The τ-Function of the Universal Whitham Hierarchy, Matrix Models and Topological Field Theories

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Abstract

The universal Whitham hierarchy is considered from the viewpoint of topological field theories. The τ-function is defined for this hierarchy. It is proved that the algebraic orbits of Whitham hierarchy can be identified with various topological matter models coupled with topological gravity. © 1994 John Wiley & Sons, Inc.

1. Introduction

The breakthrough in low-dimensional string theory that took place in the last two years is one of the most exciting results in modern mathematical and theoretical physics. In [1], [2], [6], [19], and [20] the remarkable connections between the non-perturbative theory of two-dimensional gravity coupled with various matter fields (see [3], [16], [22], and [23]) and the theory of integrable KdV-type systems were found. This has led to a complete solvability of double-scaling limit of the matrix-models that are used to simulate fluctuating triangulated Riemann surfaces. Shortly after that Witten (see [46]) presented some evidence of a relationship between random surfaces and the algebraic topology of moduli spaces of Riemann surfaces with punctures. His approach involved a particular field theory, known as topological gravity; see [32], [37], and [38]. Further development of his approach (especially, Kontsevich's proof in [24] and [25] of Witten's conjecture in [46] and [47] of the coincidence of the generating function for intersection numbers of moduli spaces with τ-function of the KdV hierarchy) has shown that the two-dimensional topological gravity is the cornerstone of this new subject of mathematical physics that includes two-dimensional quantum field theories, intersection theory on the moduli spaces of Riemann surfaces with punctures, integrable hierarchies with special Virasoro-type constraints, matrix integrals, and random surfaces.

In this paper we go on with our previous attempts in [30] and [31] to find the right place in this range of disciplines for the Whitham theory which is the most interesting part of the perturbation theory of KdV-type integrable hierarchies. They were stimulated by the results of [45], where correlation functions for topological minimal models were found. It turned out that the calculations in [45] of perturbed primary rings for A_n models can be identified with the construction of a particular solution of the first n "flows" of the dispersionless Lax hierarchy (semiclassical limit of the usual Lax hierarchy). This fact made it possible to include the corresponding deformations of primary chiral rings into a hierarchy of an infinite

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number of commuting "flows." The calculations in [45] of partition function for perturbed A_n models gave an impulse for the introduction a τ-function for dispersionless Lax equations into [30]. The truncated version of Virasoro constraints for the corresponding τ-function was proved. Their comparision with [46] shows that they are the necessary conditions for identification of "generators" of "higher" flows with gravitational descendants of primary fields after the model is coupled with gravity. They are not sufficient for $n > 2$. (The problem is the same as for τ-function of multi-matrix models. As was shown in [17] and [18], the τ-function of multi-matrix models satisfies the higher W-constraints that uniquely define it.) In Section 4 we prove the truncated version of W-constraints for τ-function of dispersionless Lax equations. Therefore, the full dispersionless Lax hierarchy can be really identified with topological A_n minimal model coupled with gravity.

The results of [30] were generalized in [31] and [13] for higher genus case. In [31] it was shown that self-similar solutions of the Whitham equations on the moduli space of genus g Riemann surfaces are related to "multi-cut" solutions of loop-equations of matrix models. In [13] the generalization of topological Landau-Ginsburg models on Riemann surfaces of special type was advanced and their primary rings and correlation functions were found. In [13] the Hamiltonian formulation (see [8], [9], [10], and [39]) of the Whitham averaging procedure was used. In [12] the integrability of general Witten-Dijgraagh-Verlinder-Verlinder (WDVV) equations (at tree-level) was proved with the use of the Hamiltonian approach to the Whitham theory.

Two- and three-points correlation functions

$$\langle\phi_\alpha\phi_\beta\rangle = \eta_{\alpha\beta}\,, \qquad c_{\alpha\beta\gamma} = \langle\phi_\alpha\phi_\beta\phi_\gamma\rangle\,, \tag{1.1}$$

of any topological field theory with N primary fields $\phi_1, \dots, \phi_N$ define associative algebra

$$\phi_\alpha\phi_\beta = c_{\alpha\beta}^{\gamma}\phi_\gamma\,, \quad c_{\alpha\beta}^{\gamma} = c_{\alpha\beta\mu}\eta^{\gamma\mu}\,, \quad \eta_{\alpha\mu}\eta^{\mu\beta} = \delta_\alpha^\beta\,, \tag{1.2}$$

with a unit ϕ_1

$$\eta_{\alpha\beta} = c_{1\alpha\beta}\,. \tag{1.3}$$

It turns out that there exists N parametric deformation of the theory such that "metric" $\eta_{\alpha\beta}$ is a constant and three-point correlators are given by the derivatives of free energy $F(t)$ of the deformed theory

$$c_{\alpha\beta\gamma}(t) = \partial_{\alpha\beta\gamma}F(t)\,, \qquad \eta_{\alpha\beta} = \partial_{1\alpha\beta}F(t) = \text{const.} \tag{1.4}$$

The associativity conditions of algebra (1.2) with structural constants (1.4) are equivalent to a system of partial differential equations on F (WDVV equations). In [12] "spectral transform" was proposed for these equations. It proves their integrability, however (as it seems to us), the explicit representation of all corresponding models remains an open problem.

In Section 5 we will show that each "algebraic" orbit of the universal Whitham hierarchy gives an exact solution of WDVV equations. Moreover, the generalization of W-constraints for corresponding τ-functions, that are proved in Section 4, provide the evidence that the universal Whitham hierarchy can be considered as a universal (at tree-level) topological field theory coupled with gravity.

In this introduction we present a definition of the Whitham hierarchy in a most general form. All the "integrable" partial differential equations, that are considered in the framework of the "soliton" theory, are equivalent to compatibility conditions of auxiliary linear problems. The general algebraic-geometrical construction of their exact periodic and quasi-periodic solutions was proposed in [26] and [27]. There the concept of the Baker-Akhiezer functions were introduced. (The analytical properties of the Baker-Akhiezer functions are the generalization of properties of the Bloch solutions of the finite-gap Sturm-Liouville operators, which were found in a series of papers by Novikov, Dubrovin, Matveev, and Its; see [7] and [49]).

The "universal" set of algebraic-geometrical data is as follows. Consider the space $\hat{M}_{g,N}$ of smooth algebraic curves Γ_g of genus g with local coordinates $k_\alpha^{-1}(P)$ in neighborhoods of N punctures P_α , $(k_\alpha^{-1}(P_\alpha) = 0)$

$$\hat{M}_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}(P),\ \alpha = 1, \ldots, N\}\ . \tag{1.5}$$

This space is a natural bundle over the moduli space $M_{g,N}$ of smooth algebraic curves Γ_g of genus g with N punctures

$$\hat{M}_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}(P)\} \longmapsto M_{g,N} = \{\Gamma_g, P_\alpha\}\ . \tag{1.6}$$

For each set of the data (1.5) and each set of g points $\gamma_1 \ldots, \gamma_g$ on Γ_g in a general position (or, equivalently, for a point of the Jacobian $J(\Gamma_g)$) the algebraic-geometrical construction gives a quasi-periodic solution of some integrable PNDE. (For the given non-linear integrable equation the corresponding set of the data has to be specified. For example, the solutions of the Kadomtsev-Petviashvili (KP) hierarchy correspond to the case $N = 1$. The solutions of the two-dimensional Toda lattice correspond to the case $N = 2$.)

The data (1.5) are "integrals" of the infinite "hierarchy" of integrable non-linear differential equations, that can be represented as a set of commuting "flows" on a phase space. Let t_A be a set of all the corresponding "times." In the framework of the "finite-gap" (algebraic-geometrical) theory of integrable equations each time t_A is coupled with a meromorphic differential $d\Omega_A(P|\mathcal{M})$, $\mathcal{M} \in \hat{M}_{g,N}$

$$t_A \longmapsto d\Omega_A(P|\mathcal{M}) \tag{1.7}$$

that is "responsible" for the flow. ($d\Omega_A(P|\mathcal{M})$ is a differential with respect to the variable $P \in \Gamma$ depending on the data (1.5) as on external parameters.)

In [28] the algebraic-geometrical perturbation theory for integrable non-linear (soliton) equations was developed. It was stimulated by the application of the Whitham approach for $(1 + 1)$ integrable equations of the KdV type; see [4],

[15], and [21]. As usual, in the perturbation theory "integrals" of an initial equation become functions of the "slow" variables εt_A (ε is a small parameter). *The Whitham equations* is a name given to equations that describe "slow" variations of "adiabatic" integrals. (We would like to emphasize that the algebraic-geometrical approach represents only one side of the Whitham theory. In [8], [9], [10], and [39] a deep differential-geometrical structure that is associated with the Whitham equations was developed.)

Let $\Omega_A(k,T)$ be a set of holomorphic functions of the variable k (which is defined in some complex domain D), depending on a finite or infinite number of variables t_A, $T = \{t_A\}$. (We keep the same notation t_A for slow variables εt_A because we are not going to consider "fast" variables in this paper.) Let us introduce a one-form

$$\omega = \sum_A \Omega_A(k,T)\,dt_A\,. \tag{1.8}$$

onto the space with coordinates (k, t_A). Its full external derivative equals

$$\delta\omega = \sum_A \delta\Omega_A(k,T)\wedge dt_A\,, \tag{1.9}$$

where

$$\delta\Omega_A = \partial_k\Omega_A\,dk + \sum_B \partial_B\Omega_A\,dt_B\,,\quad \partial_k = \partial/\partial k\,,\quad \partial_A = \partial/\partial t_A\,. \tag{1.10}$$

The following equation

$$\delta\omega\wedge\delta\omega = 0 \tag{1.11}$$

we shall call *the Whitham hierarchy* by definition .

The "algebraic" form (1.11) of the Whitham equations is equivalent to a set of partial differential equations that have to be fulfilled for any triple A, B, C

$$\sum_{\{A,B,C\}} \varepsilon^{\{A,B,C\}}\partial_A\Omega_B\partial_k\Omega_C = 0 \tag{1.12}$$

(summation in (1.12) is taken over all permutations of indices A, B, C and $\varepsilon^{\{A,B,C\}}$ is a sign of permutation).

The equations (1.11) are invariant with respect to an invertible change of variable

$$k = k(p,T)\,,\qquad \partial_p k \neq 0\,. \tag{1.13}$$

Let us fix an index A_0 and denote the corresponding function by

$$p(k,T) = \Omega_{A_0}(k,T)\,. \tag{1.14}$$

At the same time we introduce a special notation for the corresponding "time"

$$t_{A_0} = x\,. \tag{1.15}$$

After that all Ω_A can be considered as the functions of the new variable p, $\Omega_A = \Omega_A(p,T)$. The equations (1.12) for $A, B, C = A_0$ have the form

$$\partial_A \Omega_B - \partial_B \Omega_A + \{\Omega_A, \Omega_B\} = 0 , \tag{1.16}$$

where $\{f,g\}$ stands for the usual Poisson bracket on the space of functions of the two variables x, p

$$\{f,g\} = \partial_x f \partial_p g - \partial_x g \partial_p f . \tag{1.17}$$

The Whitham equations were obtained in [28] in the form (1.12). In [29] it was found that they can be represented in the algebraic form (1.11). (We would like to mention here the papers [41] and [42] where it was shown that the algebraic form of the Whitham equations leads directly to semiclassical limit of "strings" equations.)

The Whitham equations in the form (1.12) are equations on the set of functions $\Omega_A(p,T)$ and give but a certain "shape" that has to be filled with a real content. It is necessary to show that they do define correct systems of equations on the spaces $\hat{M}_{g,N}$. For zero-genus case ($g = 0$) this will be done in the next section. In the same section the construction (see [28]) of exact solutions of the zero-genus hierarchy corresponding to its "algebraic" orbits is presented. The key element of the scheme in [28] is a construction of a potential $S(p,T)$ and a "connection" $E(p,T)$ such that after the change of variable

$$p = p(E,T) , \qquad \Omega_A(E,T) = \Omega_A(p(E,T),T) , \tag{1.18}$$

the following equalities

$$\Omega_A(E,T) = \partial_A S(E,T) \tag{1.19}$$

are valid.

In Section 3 the τ-function for the Whitham equations on the spaces $\hat{M}_{0,N}$ is introduced. For all genera (the case $g > 0$ is considered in Section 7) τ-function can be represented in the following "field theory" form

$$\ln \tau = \int_\Gamma \bar{d}S \wedge dS . \tag{1.20}$$

Important remark. The integral (1.20) is not equal to zero, because $S(p,T)$ is holomorphic on Γ except for the punctures P_α and some contours, where it has "jumps." Therefore, the integral over Γ equals a sum of the residues at P_α and the contour integrals of the corresponding one-form.

The τ-function is a function of the variables t_A, $\tau = \tau(T)$. As will be shown in Section 3, it contains full information about the corresponding solutions $\Omega_A(p,T)$ of the Whitham equations. For $g > 0$ in τ a geometry of moduli spaces is incoded.

In Section 4 zero-genus Virasoro and W-constraints for τ-function are proved. In Section 5 the primary chiral rings corresponding to algebraic orbits of Whitham hierarchy are considered. The last section is preceded by Section 6 where, using the ideas of [41] and [42], a "direct transform" for general Whitham hierarchy is discussed. It turns out that the existence of a potential S is not a special property of the construction of solutions. In a hidden form it is contained in the definition of Whitham equations.

All the results that are proved for genus-zero Whitham hierarchy in the first five sections are generalized for the arbitrary genus case in Section 7. We present them without proofs mainly because they can be obtained more or less in the same way as in genus zero case but require greater length due to pure technical complexity.

2. Whitham Hierarchy: Zero Genus Case

In zero genus case a point of the "phase space" $\hat{M}_{g=0,N}$ is a set of points p_α, $\alpha = 1, \ldots, N$, and a set of formal local coordinates $k_\alpha^{-1}(p)$

$$k_\alpha(p) = \sum_{s=-1}^{\infty} v_{\alpha,s}(p - p_\alpha)^s \tag{2.1}$$

("formal local coordinate" means that r.h.s of (2.1) is considered as a formal series without any assumption of its convergency). Hence, $\hat{M}_{0,N}$ is a set of sequences

$$\hat{M}_{0,N} = \{p_\alpha,\ v_{\alpha,s},\ \alpha = 1, \ldots, N,\ s = -1, 0, 1, 2, \ldots\}\ . \tag{2.2}$$

The Whitham equations define a dependence of points of $\hat{M}_{0,N}$ with respect to the variables t_A where the set of indices $\mathcal{A}$ is as follows

$$\mathcal{A} = \{A = (\alpha, i),\ \alpha = 1, \ldots, N,\ i = 1, 2, \ldots \text{ and for } i = 0, \alpha \neq 1\}\ . \tag{2.3}$$

As it was explained in the Introduction, we can fix one of the points p_α with the help of an appropriate change of the variable p. Let us choose: $p_1 = \infty$.

Let us introduce meromorphic functions $\Omega_{\alpha,i}(p)$ for $i > 0$ with the help of the following conditions:

$\Omega_{\alpha,i}(p)$ has a pole only at p_α and coincides with the singular part of an expansion of $k_\alpha^i(p)$ near this point, i.e.,

$$\begin{aligned} \Omega_{\alpha,i}(p) &= \sum_{s=1}^{i} w_{\alpha,i,s}(p - p_\alpha)^{-s} = k_\alpha^i(p) + O(1)\ , \\ \Omega_{\alpha,i}(\infty) &= 0\ , \qquad \alpha \neq 1\ . \end{aligned} \tag{2.4}$$

$$\Omega_{1,i}(p) = \sum_{s=1}^{i} w_{1,i,s}p^s = k_1^i(p) + O(k_1^{-1})\ . \tag{2.5}$$

These polynomials can be written in the form of the Cauchy integrals

$$\Omega_{\alpha,i}(p,T) = \frac{1}{2\pi i} \oint_{C_\alpha} \frac{k_\alpha^i(z_\alpha, T)\, dz_\alpha}{p - z_\alpha} . \tag{2.6}$$

Here C_α is a small cycle around the point p_α.

The functions $\Omega_{\alpha,i=0}(p)$, $\alpha \neq 1$ are equal to

$$\Omega_{\alpha,0}(p) = -\ln(p - p_\alpha) . \tag{2.7}$$

Remark. The asymmetry of the definitions of $\Omega_{\alpha,i}$ reflects our intention to choose the index $A_0 = (1,1)$ as a "marked" index.

The coefficients of $\Omega_{\alpha,i>0}(p)$ are polynomial functions of $v_{\alpha,s}$. Therefore, the Whitham equations (1.10) (or (1.16)) can be rewritten as equations on $\hat{M}_{0,N}$. But still it has to be shown that they can be considered as a correctly defined system.

THEOREM 2.1. *The zero-curvature form* (1.16) *of the Whitham hierarchy in the zero-genus case is equivalent to the Sato-form that is a compatible system of evolution equations*

$$\partial_A k_\alpha = \{k_\alpha, \Omega_A\} . \tag{2.8}$$

Proof: Consider the equations (1.16) for $B = (\alpha, j > 0)$. From the definition of Ω_A it follows that

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = \partial_B \Omega_A - \{\Omega_A, \Omega_B^-\} + O(1) . \tag{2.9}$$

Here and below we use the notation

$$\begin{aligned} \Omega_A^- &= \Omega_A - k_\alpha^i , \qquad \text{for } A = (\alpha, i > 0) , \\ \Omega_{\alpha,0}^- &= \Omega_{\alpha,0} - \ln k_\alpha . \end{aligned} \tag{2.10}$$

Hence,

$$\partial_A k_\alpha^j - \{k_\alpha^j, \Omega_A\} = 0(k^{i-1}) . \tag{2.11}$$

Therefore,

$$\partial_A k_\alpha - \{k_\alpha, \Omega_A\} = 0(k^{i-j}) . \tag{2.12}$$

The limit of (2.12) for $j \to \infty$ proves (2.8). The inverse statement that (2.8) is a correctly defined system can be proved in a standard way. So we shall skip it.

Let us demonstrate a few examples.

Example 1. *The Khokhlov-Zabolotskaya Hierarchy*

The Khokhlov-Zabolotskaya hierarchy is the particular $N = 1$ case of our considerations. Any local coordinate $K^{-1}(p)$ near the infinity ($p_1 = \infty$)

$$K(p) = p + \sum_{s=1}^{\infty} v_s p^{-s} \tag{2.13}$$

defines a set of polynomials:

$$\Omega_i(p) = \left[K^i(p)\right]_+ , \tag{2.14}$$

here $[...]_+$ denotes a non-negative part of Laurent series. For example,

$$\Omega_2 = k^2 + u , \quad \Omega_3 = k^3 + \frac{3}{2}uk + w , \quad \text{where } u = 2v_1 , \quad w = 3v_2 . \tag{2.15}$$

If we denote $t_2 = y$, $t_3 = t$, then the equation (1.16) for $A = 2$, $B = 3$ gives

$$w_x = \frac{3}{4}u_y , \qquad w_y = u_t - \frac{3}{2}uu_x , \tag{2.16}$$

from which the dispersionless KP (dKP) equation (which is also called the Khokhlov-Zabolotskaya equation) is derived:

$$\frac{3}{4}u_{yy} + \left(u_t - \frac{3}{2}uu_x\right)_x = 0 . \tag{2.17}$$

The Khokhlov-Zabolotskaya equation is a partial differential equation and though it has no a pure evolution form, one can expect that its solutions are to be uniquely defined by their Cauchy data $u(x, y, t = 0)$, that is a function of the two variables x, y. Up to now it is not clear if this two-dimensional equation can be considered as the third equivalent form of the Whitham hierarchy (we remind that solutions of the hierarchy (2.8) formally depend on an infinite number of functions of one variable).

Example 2. *The Longwave Limit of the* 2*-d Toda Lattice*

The hierarchy of the longwave limit of the two-dimensional Toda equation is the particular $N = 2$ case of our considerations. There are two local parameters. One of them is near the infinity $p_1 = \infty$ and one is near a point $p_2 = a$. They depend on two sets of the variables $t_{\alpha,s}$, $\alpha = 1, 2$, $s = 1, 2, \ldots$ and also on the variable t_0. We shall present here only the basic two-dimensional equation of this hierarchy (an analogue of the Khokhlov-Zabolotskaya equation).

Consider three variables $t = t_0, x = t_{1,1}, y = t_{2,1}$. The corresponding functions are

$$\Omega_0 = \ln(p - a) , \quad \Omega_{1,1} = p , \quad \Omega_{2,1} = \frac{v}{p - a} . \tag{2.18}$$

Their substitution into the zero-curvature equation (1.16) gives

$$v_x = a_t v\ , \quad v_t + a_y = 0\ , \quad w_t = 0\ . \tag{2.19}$$

From (2.19) it follows

$$\partial^2_{xy}\phi + \partial^2_t e^{\phi} = 0\ , \qquad \text{where } \phi = \ln v\ . \tag{2.20}$$

This is the longwave limit of the 2-d Toda lattice equation

$$\partial^2_{xy}\varphi_n = e^{\varphi_{n-1}-\varphi_n} - e^{\varphi_n-\varphi_{n+1}} \tag{2.21}$$

corresponding to the solutions that are slow functions of the discrete variable n, which is replaced by the continuous variable t. The equation (2.20) has arisen independently in the general relativity, the theory of wave phenomena in shallow water, long radio-relay lines, and so on. A bibliography can be found in [40] where a representation of solutions of (2.20) in terms of convergent series was proposed.

Example 3. *N-layer Solutions of the Benny Equation*

This example corresponds to a general $N+1$ points case, but we consider only one zero-curvature equation. Let us choose three functions

$$\Omega_1 = p\ , \quad \Omega_2 = p + \sum_1^N \frac{v_i}{p-p_i}\ , \quad \Omega_3 = p^2 + u\ , \tag{2.22}$$

which are coupled with the variables x, y, t, respectively. (In our standart notations they are

$$x = t_{1,1}\ , \quad y = \sum_{\alpha=1}^{N+1} t_{\alpha,1}\ , \quad t = t_{1,2}\ .) \tag{2.23}$$

The zero-curvature equation (1.16) gives the system

$$p_{it} - (p_i^2)_x + u_x = 0\ , \qquad v_{it} = 2(v_i p_i)_x, u_y - u_x + 2\sum_i v_{ix} = 0\ . \tag{2.24}$$

Solutions of this system that do not depend on y are N-layer solutions of the Benny equation. As was noticed in [48], the corresponding system

$$p_{it} - (p_i^2)_x + u_x = 0\ , \quad v_{it} = 2(v_i p_i)_x\ , \quad u = 2\sum_i v_i \tag{2.25}$$

is a classical limit of the vector non-linear Schrödinger equation

$$i\psi_{it} = \psi_{i,xx} + u\psi_i\ , \qquad u = \sum_i |\psi_i|^2\ . \tag{2.26}$$

(Using this observation in [48] the integrals of (2.25) were found.)

In the second part of this section we consider "algebraic" orbits of the genus-zero Whitham equations. By definition they are specified with the help of the constraint: *there exists a meromorphic solution $E(p,T)$ of the equations*

$$\partial_A E = \{E, \Omega_A\} , \tag{2.27}$$

such that

$$\{E(p,T), k_\alpha(p,T)\} = 0 . \tag{2.28}$$

The last equality implies that there exist functions $f_\alpha(E)$ of one variable such that

$$k_\alpha(p,T) = f_\alpha(E(p,T)) . \tag{2.29}$$

In order not to be lost in a too general setting right at the beginning, let us start with an example.

Example. Lax Reductions ($N = 1$)

Consider solutions of the dKP hierarchy such that some power of local parameter (2.13) is a polynomial in p, i.e.,

$$E(p,T) = p^n + u_{n-2}p^{n-2} + \ldots + u_0 = k_1^n(p,T) . \tag{2.30}$$

The relation (2.30) implies that only a few first coefficients of the local parameter are independent. All of them are polynomials with respect to the coefficients u_i of the polynomial $E(p,T)$. The corresponding solutions of dKP hierarchy can be described in terms of dispersionless Lax equations

$$\partial_i E(p,T) = \{E(p,T), \Omega_i(p,T)\} , \tag{2.31}$$

where

$$\Omega_i(p,T) = [E^{i/n}(p,T)]_+ \tag{2.32}$$

(as before, $[\ldots]_+$ denotes a non-negative part of corresponding Laurent series). These solutions of KP hierarchy can also be characterized by the property that they do not depend on the variables $t_n, t_{2n}, t_{3n}, \ldots$. We are going to construct an analogue of the "hodograph" transform for the solution of these equations. It is a generalization and effectivization of a scheme (see [43]), where "hodograph-type" transform was proposed for hydrodynamic-type diagonalizable Hamiltonian systems; see [8], [9], [10], and [39].

Let us introduce a generating function

$$S(p) = \sum_{i=1}^{\infty} t_i \Omega_i(p) = \sum_{i=1}^{\infty} t_i K^i + O(K^{-1}) , \tag{2.33}$$

where Ω_i are given by (2.14) and $K = E^{1/n}$. (If there is only a finite number of t_i that are not equal to zero, then $S(p)$ is a polynomial.) The coefficients of S are linear functions of t_i and polynomials in u_i. We introduce a dependence of u_j on the variables t_i with the help of the following algebraic equations:

$$\frac{dS}{dp}(q_s) = 0\,, \tag{2.34}$$

where q_s are zeros of the polynomial

$$\frac{dE}{dp}(q_s) = 0\,. \tag{2.35}$$

Remark. It is not actually necessary to solve equation (2.35) in order to find q_s. We can choose q_s, $s = 1, \dots, n-1$ and u_0 as a new set of unknown functions, due to the equality

$$\begin{aligned} \frac{dE}{dp} &= np^{n-1} + (n-2)u_{n-2}p^{n-3} + \dots + u_1 = \prod_{s=1}^{n}(p - q_s)\,, \\ q_n &= -\sum_{s=1}^{n-1} q_s\,. \end{aligned} \tag{2.36}$$

Let us prove that if the dependence of $E = E(p, T)$ with respect to the variables t_i is defined by (2.34), then

$$\partial_i S(E, T) = \Omega_i(E, T)\,. \tag{2.37}$$

Consider the function $\partial_i S(E, T)$. From (2.33) it follows that

$$\partial_i S(E) = K^i + O(K^{-1}) = \Omega_i(E) + O(K^{-1})\,. \tag{2.38}$$

Hence, it is enough to prove that $\partial_i S(E)$ is a polynomial in p, because by definition Ω_i is the only polynomial in p such that

$$\Omega_i(p) = K^i + O(K^{-1})\,. \tag{2.39}$$

The function $\partial_i S(E, T)$ is holomorphic everywhere, except at $q_s(T)$, probably. In a neighborhood of $q_s(T)$ a local coordinate is

$$(E - E_s(T))^{1/2}\,, \qquad E_s(T) = E(q_s(T)) \tag{2.40}$$

(if q_s is a simple root of (2.35)). Hence, a priori S has the expansion

$$S(E, T) = \alpha_s(T) + \beta_s(T)(E - E_s(T))^{1/2} + \dots \tag{2.41}$$

and the derivative $\partial_i S(E, T)$ might be singular at the points q_s. The defining relations (2.34) imply that $\beta_s = 0$. Therefore, $\partial_i S(E)$ is regular everywhere except at the infinity and, hence, is a polynomial. The equations (2.37) are proved.

Let us present this scheme in another form. For each polynomial $E(p)$ of the form (2.30) and each formal series

$$Q(p) = \sum_{j=1}^{\infty} b_j p^j \tag{2.42}$$

the formula

$$t_i = \frac{1}{i}\operatorname{res}_\infty(K^{-i}(p)Q(p)dE(p)) \tag{2.43}$$

defines the variables

$$t_k = t_k(u_i, b_j)\,, \quad i = 0,\ldots,n-2\,, \quad j = 0,\ldots \tag{2.44}$$

as functions of the coefficients of E, Q. Consider the inverse functions

$$u_i = u_i(t_1,\ldots)\,, \qquad b_j = b_j(t_1,\ldots)\,. \tag{2.45}$$

Remark. In order to be more precise let us consider a case when Q is a polynomial, i.e., $b_j = 0$, $j > m$. From (2.43) it follows that $t_k = 0$, $k > n+m-1$. Therefore, we have $n+m-1$ "times" t_k, $k = 1,\ldots,n+m-1$ that are linear functions of b_j, $j = 1,\ldots,m$ and polynomials in u_i, $i = 0,\ldots,n-2$. So, locally the inverse functions (2.45) are well-defined.

THEOREM 2.2. *The functions $u_i(T)$ are solutions of the dispersionless Lax equation* (2.31). *Any other solutions of* (2.31) *are obtained from this particular one with the help of translations, i.e., $\tilde{u}(t_i) = u(t_i - t_i^0)$.*

Consider now the general N case. Let $E(p)$ be a meromorphic function with a pole of order n at the infinity and with poles of orders n_α at points p_α, $\alpha = 2,\ldots,M$. (We would like to mention that the case of negative $n_\alpha \neq 0$, which means that E has a zero of the order $-n_\alpha$, can be considered as well, but we are not going to do it here in order to avoid some technical complications.)

$$E = p^n + u_{n-2}p^{n-2} + \ldots + u_0 + \sum_{\alpha=2}^{M}\sum_{s=1}^{n_\alpha} v_{\alpha,s}(p-p_\alpha)^{-s}\,. \tag{2.46}$$

Consider a linear space of such functions, i.e., the space of sets

$$\begin{aligned} \mathcal{N}(n_\alpha) &= \{u_i, i = 0,\ldots,n-2;\ v_{\alpha,s}, s = 1,\ldots,n_\alpha)\}\,, \\ \alpha &= 1,\ldots,M\,, \qquad n_1 = n\,. \end{aligned} \tag{2.47}$$

If $N \leqq M$, the function $E(p)$ defines the local coordinates at the points p_α with the help of the formula

$$k_\alpha^{n_\alpha}(p) = E(p)\,, \qquad \alpha = 1,\ldots,N\,. \tag{2.48}$$

Therefore, $\mathcal{N}(n_\alpha)$ can be identified with a subspace of $\hat{M}_{0,N}$

$$\mathcal{N} \subset \hat{M}_{0,N} \; . \tag{2.49}$$

THEOREM 2.3. *The subspaces $\mathcal{N}_M(n_\alpha)$ are invariant with respect to the Whitham equations on $\hat{M}_{0,N}$ that coincide with the flows*

$$\partial_A E(p,T) = \{E(p,T), \Omega_A(p,T)\} \; , \tag{2.50}$$

where k_α and Ω_A are defined with the help of formulae (2.48), (2.6), *respectively. General solutions of* (2.50) *are given in an implicit form with the help of the following algebraic equations*

$$\frac{dS}{dp}(q_s,T) = 0 \; , \quad \text{where} \quad S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) \; . \tag{2.51}$$

which have to be fulfilled for all zeros q_s of the function

$$\frac{dE}{dp}(q_s) = 0 \; . \tag{2.52}$$

The proof is the same as the proof of the previous theorem. Its main step is the proof that from the defining relations (2.51) it follows that

$$\partial_A S(E,T) = \Omega_A(E,T). \tag{2.53}$$

DEFINITION. The particular solutions of the Whitham hierarchy that correspond to the algebraic orbits (2.50) and for which $t_A^0 = 0$ will be called *homogeneous solutions.*

An alternative formulation of this theorem can be done in the following form. Let $Q(p)$ be a meromorphic function with its poles at the points p_α, i.e.,

$$Q(p) = \sum_{j=1}^{\infty} b_{1,j} p^j + \sum_{\alpha=2}^{M} \sum_{j=1}^{\infty} b_{\alpha,j}(p - p_\alpha)^{-j} \; . \tag{2.54}$$

The formulae

$$\begin{aligned} t_{\alpha,i} &= \frac{1}{i}\operatorname{res}_\alpha(k_\alpha^{-i}(p)Q(p)dE(p))) \; , \qquad i > 0 \; ; \\ t_{\alpha,0} &= \operatorname{res}_\alpha(Q(p)dE(p)) \end{aligned} \tag{2.55}$$

define "times" $t_{\alpha,i}$ as functions of the coefficients of $Q(p)$ and $E(p)$ (which has the form (2.46)). Consider the inverse functions

$$v_{\alpha,s} = v_{\alpha,s}(t_{\beta,i}) \; , \qquad b_{\alpha,j} = b_{\alpha,j}(t_{\beta,i}) \; . \tag{2.56}$$

(recall that index $\alpha = 1$ corresponds to the infinity $p_1 = \infty$).

COROLLARY 2.1. *The inverse functions $v_{\alpha,s}(t_{\beta,i})$ are solutions of Whitham equations. In particular, (for all N)*

$$u(x,y,t) = \frac{2}{n} u_{n-2}(t_{1,1} = x, t_{2,1} = y, t_{3,1} = t, \dots) \tag{2.57}$$

is a solution of the Khokhlov-Zabolotskaya equation (2.17) *and*

$$\phi(x,y,t) = \frac{1}{n_2} \ln v_{2,n_2}(x = t_{1,1}, y = t_{2,1}, t = t_{2,0}, \dots) \tag{2.58}$$

is a solution of the longwave limit of the 2-d Toda lattice equation (2.20).

(We would like to emphasize that in the formulae (2.57), (2.58) all the "times" except the first ones are parameters.)

The inverse functions (2.56) define the dependence of the functions $E(p,T)$ and $Q(p,T)$ on the variables t_A. The differential of the potential $S(p,T)$ equals

$$dS(p,T) = Q(p,T) dE(p,T) \, . \tag{2.59}$$

From (2.53) it follows that

$$\partial_A Q(E,T) = \frac{d\Omega_A(E,T)}{dE} \, . \tag{2.60}$$

In particular,

$$\partial_x Q(E,T) = \frac{dp(E,T)}{dE} \, . \tag{2.61}$$

The derivatives with fixed E and p are interelated with the help of the following formula:

$$\partial_A f(p,T) = \partial_A f(E,T) + \frac{df}{dE} \partial_A E(p,T) \, . \tag{2.62}$$

Using (2.61) and (2.62) we obtain

COROLLARY 2.2. *The functions $Q(p,T)$ and $E(p,T)$ satisfy the quasi-classical "string equation"*

$$\{Q,E\} = 1 \, . \tag{2.63}$$

(This corollary was prompted by [41] and [42], and we shall return to it at greater length in Section 6.)

3. The τ-Function

In the previous section the "algebraic" solutions of the Whitham hierarchy in zero-genus case were constructed. It was shown that for their "potentials"

$$S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) \tag{3.1}$$

(where t_A^0 are corresponding constants) the following equalities

$$\Omega_B(k_\alpha, T) = \partial_B S(k_\alpha, T)\,, \quad B = (\beta, i) \tag{3.2}$$

are fulfilled. In this section we define with the help of $S(p,T)$ the τ-function corresponding to the algebraic solutions of the Whitham equations.

The τ-function of the universal Whitham hierarchy (in zero genus case) would be by definition

$$\ln \tau(T) = F(T)\,,$$

$$F = \frac{1}{2}\sum_{\alpha=1}^{N} \mathrm{res}_\alpha \left(\sum_{i=1}^{\infty} \tilde{t}_{\alpha,i} k_\alpha^i dS(p,T)\right) + \tilde{t}_{\alpha,0} s_\alpha(T)\,, \tag{3.3}$$

$$\tilde{t}_{\alpha,i} = t_{\alpha,i} - t_{\alpha,i}^0\,,$$

where res_α denotes a residue at the point p_α and s_α is the coefficient of the expansion

$$S(p,T) = \sum_{\alpha=1}^{N} \left(\sum_{i=1}^{\infty} \tilde{t}_{\alpha,i} k_\alpha^i\right) + t_{\alpha,0} \ln k_\alpha + s_\alpha + O(k^{-1})\,. \tag{3.4}$$

Here and below we use the notation

$$t_{1,0} = -\sum_{\alpha=2}^{N} t_{\alpha,0}\,. \tag{3.5}$$

The τ-function can be rewritten in a more compact form. Let us make cuts connecting the point $p_1 = \infty$ with the points p_α. After that we can choose a branch of the function $S(p,T)$. The coefficient s_α equals

$$s_\alpha = \frac{1}{2\pi i}\oint_{\sigma_\alpha} \ln(p - p_\alpha)\, dS\,, \tag{3.6}$$

where σ_α is a contour around the corresponding cut. The function S has jumps on the cuts. Its $\bar\partial$-derivative is a sum of delta-functions and their derivatives at the points p_α and one-dimensional delta-functions on cuts. Therefore, (3.3) can be represented in the form:

$$F = \int \bar{d}S \wedge dS\,. \tag{3.7}$$

(The integral in (3.7) is taken over the whole complex plane of the variable p. It is non-zero because $S(p,T)$ is holomorphic outside punctures and cuts, only.)

THEOREM 3.4. *For the above defined τ-function the following equalities are fulfilled:*

$$\partial_{\alpha,i}F(T) = \mathrm{res}_\alpha(k_\alpha^i\, dS(p,T)), \qquad i>0, \tag{3.8}$$

$$\partial_{\alpha,0}F(T) = s_\alpha . \tag{3.9}$$

Proof: Let us consider the derivative ∂_A for $A=(\alpha, i>0)$. It equals

$$2\partial_A F = \mathrm{res}_\alpha(k_\alpha^i dS) + \sum_{\beta=1}^{N}\sum_{j=1}^{\infty}\mathrm{res}_\beta(\tilde t_{\beta,i} k_\beta^j d\Omega_A) + \tilde t_{\beta,0}\Omega_A(p_\beta) . \tag{3.10}$$

We use in (3.10) the equality

$$\partial_A s_\beta = \Omega_A(p_\beta) . \tag{3.11}$$

From

$$\sum_{\alpha=1}^{N}\mathrm{res}_\alpha(\Omega_A d\Omega_B) = 0 \tag{3.12}$$

it follows that

$$\mathrm{res}_\beta(k_\beta^j d\Omega_{\alpha,i}) = \mathrm{res}_\alpha(k_\alpha^i d\Omega_{\beta,j}), \qquad j>0 . \tag{3.13}$$

Besides this,

$$\Omega_A(p_\beta) = \mathrm{res}_\beta(\Omega_A d\ln(p-p_\beta)) = \mathrm{res}_\alpha(k_\alpha^i d\Omega_{\beta,0}) . \tag{3.14}$$

The substitution of (3.13), (3.14) into (3.10) proves (3.8). The proof of (3.9) is absolutely analogous.

The formulae (3.8), (3.9) show that the expansion of $S(p,T)$ at the point p_α has the form

$$S(p,T) = \sum_{\alpha=1}^{N}\sum_{i=1}^{\infty}\tilde t_{\alpha,i}k_\alpha^i + \tilde t_{\alpha,0}\ln k_\alpha + \partial_{\alpha,0}F + \sum_{j=1}^{\infty}\frac{1}{j}\partial_{\alpha,j}F k_\alpha^{-j} . \tag{3.15}$$

From (3.8), (3.9) follows:

COROLLARY 3.3. *The second derivatives of F are equal to*

$$\partial^2_{A,B}F(T) = res_\alpha(k_\alpha^i\, d\Omega_B), \qquad A=(\alpha, i>0), \tag{3.16}$$

$$\partial_{\alpha,0}\partial_{\beta,0}F(T) = \ln(p_\alpha - p_\beta)\ . \tag{3.17}$$

Hence, the expansion of the non-positive part (2.10) of $\Omega_A(p)$ at the point p_β has the form

$$\Omega_A^- = \partial_{\beta,0}F + \sum_{j=0}^{\infty}\frac{1}{j}(\partial_A\partial_{\beta,j}F)k_\beta^j\ . \tag{3.18}$$

Therefore, the τ-function that depends on the "times" only, contains complete information on the functions Ω_A.

4. Truncated Virasoro and W-Constraints

In Section 3 it was shown that any solution of the Whitham equations ($g = 0$) corresponding to an algebraic orbit can be obtained from the "homogeneous" solution with the help of translations $\tilde{t}_A = t_A - t_A^0$. In this section we consider τ-functions of homogeneous solutions, only.

The truncated Virasoro constraints for the τ-function of the dispersionless Lax equations (2.31) were proved in [30]. The proof was based on an invariance of residues with respect to a change of variables. The same approach can be applied for the general N-case, also. In this paper we use another way that was inspired by the $N \to \infty$ limit of loop-equations for the one-matrix model (a review of recent developments of the loop-equations technique can be found in [35]).

The function $E(p)$ of the form (2.46) represents the complex plane of the variable p as D-sheet branching covering of the complex plane of the variable E, $D = \sum_\alpha n_\alpha$. The zeros q_s of the differential dE, $dE(q_s) = 0$, are branching points of the covering. Hence, any function $f(p)$ can be considered as a multi-valued function of the variable E. Let $p_i(E)$, $i = 1, \dots, D$ be roots of the equation

$$E(p_i) = E\ . \tag{4.1}$$

The symmetric combination of the values $f(p_i)$

$$\tilde{f}(E) = \sum_{i=1}^{D} f(p_i) \tag{4.2}$$

is a single-valued function of E. Let us apply this argument to the function $Q^K(p)$, where

$$Q(p) = \frac{dS(p)}{dE(p)}\ ; \tag{4.3}$$

$S(p)$ is the potential of the homogeneous solution of Whitham equations. The defining algebraic relations (2.51) imply that $Q(p)$ is holomorphic outside the punctures p_α (that are "preimages" of the infinity $E(p_\alpha) = \infty$). Therefore, the function

$$\tilde{Q}^K(E) = \sum_{i=1}^{N} Q^K(p_i) \tag{4.4}$$

is an entire function of the variable E. In other words, the Laurent expansion of the function $\tilde{Q}^K(E)$ contains only positive powers of E, i.e.,

$$\operatorname{res}_\infty \left(\sum_{i=1}^{N} Q^K(p_i) E^{m+1}\, dE \right) = 0\,, \qquad m = -1, 0, 1, \dots\,. \tag{4.5}$$

The residue (4.5) at the infinity of the complex E-plane is equal to a sum of residues at the points p_α, i.e.,

$$\sum_{i=1}^{N} \operatorname{res}_\alpha \left((Q^K(k_\alpha) E^{m+1}\, dE \right) = 0\,, \qquad E = k_\alpha^{n_\alpha}\,. \tag{4.6}$$

From (3.15) it follows that the function $Q(p)$ has the expansion

$$Q(k_\alpha) = \frac{1}{n_\alpha} \sum_{i=1}^{\infty} i t_{\alpha,i} k_\alpha^{i-n_\alpha} + t_{\alpha,0} k_\alpha^{-n_\alpha} + \sum_{j=1}^{\infty} \partial_{\alpha,j} F k_\alpha^{-j-n_\alpha} \tag{4.7}$$

at the point p_α. The substitution of (4.7) into (4.6) for $K = 1$ gives obvious identities:

$$\begin{aligned} \sum_{\alpha=1}^{N} t_{\alpha,0} &= 0\,, \qquad m = -1\,, \\ \sum_{\alpha=1}^{N} \partial_{\alpha, m n_\alpha} F &= 0\,, \qquad m = 0, 1, \dots\,. \end{aligned} \tag{4.8}$$

(For Lax reductions, $N = 1$, the equalities (4.8) imply that F does not depend on $t_n, t_{2n}, \dots$.) For $K > 1$ the relations (4.6) lead to highly non-trivial equations. For example, the case $K = 2$ corresponds to the truncated Virasoro-constraints.

THEOREM 4.5. *The τ-function of the homogeneous solution of the Whitham equations (corresponding to the orbit $\mathcal{N}(n_\alpha)$) is a solution of the equations*

$$\sum_{\alpha=1}^{N} \frac{1}{n_\alpha} \left(\sum_{i=n_\alpha+1}^{\infty} i t_{\alpha,i} \partial_{\alpha,i-n_\alpha} F + n_\alpha t_{\alpha,0} t_{\alpha,n_\alpha} + \frac{1}{2} \sum_{j=1}^{n_\alpha-1} j(n_\alpha - j) t_{\alpha,j} t_{\alpha,n_\alpha-j} \right) = 0\,; \tag{4.9}$$

(4.10) $$\sum_{\alpha=1}^{N} \frac{1}{n_\alpha} \sum_{i=1}^{\infty} i t_{\alpha,i} \partial_{\alpha,i} F + \frac{1}{2} t_{\alpha,0}^2 = 0 ;$$

(4.11) $$\sum_{\alpha=1}^{N} \frac{1}{n_\alpha} \left(\sum_{i=1}^{\infty} i t_{\alpha,i} \partial_{\alpha,i+mn_\alpha} F + \frac{1}{2} \sum_{j=1}^{mn_\alpha - 1} \partial_{\alpha,j} F \partial_{\alpha,mn_\alpha - j} F \right) = 0 , \qquad m = 1, \dots .$$

The equalities (4.5) for any K can be written in the form

(4.12) $$\sum_{\alpha=1}^{N} n_\alpha^{1-K} \sum_{I,J} [i_1] t_{\alpha,i_1} \cdots [i_s] t_{\alpha,i_s} \partial_{\alpha,j_{s+1}} F \cdots \partial_{\alpha,j_K} F = 0 ,$$

where the second sum is taken over all sets of indices $I = \{i_k\}$, $J = \{j_k\}$ such that

(4.13) $$\sum_{k=1}^{s} i_k = \sum_{k=s+1}^{K} j_k - (m + K - 2) n_\alpha , \qquad m > -1 .$$

and $[i]$ denotes

(4.14) $$[i] = i , \quad \text{if} \quad i \neq 0 \quad [0] = 1 .$$

For $N = 1$ the equations (4.12) coincide with a nonlinear part of the W_K constraints.

At the end of this section we present the truncated Virasoro constraints for $N = 1$ and $n = 2$ in the form of the planar limit of loop-equations for a one-matrix hermitian model.

Consider the negative part of $Q(k)$ $(N = 1, n = 2)$

(4.15) $$-\mathscr{W}_0 = Q^-(k) = \frac{1}{2} \sum_{i=1}^{\infty} t_1 k^{-1} + \sum_{j=1}^{\infty} \partial_{2j-1} F k^{-2j-1}$$

and introduce

(4.16) $$V(k) = \sum_{i=1}^{\infty} \tilde{t}_{2i} k^{2i} ,$$

where

(4.17) $$\tilde{t}_{2i} = \frac{2i+1}{2i} t_{2i+1} .$$

Then

$$Q(k) = V'(k) - \mathcal{W}_0 \tag{4.18}$$

and (4.6) is equivalent to the equation

$$\oint_C \frac{V'(\xi)\mathcal{W}_0(\xi)}{k-\xi}\, d\xi = \frac{1}{2}\mathcal{W}_0^2 , \tag{4.19}$$

where C is a small contour around the infinity. Equation (4.19) is a planar limit of the loop-equation for a one-matrix hermitian model (see [35])

$$\oint_C \frac{V'(\xi)\mathcal{W}(\xi)}{k-\xi}\, d\xi = \frac{1}{2}\mathcal{W}^2 + \sum_{i=1}^{\infty} k^{-2i-1}\frac{\partial \mathcal{W}}{\partial t_{2i}} . \tag{4.20}$$

In (4.20) $\mathcal{W}(k)$ is the Wilson loop-correlator that by definition is equal to

$$\mathcal{W} = \left\langle \operatorname{tr}\frac{1}{k-X} \right\rangle = \int \operatorname{tr}\frac{1}{k-X} e^{-\operatorname{tr}V(X)}\, dX , \tag{4.21}$$

where X is a hermitian $M \times M$ matrix.

Remark. As was shown in [5], the double-scaling limit of the $n-1$ matrix chain model is related to the n-th reduction of the KP-equation. The dispersionless Lax equations (2.30), (2.31) are their classical limit. Therefore, the negative part $\mathcal{W}_0$ of the series (4.7) for $N = 1$ and arbitrary n has to be related to the planar limit of some Wilson-type correlators for multi-matrix models. Therefore, higher "loop-equations" (corresponding to $K > 2$) are to be fulfilled for them. It would be interesting to find a direct way to produce the corresponding equations in the framework of the multi-matrix models.

5. Primary Rings of the Topological Field Theories

Topological minimal models were introduced in [14] and were considered in [34]. They are a twisted version of the discrete series of $N = 2$ superconformal Landau-Ginzburg (LG) models. A large class of the $N = 2$ superconformal LG models has been studied in [33], [36], and [44]. It was shown that a finite number of states are topological, which means that their operator products have no singularities. These states form a closed ring $\mathcal{R}$, which is called a primary chiral ring. It can be expressed in terms of the superpotential $E(p_i)$ of the corresponding model

$$\mathcal{R} = \frac{C[p_i]}{dE = 0} , \qquad dE = \frac{\partial E}{\partial p_i}\, dp_i . \tag{5.1}$$

In topological models these primary states are the only local physical excitations.

In [45], it was shown that correlation functions of primary chiral fields and integrals of their second descendants can be expressed in terms of perturbed superpotentials $E(p_i, t_1, t_2, ...)$. For the A_{n-1} model the unperturbed superpotential has the form:

$$E_0 = p^n . \tag{5.2}$$

The coefficients of a perturbed potential

$$E(p) = p^n + u_{n-2}p^{n-2} + ... + u_0 \tag{5.3}$$

can be considered as coordinates on the space of deformed topological minimal models. In [45] the dependence of u_i on the coordinates $t_1, \dots, t_{n-1}$ that are "coupled" with primary fields ϕ_i was found. It was shown that the deformation of the ring $\mathcal{R}$

$$\mathcal{R}(t_1, \dots, t_{n-1}) = C[p]/(dE(p, t_1, \dots, t_{n-1}) = 0) \tag{5.4}$$

is a potential deformation of the Fröbenius algebra (in the sense that was explained in the Introduction).

In this section we consider the application of the general Whitham equations on $\hat{M}_{0,N}$ to the theory of potential deformations of Fröbenius algebras. They are based on the following formula for the third logarithmic derivatives of the τ-function. Let $E(p, T)$ be the homogeneous solution of the Whitham equations (2.50) corresponding to an algebraic orbit $\mathcal{N}(n_\alpha)$, i.e., $E(p)$ has the form (2.46)

$$E = p^n + u_{n-2}p^{n-2} + ... + u_0 + \sum_{\alpha=2}^{M} \sum_{s=1}^{n_\alpha} v_{\alpha,s}(p - p_\alpha)^{-s} .$$

The formulae (2.55), (2.56) define the dependence of $E(p)$ and of the "dual" function $Q(p)$ on the variables t_A.

THEOREM 5.6. *The third logarithmic derivatives of the τ-function of the homogeneous solution of the Whitham equation corresponding to an algebraic orbit $\mathcal{N}(n_\alpha)$ are equal to*

$$\partial^3_{ABC} F = \sum_{q_s} \operatorname{res}_{q_s} \left(\frac{d\Omega_A \, d\Omega_B \, d\Omega_C}{dQ \, dE} \right), \tag{5.5}$$

where q_s are zeros of the differential $dE(q_s) = 0$.

Proof: Let us suppose that $A = (\alpha, i > 0)$. (The case when A, B, C are equal to $(\alpha, 0), (\beta, 0), (\gamma, 0)$ can be considered in the same way.) From (3.8) it follows that

$$\partial_C \partial^2_{AB} F = \operatorname{res}_\alpha (k^i_\alpha \, d\partial_C \Omega_B) = -\operatorname{res}_\alpha (\partial_C \Omega_B \, d\Omega_A) . \tag{5.6}$$

Here the derivative $\partial_C \Omega_A(E,T)$ is taken for the fixed E. As was explained in Section 2, the function $E(p,T)$ is a "good" coordinate except at the points $q_s(T)$ where local coordinates have the form (2.40). Hence, at the point $q_s(T)$ the function Ω_A has the expansion

$$(5.7)\qquad \begin{aligned} \Omega_B(E,T) &= w_{B,0}(T) + w_{B,1}(T)(E - E_s(T))^{1/2} + \cdots , \\ E_s(T) &= E(q_s(T),T) . \end{aligned}$$

The sum of all residues of a meromorphic differential equals zero. Therefore,

$$(5.8)\qquad \partial^3_{ABC}F = \sum_{q_s} \mathrm{res}_{q_s}(\partial_C \Omega_B \, d\Omega_A) .$$

From (5.7) it follows that in a neighborhood of the point q_s

$$(5.9)\qquad \partial_C \Omega_B = -\partial_C E_s \frac{d\Omega_B}{dE} + O(1) .$$

Therefore,

$$(5.10)\qquad \mathrm{res}_{q_s}(\partial_C \Omega_B \, d\Omega_A) = -\mathrm{res}_{q_s}\left(\partial_C E_s \frac{d\Omega_A \, d\Omega_B}{dE}\right) .$$

From (2.50) it follows that

$$(5.11)\qquad \partial_C E_s = \partial_x E_s \frac{d\Omega_C}{dp}(q_s) .$$

The string equation (2.63) implies

$$(5.12)\qquad \partial_x E_s = -\frac{dp}{dQ}(q_s) .$$

Substitution of (5.11) and (5.12) into (5.10) proves the theorem.

For each algebraic orbit $\mathcal{N}(n_\alpha)$ let us define a "small phase space" (see motivation in [46]). It will be a space of times t_a with the indices a belonging to the subset $\mathcal{A}_{sm}$,

$$(5.13)\qquad \mathcal{A}_{sm} = \{(\alpha,i) | \alpha = 1, i = 1,\dots,n-1; \alpha = 2,\dots,N, i = 0,\dots,n_\alpha\} .$$

Let us fix all the other times t_A:

$$(5.14)\qquad \begin{aligned} t_{1,n} &= 0 , \quad t_{1,n+1} = \tfrac{n}{n+1} , \quad t_{1,i} = 0 , \quad i > n+1 ; \\ t_{\alpha,i} &= 0 , \qquad \alpha = 2,\dots,N , \quad i > n_\alpha . \end{aligned}$$

Comparision with (4.7) shows that in this case

$$(5.15)\qquad Q(p) = p$$

in the formula (2.55). In other words, t_a as functions of u_i, $v_{\alpha,s}$ are given by the formulae

$$(5.16)\qquad \begin{aligned} t_{\alpha,i} &= \frac{1}{i}\operatorname{res}_\alpha(k_\alpha^{-i}(p)pdE(p))), \qquad i>0\,; \\ t_{\alpha,0} &= \operatorname{res}_\alpha(pdE(p))\,. \end{aligned}$$

Inverse functions define the dependence of the coefficients of E on the variables t_a

COROLLARY 5.4. *Let*

$$(5.17)\qquad F(t_a) = F\left(t_a, t_{1,n+1} = \frac{n}{n+1}\,,\quad t_A = 0\,,\quad A \notin \mathscr{A}_{sm}\right)$$

be the restriction of $F = \ln\tau$ *on the affine space that is* $\mathscr{A}_{sm}$ *shifted by* $t_{1,n+1} = \frac{n}{n+1}$. *Then*

$$(5.18)\qquad \partial^3_{abc}F = \sum_{q_s}\operatorname{res}_{q_s}\left(\frac{d\Omega_a\, d\Omega_b\, d\Omega_c}{dp\,dE}\right).$$

Let us summarize the results. Each meromorphic function $E(p)$ of the form (2.46) defines a factor-ring

$$(5.19)\qquad \mathscr{R}_E = \hat{\mathscr{R}}/(dE = 0)$$

of the ring $\hat{\mathscr{R}}$ of all meromorphic functions that are regular at the zeros q_s of the differential dE. The formula

$$(5.20)\qquad \langle f, g\rangle = \sum_{q_s}\operatorname{res}_{q_s}\left(\frac{f(p)g(p)}{E_p}dp\right),\quad f(p)\,,\quad g(p)\subset\hat{\mathscr{R}}\,,$$

defines a non-degenerate scalar product on $\mathscr{R}_E$. The scalar product (5.20) supplies $\mathscr{R}_E$ by the structure of the Fröbenius algebra. In the basis

$$(5.21)\qquad \phi_a = \frac{d\Omega_a}{dp}$$

the scalar product has the form

$$(5.22)\qquad \langle\phi_a\phi_b\rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha}\delta_{\alpha,\beta}\delta_{i+j,n_\alpha}\,,$$

where $[i]$ is the same as in (4.14). Our last statement is that the formulae (5.17) define in an implicit form the potential deformations of these Fröbenius algebras.

The case $N = 1$ covers the results of [45]. As was mentioned in the Introduction an integrability of WDVV equations was proved in [12]. The results of this section can be considered as an explicit construction of their particular solutions.

The "coupling" process of the ring $\mathcal{R}_E$ with topological gravity corresponds to the process of "switching on" of *all* the times of the Whitham hierarchy. It follows from the recurrent formula for the third derivatives of τ-function. First of all, let us present the formula

$$\Omega_{\alpha,i>0}\, dE = \frac{n_\alpha}{i+n_\alpha}\, d\Omega_{\alpha,i+n_\alpha} + \sum_{b=(\beta,j)\in\mathcal{A}_{sm}} \frac{n_\beta}{[n_\beta - j][j]}(\partial^2_{A,b}F)\, d\Omega_{\beta,n_\beta-j}\ . \tag{5.23}$$

It can be proved in the following way. The right- and left-hand sides of (5.23) are holomorphic outside the punctures. Hence, it is enough to compare their expansions at the points p_α. The coefficients of the expansion of Ω_A are given by the second derivatives of F (3.18). Therefore, (5.23) is fulfilled.

Let us define for each $a = (\alpha, i > 0) \in \mathcal{A}_{sm}$ the fields

$$\sigma_p(\phi_a) = \frac{d\Omega_{\alpha,pn_\alpha+i}}{dQ}\ . \tag{5.24}$$

The substitution of (5.23) into (5.8) proves the recurrent formula for the correlation functions for the gravitational descendants (see [46])

$$\langle\sigma_p(\phi_a)\sigma_B\sigma_C\rangle = \langle\sigma_{p-1}(\phi_a)\phi_b\rangle\eta^{bc}\langle\phi_c\sigma_B\sigma_C\rangle\ , \tag{5.25}$$

where σ_B, σ_C are any other states. (The integrability of general descendant equations was proved in [12].)

Remark. This paper had already been written when the author got a preprint (see [11]) in which the Fröbenius algebras and their "small phase" deformations corresponding to the Whitham hierarchy for the multi-puncture case had been considered.

6. Generating Form of the Whitham Equations

In this short section (or rather long remark) we would like to clarify our construction of the algebraic solutions of the Whitham equations and the definition of the corresponding τ-function. It was stimulated by the papers [41] and [42] where, with the use our approach in [30], the τ-function for the longwave limit of 2-d Toda lattice was introduced.

Let $\Omega_A(k, T)$ be a solution of the general zero-curvature equation (1.16)

$$\partial_A\Omega_B - \partial_B\Omega_A + \{\Omega_A, \Omega_B\} = 0\ . \tag{6.1}$$

They are compatibility conditions for the equation

$$\partial_A E = \{E, \Omega_A\}\ . \tag{6.2}$$

Therefore, an arbitrary function $E(p,x)$ defines (at least locally) the corresponding solution $E(p,T)$ of (6.2), $E(p,x) = E(p, t_{A_0} = x, t_A = 0, A \neq A_0)$. In the domain where $\partial_p E(p,T) \neq 0$ we can use a variable E as a new coordinate, $p = p(E,t)$. From (2.62) it follows that in the new coordinate the equations (6.1) are equivalent to the equations

$$\partial_A \Omega_B(E,T) = \partial_B \Omega_A(E,T) . \tag{6.3}$$

Hence, there exists a potential $S(E,T)$ such that

$$\Omega_A(E,T) = \partial_A S(E,T) . \tag{6.4}$$

Due to this potential the one-form ω (1.8) can represented as

$$\omega = \delta S(E,T) - Q(E,T) dE , \tag{6.5}$$

where

$$Q(E,T) = \frac{\partial S(E,T)}{\partial E} . \tag{6.6}$$

Hence,

$$\delta\omega = \delta E \wedge \delta Q . \tag{6.7}$$

The formulae (2.59)–(2.62) that are valid in the general case prove that the functions E and Q as functions of two variables p, x satisfy the classical string equation

$$\{Q, E\} = 1 . \tag{6.8}$$

They show that

$$\partial_A Q = \{Q, \Omega_A\} . \tag{6.9}$$

A set of the pairs of functions $Q(p,x), E(p,x)$ satisfying the string equation is a group with respect to the composition, i.e., if $Q(p,x), E(p,x)$ and $Q_1(p,x), E_1(p,x)$ are solutions of (6.8), then the functions

$$\tilde{Q}(p,x) = Q_1(Q(p,x), E(p,x); \tilde{E}(p,x) = E_1(Q(p,x), E(p,x)) \tag{6.10}$$

are the solution of (6.8), as well. The Lie algebra of this group is the algebra $SDiff(T^2)$ of two-dimensional vector-fields preserving an area. The action of this algebra on the potential, τ-function (and so on) in the framework of the longwave limit of 2-d Toda lattice was considered in [41] and [42].

The previous formulae can be used in the inverse direction. Let $E(p,x)$ and $Q(p,x)$ be any solution of the equation (6.8). Using them as Cauchy data for the equations (6.2), (6.9) we define the functions $E(p,T)$, $Q(p,T)$ that satisfy (6.8) for all T. After that the potential $S(p,T)$ can be found with the help of the formula

$$S(p,T) = \int^p Q(p,T) dE(p,T) . \tag{6.11}$$

Let us revise from this general point of view the definition of the τ-function corresponding to the solutions of the Whitham equations on $\hat{M}_{0,N}$. As was shown in Theorem 2.1, the local parameters k_α proper are solutions of the equations (6.2). Therefore, they define a set of local potentials $S_\alpha(k_\alpha)$ such that the relations (3.2)

$$\Omega_B(k_\alpha, T) = \partial_B S_\alpha(k_\alpha, T) , \qquad B = (\beta, i) ,$$

are fulfilled. On the other hand, let us consider the solutions $E(p,T), Q(p,T)$ of the equations (6.2) with the initial data

$$E(p,x) = p , \qquad Q(p,x) = x . \tag{6.12}$$

They are holomorphic outside the punctures $p_\alpha(T)$. Hence, a "global" differential $dS_0(p,T)$ exists that is also holomorphic outside the punctures $p_\alpha(T)$. Let us define a one-form on the space with the coordinates t_A

$$\delta \ln \tau = \frac{1}{2}\sum_{\alpha=1}^{N}\left(\sum_{i=1}^{\infty} \operatorname{res}_\alpha (k_\alpha^i \, dS_\alpha(p,T))\, dt_{\alpha,i} + \frac{1}{2\pi i}\oint_{\sigma_\alpha} \ln(p-p_\alpha)\, dS_0(p,T)\, dt_{\alpha,0}\right) , \tag{6.13}$$

where σ_α is a contour around the cut connecting $p_1 = \infty$ and p_α. It is easy to check with the help of the formulae (3.11)–(3.14) that $\delta \ln \tau$ is a closed form. Therefore, locally there exists a τ-function. What are advantages of algebraic solutions?

As was shown in Section 3, for algebraic solutions there exist constants t_A^0 such that the sum

$$S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) \tag{6.14}$$

is a "global" potential coinciding in neighborhoods of p_α with local potentials S_α. This provides the possibility of defining explicitly with the help of formula (3.3) the τ-function and not only its full external differential (6.13).

7. The Arbitrary Genus Case

7.1. Definition

The moduli space $\hat{M}_{g,N}$ is "bigger" then $\hat{M}_{0,N}$. In the approach in which the "times" of the Whitham hierarchy are considered as a new system of coordinates on the phase space it is natural to expect that there should be more flows in the Whitham hierarchy on $\hat{M}_{g,N}$. We shall increase their number in a few steps. But at the beginning let us consider the *basic* Whitham hierarchy in the form that has arisen as a result of the averaging procedure for the algebraic-geometrical solutions of two-dimensional integrable equations. In this hierarchy there is the same set of the "times" (2.3) and this is the only part of universal Whitham hierarchy on $\hat{M}_{g,N}$ that has a smooth degeneration to the zero-genus hierarchy.

Let Γ_g be a smooth algebraic curve of genus g with local coordinates $k_\alpha^{-1}(P)$ in neighborhoods of N punctures P_α, $(k_\alpha^{-1}(P_\alpha) = 0)$. Let us introduce meromorphic differentials $d\Omega_A$ on Γ_g such that:

1. $d\Omega_{\alpha,i>0}$ is holomorphic outside P_α and has the form

$$d\Omega_{\alpha,i} = d(k_\alpha^i + O(k_\alpha^{-1})) \tag{7.1}$$

in a neighbourhood of P_α;

2. $\Omega_{\alpha,0}$, $\alpha \neq 1$ is a differential with simple poles at the points P_1 and P_α with residues 1 and -1, respectively

$$\begin{aligned} d\Omega_{\alpha,0} &= dk_\alpha(k_\alpha^{-1} + O(k_\alpha^{-1})) \\ d\Omega_{\alpha,0} &= -dk_1(k_1^{-1} + O(k_1^{-1}))\ ; \end{aligned} \tag{7.2}$$

3. The differentials $d\Omega_A$ are uniquely normalized by the condition that *all* their periods are real, i.e.,

$$\operatorname{Im} \oint_c d\Omega_A = 0\ , \qquad c \in H_1(\Gamma_g, Z)\ . \tag{7.3}$$

The normalization (7.1) does not depend on the choice of basic cycles on Γ_g. Therefore, $d\Omega_A$ is indeed defined by data $\hat{M}_{g,N}$.

Below, for the simplification of formulae we consider the complexation of the Whitham hierarchy on $\hat{M}_{g,N}$ that is the hierarchy on the moduli space

$$\hat{M}^*_{g,N} = \{\Gamma_g, P_\alpha, k_\alpha^{-1}(P),\ a_i, b_i \in H_1(\Gamma_g, Z)\}\ , \tag{7.4}$$

where a_i, b_i is a canonical basis of cycles on Γ_g, i.e., the cycles with the intersection matrix of the form $a_i a_j = b_i b_j = 0$, $a_i b_j = \delta_{i,j}$. In this case the differentials $d\Omega_A$ should be normalized by the usual conditions

$$\oint_{a_i} d\Omega_A = 0\ , \qquad i = 1, \ldots, g\ . \tag{7.5}$$

Both types of hierarchies can be considered absolutely in a parallel way.

Now we are going to show that generating equations (1.11) in which Ω_A are integrals of the above-defined differentials are equivalent to a set of commuting evolution equations on $\hat{M}^*_{g,N}$ (or $\hat{M}_{g,N}$, respectively). Let us fix one point P_1 and choose as the "marked" index $A_0 = (1, 1)$. The multi-valued function

$$p(P) = \Omega_{1,1}(P) = \int^P d\Omega_{1,1}\ , \qquad P \in \Gamma_g \tag{7.6}$$

can be used as a coordinate on Γ everywhere except for the points Π_s, where $dp(\Pi_s) = 0$. The parameters (2.2), i.e.,

$$\{p_\alpha = p(P_\alpha),\ v_{\alpha,s},\ \alpha = 1,\dots,N,\ s = -1,0,1,2,\dots\}$$

and additional parameters

$$\pi_s = p(\Pi_s)\,, \qquad s = 1,\dots,2g\,, \tag{7.7}$$

$$U_i^p = \oint_{b_i} dp\,, \qquad i = 1,\dots,g\,, \tag{7.8}$$

are a full system of local coordinates on $\hat{M}^*_{g,N}$.

THEOREM 7.7. *The zero-curvature form* (1.20) *of the Whitham hierarchy on* $\hat{M}^*_{g,N}$ *is equivalent to the compatible system of evolution equations*

$$\partial_A k_\alpha(p,T) = \{k_\alpha(p,T), \Omega_A(p,T)\}\,, \tag{7.9}$$

$$\partial_A U_i^p = \partial_x U_i^A\,, \quad \text{where} \quad U_i^A = \oint_{b_i} d\Omega_A\,, \tag{7.10}$$

$$\partial_A \pi_s = \partial_A p(\Pi_s) = \partial_x \Omega_A(\Pi_s)\,. \tag{7.11}$$

In [12] where the application of the Whitham equations for generalized Landau-Ginsburg models was considered for the first time, it was noticed that the construction of solutions of the Whitham equations that was proposed by the author in [28] can be reformulated in the form that actually includes new "additional" flows commuting with the basic ones (7.11). It is to be mentioned that only g of them are universal. Let us introduce a set of g new times $t_{h,1},\dots,t_{h,g}$ that are coupled with normalized holomorphic differentials $d\Omega_{h,k}$

$$\oint_{a_i} d\Omega_{h,k} = \delta_{i,k}\,, \qquad i,k = 1,\dots,g\,. \tag{7.12}$$

THEOREM 7.8. *The basic Whitham hierarchy* (7.11) *is compatible with the system that is defined by the same equations but with new "Hamiltonians"* $d\Omega_{h,k}$.

The proofs of the two theorems above do not differ seriously from the usual considerations in the Sato approach, so we shall skip them.

7.2. Algebraic Orbits and Exact Solutions

Let us introduce finite-dimensional subspaces of $\hat{M}^*_{g,N}$ that are invariant with respect to the Whitham hierarchy. Consider a normalized meromorphic differential

dE of the second kind (i.e., dE has no residues at any point of Γ_g) that has poles of orders $n_\alpha + 1$ at the points P_α. (Normalized means that

$$\oint_{a_i} dE = 0 \tag{7.13}$$

for the hierarchy on $\hat{M}^*_{g,N}$ and that dE has real periods for the hierarchy on $\hat{M}_{g,N}$.) The integral $E(p)$ of this differential has the expansions of the form

$$E(p) = p^n + u_{n-2}p^{n-2} + \cdots + u_0 + O(p^{-1}) , \tag{7.14}$$

$$E(p) = \sum_{s=1}^{n_\alpha} v_{\alpha,s}(p - p_\alpha)^{-s} + O(1) \tag{7.15}$$

at the point P_1 and the points P_α, $\alpha \neq 1$, respectively. The formula (2.48), i.e.,

$$k_\alpha^{n_\alpha}(p) = E(p) , \qquad \alpha = 1, \ldots, N ,$$

defines local coordinates k_α^{-1} in neighborhoods of P_α. Therefore, we have defined the embedding of the moduli space $\mathcal{N}_g(n_\alpha)$ of curves with fixed normalized meromorphic differential dE into $\hat{M}^*_{g,N}$

$$\mathcal{N}_g(n_\alpha) \subset \hat{M}^*_{g,N} . \tag{7.16}$$

The dimension of this subspace equals

$$D = \dim \mathcal{N}_g(n_\alpha) = 3g - 2 + \sum_{\alpha=1}^{N}(n_\alpha + 1) . \tag{7.17}$$

There are two systems of local coordinates on $\mathcal{N}_g(n_\alpha)$. The first system is given by the coefficients of the expansions (7.15), (7.15)

$$\{u_i, i = 0, \ldots, n-2; \ \ p_\alpha, \ \ v_{\alpha,s}, s = 1, \ldots, n_\alpha)\} , \tag{7.18}$$

and by the variables (7.7), (7.8), i.e.,

$$\pi_s = p(\Pi_s) , \quad s = 1, \ldots, 2g ; \qquad U_i^p = \oint_{b_i} dp , \quad i = 1, \ldots, g .$$

The second system is given by the following parameters

$$U_i^E = \oint_{b_i} dE , \qquad i = 1, \ldots, g , \tag{7.19}$$

$$E_s = E(q_s) , \quad \text{where} \quad dE(q_s) = 0 , \quad s = 1, \ldots, D - g . \tag{7.20}$$

Using the first system of coordinates it is easy to show that

THEOREM 7.9. *The restriction of the Whitham hierarchy on $\mathcal{N}_g(n_\alpha)$ is given by the compatible system of equations* (2.50)

$$\partial_A E(p,T) = \{E(p,T), \Omega_A(p,T)\} .$$

(We would like to recall now that besides $t_{\alpha,i}$ the set of "times" t_A includes the times $t_{h,k}$ that are coupled with the normalized holomorphic differentials $d\Omega_{h,k}$.)

Let dH_i be a normalized differential that is defined on the cycle a_i, i.e.,

$$\oint_{a_i} dH_i = 0 . \tag{7.21}$$

For each set $H = \{dH_i\}$ of such differentials there exists a unique differential dS_H such that dS_H is holomorphic on Γ_g except for the cycles a_i where it has "jumps" that are equal to

$$dS_H^+(P) - dS_H^-(P) = dH_i(P) \qquad P \in a_i , \tag{7.22}$$

$$\oint_{a_i} dS = 0 . \tag{7.23}$$

THEOREM 7.10. *For any solution of the Whitham equations on $\mathcal{N}_g(n_\alpha)$ there exist constants t_A^0 and constant differentials dH_i (i.e., they do not depend on T) such that this solution is given in an implicit form with the help of equations*

$$\frac{dS}{dp}(q_s, T) = 0 , \tag{7.24}$$

$$S(p,T) = \sum_A (t_A - t_A^0)\Omega_A(p,T) + dS_H . \tag{7.25}$$

The relations (7.25) imply that

$$dS = Q\,dE , \tag{7.26}$$

where $Q(p)$ is holomorphic on Γ_g outside the punctures P_α and has "jumps"

$$Q^+(E) - Q^-(E) = \frac{dH_i(E)}{dE} , \qquad E \in a_i , \tag{7.27}$$

on cycles a_i.

In this section we consider the solutions of the Whitham hierarchy corresponding to the constant jumps only, i.e.,

$$dH_i(P) = t_{Q,i} dE(P) . \tag{7.28}$$

In that case dQ is a single-valued differential on Γ_g. Let us present an alternative formulation of the construction of such solutions.

Consider the moduli space

$$\widetilde{\mathcal{N}}_g(n_\alpha) = \{\Gamma_g\ ,\ dQ\ ,\ dE\} \tag{7.29}$$

of curves with a fixed canonical basis of cycles, with a fixed normalized meromorphic differential dE having poles of orders $n_\alpha + 1$ at points P_α and with a fixed normalized differential $dQ(P)$ that is holomorphic outside the punctures.

The coordinates on this space are the variables (7.7), (7.8), (7.18)

$$\{\pi_s\ ,\ U_i^p\ ,\ u_i\ ,\ p_\alpha\ ,\ v_{\alpha,s}\ ,\}$$

and the coefficients of singular terms in the expansions

$$\begin{aligned} Q(p) &= \sum_{j=1}^{\infty} b_{1,j} p^j + O(p^{-1})\ , \\ Q(p) &= \sum_{j=1}^{\infty} b_{\alpha,j}(p - p_\alpha)^{-j} + O(p - p_\alpha)\ . \end{aligned} \tag{7.30}$$

The formulae (2.55), i.e.,

$$\begin{aligned} t_{\alpha,i} &= \frac{1}{i}\mathrm{res}_\alpha(k_\alpha^{-i}(p)Q(p)\,dE(p)))\ , \qquad i > 0; \\ t_{\alpha,0} &= \mathrm{res}_\alpha(Q(p)\,dE(p)) \end{aligned} \tag{7.31}$$

and the formulae

$$t_{h,i} = \oint_{a_i} dS\ , \quad i = 1,\dots,g\ , \quad dS = Q\,dE\ , \tag{7.32}$$

$$t_{Q,i} = -\oint_{b_i} dE\ , \quad t_{E,i} = \oint_{b_i} dQ\ , \quad i = 1,\dots,g\ . \tag{7.33}$$

define times t_A as the functions on the space $\widetilde{\mathcal{N}}_g(n_\alpha)$.

The differentials $d\Omega_{E,i}$, $d\Omega_{Q,i}$ that are coupled with the times $t_{E,i}$, $t_{Q,i}$ are uniquely defined with the help of the following analytical properties:

1. The differentials $d\Omega_{E,i}$, $d\Omega_{Q,i}$ are holomorphic on the curve Γ_g everywhere except for the a-cycles, where they have "jumps". Their boundary values on a_j cycle satisfy the relations

$$\begin{aligned} d\Omega_{E,i}^+ - d\Omega_{E,i}^- &= \delta_{i,j}\,dE\ , \\ d\Omega_{Q,i}^+ - d\Omega_{Q,i}^- &= \delta_{i,j}\,dQ\ ; \end{aligned} \tag{7.34}$$

2.

$$\oint_{a_j} d\Omega_{E,i} = \oint_{a_j} d\Omega_{Q,i} = 0\,, \qquad j = 1, \ldots, g\,. \tag{7.35}$$

In the same way as was done in Section 2, it can be shown that the number of "times" is equal to the dimension of $\widetilde{\mathcal{N}}_g(n_\alpha)$. Therefore, the "times" t_A can be considered as new coordinates on $\widetilde{\mathcal{N}}_g(n_\alpha)$, i.e.,

$$\Gamma_g = \Gamma_g(T)\,, \quad dQ = dQ(T)\,, \quad dE = dE(T)\,. \tag{7.36}$$

THEOREM 7.11. *For the differential*

$$dS(E,T) = Q(E,T)\,dE \tag{7.37}$$

the following equalities

$$\partial_A S(E,T) = \Omega_A(E,T) \tag{7.38}$$

are fulfilled.

Remark. From the definition of the times (7.31), (7.32), (7.33) it follows that

$$dS = \sum_{\alpha=1}^{N} \sum_{i=0}^{\infty} t_{\alpha,i}\, d\Omega_{\alpha,i} + \sum_{k=1}^{g} t_{h,k}\, d\Omega_{h,k} + t_{E,k}\Omega_{E,k}\,. \tag{7.39}$$

We shall give here a brief sketch of the proof (7.38) for $A = (Q,k)$ only, because for all the other A the proof is essentially the same as the proof of Theorem 2.2. Consider the derivative $\partial_{Q,k}S(E,T)$. From the definition (7.37) it follows that $\partial_{Q,k}S(E,T)$ is holomorphic everywhere except for the cycle a_k. On different sides of this cycle the coordinates are E^- and $E^+ = E^- - t_{Q,k}$. Hence, taking the derivative of the equality

$$Q(E^- - t_{Q,k}) - Q(E^-) = t_{E,k}\,, \qquad E^- \in a_k\,, \tag{7.40}$$

we obtain

$$\partial_{Q,k}Q^+ - \partial_{Q,k}Q^- = \frac{dQ}{dE}\,. \tag{7.41}$$

Therefore, $\partial_{Q,k}S(E,T) = \Omega_{Q,k}$.

COROLLARY 7.5. *The integrals $E(p,T)$ and $Q(p,T)$ as functions of the variable $p = \Omega_{1,1}$ satisfy the Whitham equations* (2.50) *and the classical string equation* (2.63).

(In both the theorems the set of times t_A includes all the times $t_{\alpha,i}$, $t_{h,i}$, $t_{E,i}$, $t_{Q,i}$.)

7.3. The τ-Function

The τ-function of the particular solution of the Whitham equation on $\widetilde{\mathcal{N}}_g(n_\alpha)$ that was constructed above is defined by the formula

$$\ln \tau(T) = F(T),$$

$$F = F_0(T) + \frac{1}{4\pi i} \sum_{k=1}^{g} \oint_{a_k^-} t_{E,k} E\, dS - \oint_{b_k} t_{h,k}\, dS + t_{h,k} t_{E,k} E_k, \tag{7.42}$$

where $F_0(T)$ is given by (3.3), i.e.,

$$F_0 = \frac{1}{2} \sum_{\alpha=1}^{N} \operatorname{res}_\alpha \left(\sum_{i=1}^{\infty} t_{\alpha,i} k_\alpha^i\, dS(p,T) \right) + t_{\alpha,0} s_\alpha(T),$$

(the first integral in (7.42) is taken over the left side of the a_k cycle and $E_k = E(P_k)$ where P_k is the intersection point of a_k and b_k cycles).

Remark. The differential dS is discontinuous. Therefore, its integral over b_k-cycle depends on the choice of the cycle. The last term in (7.42) restores the invariance (i.e., F depends on the homology class of cycles, only).

THEOREM 7.12. *For the above-defined τ-function the equalities* (3.8), (3.9) *are fulfilled. Besides this,*

$$\partial_{h,k} F = \frac{1}{2\pi i} \left(t_{E,k} E_k - \oint_{b_k} dS \right), \tag{7.43}$$

$$\partial_{E,k} F = \frac{1}{2\pi i} \left(\oint_{a_k} E\, dS \right), \tag{7.44}$$

$$\partial_{Q,k} F = \frac{1}{4\pi i} \left(\oint_{a_k} Q\, dS - 2 t_{E,k} t_{h,k} \right). \tag{7.45}$$

The proof of all these equalities is analogous to the proof of (3.8), (3.9) and it uses different types of identities that can be proved with the help of usual considerations of contour integrals.

COROLLARY 7.6. *For $A = (\alpha, i)$ the second derivatives $\partial^2_{A,B} F$ are given by the formulae* (3.16), (3.17). *Besides this,*

$$\partial^2_{(h,k);A} F = \frac{1}{2\pi i} \left(E_k \delta_{(E,k);A} + Q_k \delta_{(Q,k);A} - \oint_{b_k} d\Omega_A \right), \tag{7.46}$$

$$\partial^2_{(E,k);A} = \frac{1}{2\pi i} \left(\oint_{a_k} E\, d\Omega_A \right), \tag{7.47}$$

$$\partial^2_{(Q,k);A}F = \frac{1}{2\pi i}\left(\oint_{a_k} Q\,d\Omega_A - \partial_A(t_{E,k}t_{h,k})\right). \tag{7.48}$$

We would like to mention that in particular the formula (7.46) gives a matrix of b-periods of normalized holomorphic differentials on Γ_g

$$\partial^2_{(h,i);(h,j)}F = -\oint_{b_i} d\Omega_{h,j}\,. \tag{7.49}$$

(for the particular case this relation was obtained for the first time in [13]).

THEOREM 7.13. *The third derivatives of $F(T)$ are equal to*

$$\partial^3_{ABC}F = \sum_{q_s} \mathrm{res}_{q_s}\left(\frac{d\Omega_A\,d\Omega_B\,d\Omega_C}{dQ\,dE}\right) + \eta_{ABC}\,, \tag{7.50}$$

where

$$\begin{aligned} \eta_{ABC} &= 0 && \text{if } A,B,C \neq (Q,k)\,, \\ \eta_{AB(Q,k)} &= \frac{1}{2\pi i}\oint_{a_k}\frac{d\Omega_A\,d\Omega_B}{dE} && \text{if } A,B \neq (Q,k)\,. \end{aligned}$$

7.4. Virasoro Constraints

In this subsection we present "L_0, L_{-1}" constraints for the τ-function of the homogenious solution of the Whitham hierarchy on $\widetilde{\mathcal{N}}_g(n_\alpha)$.

Consider the differential Q^2dE. It is holomorphic on Γ_g outside the punctures and cycles a_k where it has jumps

$$(Q^2\,dE)^+ - (Q^2\,dE)^- = 2t_{E,k}Q\,dE = 2t_{E,k}\,dS\,. \tag{7.51}$$

Therefore,

$$\sum_{\alpha=1}^{N}\mathrm{res}_\alpha(Q^2\,dE) + \frac{1}{\pi i}\sum_{k=1}^{g} t_{E,k}t_{h,k} = 0\,. \tag{7.52}$$

The expansion of Q near the puncture P_α has the form (4.7). Its substitution into (7.52) gives

$$\begin{aligned} &\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\left(\sum_{i=n_\alpha+1}^{\infty} it_{\alpha,i}\partial_{\alpha,i-n_\alpha}F + n_\alpha t_{\alpha,0}t_{\alpha,n_\alpha}\right. \\ &\qquad\left. + \frac{1}{2}\sum_{j=1}^{n_\alpha-1} j(n_\alpha - j)t_{\alpha,j}t_{\alpha,n_\alpha-j}\right) + \frac{1}{2\pi i}\sum_{k=1}^{g} t_{E,k}t_{h,k} = 0\,. \end{aligned} \tag{7.53}$$

In the same way the consideration of the differential $Q^2E\,dE$ proves an analogue of L_{-1} constraint:

$$\sum_{\alpha=1}^{N}\frac{1}{n_\alpha}\sum_{i=1}^{\infty} i t_{\alpha,i}\partial_{\alpha,i}F + \frac{1}{2\pi i}\sum_{k=1}^{g} t_{E,k}\partial_{E,k}F + \frac{1}{2}t_{\alpha,0}^2 = 0 \; ; \tag{7.54}$$

Remark. In order to obtain higher "$L_{n>0}$" Virasoro constraints one has to introduce p-gravitational descendants of the "fields" $d\Omega_{E,k}$ that are holomorphic differentials on Γ except for the a_k-cycle where they have "jumps" that are equal to $E^p dE$.

7.5. Landau-Ginzburg-Type Models on Riemann Surfaces

In this subsection we present the generalization of the results of Section 5 for the case of Riemann surfaces of an arbitrary genus. Let us consider a genus g Riemann surface Γ_g with fixed canonical basis of cycles and with fixed meromorphic normalized differential dE, i.e., a point of the moduli space $\mathcal{N}_g(n_\alpha)$. The same formulae (5.19), (5.20), as in genus zero case, define a Fröbenius algebra $\mathcal{R}_{\Gamma_g,dE}$

$$\mathcal{R}_{\Gamma_g,dE} = \hat{\mathcal{R}}/(dE = 0)\,, \tag{7.55}$$

where $\hat{\mathcal{R}}$ is a ring of all meromorphic functions that are regular at the zeros q_s of the differential dE. The formula

$$\langle f,g\rangle = \sum_{q_s}\operatorname{res}_{q_s}\left(\frac{f(p)g(p)}{E_p}dp\right), \quad f(p)\,, \quad g(p)\subset\hat{\mathcal{R}}\,, \tag{7.56}$$

defines a non-degenerate scalar product on $\mathcal{R}_{\Gamma_g,dE}$.

For any g a "small phase" space is the space of times t_a with indices $a\in\mathcal{A}_{sm}^g$, where $\mathcal{A}_{sm}^g$ is a union of $\mathcal{A}_{sm}$ (that was defined in Section 5)

$$\mathcal{A}_{sm} = \{a=(\alpha,i)\,|\,\alpha=1,\ i=1,\dots,n-1;\ \alpha=2,\dots,N,\ i=0,\dots,n_\alpha\}$$

and indices $(h,k),(E,k)$. In the basis

$$\phi_a = \frac{d\Omega_a}{dp} \tag{7.57}$$

the scalar products have the form:

$$\langle\phi_a\phi_b\rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha}\delta_{\alpha,\beta}\delta_{i+j,n_\alpha}\,, \qquad a,b\in\mathcal{A}_{sm} \tag{7.58}$$

$$\langle\phi_{E,k}\phi_{h,s}\rangle = \delta_{k,s}\,, \tag{7.59}$$

otherwise it is zero (here $[i]$ is the same as in (4.14)).

Let us consider the Whitham "times" t_a that were defined in (7.31), (7.32), (7.33) for the choice $dQ = dp$, i.e.,

$$
\begin{aligned}
t_{\alpha,i} &= \frac{1}{i}\mathrm{res}_\alpha(k_\alpha^{-i}(p)pdE(p))\,, \qquad \alpha, i > 0 \in \mathcal{A}_{sm}\,; \\
t_{\alpha,0} &= \mathrm{res}_\alpha(pdE(p))\,;
\end{aligned} \tag{7.60}
$$

$$
t_{h,k} = \oint_{a_k} pdE\,, \qquad k = 1,\dots,g\,; \tag{7.61}
$$

$$
t_{p,i} = -\oint_{b_k} dE\,, \quad t_{E,k} = \oint_{b_k} dp\,, \quad k = 1,\dots,g\,. \tag{7.62}
$$

Remark. This paper had already been written when the following result was proved. Let us consider the one-point case, i.e., $\mathcal{N}_g(n) \subset \hat{M}^*_{g,N=1}$ and let us suppose that we restrict our consideration on the space of real M-curves Γ. That means that we consider curves with anti-holomorphic involution $\tau : \Gamma \to \Gamma$ and $\tau(P_1) = P_1$. Then the following statement is valid.

THEOREM 7.14. *The restriction of the map*

$$
\mathcal{N}_g(n_\alpha) \longmapsto \{t_a, a \in \mathcal{A}^g_{sm}\} \tag{7.63}
$$

onto the space of M-curves with one puncture is one-to-one correspondence with some domain in real space with real coordinates t_a.

The proof of this statement will be published later.

Let us fix the values $t_{p,k} = t^0_{p,k}$ and consider the restriction of $F = \ln\tau$ on the affine space that is $\mathcal{A}^g_{sm}$ shifted by

$$
t_{1,n+1} = \frac{n}{n+1}\,, \qquad t_{p,k} = t^0_{p,k}\,.
$$

Then from the statement of Theorem 7.7 it follows that

$$
\partial^3_{abc}F = \sum_{q_s} \mathrm{res}_{q_s}\left(\frac{d\Omega_a d\Omega_b d\Omega_c}{dp\,dE}\right), \qquad a, b, c \in \mathcal{A}^g_{sm}\,. \tag{7.64}
$$

COROLLARY 7.7. *The dependence of the Fröbenius algebra corresponding to $\mathcal{N}_g(n_\alpha)$ on the coordinates t_a, $a \in \mathcal{A}^g_{sm}$ is a potential deformation.*

In [13] the particular case of this statement was proved. It corresponds to the Whitham hierarchy on moduli space of genus g curves with fixed function $E(P)$

having a pole of order n at only one point P_1. (This moduli space is a subspace of $\mathcal{N}_g(n)$ that is specified by the conditions $t_{p,k} = 0$.) The differential-geometrical interpretation of Whitham coordinates that was proposed in [13] is valid in a general case as well.

Let us denote a subspace of $\mathcal{N}_g(n_\alpha)$ corresponding to the fixed values of $t_{p,k} = t^0_{p,k}$ by $\mathcal{N}_g(n_\alpha | t^0_{p,k})$. A system of local coordinates on its open submanifold $\mathcal{D}$ is given by (7.20), i.e.,

$$E_s = E(q_s)\ , \quad \text{where} \quad dE(q_s) = 0\ , \quad s = 1, \ldots, D - g = 2g - 2 + \sum_{\alpha=1}^{N} (n_\alpha + 1)\ .$$

Submanifold $\mathcal{D}$ can be defined as a submanifold on which the values E_s are distinct. The formula

$$ds^2 = \sum_{s=1}^{D-g} \operatorname{res}_{q_s} \left(\frac{dp\,dp}{dE} \right) \left(dE_s \right)^2 \tag{7.65}$$

defines a metric on $\mathcal{D} \subset \mathcal{N}_g(n_\alpha | t^0_{p,k})$. The scalar products of the vector-fields $\partial_a = \frac{\partial}{\partial_a}$ $a \in \mathcal{A}^g_{sm}$ with respect to this metric have the form:

$$\langle \partial_a \partial_b \rangle = \eta_{ab} = \frac{[i][j]}{n_\alpha} \delta_{\alpha,\beta} \delta_{i+j,n_\alpha}\ , \qquad a, b \in \mathcal{A}_{sm} \tag{7.66}$$

$$\langle \partial_{E,k} \partial_{h,s} \rangle = \delta_{k,s}\ , \tag{7.67}$$

otherwise they are zeros. The proof of (7.66), (7.67) is based on the formula (5.11)

$$\partial_A E_s = \frac{d\Omega_A}{dp}. \tag{7.68}$$

and formulae (7.58), (7.59). Consequently, in the Whitham coordinates t_a, $a \in \mathcal{A}^g_{sm}$ the metric (7.65) has constant coefficients (i.e., ds^2 is a flat metric and the t_a are flat coordinates).

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