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THE TANGENT BUNDLES OVER EQUIVARIANT REAL PROJECTIVE SPACES

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Abstract. let G be a nontrivial cyclic group of odd order. In the present paper, we will prove that the fourfold Whitney sum of the tangent bundle of real projective plane of any three dimensional nontrivial real G-representation is equivariantly a product bundle.

1. Introduction

In this paper, let G be a finite group. For a real G-module V (of finite dimension), let $S(V)$ denote the unit sphere of V with respect to some G-invariant inner product, and set $P(V) = S(V)/\{\pm 1\}$ the equivariant real projective space. For a G-space M and a real G-module V, let $\varepsilon_M(V)$ denote the product bundle with total space $M \times V$ over M, and let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space with trivial G-action. When M is a smooth manifold, let $T(M)$ denote the tangent bundle of M. For $M = P(V)$, let γ_M denote the equivariant canonical line bundle over M and $\gamma_M^\perp$ the orthogonal complement of γ_M in $\varepsilon_M(V)$, therefore the total space $E(\gamma_M)$ of γ_M is

$$\{(\{\pm x\}, v) \mid x \in S(V),\, v \in \mathbb{R} \cdot x (\subset V)\},$$

and $\gamma_M \oplus \gamma_M^\perp = \varepsilon_M(V)$. We obtain the following three theorems.

Theorem 1. *For a G-module V, let $M = P(V)$ and $\gamma = \gamma_M$. Then the following hold.*

1. $\mathrm{Hom}(\gamma, \gamma) = \varepsilon_M(\mathbb{R})$.
2. $\mathrm{Hom}(\gamma, \varepsilon_M(\mathbb{R})) = \gamma$.
3. $T(M) = \mathrm{Hom}(\gamma, \gamma^\perp)$.
4. $T(M) \oplus \varepsilon_M(\mathbb{R}) = \mathrm{Hom}(\gamma, \varepsilon_M(V))$.
5. $\mathrm{Hom}(\gamma, \varepsilon_M(V)) = \gamma \otimes \varepsilon_M(V)$.

Theorem 2. *Let G be a cyclic group of odd order and V a 2-dimensional real G-module with free G-action except the origin. Let $M = P(\mathbb{R} \oplus V)$ and $\gamma = \gamma_M$. Then the 4-fold Whitney sum $\gamma^{\oplus 4}$ of γ is G-isomorphic to the G-vector bundle $\varepsilon_M(\mathbb{R}^4)$.*

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Remark For any natural number k, (even if forgetting the G-action) $\gamma^{\oplus 2} \oplus \varepsilon_M(\mathbb{R}^k)$ is not isomorphic to $\varepsilon_M(\mathbb{R}^{k+2})$ as real vector bundles. This fact follows from that the total Stiefel-Whitney class of $\gamma^{\oplus 2}$ is not trivial (cf. Chapter 4 in [1]).

Theorem 3. *Let G be a cyclic group of odd order and V a 2-dimensional real G-module with free G-action except the origin. Let $T(M)$ denote the tangent bundle of $M = P(\mathbb{R} \oplus V)$. Then $T(M)^{\oplus 4} \oplus \varepsilon_M(\mathbb{R}^4)$ is G-isomorphic to $\varepsilon_M(V^{\oplus 4}) \oplus \varepsilon_M(\mathbb{R}^4)$. Hence,*

$$4[T(M)] = 0 \quad \text{in } \widetilde{KO}_G(M).$$

Since these theorems are important to study of G-actions on spheres, especially study of Smith Problem (see [2]), we describe concrete proofs in the present paper. We also show in Theorem 5 that Theorems 2 and 3 do not hold for cyclic groups G of even order.

2. Proof of Theorem 1

Theorem 1 will be proved in this section. Let G be a finite group and V a real G-module (of finite dimension). Set $M = P(V)$. For $x \in S(V)$, let $L_{[x]}$ denote the line through the points $\pm x$ in V, and let $L_{[x]}^{\perp}$ be orthogonal complement.

2.1 G-vector bundle $\mathrm{Hom}(\gamma_M, \gamma_M)$. The total space $E(\xi)$ of the vector bundle $\xi = \mathrm{Hom}(\gamma_M, \gamma_M)$ is $\bigcup_{[x] \in M} \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\gamma_M))$. Now for $x \in S(V)$, let $F_{[x]}(\gamma_M)$ denote the fiber of γ_M over the point $[x]$. The projection map $\pi : E(\xi) \to M$ of vector bundle ξ is defined by $\pi(f) = [x]$ for $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\gamma_M))$. The G-action on $E(\xi)$ is given by $gf \in \mathrm{Hom}(F_{[gx]}(\gamma_M), F_{[gx]}(\gamma_M))$;

$$(gf)([gx], gv) = ([gx], gf(g^{-1}(gv)) = ([gx], gf(v))$$

for $g \in G$ and $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\gamma_M))$. Hence ξ is a G-vector bundle.

2.2 G-vector bundle $\varepsilon_M(\mathbb{R}^n)$. For a natural number n, the total space $E(\varepsilon)$ of the vector bundle $\varepsilon = \varepsilon_M(\mathbb{R}^n)$ is $\{([x], b) \mid [x] \in M,\ b \in \mathbb{R}^n\}$. The projection map $\pi : E(\varepsilon) \to M$ of ε is $\pi([x], b) = [x]$. For $[x] \in M$, $b \in \mathbb{R}^n$. The G-action on $E(\varepsilon)$ is given by $g([x], b) = ([gx], b)$ for $g \in G$ and $([x], b) \in E(\varepsilon)$. Therefore ε is a G-vector bundle.

2.3 Isomorphism $\mathbf{Hom}(\gamma_M, \gamma_M) \to \varepsilon_M(\mathbb{R})$. We define a bundle map φ : $\mathrm{Hom}(\gamma_M, \gamma_M) \to \varepsilon_M(\mathbb{R})$ as follows. Let $x \in S(V)$ and $f \in \mathrm{Hom}(F_{[x]}\gamma_M, F_{[x]}\gamma_M)$. Then a unique real number t is determined by $f(x) = tx$. In fact, using a certain G-equivariant inner product $\langle -, - \rangle$ on V, we have $t = \langle f(x), x \rangle$. Let

$$\varphi(f) = ([x], t).$$

It is easy to check that the map $\varphi : \mathrm{Hom}(\gamma_M, \gamma_M) \to \varepsilon_M(\mathbb{R})$ is an isomorphism of G-vector bundles.

2.4 G-vector bundle $\mathbf{Hom}(\gamma_M, \varepsilon_M(\mathbb{R}^n))$. For a natural number n, the total space $E(\eta)$ of the vector bundle $\eta = \mathrm{Hom}(\gamma_M, \varepsilon_M(\mathbb{R}^n))$ is given by

$$\bigcup_{[x]\in M} \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(\mathbb{R}^n))).$$

The projection map $\pi : E(\eta) \to M$ of η is $\pi(f) = [x]$. The G-action on the total space $E(\eta)$ is given by $gf \in \mathrm{Hom}(F_{[gx]}(\gamma_M), F_{[gx]}(\varepsilon_M(\mathbb{R}^n)))$;

$$(gf)([gx], gv) = ([gx], f(g^{-1}(gv))) = ([gx], f(v)) \ \ (v \in F_{[x]}(\gamma_M))$$

for $g \in G$ and $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(\mathbb{R}^n)))$. With this structure, clearly η is a G-vector bundle.

2.5 Isomorphism $\mathbf{Hom}(\gamma_M, \varepsilon_M(\mathbb{R})) \to \gamma_M$. For $f \in \mathrm{Hom}(F_{[x]}\gamma_M, F_{[x]}\varepsilon_M(\mathbb{R}))$, $x \in S(V)$, let the bundle map $\varphi : \mathrm{Hom}(\gamma_M, \varepsilon_M(\mathbb{R})) \to \gamma_M$ be defined by

$$\varphi(f) = ([x], (f(x))x).$$

It is easy to check that the map $\varphi : \mathrm{Hom}(\gamma_M, \varepsilon_M(\mathbb{R})) \to \gamma_M$ is an isomorphism of G-vector bundles.

2.6 G-vector bundle $T(M)$. Let M be a G-manifold equivariantly embedded in a real G-module W. Then the tangent vector space $T_{[x]}(M)$ at $[x] \in M$ can be regarded as a subspace of W. The total space $E(\tau)$ of the tangent bundle $\tau = T(M)$ over M is $\{([x], v) \mid [x] \in M,\ v \in T_{[x]}(M)\}$, where $T_{[x]}(M)$ is the tangent space at $[x] \in M$. The projection map $\pi : E(\tau) \to M$ of τ is $\pi([x], v) = [x]$. For $g \in G$, $v \in T_{[x]}(M)$, the G-action on the total space $E(\tau)$ is given by $g([x], v) = ([gx], gv)$. For $v \in W$, gv belongs to W. Thus $T(M)$ is a G-vector bundle.

2.7 G-vector bundle $\mathbf{Hom}(\gamma_M, \gamma_M^{\perp})$. The total space $E(\gamma')$ of the vector bundle $\gamma' = \gamma_M^{\perp}$ is $\{([x], v) \in M \times V \mid \langle x, v \rangle = 0\}$, and the projection map is $([x], v) \mapsto [x]$. Meanwhile the total space $E(\xi)$ of the vector bundle $\xi = \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$ is

$$\bigcup_{[x]\in M} \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\gamma_M^{\perp}))$$

and the projection map $\pi : E(\xi) \to M$ is $\pi(f) = [x]$. The G-action on the total space $E(\xi)$ is given by $gf \in \mathrm{Hom}(F_{[gx]}(\gamma_M), F_{[gx]}(\gamma_M^{\perp}))$; $(gf)([gx], gv) = ([gx], gf(g^{-1}(gv)) = ([gx], gf(v))$, $v \in F_{[x]}(\gamma_M)$ for $g \in G$ and $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(\mathbb{R})))$. Therefore ξ is a G-vector bundle.

2.8 Isomorphism $T(M) \to \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$. Consider the two total spaces $E(\tau)$ and $E(\tau')$ of the vector bundles $\tau = T(M)$ and $\tau' = T(S(V))$, respectively. For two points (x, v), $(y, w) \in E(\tau')$, $(x, y \in S(V)$, $v \in T_x(S(V))$, $w \in T_y(S(V))$, if we define that

$$(x, v) \sim (y, w) \Leftrightarrow \{x = y \text{ and } v = w\} \text{ or } \{x = -y \text{ and } v = -w\},$$

then a natural identification $E(\tau) = E(\tau')/\sim$ can be obtained. In brief, $E(\tau) = \{\pm\{(x, v)\} | x \in S(V), v \in T_x S(V)\}$. Let the bundle map $\varphi : T(M) \to \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$ be defined by

$$\varphi(\{\pm(x, v)\})(\alpha x) = \alpha v \quad (\alpha \in \mathbb{R}).$$

It is easy to check that $\varphi : T(M) \to \mathrm{Hom}(\gamma_M, \gamma_M^{\perp})$ is an isomorphism of G-vector bundles.

2.9 G- vector bundle $\mathrm{Hom}(\gamma_M, \varepsilon_M(V))$. The total space $E(\xi')$ of the vector bundle $\xi' = \mathrm{Hom}(\gamma_M, \varepsilon_M(V))$ is given by

$$\bigcup_{[x]\in M} \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(V))).$$

For $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(\mathbb{R})))$, the projection map $\pi : E \to M$ is $\pi(f) = [x]$. The G-action on the total $E(\xi')$ is given by $gf \in \mathrm{Hom}(F_{[gx]}(\gamma_M), F_{[gx]}(\varepsilon_M(V)))$;

$$(gf)([gx], gv) = ([gx], gf(g^{-1}(gv)) = ([gx], gf(v)) \quad (v \in L_{[x]})$$

for $g \in G$ and $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(\mathbb{R})))$. Hence ξ' is a G-vector bundle.

2.10 Isomorphism $T(M) \oplus \varepsilon_M(\mathbb{R}) \to \mathrm{Hom}(\gamma_M, \varepsilon_M(V))$. According to 2.3, 2.8, we have

$$\begin{aligned} T(M) \oplus \varepsilon_M(\mathbb{R}) &\cong \mathrm{Hom}(\gamma_M, \gamma_M^{\perp}) \oplus \mathrm{Hom}(\gamma_M, \gamma_M) \\ &\cong \mathrm{Hom}(\gamma_M, \gamma_M^{\perp} \oplus \gamma_M) \\ &\cong \mathrm{Hom}(\gamma_M, \varepsilon_M(V)). \end{aligned}$$

2.11 Isomorphism $\mathbf{Hom}(\gamma_M, \varepsilon_M(V)) \to \gamma_M \otimes \varepsilon_M(V)$. Let

$$\varphi : \mathrm{Hom}(\gamma_M, \varepsilon_M(V)) \to \gamma_M \otimes \varepsilon_M(V)$$

be the bundle map defined by the formula

$$\varphi(f) = ([x], x \otimes f(x)),$$

for $f \in \mathrm{Hom}(F_{[x]}(\gamma_M), F_{[x]}(\varepsilon_M(V)))$. It is easy to check that $\varphi : \mathrm{Hom}(\gamma_M, \varepsilon_M(V)) \to \gamma_M \otimes \varepsilon_M(V)$ is an isomorphism of G-vector bundles.

3. Proof of Theorem 2

We will prove Theorem 2 in this section, let us begin with some preparations.

Let G be a finite group and X a G-space with $X = Y \cup Z$, where Y and Z are closed G-subsets of X. Suppose that there exists a closed neighborhood N of $Y \cap Z$ in Y with a G-homeomorphism

$$\phi : N \to (Y \cap Z) \times [0, 1]$$

such that

$$\phi(y) = (\phi_1(y), \phi_2(y)) \ \ (y \in N,\ \phi_1(y) \in Y \cap Z,\ \phi_2(y) \in [0, 1])$$

and that $\phi^{-1}((Y \cap Z) \times [0, 1))$ is an open set of Y and that the equality $\phi(y) = (y, 0)$ holds for $y \in Y \cap Z$, where G acts on the closed interval $[0, 1]$ by the trivial action.

Let ξ be a G-vector bundle over X of fiber dimension n with some G-invariant Euclidean metric. Suppose that $\xi|_Y$ and $\xi|_Z$ are isomorphic to $\varepsilon_Y(\mathbb{R}^n)$ and $\varepsilon_Z(\mathbb{R}^n)$, respectively. Let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ denote the standard orthonormal framings of $\xi|_Y$ and $\xi|_Z$, respectively. Then $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are G-equivariant, that is, $e_j(gy) = ge_j(y)$ and $f_j(gz) = gf_j(z)$ for any $g \in G$, $y \in Y$, $z \in Z$. Let A denote the transition matrix function from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$ on $Y \cap Z$, that is, $A(x) = [a_{ij}(x)]$ is a real $n \times n$-matrix such that the following holds for $x \in Y \cap Z$:

$$f_i(x) = \sum_{j=1}^{n} a_{ji}(x) e_j(x) \quad (i = 1, \ldots, n).$$

Since $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are orthonormal, the matrix $A(x)$ lies in $O(n)$. Thus A is a map from $Y \cap Z$ to $O(n)$. It is easy to see

$$A(gx) = A(x) \quad (g \in G, x \in Y \cap Z).$$

Thus A is G-invariant. In other words, A can be regarded as a G-map $Y \cap Z \to O(n)$, where $O(n)$ has the trivial G-action.

Lemma 4. *If $A : Y \cap Z \to O(n)$ is G-homotopic to the constant map with value I, then ξ is isomorphic to the real G-vector bundle $\varepsilon_X(\mathbb{R}^n)$.*

Proof. Let $B : Y \cap Z \to O(n)$ be the constant map with value I. In the below we will show that $(f_1, \ldots, f_n)$ can be extended over X as a G-equivariant framing by using the fact that A is G-homotopic to B.

Since A is G-homotopic to B, there exists a continuous map $H : Y \cap Z \times [0,1] \to O(n)$; $H(x,t) = (h_{ij}(x,t))$, $x \in Y \cap Z$, $t \in [0,1]$, satisfying the following:

$$\begin{cases} H(x,0) = A(x) \ \ (x \in Y \cap Z) \\ H(x,1) = B(x) \ \ (x \in Y \cap Z) \\ H(gx,t) = H(x,t) \ \ (x \in Y \cap Z, t \in [0,1]). \end{cases}$$

Now let the framing $(k_1, \ldots, k_n)$ over N be defined by

$$(k_1(x), \ldots, k_n(x)) = (e_1(x), \ldots, e_n(x))H(\phi_1(x), \phi_2(x)).$$

In other words,

$$k_i(x) = \sum_{j=1}^{n} h_{ji}(\phi_1(x), \phi_2(x))e_j(x).$$

Then the following can be checked.

(1) The G-equivariance of $(k_1, \ldots, k_n)$:

$$\begin{aligned} (k_1(gx), \ldots, k_n(gx)) &= (e_1(gx), \ldots, e_n(gx))H(\phi_1(gx), \phi_2(gx)) \\ &= (ge_1(x), \ldots, ge_n(x))H(g\phi_1(x), g\phi_2(x)) \\ &= (ge_1(x), \ldots, ge_n(x))H(\phi_1(x), \phi_2(x)) \\ &= (gk_1(x), \ldots, gk_n(x)). \end{aligned}$$

(2) When $\phi_2(x) = 0$:

$$\begin{aligned} (k_1(x), \ldots, k_n(x)) &= (e_1(x), \ldots, e_n(x))H(x,0) \\ &= (e_1(x), \ldots, e_n(x))A(x) \\ &= (f_1(x), \ldots, f_n(x)). \end{aligned}$$

(3) When $\phi_2(x) = 1$:

$$\begin{aligned} (k_1(x), \ldots, k_n(x)) &= (e_1(x), \ldots, e_n(x))H(\phi_1(x), 1) \\ &= (e_1(x), \ldots, e_n(x))B(\phi_1(x)) \\ &= (e_1(x), \ldots, e_n(x)). \end{aligned}$$

These show that the G-equivariant framing $(f_1, \ldots, f_n)$ can be extended over X. Hence the G-vector bundle ξ is isomorphic to the trivial G-vector bundle $\varepsilon_X(\mathbb{R}^n)$. □

In the remainder of this section, let G be a cyclic group of odd order p and V a 2-dimensional real G-module with free G-action except the origin. Set $W = \mathbb{R} \oplus V$, and let γ denote the canonical line bundle over the real projective space $M = P(W)$, and let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Choose g_0 the generator of G and $\{v_1, v_2\}$ a basis of V such that they satisfy

$$g_0(v_1, v_2) = (v_1, v_2)\begin{pmatrix} \cos 2\pi/p & -\sin 2\pi/p \\ \sin 2\pi/p & \cos 2\pi/p \end{pmatrix}.$$

Therefore if V is regarded as a complex vector space $\mathbb{C}$, then

$$g_0 z = \exp(2\pi\sqrt{-1}/p)z \quad (z \in \mathbb{C}).$$

The point $[x]$ in the real projective space $M = P(W)$ can be written by

$$[x] = [\cos t, (\sin t)z],$$

where $z \in S^1$,$t \in \mathbb{R}$, $0 \leq t \leq \dfrac{\pi}{2}$. Now let

$$Y = \{[\cos t, (\sin t)z] \mid z \in S^1,\ 0 \leq t \leq \pi/4\},$$
$$Z = \{[\cos t, (\sin t)z] \mid z \in S^1,\ \pi/4 \leq t \leq \pi/2\}.$$

Then Y is G-homeomorphic to the disk D^2, and Z is G-homeomorphic to the Möbius strip. Let G act on S^1 by $g_0 z = \exp(2\pi\sqrt{-1}/p)z$. Since

$$Y \cap Z = \left\{ \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}z \right] \;\middle|\; z \in S^1 \right\},$$

we define the map $\psi : Y \cap Z \to S^1$ by

$$\psi\left(\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}z \right] \right) = z.$$

Then ψ is clearly a G-homeomorphism. Next we define the G-bundle map $\varphi_Y : \varepsilon_Y(\mathbb{R}^2) \to (\gamma \oplus \gamma)|_Y$ by

$$(b, r_1, r_2) \to (b, r_1 a, r_2 a),$$

where $a = (\cos t, (\sin t)z)$, $0 \leq t \leq \pi/4$, $z \in S^1$, and $b = [a] \in M$. Further we define the G-bundle map $\varphi_Z : \varepsilon_Z(\mathbb{R}^2) \to (\gamma \oplus \gamma)|_Z$ by

$$(b, r_1, r_2) \to (b, (r_1 \cos(p\theta) - r_2 \sin(p\theta))a,\ (r_1 \sin(p\theta) + r_2 \cos(p\theta))a),$$

where $a = (\cos t, (\sin t)z)$, $\pi/4 \leq t \leq \pi/2$, $z = \cos\theta + i \sin\theta \in S^1$, $b = [a] \in M$. Then the canonical framings are

$$\begin{cases} e_1(b) = \varphi_Y(b, 1, 0) \\ e_2(b) = \varphi_Y(b, 0, 1) \end{cases} \quad \text{and} \quad \begin{cases} f_1(b) = \varphi_Z(b, 1, 0) \\ f_2(b) = \varphi_Z(b, 0, 1). \end{cases}$$

Clearly both $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are G-equivariant. From now on, for $b = [a] = [\cos t, (\sin t)z] \in Y \cap Z$, $t = \pi/4$, $z = cos\theta + i\sin\theta$, we will determine the matrix function $A_{\gamma^{\oplus 2}} = [a_{ij}] : Y \cap Z \to O(2)$ such that

$$(f_1(b), f_2(b)) = (e_1(b), e_2(b))A_{\gamma^{\oplus 2}}(b).$$

Easily we obtain

$$A_{\gamma^{\oplus 2}}(b) = \begin{pmatrix} \cos(p\theta) & -\sin(p\theta) \\ \sin(p\theta) & \cos(p\theta) \end{pmatrix} \ (b = [a], a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(\cos\theta + i\sin\theta))).$$

By this form, for any $b \in Y \cap Z$, it is known that $A_{\gamma^{\oplus 2}}(b)$ belongs to $SO(2)$. Hence $A_{\gamma^{\oplus 4}} : Y \cap Z \to SO(4)$ is of the form

$$A_{\gamma^{\oplus 4}}(b) = \begin{pmatrix} A_{\gamma^{\oplus 2}}(b) & 0 \\ 0 & A_{\gamma^{\oplus 2}}(b) \end{pmatrix} \ (b \in Y \cap Z).$$

In the below, we will show that $A_{\gamma^{\oplus 4}}$ is G-homotopic to the constant map with value I. By virtue of the commutative diagram

$$\begin{array}{ccc} Y \cap Z & \xrightarrow{A_{\gamma^{\oplus 4}}} & SO(4) \\ \downarrow & & \downarrow \\ (Y \cap Z)/G & \xrightarrow{A_{\gamma^{\oplus 4}}/G} & SO(4)/G = SO(4), \end{array}$$

we can determine the G-homotopy class $[A_{\gamma^{\oplus 4}}] \in [Y \cap Z, SO(4)]^G$ by the corresponding homotopy class $[A_{\gamma^{\oplus 4}}/G] \in [(Y \cap Z)/G, SO(4)]$. The G-manifold $Y \cap Z$ is identified with S^1 by the map $\psi : Y \cap Z \to S^1$ given above. Thus we have the commutative diagram

$$\begin{array}{ccc} Y \cap Z & \xrightarrow{\psi} & S^1 \\ \downarrow & & \downarrow \\ (Y \cap Z)/G & \xrightarrow{\psi/G} & S^1/G, \end{array}$$

where the orbit space $(Y \cap Z)/G$ is homeomorphic to $S^1/G \cong S^1$. Hence we obtain $[Y \cap Z, SO(4)]^G \cong [(Y \cap Z)/G, SO(4)] \cong \pi_1(SO(4)) \cong \mathbb{Z}/2$. By the equalities

$$\begin{aligned}[A_{\gamma^{\oplus 4}}/G] &= \left[\begin{pmatrix} A_{\gamma^{\oplus 2}}/G & 0 \\ 0 & A_{\gamma^{\oplus 2}}/G \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} A_{\gamma^{\oplus 2}}/G & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ 0 & A_{\gamma^{\oplus 2}}/G \end{pmatrix}\right] \\ &= 2\left[\begin{pmatrix} A_{\gamma^{\oplus 2}}/G & 0 \\ 0 & I \end{pmatrix}\right] \\ &= 0\end{aligned}$$

in $[(Y \cap Z)/G, SO(4)]$, we conclude $[A_{\gamma^{\oplus 4}}] = 0$ in $[Y \cap Z, SO(4)]^G$. That is, $A_{\gamma^{\oplus 4}}$ is G-homotopic to the constant map with value I. By Lemma 4, the G-vector bundle $\gamma^{\oplus 4}$ is isomorphic to the G-vector bundle $\varepsilon_M(\mathbb{R}^4)$.

Next we prove Theorem 3. It follows from Theorem 1 (4) that

$$T(M) \oplus \varepsilon_M(\mathbb{R}) = \mathrm{Hom}(\gamma, \varepsilon_M(V \oplus \mathbb{R})).$$

Thus the following equalities hold:

$$\begin{aligned}(T(M) \oplus \varepsilon_M(\mathbb{R}))^{\oplus 4} &= \mathrm{Hom}(\gamma^{\oplus 4}, \varepsilon_M(V \oplus \mathbb{R})) \\ &= \gamma^{\oplus 4} \otimes \varepsilon_M(V \oplus \mathbb{R}) \\ &= \varepsilon_M(\mathbb{R}^4) \otimes \varepsilon_M(V \oplus \mathbb{R}) \\ &= (\varepsilon_M(\mathbb{R}^4) \otimes \varepsilon_M(V)) \oplus (\varepsilon_M(\mathbb{R}^4) \otimes \varepsilon_M(\mathbb{R})) \\ &= \varepsilon_M(V^{\oplus 4}) \oplus (\varepsilon_M(\mathbb{R}^4)).\end{aligned}$$

This completes the proof.

4. The case where G is of even order

In Theorems 2 and 3, we assume that G is a cyclic group of odd order. This section is devoted to showing these theorems are not valid for G of even order.

Theorem 5. *Let G be a cyclic group of even order $2n$ and V a $2m$-dimensional real G-module with free G-action except the origin, where $m \geq 1$. Let γ_M denote the canonical line bundle over $M = P(\mathbb{R} \oplus V)$. Then for any $k \in \mathbb{N}$ and any real G-modules U, W, the bundle $\gamma_M^{\oplus k} \oplus \varepsilon_M(U)$ is not G-isomorphic to $\varepsilon_M(W)$. In addition, $T(M)^{\oplus k} \oplus \varepsilon_M(U)$ is not G-isomorphic to $\varepsilon_M(W)$.*

Proof. Let G be the cyclic group of order $2n$ with specified generator g_0. Set $a = g_0^n$. Clearly the group $\langle a \rangle$ generated by a has order 2. In order to prove the theorem, it suffices to show

$$\mathrm{Res}^G_{\langle a \rangle}(\gamma_M^{\oplus k} \oplus \varepsilon_M(U)) \not\cong_{\langle a \rangle} \mathrm{Res}^G_{\langle a \rangle}\, \varepsilon_M(W)$$

and

$$\mathrm{Res}^G_{\langle a \rangle}(T(M)^{\oplus k} \oplus \varepsilon_M(U)) \not\cong_{\langle a \rangle} \mathrm{Res}^G_{\langle a \rangle}\, \varepsilon_M(W).$$

By the context in the theorem, we have $ar = r$ and $av = -v$ for $r \in \mathbb{R}$ and $v \in V$.

For the points $x_0 = (1, 0, \ldots, 0)$ and $x_1 = (0, 1, 0, \ldots, 0) \in \mathbb{R} \oplus V$, we see without difficulties that the eigenvalue of the transformation $a : F_{[x_0]}(\gamma_M) \to F_{[x_0]}(\gamma_M)$ is 1, but that of the transformation $a : F_{[x_1]}(\gamma_M) \to F_{[x_1]}(\gamma_M)$ is -1, where $F_{[x_0]}(\gamma_M)$ and $F_{[x_1]}(\gamma_M)$ denote the fibers of γ_M over $[x_0]$ and $[x_1]$, respectively. Thus $\mathrm{Res}^G_{\langle a \rangle}(\gamma_M^{\oplus k} \oplus \varepsilon_M(U))$ is not $\langle a \rangle$-isomorphic to $\mathrm{Res}^G_{\langle a \rangle}\, \varepsilon_M(W)$. The eigenvalues of a on $T_{[x_0]}(M)$ are $-1, \ldots, -1$, whereas the eigenvalues of a on $T_{[x_1]}(M)$ are $-1, 1, \ldots, 1$. Thus $\mathrm{Res}^G_{\langle a \rangle}(T(M)^{\oplus k} \oplus \varepsilon_M(U))$ is not $\langle a \rangle$-isomorphic to $\mathrm{Res}^G_{\langle a \rangle}\, \varepsilon_M(W)$. ∎

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