed coefficient sets. System-level simulations show that the proposed NTF synthesis method yields equal performance as the Chebyshev-II design method, in terms of SNR of the $\Delta\Sigma$ modulator with sinusoidal inputs. A design example is also demonstrated, where a binary 12th-order tunable bandpass $\Delta\Sigma$ modulator is shown to operate correctly over the whole Nyquist range.

Table I gathers the NTF equations derived in the text.

References

[1] R. Schreier and G. C. Temes, *Understanding Delta-Sigma Data Converters*. Hoboken (NJ): Wiley, 2005.

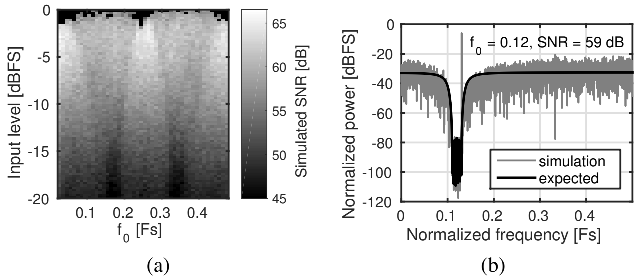


Fig. 6. Demonstration of a binary 12th-order tunable bandpass $\Delta\Sigma$ modulator synthesized with the proposed NTF design method, assuming OSR = 24 and $||H||_\infty = 3.5$ dB. (a) Simulated SNR as a function of center frequency and input sine amplitude. (b) Example spectrum with half-scale sinusoidal input, compared against the expected quantization noise power density.

TABLE I. NTF DESIGN EQUATIONS

Input constraints	order $2M$, OSR, ω_0, $\|\|H\|\|_\infty$
NTF	$H(z) = \prod_{i=1}^{M} \frac{1-(2\cos\omega_i)z^{-1}+z^{-2}}{1-(2r\cos\omega_i)z^{-1}+r^2z^{-2}}$
Zero locations	$\omega_i = \omega_0 + z_i \cdot \pi/(2\cdot \mathrm{OSR})$
$M=1$	$z_1 = 0$
$M=2$	$z_{1,2} = \pm 1/\sqrt{3}$
$M=3$	$z_{1\text{-}3} = 0, \pm\sqrt{3/5}$
$M=4$	$z_{1\text{-}4} = \pm\sqrt{3/7 \pm \sqrt{(3/7)^2 - 3/35}}$
$M=5$	$z_{1\text{-}5} = 0, \pm\sqrt{5/9 \pm \sqrt{(5/9)^2 - 5/21}}$
$M=6$	$z_{1\text{-}6} = \pm 0.23862, \pm 0.66121, \pm 0.93247$
Pole radius (specific ω_0)	$r \approx \sqrt{\cos^2\omega_0 + 2\frac{1+\lvert\cos\omega_0\rvert}{\sqrt[M]{\|\|H\|\|_\infty}} - 1} - \lvert\cos\omega_0\rvert$
Pole radius (any ω_0)	$r \approx \frac{2}{\sqrt[2M]{\|\|H\|\|_\infty}} - 1$

[2] T.-H. Kuo, K.-D. Chen, and J.-R. Chen, "Automatic coefficients design for high-order sigma-delta modulators," *IEEE Trans. Circuits Syst. II*, vol. 46, no. 1, pp. 6–15, Jan. 1999.

[3] M. Snoeij, O. Bajdechi, and J. Huijsing, "A 4th-order switched-capacitor sigma-delta A/D converter using a high-ripple Chebyshev loop filter," in *IEEE Int. Solid-State Circuits Conf. Dig. Tech. Papers*, May 2001, pp. 615–618.

[4] Z.-M. Lin and W.-H. Sheu, "A generic multiple-feedback architecture and method for the design of high-order $\Sigma\Delta$ modulators," *IEEE Trans. Circuits Syst. II*, vol. 49, no. 7, pp. 465–473, Jul. 2002.

[5] K. Yamamoto, A. Carusone, and F. Dawson, "A delta-sigma modulator with a widely programmable center frequency and 82-dB peak SNDR," *IEEE J. Solid-State Circuits*, vol. 43, no. 8, pp. 1772–1782, Aug. 2008.

[6] S. Asghar, R. del Rio, and J. de la Rosa, "A 0.2-to-2MHz BW, 50-to-86dB SNDR, 16-to-22mW flexible 4th-order $\Sigma\Delta$ modulator with DC-to-44MHz tunable center frequency in 1.2-V 90-nm CMOS," in *IEEE/IFIP Int. Conf. VLSI-SoC*, Oct. 2012, pp. 47–52.

[7] L. Cardelli, L. Fanucci, V. Kempe, F. Mannozzi, and D. Strle, "Tunable bandpass sigma delta modulator using one input parameter," *Electronics Letters*, vol. 39, no. 2, pp. 187–189, Jan. 2003.

[8] J. Lindeberg, J. Vankka, J. Sommarek, and K. Halonen, "A 1.5-V direct digital synthesizer with tunable delta-sigma modulator in 0.13-μm CMOS," *IEEE J. Solid-State Circuits*, vol. 40, no. 9, pp. 1978–1982, Sep. 2005.

[9] E. Roverato, M. Kosunen, J. Lemberg, K. Stadius, and J. Ryynänen, "RX-band noise reduction in all-digital transmitters with configurable spectral shaping of quantization and mismatch errors," *IEEE Trans. Circuits Syst. I*, vol. 61, no. 11, pp. 3256–3265, Nov. 2014.

[10] S. K. Mitra, *Digital Signal Processing - A Computer-Based Approach*, 2nd ed. McGraw-Hill, 2001.