The tautological ring of the moduli space of curves

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Abstract

The tautological ring of the moduli space of curves $\mathcal{M}_g$ is a subring $R^*(\mathcal{M}_g)$ of the Chow ring $A^*(\mathcal{M}_g)$. The tautological ring can also be defined for other moduli spaces of curves, such as the moduli space of curves of compact type $\mathcal{M}_g^c$ or the moduli space of Deligne-Mumford stable pointed curves $\overline{\mathcal{M}}_{g,n}$. We conjecture and prove various results about the structure of the tautological ring.

In particular, we give two proofs of the Faber-Zagier relations, a large family of relations between the kappa classes in $R^*(\mathcal{M}_g)$ that contains all known relations. The first proof (joint work with R. Pandharipande) uses the virtual geometry of the moduli space of stable quotients developed by Marian, Oprea, and Pandharipande. The second proof (joint work with R. Pandharipande and D. Zvonkine) uses Witten's class on the moduli space of 3-spin curves and the classification of semisimple cohomological field theories by Givental and Teleman. The second proof has the disadvantage that it only proves the image of the Faber-Zagier relations in cohomology, but the advantage that it also proves an extension of the relations to $\overline{\mathcal{M}}_{g,n}$ that was conjectured by the author. These relations on $\overline{\mathcal{M}}_{g,n}$ and their restrictions to smaller moduli spaces of curves seem to describe all known relations in the tautological ring.

We also prove several combinatorial results about the structure of the Gorenstein quotient rings of $R^*(\mathcal{M}_g)$ and $R^*(\mathcal{M}_g^c)$. This includes several new families of relations that are similar to the Faber-Zagier relations, as well as joint work with F. Janda giving formulas for ranks of restricted socle pairings in $R^*(\mathcal{M}_g^c)$.

The appendix presents data obtained by computer calculations of the tautological relations on $\overline{\mathcal{M}}_{g,n}$ and their restrictions to $\mathcal{M}_{g,n}^c$ and $\mathcal{M}_{g,n}^{rt}$ for small values of g and n. The data suggests several new locations in which the tautological ring might not be a Gorenstein ring.

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Contents

Chapter 1

Introduction

The moduli space $\mathcal{M}_g$ of smooth curves of genus g and its compactification $\overline{\mathcal{M}}_g$, the Deligne-Mumford moduli space of stable curves, are fundamental objects in algebraic geometry. Many questions about curves and families of curves can be rephrased in terms of these moduli spaces. The Chow ring $A^*(X)$ of an algebraic variety X encodes information about how its subvarieties intersect each other. In the case $X = \mathcal{M}_g$ or $X = \overline{\mathcal{M}}_g$, the Chow ring is incompletely understood and yet its structure has important ramifications for enumerative questions about curves.

The topic of this thesis is the *tautological ring*, a naturally defined subring $R^*(M)$ of the Chow ring $A^*(M)$ of some moduli space of curves M. The tautological ring should be viewed as the subring consisting of the classes that appear most naturally in geometry. The tautological ring was originally studied by Mumford [26] in the case $M = \mathcal{M}_g$, where the tautological ring is the subring generated by the kappa classes $\kappa_1, \kappa_2, \ldots$. Mumford's conjecture, proved in 2002 by Madsen and Weiss [22], states that these κ classes freely generate the stable cohomology of $\mathcal{M}_g$:

$$\lim_{g\to\infty} H^*(\mathcal{M}_g, \mathbb{Q}) \ = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots],$$

For fixed g, however, there are polynomial relations between the kappa classes. The situation should be viewed as being analogous to that of the cohomology of the Grassmannian varieties: the relations are unstable but have a rich combinatorial structure.

The tautological ring of the moduli space of stable curves $\overline{\mathcal{M}}_g$ is substantially more complicated because there are many more tautological classes than just polynomials in the kappa classes: in particular, there are classes of boundary strata, which are in bijection with certain weighted graphs. In conjunction with this increase in complexity, though, there are also new applications. Many moduli spaces in algebraic geometry map to $\overline{\mathcal{M}}_g$, and relations between tautological classes on $\overline{\mathcal{M}}_g$ pull back to constrain the intersection theory of the larger moduli space. One example of this phenomenon is Gromov-Witten theory, where relations in the tautological ring give rise to topological recursion relations between Gromov-Witten invariants.

1.1 The Faber-Zagier relations

The Faber-Zagier relations are an explicit combinatorial family of finitely many kappa polynomials for each $g \geq 2$ and $d \geq 0$. They are defined (see Section 2.4) in terms of the fundamental power series

$$A(T) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(2n)!}T^n, \ B(T) = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(2n)!}\frac{6n+1}{6n-1}T^n$$

by a procedure involving taking a logarithm, inserting kappa classes, and then exponentiating. Faber and Zagier conjectured that these polynomials are relations in $R^d(\mathcal{M}_g)$. In fact, all known relations in the tautological ring are linear combinations of the Faber-Zagier relations. It is thus natural to conjecture that the Faber-Zagier relations completely determine the structure of $R^*(\mathcal{M}_g)$. This would however contradict Faber's celebrated conjecture that the tautological ring is a Gorenstein ring (see Section 2.3); both conjectures are true for $g \leq 23$ and open for $g \geq 24$, when at most one of them can be true. The Faber-Zagier relations and the series A and B will play a central role in this paper.

1.2 Results

After reviewing basic facts and notation dealing with the tautological ring in Chapter 2, we begin in Chapter 3 by collecting several new results about the tautological ring from a purely combinatorial perspective. This perspective is founded on the existence of explicit isomorphisms

$$\epsilon : R^{g-2}(\mathcal{M}_g) \to \mathbb{Q}, \ \epsilon^c : R^{2g-3}(\mathcal{M}_g^c) \to \mathbb{Q},$$

where here $\mathcal{M}_g^c$ is the moduli space of curves of compact type, an intermediate space between $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$.

In Sections 3.1–3.4, we construct several new families of relations that are similar in form to the Faber-Zagier relations and are consistent with the maps ϵ and ϵ^c. Then in Sections 3.5–3.7 we present joint work with F. Janda [19] that connects the combinatorics of the compact type tautological ring

$R^*(\mathcal{M}_g^c)$ with that of the smooth tautological ring $R^*(\mathcal{M}_g)$. The result is new information about the relations between the kappa classes inside $R^*(\mathcal{M}_g^c)$, building on results of Pandharipande [27].

Chapter 4 presents joint work with R. Pandharipande [28]. We give the first proof that the classical Faber-Zagier relations do in fact hold in the tautological ring of $\mathcal{M}_g$. The geometric source used to construct these relations is the virtual geometry of the moduli space of stable quotients, the theory of which was developed by Marian, Oprea, and Pandharipande [24]. The relations constructed in this way do not naturally arise in the same form as the Faber-Zagier relations. Instead, there are several complicated algebraic transformations that must be applied before the series A and B are visible.

The proof of the Faber-Zagier relations via stable quotients actually establishes a stronger result: the Faber-Zagier relations are restrictions to the interior of tautological relations on $\overline{\mathcal{M}}_g$. In Chapter 5, we first make rigorous the notion of a tautological relation on $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable n-pointed curves of genus g, by defining the strata algebra $\mathcal{S}_{g,n}$; this is a combinatorially defined algebra that surjects onto the tautological ring. Then we conjecture a large family of relations on $\overline{\mathcal{M}}_{g,n}$, constructed as a sum over dual graphs corresponding to boundary strata. When $n = 0$, these relations restrict to the Faber-Zagier relations. It is possible that these conjectured relations generate *all* relations in the tautological rings.

In joint work with R. Pandharipande and D. Zvonkine [30] in Chapter 6, we prove that the conjectured relations of Chapter 5 hold after mapping to cohomology via the map

$$R^*(\overline{\mathcal{M}}_{g,n}) \hookrightarrow A^*(\overline{\mathcal{M}}_{g,n}) \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

This gives a second proof that the Faber-Zagier relations hold, though only in cohomology. The geometric source used in this proof is Witten's class on the moduli space of 3-spin curves. The proof proceeds by constructing a semisimple cohomological field theory based on this class and then using the classification of such theories by Givental and Teleman [36] to give an explicit formula for the theory. Vanishing coming from the purity of Witten's class then gives tautological relations. The series A and B appear in simple fashion as solutions of differential equations.

The same construction can be applied using the r-spin Witten class for $r > 3$. Although the computations become much more complicated, this procedure recovers some of the families of Faber-Zagier-like relations described in Section 3.2.

Finally, we present some data in Appendix A obtained by computer calculations of the relations of Chapters 5 and 6. In particular, we predict several moduli spaces of curves where we expect the tautological ring is not a Gorenstein ring (because our relations are insufficient to cut it down to one).

Chapter 2

Basic facts and notation

2.1 Moduli spaces of curves

All curves will be assumed to be connected. Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves of genus g with n distinct marked points $x_1, \ldots, x_n$ and let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification, the moduli space of stable nodal curves of arithmetic genus g with n distinct nonsingular marked points $x_1, \ldots, x_n$. We will study the intersection theory of these spaces.

The $\overline{\mathcal{M}}_{g,n}$ have two types of natural maps between them: the forgetful morphisms

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

given by forgetting one of the marked points and the gluing morphisms

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

$$\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

given by gluing two marked points together. Together, we call these maps the *tautological morphisms*.

We also define two intermediate moduli spaces lying between $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. The first, the moduli space of curves with rational tails, $\mathcal{M}^{rt}_{g,n}$ is defined to be the preimage of $\mathcal{M}_g$[1] under the composition of forgetful morphisms $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ forgetting all the markings. Unlike $\mathcal{M}_{g,n}$, $\mathcal{M}^{rt}_{g,n}$ is proper over $\mathcal{M}_g$. For the second intermediate moduli space, we must first discuss the stratification of $\overline{\mathcal{M}}_{g,n}$ by topological type.

[1]When $g < 2$, we use $\mathcal{M}_{0,3}$ or $\mathcal{M}_{1,1}$ here instead.

2.1.1 Dual graphs

Given a point $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$, its *dual graph* is defined by replacing each irreducible component of C by a vertex (labeled by its genus), replacing each node of C by an edge connecting the vertices corresponding to the two sides of the node, and for each $i = 1, \ldots, n$ attaching a half-edge with label i to the vertex corresponding to the irreducible component containing x_i. The result is a *stable graph*

$$\Gamma = (\mathrm{V}, \mathrm{H}, \mathrm{L},\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0},\ v : \mathrm{H} \to \mathrm{V},\ \iota : \mathrm{H} \to \mathrm{H})$$

satisfying the following properties:

(i) V is a vertex set with a genus function $\mathrm{g} : V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution ι,

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of ι and endowed with a bijective correspondence with the set of markings $\{1, \ldots, n\}$,

(v) the pair (V, E) defines a *connected* graph,

(vi) the genus of $\Gamma, \mathrm{g}(\Gamma) = \sum_{v \in V} \mathrm{g}(v) + h^1(\Gamma)$, is equal to g.

(vii) for each vertex v, the stability condition holds:

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

where $\mathrm{n}(v)$ is the valence of Γ at v, the number of half-edges incident with v.

We can now define the moduli space of curves of compact type, $\mathcal{M}^c_{g,n}$, as the locus of marked curves $[C, x_1, \ldots, x_n] \in \overline{\mathcal{M}}_{g,n}$ with dual graph that is a tree. We also observe that $\mathcal{M}_{g,n}$ is the locus with dual graph having a single vertex and that $\mathcal{M}^{rt}_{g,n}$ is the locus with dual graph that is a tree with at most one vertex with positive genus, so

$$\mathcal{M}_{g,n} \subseteq \mathcal{M}^{rt}_{g,n} \subseteq \mathcal{M}^c_{g,n} \subseteq \overline{\mathcal{M}}_{g,n},$$

with equality in the first inclusion when $n = 0$ and equality in the second and third inclusions when $g = 0$.

2.2 Tautological classes

For each $\mathcal{M} = \mathcal{M}_{g,n}, \mathcal{M}^{rt}_{g,n}, \mathcal{M}^{c}_{g,n}, \overline{\mathcal{M}}_{g,n}$, the *tautological ring* of $\mathcal{M}$ is a subring $R^*(\mathcal{M}) \subseteq A^*(\mathcal{M})$ of the Chow ring (with rational coefficients). The tautological ring was originally defined by Mumford [26] in the case of $\mathcal{M}_g$, but we follow Faber and Pandharipande [10] and first define the tautological ring of $\overline{\mathcal{M}}_{g,n}$: the tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are simultaneously defined for all g, n to be the smallest $\mathbb{Q}$-subalgebras of the Chow rings that are closed under pushforward[2] via the tautological morphisms described in Section 2.1. The other tautological rings are then defined by restriction.

This definition can be made more explicit: we will describe a set of additive generators for the tautological rings. First, the most basic tautological classes are the cotangent line classes ψ_i. For each marking $i = 1, 2, \ldots, n$, there is a line bundle $\mathbb{L}_i$ on $\overline{\mathcal{M}}_{g,n}$ defined by taking the relative dualizing sheaf of the map $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ forgetting that point: this is the cotangent line to the curve at the marked point x_i. The class $\psi_i \in A^1(\overline{\mathcal{M}}_{g,n})$ is then the first Chern class of $\mathbb{L}_i$. Raising this class to some power and pushing forward along π gives another basic type of tautological class:

$$\kappa_a = \pi_*(\psi^{a+1}) \in A^a(\overline{\mathcal{M}}_{g,n}).$$

The tautological ring of the smooth locus, $R^*(\mathcal{M}_{g,n})$, is simply the subring of polynomials in the ψ_i and κ_a for $1 \leq i \leq n$ and $a > 0$.

There are many more tautological classes on $\overline{\mathcal{M}}_{g,n}$. The most basic new ones are the classes of *boundary strata*, closures of the loci of curves with fixed dual graphs. These classes can be interpreted in terms of gluing maps as follows. Let Γ be any stable graph of genus g with n legs. Define

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)},$$

and let

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n} \tag{2.1}$$

be defined by gluing together marked points at each edge of Γ. Then the image of ξ_Γ is the boundary stratum corresponding to Γ, and thus the class of this boundary stratum is equal to

$$\frac{1}{|\operatorname{Aut}(\Gamma)|}\xi_{\Gamma*}(1)$$

as the degree of ξ_Γ onto its image is $|\operatorname{Aut}(\Gamma)|$.

[2]It turns out that the tautological rings are also closed under pullbacks.

It is well known (see [17] for instance) that if we repeat this construction but take the pushforward of an arbitrary monomial in the kappa and psi classes instead of just 1, then we get a set of additive generators for $R^*(\overline{\mathcal{M}}_{g,n})$. In other words, choose a stable graph Γ with genus g and n legs, and let θ be a product of kappa and psi classes on the components $\overline{\mathcal{M}}_{g_v,n_v}$ of $\overline{\mathcal{M}}_\Gamma$; then take

$$\xi_{\Gamma*}\theta \tag{2.2}$$

as a generator. For a generating set for $R^*(\mathcal{M}^c_{g,n})$ or $R^*(\mathcal{M}^{rt}_{g,n})$, we can simply restrict which graphs Γ we use.

In Section 5.2, we will define a formal algebra $\mathcal{S}_{g,n}$ based on this generator set and view $\mathcal{R}^*(\mathcal{M}^c_{g,n})$ as a quotient of $\mathcal{S}_{g,n}$.

We will implicitly use the following well-known result about the forgetful maps at many times when working with tautological classes.

Lemma 2.1. *Consider the following commutative square of forgetful maps:*

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+k+m} \arrow[r, "P_k"'] \arrow[d, "P_m"] & \overline{\mathcal{M}}_{g,n+m} \arrow[d, "p_m"'] \\
\overline{\mathcal{M}}_{g,n+k} \arrow[r, "p_k"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

The relation $(p_k)^*(p_m)_* = (P_m)_*(P_k)^*$ holds in cohomology.

Proof. Let X be the fiber product of p_m and p_k, with maps

$$a : X \to \overline{\mathcal{M}}_{g,n+m}\ , \quad b : X \to \overline{\mathcal{M}}_{g,n+k}\ , \quad f : \overline{\mathcal{M}}_{g,n+k+m} \to X\ .$$

Then $(p_k)^*(p_m)_* = b_*a^*$ is immediate, and also

$$(P_m)_*(P_k)^* = (b_*f_*)(f^*a^*) = b_*(f_*f^*)a^* = b_*a^*$$

by birationality of f. $\square$

2.3 Faber's conjectures

Faber [9] made conjectures which completely determine the structure of the tautological ring of $\mathcal{M}_g$ (and later made analogous conjectures for $\mathcal{M}^{rt}_{g,n}, \mathcal{M}^{c}_{g,n}$, and $\overline{\mathcal{M}}_{g,n}$).

Conjecture 2.2 (Faber's conjectures)**.** *For any $g \geq 2$, $R^*(\mathcal{M}_g)$ has the following structure:*

1. *$R^d(\mathcal{M}_g) = 0$ for $d > g-2$;*
2. *$R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$;*
3. *the multiplication pairing*

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

is perfect.

The first two parts of Conjecture 2.2 were proven by Looijenga [21] and Faber [8]. The third part, which states that the tautological ring is Gorenstein, was verified by Faber for $g \leq 23$ by constructing sufficiently many relations. The conjecture remains open for $g \geq 24$, where all known methods for constructing relations have failed to produce enough relations to make the ring Gorenstein. In particular, the Faber-Zagier relations described in the next section are not enough to render the tautological ring Gorenstein for $g \geq 24$.

Faber also conjectured a remarkable formula for the proportionalities between kappa polynomials of degree $g-2$ given by the isomorphism $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$. The formula is most easily stated in terms of an alternative basis for the kappa polynomials, not the monomial basis.

Suppose $\alpha_1, \alpha_2, \ldots, \alpha_m$ are nonnegative integers, and let $\pi : \overline{\mathcal{M}}_{g,m} \to \overline{\mathcal{M}}_g$ be the map forgetting m markings. Then there is the basic pushforward formula

$$\pi_*(\psi_1^{\alpha_1+1} \cdots \psi_m^{\alpha_m+1}) = \sum_{\sigma \in S_m} \kappa_\sigma,$$

where the sum on the right runs over all elements of the symmetric group S_m and

$$\kappa_\sigma = \prod_{c \text{ cycle of } \sigma} \kappa_{\alpha_c},$$

where α_c is the sum of the α_i over all i in the cycle c.

It will be convenient to have notation for these kappa polynomials: we write

$$\overbrace{\kappa_{\alpha_1}\cdots\kappa_{\alpha_m}} := \sum_{\sigma\in S_m} \kappa_\sigma. \tag{2.3}$$

We view $\overbrace{\cdot}$ as an automorphism of the $\mathbb{Q}$-vector space of formal polynomials in $\kappa_0, \kappa_1, \ldots$. Note that it is not a ring homomorphism. When applied to monomials in $\kappa_1, \kappa_2, \ldots$, this automorphism gives an alternative basis for the kappa polynomials, which we will call the *pushforward kappa polynomial basis*.

Faber's proportionality formula for $R^{g-2}(\mathcal{M}_g)$ is then

$$\overbrace{\kappa_{\alpha_1}\cdots\kappa_{\alpha_m}} = \frac{(2g-3+m)!(2g-1)!!}{(2g-1)!\prod_{i=1}^m(2\alpha_i+1)!!}\kappa_{g-2}.$$

We define the *socle evaluation* $\epsilon : R^{g-2}(\mathcal{M}_g) \to \mathbb{Q}$ by $\epsilon(x\kappa_{g-2}) = x$, so we can write

$$\epsilon\left(\overbrace{\kappa_{\alpha_1}\cdots\kappa_{\alpha_m}}\right) = \frac{(2g-3+m)!(2g-1)!!}{(2g-1)!\prod_{i=1}^m(2\alpha_i+1)!!}. \tag{2.4}$$

This formula is a consequence of the Virasoro constraints for surfaces, as explained in [13]. The socle evaluation formula for $R^{g-2+n}(\mathcal{M}_{g,n}^{rt})$ is similar.

Let $\mathcal{I}_g \subset R^*(\mathcal{M}_g)$ be the ideal given by the kernel of the pairing in Faber's conjecture. Define the *Gorenstein quotient*

$$\mathrm{Gor}^*(\mathcal{M}_g) = \frac{R^*(\mathcal{M}_g)}{\mathcal{I}_g}\ .$$

If Faber's conjecture is true for g, then $\mathcal{I}_g = 0$ and $\mathrm{Gor}^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$.

The Gorenstein quotient $\mathrm{Gor}^*(\mathcal{M}_g)$ is completely determined by the κ evaluations. The ring $\mathrm{Gor}^*(\mathcal{M}_g)$ can therefore be studied as a purely algebro-combinatorial object, as we will do in Chapter 3.

It turns out that the situation with $\mathcal{M}_{g,n}^c$, the moduli space of curves of compact type, is quite similar. The socle evaluation formula is

$$\epsilon^c\left(\overbrace{\kappa_{\alpha_1}\cdots\kappa_{\alpha_m}}\psi_1^{\beta_1}\cdots\psi_n^{\beta_n}\right) = \frac{(2g-3+m+n)!}{\prod_{i=1}^m(\alpha_i+1)!\prod_{i=1}^n\beta_i!}, \tag{2.5}$$

where $\epsilon^c : R^{2g-3+n}(\mathcal{M}_{g,n}^c) \to \mathbb{Q}$ is normalized by $\epsilon^c(\kappa_{,g-3+n}) = 1$.

In both of these cases, the only unresolved part of Faber's conjectures is the question of whether the tautological ring is Gorenstein. It is also natural to state these conjectures for $\overline{\mathcal{M}}_{g,n}$, but the

Gorenstein property is known to fail in this case by work of Petersen and Tommasi [31].

In Appendix A, we will discuss computations that suggest locations where the Gorenstein conjecture might fail for $\mathcal{M}_{g,n}^{rt}$ or $\mathcal{M}_{g,n}^{c}$ (analogously to the $g = 24$ case for $\mathcal{M}_g$).

2.4 The FZ relations

Faber and Zagier conjectured a remarkable set of relations among the κ classes in $R^*(\mathcal{M}_g)$.

We will first write the Faber-Zagier relations as they have customarily been described, and then we will give a couple of equivalent descriptions. We will initially use the following notation. Let the variable set

$$\mathbf{p} = \{\ p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots\ \}$$

be indexed by positive integers *not* congruent to 2 modulo 3. Define power series

$$A(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i, B(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i.$$

The power series A and B will play a fundamental role in this paper. Here we use them to form the series

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \cdots)A(t) + (p_1 + tp_4 + t^2 p_7 + \cdots)B(t)\ .$$

Since Ψ has constant term 1, we may take the logarithm. Define the constants $C_r^{\mathsf{FZ}}(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ t^r \mathbf{p}^\sigma\ .$$

The above sum is over all partitions σ of size $|\sigma|$ which avoid parts congruent to 2 modulo 3. The empty partition is included in the sum. To the partition $\sigma = 1^{n_1} 3^{n_3} 4^{n_4} \cdots$, we associate the monomial $\mathbf{p}^\sigma = p_1^{n_1} p_3^{n_3} p_4^{n_4} \cdots$. Let

$$\gamma^{\mathsf{FZ}} = \sum_{\sigma} \sum_{r=0}^{\infty} C_r^{\mathsf{FZ}}(\sigma)\ \kappa_r t^r \mathbf{p}^\sigma\ .$$

For a series $\Theta \in \mathbb{Q}[\kappa][[t, \mathbf{p}]]$ in the variables κ_i, t, and p_j, let $[\Theta]_{t^d \mathbf{p}^\sigma}$ denote the coefficient of $t^d \mathbf{p}^\sigma$ (which is a polynomial in the κ_i).

Then the Faber-Zagier (FZ) relations are

$$\big[\exp(-\gamma^{\mathsf{FZ}})\big]_{t^d \mathbf{p}^\sigma} = 0 \tag{2.6}$$

in $R^*(\mathcal{M}_g)$ for $3d \geq g+1+|\sigma|$ and $d \equiv g+1+|\sigma| \mod 2$.

The dependence upon the genus g in the Faber-Zagier relations occurs in the inequality, the modulo 2 restriction, and via $\kappa_0 = 2g-2$. For a given genus g and codimension r, the FZ construction provides only *finitely* many relations. While not immediately clear from the definition, the $\mathbb{Q}$-linear span of the Faber-Zagier relations determines an ideal in $\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]$ — the reason for this is the simple identity

$$\kappa_a \,\mathsf{FZ}(g,d,\sigma) = \mathsf{FZ}(g,d+a,\sigma \sqcup \{3a\}) - \sum_{\tau} \mathsf{FZ}(g,d+a,\tau),$$

where $\mathsf{FZ}(g,d,\sigma)$ denotes the left hand side of (2.6) and the sum runs over partitions τ formed by increasing one part of σ by $3a$. Using this identity, we see that it is sufficient to take only those FZ relations given by σ with no parts divisible by 3 if we just want a set of generators for the FZ ideal.

Faber and Zagier conjectured the relations (2.6) from a concentrated study of the Gorenstein quotient $\mathrm{Gor}^*(\mathcal{M}_g)$. In Chapter 4, we'll give a proof that these relations hold, and in Chapter 6 we'll give a second proof that these relations hold in cohomology. It seems plausible that the FZ relations give *all* relations in $R^*(\mathcal{M}_g)$.

2.4.1 An alternate form for the FZ relations

It will be convenient to introduce the notation

$$\left\{\sum_a c_a t^a\right\}_\kappa := \sum_a c_a \kappa_a t^a.$$

In other words, $\{\cdot\}_\kappa$ multiplies the coefficient of t^a by κ_a for each a. The FZ relations can then be written in a more self-contained manner:

$$\big[\exp(-\{\log((1+tp_3+t^2p_6+t^3p_9+\cdots)A(t)+(p_1+tp_4+t^2p_7+\cdots)B(t))\}_\kappa)\big]_{t^d\mathbf{p}^\sigma} = 0.$$

We now apply a simple combinatorial lemma to switch to the pushforward kappa polynomial basis of (2.3).

Lemma 2.3. *For any power series $X(t)$ with constant term 0,*

$$\exp(-\{\log(1-X)\}_\kappa) = \overbrace{\exp(\{X\}_\kappa)}.$$

Proof. The left hand side expands to

$$\exp\left(\sum_{n\geq 1}\frac{1}{n}\{X^n\}_\kappa\right) = \sum_\mu \frac{1}{|\operatorname{Aut}(\mu)|\prod_{i\in\mu} i}X^{\{\mu\}},$$

where the sum runs over all partitions and

$$X^{\{\mu\}} := \prod_{i\in\mu}\{X^i\}_\kappa.$$

The right hand side is equal to this because

$$\overbrace{\{X\}_\kappa^n} = \sum_{\sigma\in S_n} X^{\{c(\sigma)\}},$$

where $c(\sigma)$ denotes the cycle type of the permutation σ, and the number of permutations with cycle type μ is precisely

$$\frac{|\mu|!}{|\operatorname{Aut}(\mu)|\prod_{i\in\mu} i}.$$

□

This gives a new expression for the FZ relations: up to sign, they are

$$\left[\overbrace{\exp(\{1-A(t)\}_\kappa)\{B(t)\}_\kappa^{\sigma_1}\{tA(t)\}_\kappa^{\sigma_3}\{tB(t)\}_\kappa^{\sigma_4}\{t^2A(t)\}_\kappa^{\sigma_6}\cdots}\right]_{t^d} = 0, \tag{2.7}$$

where here σ_i denotes the number of parts of size i in σ.

We will see a different reformulation of the FZ relations in Section 4.6.2.

2.5 The structure of $\kappa^*(\mathcal{M}_{g,n}^c)$

The full tautological ring of $\mathcal{M}_{g,n}^c$, the moduli space of curves of compact type, is quite complicated because it involves psi classes and boundary strata. Unlike $\mathcal{R}^*(\mathcal{M}_g)$, the compact type tautological ring $\mathcal{R}^*(\mathcal{M}_{g,n}^c)$ is not a quotient of $\mathbb{Q}[\kappa_1,\kappa_2,\ldots]$.

However, the subring generated by the kappa classes, the *kappa ring* $\kappa^*(\mathcal{M}_{g,n}^c)$, is quite well understood due to work of Pandharipande [27]. We summarize some of his results in this section.

First, he proves that if a kappa polynomial is zero in $\kappa^*(\mathcal{M}_{0,g+2n}^c)$, then it is zero in $\kappa^*(\mathcal{M}_{g,n}^c)$.

This defines surjective ring homomorphisms

$$\iota_{g,n} : \kappa^*(\mathcal{M}^c_{0,2g+n}) \to \kappa^*(\mathcal{M}^c_{g,n}),$$

which he proves are isomorphisms for $n \geq 1$ or $n = 0$ and degree at most $g - 2$.

He also computes the betti numbers of the kappa ring whenever $n > 0$: if $D = 2g - 3 + n$ is the socle degree, then

$$\dim_{\mathbb{Q}} \kappa^d(\mathcal{M}^c_{g,n}) = |P(d, D + 1 - d)|,$$

the number of partitions of d of length at most $D + 1 - d$.

Finally, he proves a partial Gorenstein property when $n > 0$: a kappa polynomial F of degree d is a relation in $\kappa^*(\mathcal{M}^c_{g,n})$ if and only if it lies in the kernel of the restricted socle pairing

$$\kappa^d(\mathcal{M}^c_{g,n}) \times R^{D-d}(\mathcal{M}^c_{g,n}) \to \mathbb{Q},$$

i.e. if and only if it holds in the Gorenstein quotient $\mathrm{Gor}^*(\mathcal{M}^c_{g,n})$.

Chapter 3

The combinatorics of the socle evaluations

3.1 Consistency of the FZ relations with the socle evaluation

In this chapter, we will attempt to understand the combinatorics of the socle evaluation formulas (2.4) and (2.5) better. For the most part, we will not be concerned with which relations are true in the tautological ring, but rather with which relations are true in the Gorenstein quotients $\mathrm{Gor}^*(\mathcal{M}_g)$.

We begin by giving a short proof that the FZ relations hold in $\mathrm{Gor}^*(\mathcal{M}_g)$. This is a fact that was certainly known to Faber and Zagier, but we are not aware of a proof in the literature.

Theorem 3.1. *The* FZ *relations hold in* $\mathrm{Gor}^*(\mathcal{M}_g)$ *for any* $g \geq 2$*.*

Proof. By definition, a kappa polynomial of degree d is zero in $\mathrm{Gor}^*(\mathcal{M}_g)$ if and only if all multiples of it of degree $g-2$ are zero in $R^{g-2}(\mathcal{M}_g)$. Since the FZ relations are additive generators for an ideal, this means that they hold in $\mathrm{Gor}^*(\mathcal{M}_g)$ if and only if the FZ relations of degree $d = g-2$ are zero in $R^{g-2}(\mathcal{M}_g)$.

These relations are parametrized by partitions σ of size $3(g-2) - g - 1 - 2e = 2g - 7 - 2e$ for some nonnegative integer e. We can compute the socle evaluation of these relations by combining our final interpretation of FZ (2.7) with the socle evaluation formula (2.4). Some additional notation will be useful here. If $C(t) = \sum_n c_n t^n$ is a power series in t, define a power series $\widehat{C}(t)$ in $\sqrt{t}$ by

$$\widehat{C}(t) = \sum_n \frac{c_n}{(2n+1)!!} t^{n+\frac{1}{2}}.$$

Then if $C_1, C_2, \ldots, C_m$ are power series, we have

$$\epsilon\left(\left[\overbrace{\{C_1\}_\kappa \cdots \{C_m\}_\kappa}\right]_{t^{g-2}}\right) = \frac{(2g-3+m)!(2g-1)!!}{(2g-1)!}\left[\left(\frac{\widehat{C}_1}{\sqrt{t}}\right)\cdots\left(\frac{\widehat{C}_m}{\sqrt{t}}\right)\right]_{t^{g-2}},$$

where ϵ is the socle evaluation map. Applying this identity, we see that the FZ relations corresponding to σ will vanish if and only if

$$\left[\left(1-\frac{\widehat{1-A}}{\sqrt{t}}\right)^{-2g+2-\ell(\sigma)}\left(\frac{\widehat{B}}{\sqrt{t}}\right)^{\sigma_1}\left(\frac{\widehat{tA}}{\sqrt{t}}\right)^{\sigma_3}\cdots\right]_{t^{g-2}} = 0.$$

Collecting the powers of t and writing $|\sigma| = 2g-7-2e$, we have the following identity left to prove:

$$\left[\widehat{A}^{-5-2e}\left(\frac{\widehat{B}}{\widehat{A}^2}\right)^{\sigma_1}\left(\frac{\widehat{tA}}{\widehat{A}^4}\right)^{\sigma_3}\left(\frac{\widehat{tB}}{\widehat{A}^5}\right)^{\sigma_4}\left(\frac{\widehat{t^2A}}{\widehat{A}^7}\right)^{\sigma_6}\cdots\right]_{t^{-1}} = 0, \tag{3.1}$$

where σ is any partition of odd length with no part congruent to 2 mod 3 and e is a nonnegative integer.

There are a couple of ways of going about proving this identity. A key observation is that the series $\widehat{A}$ and $\widehat{B}$ actually belong to an algebraic extension of $\mathbb{Q}(t)$:

$$\widehat{A} = \frac{1}{4\sqrt{6}}\sin\left(\frac{2}{3}\sin^{-1}(6\sqrt{6}\sqrt{t})\right),$$

$$\widehat{B} = -\frac{1}{8\sqrt{6}}\sin\left(\frac{4}{3}\sin^{-1}(6\sqrt{6}\sqrt{t})\right).$$

These identities are special cases of the basic Chebyshev polynomial identity

$$\frac{1}{r}\sin(r\sin^{-1}(t)) = \frac{t}{1!} + \frac{(1^2-r^2)t^3}{3!} + \frac{(1^2-r^2)(3^2-r^2)t^5}{5!} + \cdots$$

specialized to rational values of r.

Also, observe that $2\frac{d}{dt}\widehat{tC} = \widehat{C}$. This means that $\widehat{t^kA}$ and $\widehat{t^kB}$ are given by repeatedly integrating the two algebraic series above. The result is that all the series appearing in (3.1) are linear combinations of $\sin(m\theta)$ for $m \in \mathbb{N}$ and

$$\theta = \frac{2}{3}\sin^{-1}(6\sqrt{6}\sqrt{t}).$$

After rewriting everything in terms of $\sin(\theta)$ and $\cos(\theta)$ and using the parity condition on $|\sigma|$, we are left with the identity

$$\left[\frac{\cos(\theta)}{\sin^{6+2e}(\theta)}\right]_{t^{-1}} = 0.$$

Changing variables to $u = \sin(\theta)$ and using $dt = \frac{\sin(3\theta)}{108\cos(\theta)}du$, this is equivalent to

$$\left[\frac{\sin(3\theta)}{u^{6+2e}}\right]_{u^{-1}} = 0,$$

which is true because $\sin(3\theta)$ is a cubic polynomial in u. $\square$

3.2 Conjectural relations of FZ type on $R^*(\mathcal{M}_g)$

It is natural to ask how unique the FZ relations are. There are several visible parameters in the definition of the FZ relations: the condition on the parts of the partition σ, the inequality and congruence relating $|\sigma|, g$, and d, and the power series A and B. Are there other choices for these parameters that also give true relations in the tautological ring?

Experimentally, it seems that there is exactly one tautological relation in $R^d(\mathcal{M}_g)$ when $3d - g - 1 = 0$ or 1, and the A and B power series are clearly visible in these unique relations. Because of this, it seems impossible to find a different "FZ-like" family of relations that also starts in as low codimension as the FZ relations. However, there is plenty of room to find FZ-like families of relations that start in higher codimension.

We now describe precisely what we mean by an FZ-like family. Suppose r is a positive integer, and let $J \subseteq \{0, \ldots, r-1\}$ be some subset representing residues mod r, with $0 \in J$. Let $C_j \in \mathbb{Q}(t)$ for $j \in J$ be power series, with C_0 having constant term 1, and then define $C_{j+kr} = t^k C_j$ for any $j \in J$ and $k > 0$. Also pick positive integers s and a and an integer b. Then we say that the relations

$$\Big[\overbrace{\exp(\{1 - C_0(t)\}_\kappa) \prod_{\substack{n>0 \\ n \bmod r \in J}} \{C_n(t)\}_\kappa^{\sigma_n}}\Big]_{t^d} = 0, \tag{3.2}$$

for σ a partition only containing parts congruent to an element of J mod r and with g, σ, d satisfying

$$rd \geq ag + b + |\sigma|, \qquad rd \equiv ag + b + |\sigma| \pmod{s}$$

are a *family of* FZ *type*.

Thus the usual FZ relations are given by taking $r = 3, J = \{0, 1\}, C_0 = A, C_1 = B, s = 2, a = b = 1$. We are interested in finding other families of FZ type that give true relations in the Gorenstein quotient.

A family of FZ type additively generates an ideal just as the usual FZ relations do, so the same approach as used in the proof of Theorem 3.1 can be used here to translate the Gorenstein condition into a condition about residues of power series. The resulting condition is

$$\left[\widehat{C}_0^{-2g+2+\ell(\sigma)} \prod_{\substack{n>0 \\ n \bmod r \in J}} \widehat{C}_n^{\sigma_n}\right]_{t^{-1}} = 0 \tag{3.3}$$

for $|\sigma| \leq (r-a)g - 2r - b$ and congruent mod s.

There are now many choices of parameters for our family of FZ type that satisfy this condition. We describe one particularly nice infinite series of such choices.

Theorem 3.2. *For any $r \geq 3$, the family of FZ type defined as follows gives true relations in the Gorenstein quotient $\mathrm{Gor}^*(\mathcal{M}_g)$: take $r = r, J = \{0, 1, \ldots, r-2\}, s = 2, a = r-2,$ and $b = 4 - r$. For $j = 0, 1, \ldots, r-2$, take*

$$C_j = \sum_{n \geq 0} \prod_{i=1}^{n} \left((2i-1)^2 - \left(\frac{2j+2}{r}\right)^2\right) \frac{t^n}{n!}.$$

Proof. We just need to check (3.3). As in the proof of Theorem 3.1 (which was just the case $r = 3$), the $\widehat{C}_n$ are algebraic functions; up to normalization, we have

$$\widehat{C}_j = \sin\left(\frac{2j+2}{r}\sin^{-1}(\sqrt{t})\right).$$

If we write

$$\theta = \frac{2}{r}\sin^{-1}(\sqrt{t}),$$

then the identity to be proven becomes

$$\left[\frac{\cos(\theta)}{\sin^{r+3+2e}(\theta)}\right]_{t^{-1}} = 0$$

if r is odd, and

$$\left[\frac{1}{\sin^{r+2+2e}(\theta)}\right]_{t^{-1}} = 0$$

if r is even. In either case, the identity follows from substituting $u = \sin(\theta)$ as before. $\square$

In Section 6.5, we will prove that the relations of Theorem 3.2 are true in cohomology. This is weaker than the following conjecture, which however seems very likely.

Conjecture 3.3. *The relations of Theorem 3.2 hold in* $R^*(\mathcal{M}_g)$.

The methods of Section 6.5 also prove in cohomology other families of relations of FZ type given by more complicated series C_j. Because of this, a classification of families of FZ type that give relations that hold in the Gorenstein quotient would be useful.

Question. *Which families of* FZ *type give relations that hold in* $\mathrm{Gor}^*(\mathcal{M}_g)$*?*

3.3 The FZ_2 relations

The family of FZ type described by taking $r = 4$ in Theorem 3.2 are especially interesting. These relations begin at $d = \lceil \frac{g}{2} \rceil$. Since this is precisely where the usual FZ relations seem to begin to miss some relations that are true in the Gorenstein quotient, it is a priori possible that these relations might not be contained in the span of the usual FZ relations. However, computations with $g \geq 24$ indicate that this is not actually the case. For even $g \leq 800$, we have checked that the unique relation in this family of degree $\frac{g}{2}$ is a linear combination of FZ relations.

It is convenient to discard the relations in this family coming from partitions σ with at least one odd part. This corresponds to removing 1 from the set J of residues mod 4. If we then divide out by 2 and renormalize, we are left with the family of FZ type

$$\Big[\overbrace{\exp(\{1 - A_2(t)\}_\kappa)\{B_2(t)\}_\kappa^{\sigma_1}\{tA_2(t)\}_\kappa^{\sigma_2}\{tB_2(t)\}_\kappa^{\sigma_3}\{tA_2(t)\}_\kappa^{\sigma_4}\cdots}\Big]_{t^d} = 0,$$

where σ is any partition satisfying $2d \geq g + |\sigma|$ and A_2, B_2 are the hypergeometric series

$$A_2(t) = \sum_{i=0}^{\infty} \frac{(4i)!}{(2i)!\ i!} t^i, \ \ B_2(t) = \sum_{i=0}^{\infty} \frac{(4i)!}{(2i)!\ i!} \frac{4i+1}{4i-1} t^i.$$

We call this family FZ_2 due to its relative prominence among families of FZ type and its great similarity to the classical FZ relations.

Although there are many fewer FZ_2 relations than FZ relations because they start at $\frac{g}{2}$ instead of $\frac{g+1}{3}$, they are more tractable to analyze because the parameters do not have a parity condition.

Theorem 3.4. *The quotient* Q^* *of* $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ *by the genus* g FZ_2 *relations has degree* d *piece* Q^d *satisfying*

$$\dim_{\mathbb{Q}} Q^d \leq |P(d, g-1-d)|$$

for any $d \geq 0$.

Proof. First, define power series $D_0, D_1, \dots$ in terms of A_2 and B_2 by

$$D_0 = A_2, D_1 = B_2 + A_2, D_2 = tA_2, D_3 = tB_2, D_4 = t^2A_2, \dots$$

Then write

$$R(\sigma, d) = \left[\overbrace{\exp(\{1 - D_0(t)\}_\kappa)\{D_1(t)\}_\kappa^{\sigma_1}\{D_2(t)\}_\kappa^{\sigma_2}\{D_3(t)\}_\kappa^{\sigma_3}\cdots}\right]_{t^d}. \tag{3.4}$$

for any partition σ and nonnegative integer d.

It is easy to see that the relations

$$R(\sigma, d) = 0 \text{ for } 2d \geq g + |\sigma|$$

are equivalent to the FZ_2 relations. The only change is that we have replaced $B_2(t)$ with $D_1(t) = B_2(t) + A_2(t)$; our reason for doing this is that D_1 now has constant term zero, which means that κ_0 does not appear in the definition of $R(\sigma, d)$.

For any two partitions σ, τ, let $K(\sigma, \tau)$ be the coefficient of $\overbrace{\kappa^\tau}$ in $R(\sigma, d)$. Then define a matrix M with rows and columns indexed by partitions of d by setting $M_{\sigma\tau} = K(\sigma_-, \tau)$, where σ_- is the partition formed by reducing each part of σ by one and discarding the parts of size zero.

Claim. *M is invertible.*

Before proving the claim, we note that the claim implies the theorem. For if $|\sigma| = d$ and $\ell(\sigma) \geq g - d$, then $|\sigma_-| \leq d - (g - d) = 2d - g$ and thus $R(\sigma_-, d)$ is a FZ_2 relation in genus g. The claim implies that these FZ_2 relations are linearly independent, so the quotient of the space of degree d kappa polynomials by these relations has dimension at most the number of partitions of d of length at most $g - 1 - d$, as desired.

We now prove that M is invertible. We will do this by constructing another matrix A of the same size and proving that the product MA is upper-triangular with respect to any ordering of the partitions of d that places partitions containing more parts of size one after partitions containing fewer parts of size one.

First, we can easily compute the coefficient $K(\sigma, \tau)$ as a sum over injections from the set of parts of σ to the set of parts of τ describing which factors in (3.4) account for which κ classes. We write such an injection as $\phi : \sigma \hookrightarrow \tau$. The parts of τ that are not in the image of this injection are produced by the exponential factor. The result is

$$K(\sigma,\tau)=\frac{(-1)^{\ell(\tau)-\ell(\sigma)}}{|\operatorname{Aut}(\tau)|}\sum_{\phi:\sigma\hookrightarrow\tau}\prod_{i\overset{\phi}{\mapsto}j}[D_i]_{t^j}\prod_{j\in(\tau\setminus\phi(\sigma))}[D_0]_{t^j}. \tag{3.5}$$

Now for any partitions of the same size τ and μ, set

$$A_{\tau\mu}=\sum_{\substack{\psi:\tau\to\mu\\ \text{refinement}}}\frac{|\operatorname{Aut}(\tau)|}{\prod_{k\in\mu}|\operatorname{Aut}(\psi^{-1}(k))|}\prod_{k\in\mu}(\ell(\psi^{-1}(k))+2k+1)!\prod_{j\in\tau}\frac{1}{(2j+1)!!},$$

where the sum runs over all partition refinements $\psi:\tau\to\mu$, i.e. functions from the set of parts of τ to the set of parts of μ such that the preimage of each part k of μ is a partition of k.

We now can factor the sums appearing in the entries of the product matrix MA:

$$\sum_{\tau}K(\sigma,\tau)A_{\tau,\mu}=\sum_{\xi:\sigma\to\mu}\prod_{\substack{k\in\mu\\ \sigma'=\xi^{-1}(k)}}\left(\sum_{\tau'}K(\sigma',\tau')A_{\tau',(k)}\right), \tag{3.6}$$

where $\xi:\sigma\to\mu$ means a function from the set of parts of ξ to the set of parts of μ.

Thus we just need to understand the sum

$$\sum_{\tau}K(\sigma,\tau)A_{\tau,(k)}.$$

When this is expanded using formula (3.5) for the $K(\sigma,\tau)$ and the definition of $A_{\tau\mu}$, the result is

$$(-1)^{\ell(\sigma)}\sum_{\tau\vdash k}\sum_{\phi:\sigma\hookrightarrow\tau}\frac{(-1)^{\ell(\tau)}(\ell(\tau)+2k+1)!}{|\operatorname{Aut}(\tau)|}\prod_{j\in\tau}\frac{1}{(2j+1)!!}\prod_{i\overset{\phi}{\mapsto}j}[D_i]_{t^j}\prod_{j\in(\tau\setminus\phi(\sigma))}[D_0]_{t^j}.$$

We now add back in the formal variable t to keep track of the size of τ, and factor based on the values of the images of the parts of σ under ϕ. The series

$$\widehat{D}_i=\sum_{j\geq 1}[D_i]_{t^j}\frac{t^{j+\frac{1}{2}}}{(2j+1)!!}.$$

appear when doing this.

After removing some nonzero scaling factors, the result is

$$\left[\widehat{D}_0^{-2k-2}\prod_{i\in\sigma}\frac{\widehat{D}_i}{\widehat{D}_0}\right]_{t^{-1}}.$$

This is very close to the expressions appearing in the proof of Theorem 3.2 when $r = 4$, since we have D_n equal (up to scaling) to the C_{2n} described in the theorem for every $n \neq 1$, and $D_1 = C_2 - C_0$. The identity giving there that these relations are true in the Gorenstein quotient is equivalent to the statement that

$$\sum_{\tau} K(\sigma, \tau) A_{\tau,(k)} = 0$$

whenever $|\sigma| < k - 1$. Moreover, we can easily compute that this sum is nonzero when $\sigma = (k-1)$, as this is just a matter of checking that

$$\left[\frac{1}{\sin^4 \theta}\right]_{t^{-1}} \neq 0$$

for $\theta = \frac{1}{2}\sin^{-1}(\sqrt{t})$.

We now return to the matrix MA. Suppose that σ and τ are partitions of d containing S and T ones respectively, and such that $(MA)_{\sigma\tau} \neq 0$. By the identity (3.6) and the discussion above, this means that there exists some function $\xi : \sigma_- \to \tau$ such that the preimage of each part $k \in \tau$ is a partition of size at least $k - 1$. Thus we have

$$|\sigma| - \ell(\sigma) = |\sigma_-| \geq |\tau| - \ell(\tau).$$

Since $|\sigma| = |\tau| = d$, this means that $\ell(\tau) \geq \ell(\sigma)$. Moreover, comparing lengths of the partitions gives that

$$\ell(\sigma) - S = \ell(\sigma_-) \geq \ell(\tau) - T.$$

Adding these inequalities, we have that $T \geq S$. If equality holds, then the preimage of each part $k \in \tau$ must be of size $k - 1$ and have length one if $k > 1$, which means that $\tau = \sigma$. Thus the matrix MA is triangular. The nonvanishing of the diagonal entries follows from the nonvanishing of

$$\sum_{\tau} K((k-1), \tau) A_{\tau,(k)}$$

described above. $\square$

As we will see in Section 6.5, the FZ_2 relations hold in cohomology, so we have the following corollary:

Corollary 3.5. *The tautological cohomology $RH^*(\mathcal{M}_g)$ has betti numbers satisfying*

$$\dim_{\mathbb{Q}} RH^d(\mathcal{M}_g) \leq |P(d, g-1-d)|$$

for any $d \geq 0$.

These bounds are not sharp for most ranges of d, but we are not aware of stronger proven bounds.

3.4 Relations of FZ type on $\kappa^*(\mathcal{M}_{g,n}^c)$

It turns out that relations in the compact type kappa ring $\kappa^*(\mathcal{M}_{g,n}^c)$ studied by Pandharipande [27] can also be formulated in the FZ framework used in $R^*(\mathcal{M}_g)$. It will be convenient to let $D = 2g - 3 + n$ be the socle degree; recall that for $n > 0$ the structure of the kappa ring only depends on D.

Consider the family of FZ type (as in Section 3.2)

$$\Big[\overbrace{\exp(\{1 - A_c(t)\}_\kappa)\{B_c(t)\}_\kappa^{\sigma_1}\{tA_c(t)\}_\kappa^{\sigma_2}\{tB_c(t)\}_\kappa^{\sigma_3}\{t^2A_c(t)\}_\kappa^{\sigma_4}\cdots}\Big]_{t^d} = 0,$$

where σ is any partition satisfying $2d \geq D + 2 + |\sigma|$ and A_c, B_c are the series

$$A_c(t) = \sum_{i=0}^{\infty}(2i-1)!!\ t^i, B_c(t) = 1.$$

We call this family FZ^c. We can prove that these relations completely determine the structure of the kappa ring when there is at least one marked point.

Theorem 3.6. *The FZ^c relations hold in $\kappa^*(\mathcal{M}_{g,n}^c)$ for any $g, n \geq 0$ with $2g - 3 + n = D \geq 0$. Moreover, if $n > 0$ then the span of the FZ^c relations is the entire space of relations between the kappa classes.*

Proof. The proof makes heavy use of the results of Pandharipande [27] presented in Section 2.5.

For the first part of the theorem, it suffices to prove that the FZ^c relations hold in $\mathrm{Gor}^*(\mathcal{M}_{g,n}^c)$, the Gorenstein quotient of the tautological ring. In other words, we must show that if we multiply an FZ^c relation of degree d by a tautological class of degree $D - d$, then the result vanishes in the socle.

We first check the case in which the tautological class is a product of kappa and psi classes. As usual, the family of FZ^c relations is closed under multiplication by kappa classes, so we can assume that the tautological class is just a monomial in the psi classes. It is clear from (2.5) that only the degree of the monomial matters. We can now translate vanishing in the socle into a power series identity as in the proof of Theorem 3.1, but because of the different socle evaluation formula, we need to change the definition of the $\widehat{\cdot}$ operator slightly: if $C(t) = \sum_n c_n t^n$ is a power series in t, we write

$$\widehat{C}(t) = \sum_n \frac{c_n}{(n+1)!} t^{n+1}.$$

Then the identity that we need to prove for the desired vanishing when multiplying an FZ^c relation by a psi monomial is

$$\left[\left(1 - \frac{\widehat{1-A_c}}{\sqrt{t}}\right)^{-D-1-\ell(\sigma)} \left(\frac{\widehat{B_c}}{t}\right)^{\sigma_1} \left(\frac{\widehat{tA_c}}{t}\right)^{\sigma_2} \cdots\right]_{t^d} = 0,$$

for $2d \geq D + 2 + |\sigma|$.

We can collect the powers of t and rewrite this as

$$\left[\frac{t^{D-d}}{\widehat{A_c}^{D+1}} \left(\frac{\widehat{B_c}}{\widehat{A_c}}\right)^{\sigma_1} \left(\frac{\widehat{tA_c}}{\widehat{A_c}}\right)^{\sigma_2} \left(\frac{\widehat{tB_c}}{\widehat{A_c}}\right)^{\sigma_3} \cdots\right]_{t^{-1}} = 0. \tag{3.7}$$

Now, it is easy to compute that $\widehat{A_c} = 1 - \sqrt{1-2t}$ and $\widehat{B_c} = t$, and in general $\widehat{tC}$ is the antiderivative of $\widehat{C}$ with constant term 0. This means that the series F appearing in (3.7) belongs to $\mathbb{Q}(\sqrt{1-2t})$.

Let $\bar{\cdot}$ denote the Galois involution $\sqrt{1-2t} \mapsto -\sqrt{1-2t}$ for this quadratic extension of $\mathbb{Q}(t)$. We first observe that

$$[\overline{F}]_{t^{-1}} = 0,$$

as $\overline{1 - \sqrt{1-2t}} = 1 + \sqrt{1-2t}$ has leading term 2 so $\overline{F}$ is actually a power series.

Thus

$$[F]_{t^{-1}} = [F + \overline{F}]_{t^{-1}}.$$

Now, $F + \overline{F}$ is clearly a Laurent polynomial in t, since

$$\frac{1}{\widehat{A_c}} = \frac{1 + \sqrt{1-2t}}{2t}.$$

Thus it will suffice to show that $F + \overline{F}$ vanishes to order greater than 1 at infinity. This is a simple calculation:

$$-(D-d)+\frac{D+1}{2}-\sum_{i\in\sigma}\frac{i}{2}=\frac{1}{2}(2d-D+1-|\sigma|)$$
$$\geq\frac{3}{2}.$$

Thus any polynomial in the kappa and psi classes of degree $D-d$ will annihilate any FZ^c relation of degree d in the socle. We now consider multiplication by an arbitrary tautological class. Such a class is a sum of generators of the form described in (2.2), so take one such generator, which is given by a stable tree Γ with m vertices and kappa-psi monomials $\theta_1, \theta_2, \ldots, \theta_m$ assigned to the vertices $v_1, v_2, \ldots, v_m$ of Γ. Suppose that the components $\mathcal{M}^c_{g_i,n_i}$ corresponding to the vertices of Γ have socle degrees $D_1, D_2, \ldots, D_m$ respectively, so $D_1+\cdots+D_m = D-m+1$.

Now, the splitting rule for pulling back kappa classes to the boundary tells us the product of this generator with a given FZ^c relation: it is given by summing over all partitions $d = d_1+\cdots+d_m$ and $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_m$ over the m vertices, and then taking the pushforward via ξ_Γ of the product of θ_i and the (d_i, σ_i) FZ^c relation on vertex v_i for each i. Here by the (d_i, σ_i) FZ^c relation we just mean the kappa polynomial formally given by the FZ^c formula; the parameters (d_i, σ_i) and the socle degree D_i do not need to satisfy the FZ^c inequality $2d_i \geq D_i+2+|\sigma_i|$.

However, we observe that this inequality must hold for at least one value of i. For if it does not hold for any i, then we can sum up the inequality

$$2d_i \leq D_i+1+|\sigma_i|$$

to get

$$2d \leq (D-m+1)+m+|\sigma|,$$

which contradicts the inequality on the original FZ^c parameters. Thus each term of this summation includes a real FZ^c relation on one vertex, and thus has socle evaluation zero by the previous computation. This completes the proof that the FZ^c relations are true relations.

For the second part of the theorem, we check that the quotient Q^* of $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ by the FZ^c relations has degree d piece Q^d satisfying

$$\dim_{\mathbb{Q}} Q^d \leq |P(d, D+1-d)|$$

for any $d \geq 0$. Since $\dim_{\mathbb{Q}} \kappa^d(\mathcal{M}_{g,n}^c) = |P(d, D+1-d)|$ for $n > 0$ by [27], this would imply that the FZ^c relations span all the relations.

Note that these rank bounds are the same as those appearing in Theorem 3.4. We can in fact use the same approach to prove these bounds; the only differences in the proof are that the power series $D_0, D_1, \ldots$ involved are different and that we must use a slightly different formula for the $A_{\tau\mu}$ because the socle evaluation formula is different in compact type:

$$A_{\tau\mu} = \sum_{\substack{\psi:\tau\to\mu \\ \text{refinement}}} \frac{|\operatorname{Aut}(\tau)|}{\prod_{k\in\mu} |\operatorname{Aut}(\psi^{-1}(k))|} \prod_{k\in\mu} (\ell(\psi^{-1}(k)) + k)! \prod_{j\in\tau} \frac{1}{(j+1)!}.$$

$\square$

3.5 Connecting the socle evaluations on $\mathcal{M}_g$ and $\mathcal{M}_g^c$

The remainder of this chapter presents joint work with Felix Janda [19].

We will study the restriction

$$\kappa^d(\mathcal{M}_g^c) \times R^r(\mathcal{M}_g^c) \to \mathbb{Q}$$

of the compact type socle evaluation pairing for $r + d = 2g - 3$, for any $g \geq 2$. We will prove the following theorems, which were previously conjectured by Pandharipande.

Housing Theorem. *The rank of the pairing of κ classes against boundary classes*

$$\kappa^d(\mathcal{M}_g^c) \times BR^r(\mathcal{M}_g^c) \to \mathbb{Q}$$

equals the rank of the pairing of κ classes against pure boundary strata

$$\kappa^d(\mathcal{M}_g^c) \times PBR^r(\mathcal{M}_g^c) \to \mathbb{Q}.$$

Furthermore, these ranks are equal to the number of partitions of d of length less than $r+1$ plus the number of partitions of d of length $r+1$ which contain at least two even parts.

Rank Theorem. *The rank of the pairing of κ classes against general tautological classes*

$$\kappa^d(\mathcal{M}_g^c) \times R^r(\mathcal{M}_g^c) \to \mathbb{Q}$$

equals the sum of the rank of the pairing of κ classes against boundary classes

$$\kappa^d(\mathcal{M}_g^c) \times BR^r(\mathcal{M}_g^c) \to \mathbb{Q}$$

and the rank of the $\mathcal{M}_g$ socle pairing

$$\kappa^r(\mathcal{M}_g) \times \kappa^{g-2-r}(\mathcal{M}_g) \to \mathbb{Q}.$$

These theorems will be proven by direct combinatorial analysis of the socle evaluation formulas (2.4) and (2.5). In particular, we have no geometric explanation of the Rank Theorem, which connects the compact type case and the smooth case.

3.5.1 Consequences

The motivation for considering these pairings was the question of *mystery relations*: relations that hold in $\mathrm{Gor}^*(\mathcal{M}_g)$ but are not linear combinations of FZ relations. Since FZ relations extend to tautological relations in $R^*(\overline{\mathcal{M}}_g)$ (see Chapters 4 and 5), a possible reason for the existence of mystery relations might be if they do not extend tautologically to relations in $R^*(\mathcal{M}_g^c)$ or $R^*(\overline{\mathcal{M}}_g)$. The Rank Theorem can be interpreted as saying that part of the obstruction to this extension is zero: the mystery relations at least extend to classes in the tautological ring of $\mathcal{M}_g^c$ which pair to zero with the κ subring. It is an interesting question whether the mystery relations extend to classes in the tautological ring of $\mathcal{M}_g^c$ which are relations in the Gorenstein quotient.

The Rank Theorem also gives us more information about the kappa ring $\kappa^*(\mathcal{M}_g^c)$. As discussed in Section 2.5, Pandharipande [27] defined a surjective ring homomorphism

$$\iota_{g,0} : \kappa^*(\mathcal{M}_{0,2g}^c) \to \kappa^*(\mathcal{M}_g^c)$$

and proved that this is an isomorphism for degree at most $g-2$. The Rank Theorem can be viewed as a statement about the higher degree parts of $\iota_{g,0}$.

Theorem 3.7. *Let $g \geq 2$, $0 \leq e \leq g-2$, and $d = g-1+e$. Let δ_d be the rank of the kernel of the map from $\kappa^d(\mathcal{M}_g^c)$ to the Gorenstein quotient of $R^*(\mathcal{M}_g^c)$. Let γ_e be the rank of the space of κ relations of degree e in the Gorenstein quotient of $R^*(\mathcal{M}_g)$. Then the degree d part of the kernel of $\iota_{g,0}$ has rank $\gamma_e - \delta_d$.*

Proof. By [27], the rank of $\kappa^d(\mathcal{M}_{0,2g}^c)$ is equal to $|P(d, 2g-2-d)|$, the number of partitions of d of length at most $2g-2-d$. On the other side, the rank of $\kappa^r(\mathcal{M}_g^c)$ is equal to δ_r plus the rank of the first pairing appearing in the Rank Theorem. The rank of the second pairing appearing in the Rank Theorem is given by the Housing Theorem, and the rank of the third pairing appearing in the Rank Theorem is equal to $|P(e)| - \gamma_e$. Putting these pieces together gives the theorem statement. $\square$

Remark. The components γ_e and δ_d appearing in the above theorem both have conjectural values. The FZ relations give a prediction for γ_e (if they are the only relations in the first half of the Gorenstein quotient and are linearly independent):

$$\gamma_e = \begin{cases} a(3e-g-1) & \text{if } e \le \frac{g-2}{2} \\ a(3(g-2-e)-g-1) & \text{else,} \end{cases}$$

where $a(n)$ is the number of partitions of n with no parts of sizes $5, 8, 11, \ldots$.

The Gorenstein conjecture in compact type would imply that $\delta_d = 0$, though in fact this is a much weaker statement. Combining these predictions gives a conjecture for all the betti numbers of $\kappa^*(\mathcal{M}_g^c)$.

3.6 The Housing Theorem

3.6.1 Notation concerning partitions

We will use the following notation heavily in the proofs of the Housing Theorem and the Rank Theorem in the next two sections. If σ is a partition, then we let $I(\sigma)$ be a set of $\ell(\sigma)$ elements which we will use to index the parts of σ. For example we could take

$$I(\sigma) = [\ell(\sigma)] := \{1, \ldots, \ell(\sigma)\}.$$

For two partitions $\sigma, \tau \in P(n)$ and a map $\varphi : I(\sigma) \to I(\tau)$ we say that φ is a *refining function* of τ into σ if for any $i \in I(\tau)$ we have

$$\tau_i = \sum_{j \in \varphi^{-1}(i)} \sigma_j.$$

If for given σ, τ there exists a refining function φ of τ into σ we say that σ is a refinement of τ.

For a finite set S, a *set partition* P of S (written $P \vdash S$) is a set $P = \{S_1, \ldots, S_m\}$ of nonempty subsets of S such that S is the disjoint union of the S_i.

For a partition σ and a set S of subsets of $I(\sigma)$ we define a new partition σ^S indexed by the elements of S by setting $(\sigma^S)_s = \sum_{i\in s}\sigma_i$ for each $s \in S$. Usually we will take a set partition P of $I(\sigma)$ for S. For a subset $T \subseteq I(\sigma)$ we define the *restriction* $\sigma|_T$ *of* σ *to* T by σ^S, where S is the set of all 1-element subsets of T; in other words, $\sigma|_T = (\sigma_t)_{t\in T}$.

3.6.2 Socle evaluations

In (2.5), we saw a simple formula for the socle evaluations of the pushforward kappa polynomial basis. Here we will transform this formula into the standard kappa monomial basis. We denote the socle evaluations of monomials in the kappa and psi classes in compact type by

$$\vartheta(\sigma;\tau) := \epsilon^c(\kappa_\sigma \psi^\tau),$$

where σ and τ are partitions. In this equation we have used κ_σ as an abbreviation for $\prod_{i\in I(\sigma)} \kappa_{\sigma_i}$ and ψ^τ for $\prod_{i\in I(\tau)} \psi_i^{\tau_i}$ indexing the $|\tau|$ marked points by the parts of τ. We will write

$$\vartheta(\sigma) := \vartheta(\sigma;\emptyset)$$

when we just have κ classes and no ψ classes.

Lemma 3.8. *For partitions σ and τ such that $2g-3+\ell(\tau) = |\sigma|+|\tau|$ we have*

$$\vartheta(\sigma;\tau) = \sum_{P\vdash I(\sigma)} (-1)^{|P|+\ell(\sigma)} \binom{2g-3+|P|+\ell(\tau)}{((\sigma^P)_i+1)_{i\in P}, \tau}.$$

Proof. From the basic socle evaluation formula (2.5) we see that it suffices to prove the identity

$$\kappa_\sigma = \sum_{P\vdash I(\sigma)} (-1)^{|P|+\ell(\sigma)} \widehat{\kappa_{\sigma^P}}.$$

This follows from writing (2.3) as a sum over set partitions

$$\widehat{\kappa_\sigma} = \sum_{P\vdash I(\sigma)} \left(\prod_{S\in P} (|S|-1)!\right) \kappa_{\sigma^P}$$

and using partition refinement inversion. $\square$

To evaluate the more general integrals which arise when we pair κ classes with arbitrary tautological classes, we can restrict ourselves to pairing a κ monomial with the usual generators for the tautological ring given in (2.2). In this case we have to sum over the set of possible distributions of the κ classes to the vertices of Γ and then multiply the socle evaluations at each vertex.

We can calculate the $\mathcal{M}_g$ socle evaluations of κ monomials analogously to Lemma 3.8.

3.6.3 Housing partitions

Let us now study pairing κ monomials of degree d with pure boundary classes via the compact type socle pairing. Each pure boundary stratum in codimension $2g-3-d$ is determined by a tree $\Gamma = (V, E)$ with $|V| = 2g-2-d$ vertices and $|E| = 2g-3-d$ edges and a genus function $g : V \to \mathbb{Z}_{\geq 0}$ with $\sum_{v \in V} g(v) = g$. Then the class is the pushforward of 1 along the gluing map $\xi_\Gamma : \prod_{v \in V} \mathcal{M}^c_{g(v),n(v)} \to \mathcal{M}^c_g$ corresponding to the tree Γ, where $n(v)$ is the degree of the vertex v. From this data we obtain a partition of

$$\begin{aligned}\sum_{v \in V}(2g(v) - 3 + n(v)) &= 2g - 3(2g-2-d) + 2(2g-3-d) \\ &= d\end{aligned}$$

by collecting the socle dimensions $2g(v) - 3 + n(v)$ for each vertex $v \in V$ and throwing away the zeroes. We will call this partition the *housing data* of the pure boundary stratum. From the socle evaluation formula it is easy to see that the pairing of the κ ring with a pure boundary stratum is determined by its housing data.

On the other hand it is interesting to consider which partitions of d can arise as housing data corresponding to a pure boundary stratum. We will call these partitions *housing partitions*.

Lemma 3.9. *A partition σ of d is a housing partition if and only if it either has fewer than $2g-2-d$ parts or exactly $2g-2-d$ parts with at least two even.*

Proof. Only partitions of length at most $2g-2-d$ can be housing partitions because there are only that many vertices. Furthermore it is easy to see that no partition of $2g-2-d$ parts with fewer than two even parts can arise since every vertex with only one edge gives an even part (or no part if $g(v) = 1$).

Now suppose σ is a partition of d with either fewer than $2g-2-d$ parts or exactly $2g-2-d$ parts with at least two even. Let $(\tau_i)_{1 \leq i \leq 2g-2-d}$ be the tuple of nonnegative integers given by appending $2g-2-d-\ell(\sigma)$ zeroes to σ, so the sum of the τ_i is d and exactly $2k+2$ of the τ_i are even for some nonnegative integer k.

Construct a tree Γ by taking a path of $2g-2-d-k$ vertices and adding k additional leaves connected to vertices $2, 3, \ldots, k+1$ along the path respectively. Thus Γ has $2g-2-d$ vertices, each of degree at most three, and exactly $2k+2$ of the vertices of Γ have odd degree. We now choose a bijection between the τ_i and the vertices of Γ such that even τ_i are assigned to vertices of odd degree. We can then assign a genus $g_i = (\tau_i + 3 - n_i)/2$ to each vertex, where n_i is the degree of the vertex to which τ_i was assigned. The resulting stable tree has housing data σ, as desired. $\square$

3.6.4 Reduction to a combinatorial problem

We have already described the housing data of a pure boundary stratum. Let us now describe a similar notion for any class in the usual additive generating set for the tautological ring (2.2). Such a class is given by a boundary stratum corresponding to a tree $\Gamma = (V, E)$ and a genus assignment $g : V \to \mathbb{Z}_{\geq 0}$, along with assignments of monomials in κ and ψ classes (of degrees $r(v)$ and $s(v)$ respectively) to each component of the stratum. Let $k = \sum_{v \in V}(r(v) + s(v))$; then we must have $|E| = 2g - 3 - d - k$ edges in the tree in order to obtain a class of degree $2g - 3 - d$. If this class does not vanish by dimension reasons then we can obtain a partition γ of

$$\sum_{v \in V} (2g(v) - 3 + n(v) - r(v) - s(v)) = 2g - 3(2g - 2 - d - k) + 2(2g - 3 - d - k) - k = d$$

by assigning to each vertex of V the number $2g(v) - 3 + n(v) - r(v) - s(v)$. This is exactly the degree $d'(v)$ such that the socle pairing of $R^{d'(v)}(\mathcal{M}^c_{g(v),n(v)})$ with the monomial of ψ and κ classes at v is not zero for dimension reasons. Then the pairing with the boundary class is determined by the partition γ, an assignment of degrees $r(i)$ and $s(i)$ to the parts $i \in I(\gamma)$ and partitions $\tau_i \in P(r(i))$ and $\rho_i \in P(s(i))$ corresponding to the κ and ψ monomials. In particular we can leave out classes which were assigned to vertices with $2g(v) - 3 + n(v) - r(v) - s(v) = 0$ and we do not need to remember which node corresponds to each ψ. The result of the socle pairing of this class together with a κ monomial corresponding to a partition π of d is (up to scaling) given by

$$\sum_{\varphi} \prod_{j \in I(\gamma)} \vartheta\left(\pi_{\varphi^{-1}(j)}, \tau_j; \rho_j\right),$$

where the sum runs over all refining functions φ of γ into π.

When we view $\mathbb{Q}^{P(d)}$ as a ring of formal κ polynomials, this pairing gives linear forms $v_{\gamma,\{\tau_i\},\{\rho_i\}} \in \left(\mathbb{Q}^{P(d)}\right)^*$. We notice that the formulas still make combinatorial sense even if the $\gamma, \{\tau_i\}, \{\rho_i\}$ data does not come from pairing with an actual tautological class.

The special case where all the $r(i)$ and $s(i)$ are zero determines the pairing of κ classes with pure boundary classes. We get $|P(d)|$ linear forms M_λ, which we normalize such that $M_\lambda(\lambda) = 1$:

$$M_\lambda(\pi) = \frac{1}{\mathrm{Aut}(\lambda)} \sum_{\varphi} \prod_{j \in I(\lambda)} \vartheta\left(\pi_{\varphi^{-1}(j)}\right). \tag{3.8}$$

In this way we obtain a basis of $\left(\mathbb{Q}^{P(d)}\right)^*$. If we sort partitions in any way such that shorter partitions come before longer partitions, then the basis change matrix from this basis to the standard basis is triangular with ones on the diagonal. Note that this basis uses some partitions which are not housing partitions.

The housing theorem can now be reformulated as follows:

Claim. *The span of* $\{M_\lambda : \lambda$ *is a housing partition*$\}$ *in* $\left(\mathbb{Q}^{P(d)}\right)^*$ *equals the span of the* $v_{\gamma,\{\tau_i\},\{\rho_i\}}$ *for all choices of housing data.*

To prove this claim we will first express the vectors $v_{\gamma,\{\tau_i\},\{\rho_i\}}$ for any choice of housing data in terms of the basis of $\left(\mathbb{Q}^{P(d)}\right)^*$ we have described above in Section 3.6.5. We will then in Section 3.6.6 rewrite the coefficients as counts of certain combinatorial objects. This combinatorial interpretation is proved in Section 3.6.7. We conclude in Section 3.6.8 by showing that when expressing vectors v corresponding to actual housing data in terms of the M_λ, the coefficient is zero whenever λ is not a housing partition.

3.6.5 A matrix inversion

In section 3.6.4 we have seen that there are formal expansions

$$v_{\gamma,\{\tau_i\},\{\rho_i\}} = \sum_{\lambda \in P(d)} c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} M_\lambda$$

for some coefficients $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}}$.

We can calculate $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}}$ explicitly by inverting the triangular matrix given by equation (3.8). We obtain

$$c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} = \sum_{l=0}^{\infty} (-1)^l \sum_{\lambda_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_l} \lambda_l \xrightarrow{\varphi_{l+1}} \gamma} \frac{v_{\gamma,\{\tau_i\},\{\rho_i\}}(\lambda_l)}{\prod_{i=1}^{l} |\mathrm{Aut}(\lambda_i)|} \prod_{i=1}^{l} \prod_{j \in I(\lambda_i)} \vartheta\left((\lambda_{i-1})_{\varphi_i^{-1}(j)}\right),$$

where we sum over chains $\lambda = \lambda_0, \ldots \lambda_l$ of refinements of γ with corresponding refinement functions φ_i. In particular $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} = 0$ if λ is not a refinement of γ.

We can reduce to the special case in which $\gamma = (d)$ is of length one by splitting this sum based on the composition $\varphi := \varphi_{l+1} \circ \varphi_l \circ \cdots \circ \varphi_1$ and examining the contribution of the preimages of the $j \in I(\gamma)$. The result is

$$c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} = \sum_{\varphi} \prod_{j \in I(\gamma)} c_{\lambda_{\varphi^{-1}(j)},(\gamma_j),\{\tau_j\},\{\rho_j\}}, \tag{3.9}$$

summed over refinements φ of γ into λ.

When $\gamma = (d)$, we set $\tau_1 =: \tau$ and $\rho_1 =: \rho$ and we can write more compactly

$$c_{\lambda,(d),\{\tau\},\{\rho\}} = \sum_{l=0}^{\infty} (-1)^l \sum_{\lambda_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_l} \lambda_l} \frac{\vartheta(\lambda_l, \tau; \rho)}{\prod_{i=1}^{l} |\mathrm{Aut}(\lambda_i)|} \prod_{i=1}^{l} \prod_{j \in I(\lambda_i)} \vartheta\left((\lambda_{i-1})_{\varphi_i^{-1}(j)}\right). \tag{3.10}$$

3.6.6 Interpreting the coefficients combinatorially

We will interpret the coefficients $c_{\lambda,(d),\{\tau\},\{\rho\}}$ as counting certain permutations of symbols labeled by the parts of the partitions λ, τ, and ρ. We say that a symbol is *of kind* i if it is labelled by some i belonging to the disjoint union of the indexing sets of the partitions, $I(\lambda) \sqcup I(\tau) \sqcup I(\rho)$. There will in general be multiple symbols of a given kind.

Main Claim. *The coefficient* $c_{\lambda,(d),\{\tau\},\{\rho\}}$ *counts the number of permutations of*

- $\lambda_i + 1$ *symbols of kind* i *for each* $i \in I(\lambda)$,
- $\tau_i + 1$ *symbols of kind* i *for each* $i \in I(\tau)$, *and*
- ρ_i *symbols of kind* i *for each* $i \in I(\rho)$

such that:

1. *If the last symbol of some kind* i *is immediately followed by the first symbol of kind* j *with* $i, j \in I(\lambda) \sqcup I(\tau)$, *then we have* $i < j$.
2. *For* $i \in I(\lambda)$ *the last element of kind* i *is not immediately followed by a symbol of kind* j *for any* $j \in I(\lambda)$,

averaged over all total orders $<$ *of* $I(\lambda) \sqcup I(\tau)$ *such that elements of* $I(\tau)$ *are smaller than elements of* $I(\lambda)$.

It follows in particular that the coefficient $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}}$ is non-negative.

3.6.7 Proof of the main claim

Refinements of permutations of symbols

For given natural numbers d, n and a partition $\tau \in P(d)$ we will study permutations of $\tau_i + 1$ symbols of kind i for $i \in I(\tau)$ and n symbols of kind c. (The permutations of symbols appearing in the previous section are an instance of this.) We will need to construct refined permutations of this type for partition refinements $\varphi : I(\sigma) \to I(\tau)$. For this we need additional *refinement data*: for each $i \in I(\tau)$, a permutation T_i of $\sigma_j + 1$ symbols of kind j for $j \in \varphi^{-1}(i)$. Then we can obtain a permutation S' of $\sigma_i + 1$ symbols of kind i and n symbols of kind c in the following way:

For each $i \in I(\tau)$ and each $j \in \varphi^{-1}(i)$, modify T_i by gluing the last symbol of kind j with the immediately following symbol; the result is a permutation T_i' of $\tau_i + 1$ symbols. To construct S' from S, for each i we replace the symbols of kind i by T_i' and then remove the glue.

Reinterpretation

We start with a combinatorial interpretation of the number $\vartheta(\sigma; \tau)$ for partitions σ and τ.

Lemma 3.10. *Given an arbitrary total order $<$ on $I(\sigma)$, the number $\vartheta(\sigma; \tau)$ is equal to the number of permutations of*

- $\sigma_i + 1$ *symbols of kind i for each $i \in I(\sigma)$ and*
- τ_i *symbols of kind i for each $i \in I(\tau)$*

such that the following property holds:

If the last symbol of kind i is immediately followed by the first symbol of kind j for $i, j \in I(\sigma)$ then we have $i < j$.

Proof. For each permutation S of symbols as above, but not necessarily satifying the property, we can assign a set partition $Q_S \vdash I(\sigma)$ which measures in what ways it fails to satisfy the property: Q_S is the finest set partition such that if $i < j$ and the last symbol of kind i is immediately followed by the first symbol of kind j in S, then i and j are in the same part of Q_S. Thus S satisfies the given property if and only if Q_S is the set partition with all parts of size 1.

The multinomial coefficient in the summand in the formula for $\vartheta(\sigma; \tau)$ given by Lemma 3.8 corresponding to a set partition $P \vdash I(\sigma)$ counts the number of permutations S such that for $p = \{p_1, \dots, p_k\} \in P$ with $p_1 < \cdots < p_k$, the last element of kind p_i is immediately followed by the first element of kind p_{i+1} in S for $i = 1, \cdots, k-1$. These are precisely the S such that Q_S can be obtained by combining parts of P such that the largest element in one part is smaller than the smallest element of the other part.

This means that the contribution of a permutation with failure set partition $Q = \{Q_1, \dots, Q_k\}$ to the sum in Lemma 3.8 is precisely

$$\prod_{i=1}^{k} \sum_{j=0}^{|Q_k|-1} (-1)^j \binom{|Q_k| - 1}{j},$$

which is 1 for Q the set partition with all parts of size 1 and 0 otherwise. $\square$

Equipped with Lemma 3.10, the next step is to interpret the coefficient $c_{\lambda,(d),\{\tau\},\{\rho\}}$ as the sum of the values of a function f on the set $S_{\lambda,\tau,\rho}$ of permutations of $\lambda_i + 1$, $\tau_i + 1$, ρ_i symbols of kind i for $i \in I(\lambda)$, $i \in I(\tau)$ and $i \in I(\rho)$ respectively.

Fix

- a chain of partitions $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_l$ with refining maps φ_i,
- an order $<$ on $I(\lambda_l) \sqcup I(\tau)$ such that elements of $I(\tau)$ appear before elements of $I(\lambda_l)$,
- orders on $\varphi_i^{-1}(j)$ for $1 \leq i \leq l$ and $j \in I(\lambda_i)$.

Then we identify each κ socle evaluation factor

$$\vartheta\left((\lambda_{i-1})_{\varphi_i^{-1}(j)}\right)$$

with the number of permutations of $(\lambda_{i-1})_k + 1$ symbols of kind $k \in \varphi_i^{-1}(j)$ such that if the last symbol of kind k is immediately followed by the first symbol of kind k', then $k < k'$. We interpret each such permutation as refinement data corresponding to the refinement φ_i of λ_i into λ_{i+1}.

Furthermore we interpret the factor

$$\vartheta\left(\lambda_l, \tau; \rho\right)$$

as the number of permutations of $(\lambda_l)_k + 1$, $\tau_k + 1$ and ρ_k symbols of kind k with $k \in I(\lambda_l)$, $k \in I(\tau)$ and $k \in I(\rho)$ respectively such that if the last symbol of kind k is immediately followed by the first symbol of kind k' for $k, k' \in I(\lambda_l) \sqcup I(\tau)$, then $k < k'$. In order to remove the dependence on the chosen orders we will average over all choices of them.

Simplification

Given all this data, we can build a "composite permutation" by repeatedly refining the collection of symbols of kind k with $k \in I(\lambda_l)$ using the construction from Section 3.6.7 and keeping the order of the other symbols intact. The result is a permutation of $\lambda_k + 1$, $\tau_k + 1$ and ρ_k symbols of kind k for $k \in I(\lambda)$, $k \in I(\tau)$ and $k \in I(\rho)$ respectively. Any permutation obtained in this way has the property that the last symbol of any kind $j \in I(\lambda)$ is not immediately followed by the first symbol of some kind $j' \in I(\tau)$.

For any permutation in $S_{\lambda,\tau,\rho}$ we assign a set partition $P \vdash I(\lambda)$, which measures in what way it fails to satisfy condition (2) in the main claim. We define P to be the finest set partition such that if the last symbol of kind i is immediately followed by a symbol of kind j for $i, j \in I(\lambda)$ then i and j lie in the same set in P.

Now, suppose we are given a chain of partitions $\lambda, \lambda_1, \ldots, \lambda_l$ along with additional refining data and base permutation as above, and supppose the resulting composite permutation has failure set partition P that is not the partition into one-element sets.

By the definition of P, if we change the order on $I(\lambda) \sqcup I(\tau)$ such that the order on $I(\tau)$ and each element of P is preserved, all the conditions on the data are still satisfied.

On the other hand, consider the following data:

- the chain $\lambda, \lambda_1, \ldots, \lambda_l, \lambda_l^P$ with refining maps φ_i as before,
- any refining map $\varphi' : I(\lambda_l) \to I(\lambda_l^P)$ which is up to an automorphism of λ_l^P the canonical one,
- the orders and refining data corresponding to the φ_i as before,
- in addition an order on each element of P induced by the order on $I(\lambda_l) \sqcup I(\tau)$,
- refining data corresponding to φ' induced from the permutation corresponding to λ_l, τ and ρ,
- any order on $I(\lambda_l^P) \sqcup I(\tau)$ such that the restriction to $I(\tau)$ is the restriction of the order on $I(\lambda_l) \sqcup I(\tau)$ and such that elements of $I(\tau)$ appear before elements of $I(\lambda_l^P)$,
- permutations of $(\lambda_l^P)_i + 1$, $\tau_i + 1$, ρ_i symbols of kind i for $i \in P$, $i \in I(\tau)$ and $i \in I(\rho)$ respectively, defined from the permutation corresponding to λ_l by leaving out the last symbol of any kind $i \in I(\lambda_l)$ which is not the last one in a set of P and identifying symbols according to P.

It is easy to check that the refining data and the permutation still satisfy the order conditions. Furthermore, the failure set partition of the composite partition of this new data is the partition into one-element sets.

The original chain with additional data giving failure set partition P and the extended chain with additional data giving failure set partition the partition into one-element sets contribute to $c_{\lambda,(d),\{\tau\},\{\rho\}}$ in formula (3.10) with opposite signs, since the extended chain is one element longer. We claim these contributions are actually equal.

For the original chain, we have

$$\frac{(\ell(\lambda_l))!}{\prod_{j\in P} |\varphi'^{-1}(j)|!}$$

choices of orders on $I(\lambda) \sqcup I(\tau)$ in the above construction. For the extended chain, we made

$$|\mathrm{Aut}(\lambda_l^P)|(\ell(\lambda_l^P))!$$

choices in the above construction.

However, the contributions are also weighted by averaging over choices of orders and by the coefficients in (3.10). For the original chain the weight is

$$((\ell(\lambda_l))!)^{-1}$$

and for the extended chain the weight is

$$\left(|\mathrm{Aut}(\lambda_l^P)|(\ell(\lambda_l^P))! \prod_{j\in P} |\varphi'^{-1}(j)|!\right)^{-1}.$$

Thus the two contributions cancel.

The only remaining contributions come when $l = 0$ and P is the set partition into one-element sets. These are the permutations counted in the main claim.

3.6.8 Proof of the Housing Theorem

We begin with a simple lemma.

Lemma 3.11. *Suppose $r + s + \ell(\gamma) < \ell(\lambda)$. Then $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} = 0$.*

Proof. We examine the summand in formula (3.9) corresponding to some φ. A factor in this summand can only be nonzero if $r(i) + s(i) + 1 \geq \ell(\varphi^{-1}(i))$. Therefore each summand will vanish unless

$r + s + \ell(\gamma) \geq \ell(\lambda)$. □

Now let us suppose that γ, $\{\tau_i\}$, $\{\rho_i\}$ is the housing data of a boundary class of the generating set. We need to show that $c_{\lambda,\gamma,\{\tau_i\},\{\rho_i\}} = 0$ for each λ which is not a housing partition.

Let us first study the case $\ell(\lambda) > 2g - 2 - d$. Since γ is derived from a boundary stratum of at most codimension $2g - 3 - d - r - s$ (we are missing the ψ and κ classes from the components, which do not contribute to γ) by diminishing parts by their κ and ψ degrees, we have the inequality $\ell(\gamma) \leq 2g - 2 - d - r - s$. Then by Lemma 3.11 we are done in this case. The same argument settles also the case where there are components of the boundary stratum we are considering which do not appear in γ and $\ell(\lambda) = 2g - 2 - d$.

Now assume that $\ell(\lambda) = 2g-2-d$ and that λ contains no even part. Then by the same arguments if the coefficient is nonzero, we must have $\ell(\gamma) = 2g-2-d-r-s$. Then from the proof of Lemma 3.11 we see that $r(i) + s(i) = \ell(\varphi^{-1}(i)) - 1$ for each $i \in I(\gamma)$. This implies $\ell(\varphi^{-1}(i)) + r(i) + s(i) \equiv 1 \pmod 2$ and therefore for each $i \in I(\gamma)$ we have $\gamma_i + r(i) + s(i) \equiv 1 \pmod 2$. Hence each part of the housing data (for the underlying boundary stratum), which γ was obtained from by subtraction of $r(i) + s(i)$ at each part, is odd. This is a contradiction, so the coefficient must be zero, as desired.

3.7 The Rank Theorem

3.7.1 Reformulation

Let us first formulate a stronger version of the Rank Theorem.

Theorem 3.12. *For any κ polynomial F in degree $r := 2g - 3 - d$ the following two statements are equivalent:*

1. *For any $\pi \in P(g - 2 - r)$ we have $F\kappa_\pi = 0$ in $R^{g-2}(\mathcal{M}_g)$.*

2. *There is a pure boundary strata class $B \in PBR^r(\mathcal{M}_g^c)$ such that for any $\pi' \in P(2g - 3 - r)$ we have $(F - B)\kappa_{\pi'} = 0$ in $R^{2g-3}(\mathcal{M}_g^c)$.*

It will be convenient to show that we can replace the first condition in Theorem 3.12 by

3. For any $\pi \in P(g - 2 - r)$ of length at most $r + 1$ we have $F\kappa_\pi = 0$ in $R^{g-2}(\mathcal{M}_g)$.

Then Theorem 3.12 will follow from the following simple argument. Consider an F satisfying the second condition and we want to show that $F\kappa_\pi = 0$ for some given $\pi \in P(g - 2 - r)$. Notice that then also $F\kappa_\pi$ satisfies the second condition since $B\kappa_\pi$ lies in $BR^{g-2}(\mathcal{M}_g^c)$ and by the housing theorem can be replaced by some $B' \in PBR^{g-2}(\mathcal{M}_g^c)$. We then find that $F\kappa_\pi = 0$ since in this case the length condition is trivial.

As we have seen in Section 3.6.7, not only boundary classes but also every κ class can be written in terms of virtual boundary strata in the socle pairing with the kappa ring. So the second condition in the theorem is equivalent to the condition that only actual boundary strata are needed in the expansion of F. Notice that by Lemma 3.11 we only need strata corresponding to partitions of $2g-3-r$ of length at most $r+1$. However we might need terms corresponding to partitions $2g-3-r$ of length equal to $r+1$ with only odd parts, and those are the terms we are interested in. For the proof of the Rank Theorem we will need to understand the coefficients corresponding to these classes better.

Observe that partitions of $2g-3-r$ of length being equal to $r+1$ with only odd parts correspond to partitions of $g-2-r$ of length at most $r+1$. So for any $\sigma \in P(g-2-r, r+1)$ we can look at $\eta_\sigma, \mu_\sigma \in (\mathbb{Q}^{P(r)})^*$ with $\eta_\sigma(\tau) := c_{\lambda,\tau,\emptyset}$, where λ is the partition of $2g-3-r$ of length $r+1$ corresponding to σ, and $\mu_\sigma(\tau)$ is up to a factor the socle evaluation $\epsilon(\kappa_\sigma\kappa_\tau)$, namely

$$\mu_\sigma(\tau) = \sum_{P \vdash I(\sigma) \sqcup I(\tau)} (-1)^{\ell(\sigma)+\ell(\tau)+|P|} \frac{(2g-3+|P|)!}{\prod_{i \in P}(2(\sigma,\tau)_i^P + 1)!!}.$$

So what we need to show is the following:

Claim. *The $\mathbb{Q}$-subspaces of $(\mathbb{Q}^{P(r)})^*$ spanned by η_σ and μ_σ for σ ranging over all partitions of $g-2-r$ of length at most $r+1$ are equal.*

Recall from Section 3.6.7 that $\eta_\sigma(\tau)$ is the number of all permutations S of $\lambda_i + 1$ symbols of kind $i \in I(\lambda)$ and $\tau_i + 1$ symbols of kind $i \in I(\tau)$ satisfying

1. The last symbol of kind i for some $i \in I(\lambda)$ is either at the end of the sequence or immediately followed by a symbol of kind j for some $j \in I(\tau)$ which is not the first of its kind.

2. The successor of the last element of kind i is not the first element of kind j for any $i, j \in I(\tau)$ with $i < j$, where we fix some order on $I(\tau)$.

Before coming to the main part of the proof we apply an invertible transformation Φ to $(\mathbb{Q}^{P(r)})^*$ to simplify the definitions of η and μ. The inverse of the transformation we want to apply sends a linear form $\varphi' \in (\mathbb{Q}^{P(r)})^*$ to a linear form φ defined by

$$\varphi(\tau) = \sum_{P \vdash I(\tau)} (-1)^{\ell(\tau)+|P|} \varphi'(\tau^P). \tag{3.11}$$

The transformation Φ defined in this way is clearly invertible. By a similar argument as in the proof of Lemma 3.10, we can show that the image η'_σ of η_σ under Φ is defined in the same way as η_σ but leaving out Condition 2 on the permutations.

To study the action of Φ on μ we use the following lemma:

Lemma 3.13. *Let F be a function $F : P(n+m) \to \mathbb{Q}$ and define for any $\sigma \in P(n)$ functions $G_\sigma, G'_\sigma : P(m) \to \mathbb{Q}$ in terms of F by*

$$G_\sigma(\tau) = \sum_{P \vdash I(\sigma)\sqcup I(\tau)} F((\sigma \sqcup \tau)^P)$$

$$G'_\sigma(\tau) = \sum_{\substack{P \vdash I(\sigma)\sqcup I(\tau) \\ P \text{ separates } I(\tau)}} F((\sigma \sqcup \tau)^P),$$

where the second sum just runs over set partitions P such that each element of $I(\tau)$ belongs to a separate part. Then

$$G_\sigma(\tau) = \sum_{P \vdash I(\tau)} G'_\sigma(\tau^P).$$

Proof. Given set partitions P of $I(\tau)$ and Q of $I(\sigma) \sqcup I(\tau^P)$, with Q separating $I(\tau^P)$, we can alter Q by replacing each element of $I(\tau^P)$ by the elements in the corresponding part of P. Each set partition of $I(\sigma) \sqcup I(\tau)$ is obtained exactly once by this construction. $\square$

So we have that $\mu'_\sigma(\tau)$ is

$$\mu'_\sigma(\tau) = \sum_{\substack{P \vdash I(\sigma)\sqcup I(\tau) \\ P \text{ separates } I(\tau)}} (-1)^{\ell(\sigma)+\ell(\tau)+|P|} \frac{(2g-3+|P|)!}{\prod_{i\in P}(2(\sigma \sqcup \tau)^P_i + 1)!!}.$$

We can use the lemma again with the roles of σ and τ interchanged to replace the generators of the span of μ'_σ by μ''_σ with

$$\mu''_\sigma(\tau) := \sum_{\substack{P \vdash I(\sigma)\sqcup I(\tau) \\ P \text{ separates } I(\tau) \\ P \text{ separates } I(\sigma)}} (-1)^{\ell(\sigma)+\ell(\tau)+|P|} \frac{(2g-3+|P|)!}{\prod_{i\in P}(2(\sigma \sqcup \tau)^P_i + 1)!!}. \tag{3.12}$$

Therefore we have reduced the proof of the Rank Theorem to proving the following claim.

Claim. *The $\mathbb{Q}$-subspaces of $(\mathbb{Q}^{P(r)})^*$ spanned by η'_σ and μ''_σ for σ ranging over all partitions of $g-2-r$ of length at most $r+1$ are equal.*

3.7.2 Further strategy of proof

In order to prove the claim we will establish interpretations for $\eta'_\sigma(\tau)$ and $\mu''_\sigma(\tau)$ as counts of symbols of different kinds satisfying some ordering constraints. This enables us to find nonzero constants $F(i)$ for each $i \in I(\sigma)$ independent of τ such that

$$\mu''_\sigma(\tau) = \sum_{P \vdash I(\sigma)} \prod_{i \in P} F(i) \frac{\eta'_{\sigma^P}(\tau)}{(r+1-|P|)!},$$

giving a triangular transformation.

For the interpretations the notion of a *comb-like order* plays an important role. We say that symbols $i_1 \dots i_{2m+1}$ are in comb-like order if we have the relations $i_1 < i_3 < \cdots < i_{2m+1}$ and $i_{2j} < i_{2j+1}$ for $j \in [m]$. This is illustrated in Figure 3.1.

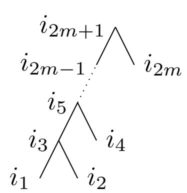

Figure 3.1: A comb-like order

Note that the number of comb-like orderings of $2m+1$ symbols is $(2m+1)!/(2m+1)!!$. More generally the number

$$\frac{(2|\pi| + \ell(\pi))!}{\prod_{i \in I(\pi)} (2\pi_i + 1)!!}$$

corresponding to a partition π counts the number of permutations of the $2|\pi| + \ell(\pi)$ symbols $\bigcup_{i \in I(\pi)} \{i_1, \dots, i_{2\pi_i+1}\}$ such that symbols corresponding to the same part of π appear in comb-like order.

3.7.3 Combing orders

We obtain a first reinterpretation of $\eta'_\sigma(\tau)$ by numbering the symbols of equal kind:

Interpretation A1. $\eta'_\sigma(\tau)$ is the number of all permutations of symbols $i_1, \dots, i_{\tau_i+1}$ for $i \in I(\tau)$ and $i_1, \dots, i_{\lambda_i+1}$ for $i \in I(\lambda)$ such that for fixed $i \in I(\tau) \sqcup I(\lambda)$ the i_j appear in order and for all $i \in I(\lambda)$ the symbol i_{λ_i+1} is either at the end of the sequence or immediately followed by some j_k for $j \in I(\tau)$ and $k \neq 1$.

Since λ has length $r+1$ and $|\tau| = r$, such a permutation gives a bijection between the j_k for $j \in I(\tau)$ with $k \neq 1$ and all but one of the i_{λ_i+1} for $i \in I(\lambda)$. After picking this bijection, we can remove the i_{λ_i+1}.

Interpretation A2. $\eta'_\sigma(\tau)$ is the sum over bijections

$$\varphi : I(\lambda) \to \{i_j \mid i \in I(\tau), j \neq 1\} \sqcup \{\text{End}\}$$

of the number of permutations of symbols $i_1, \ldots, i_{\tau_i+1}$ for $i \in I(\tau)$ and $i_1, \ldots, i_{\lambda_i}$ for $i \in I(\lambda)$ such that symbols of the same kind appear in order and all symbols i_j for $i \in I(\lambda)$ appear before $\varphi(i)$ (this condition is empty if $\varphi(i) = \text{End}$).

We can then add new symbols immediately following each i_{λ_i} for $i \in I(\lambda)$ and reindex the i_j for $i \in I(\tau)$ to create comb-like orderings.

Interpretation A3. $\eta'_\sigma(\tau)$ is the sum over bijections

$$\varphi : I(\lambda) \to \{i_j \mid i \in I(\tau), j \text{ even}\} \sqcup \{\text{End}\}$$

of the number of permutations of symbols $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$, $i_1, \ldots, i_{\lambda_i}$ for $i \in I(\lambda)$, and an additional symbol End such that the i_j for $i \in I(\tau)$ appear in comb-like order, the i_j for $i \in I(\lambda)$ appear in order, and i_{λ_i} for $i \in I(\lambda)$ is immediately followed by $\varphi(i)$.

Recall that λ is defined in terms of σ by taking the numbers $2\sigma_i + 1$ for each $i \in I(\sigma)$ and adding as many ones as needed to reach length $r + 1$. There is only one symbol i_1 of kind i for $i \in I(\lambda) \setminus I(\sigma)$ in Interpretation A3 of $\eta'_\sigma(\tau)$ and it must be immediately followed by $\varphi(i)$. For convenience set $(r + 1 - \ell(\sigma))! \cdot \eta''_\sigma := \eta'_\sigma$. Removing these symbols i_1 gives an interpretation of η''_σ.

Interpretation A4. $\eta''_\sigma(\tau)$ is a sum over all injections

$$\varphi : I(\sigma) \to \{i_j \mid i \in I(\tau), j \text{ even}\} \sqcup \{\text{End}\}$$

of the number of permutations of symbols $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$, $i_1, \ldots, i_{2\sigma_i+1}$ for $i \in I(\sigma)$, and an additional symbol End such that the i_j for $i \in I(\tau)$ appear in comb-like order, the i_j for $i \in I(\sigma)$ appear in order, and $i_{2\sigma_i+1}$ for $i \in I(\sigma)$ is immediately followed by $\varphi(i)$.

We now switch to the interpretation of $\mu''_\sigma(\tau)$, which was defined in (3.12). The coefficient corresponding to a set partition P can be interpreted as the number of permutations of symbols $i_1, \ldots, i_{2(\sigma\sqcup\tau)^P_i+1}$ for $i \in I((\sigma \sqcup \tau)^P)$ and one additional symbol $\star$ such that all $i_1, \ldots, i_{2(\sigma\sqcup\tau)^P_i+1}$ for $i \in I((\sigma \sqcup \tau)^P)$ appear in comb-like order.

Because of the restrictions in the sum, the parts of P are either singletons or contain exactly one element from each of $I(\sigma)$ and $I(\tau)$. This defines a function $\psi : I(\sigma) \to I(\tau) \sqcup \{\star\}$, injective when restricted to the preimage of $I(\tau)$. Interpreting the summands as counting comb-like orders and cutting combs into two pieces for each part of P of size two gives the following:

Interpretation B1. $\mu''_\sigma(\tau)$ is the sum over functions

$$\psi : I(\sigma) \to I(\tau) \sqcup \{\star\}$$

such that $\psi|_{\psi^{-1}(I(\tau))}$ is injective, of a sign of $(-1)^{|\psi^{-1}(I(\tau))|}$ times the number of permutations of symbols $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$, $i_1, \ldots, i_{2\sigma_i+1}$ for $i \in I(\sigma)$ and one additional symbol $\star$ such that all $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$ and all $i_1, \ldots, i_{2\sigma_i+1}$ for $i \in I(\sigma)$ appear in comb-like order and such that $i_{2\sigma_i+1}$ for $i \in I(\sigma)$ with $\psi(i) \neq \star$ is immediately followed by $\psi(i)_1$.

Now we split the set of such permutations depending on the symbols immediately following symbols $i_{2\sigma_i+1}$ for $i \in I(\sigma)$. We notice that the signed sum exactly kills those permutations where some $i_{2\sigma_i+1}$ for $i \in I(\sigma)$ is immediately followed by some j_1 for $j \in I(\tau)$ since if such a summand appears for some ψ with $\psi(i) \neq j$ we must have $\psi(i) = \star$ and we find the same summand with opposite sign in the sum corresponding to the map ψ' defined by $\psi'(i) = j$ and $\psi'(k) = \psi(k)$ for $k \neq i$ and vice versa.

Interpretation B2. $\mu''_\sigma(\tau)$ is the number of permutations of symbols $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$, $i_1, \ldots, i_{2\sigma_i+1}$ for $i \in I(\sigma)$ and one additional symbol $\star$ such that all $i_1, \ldots, i_{2\tau_i+1}$ for $i \in I(\tau)$ and all $i_1, \ldots, i_{2\sigma_i+1}$ for $i \in I(\sigma)$ appear in comb-like order and such that $i_{2\sigma_i+1}$ for $i \in I(\sigma)$ is not immediately followed by a symbol of the form j_1 with $j \in I(\tau)$.

Interpretations A4 and B2 are very close. The differences between the two of them are that the σ-type symbols are in total order rather than comb-like order in A4 and that the conditions on the elements immediately following the $i_{2\sigma_i+1}$ are different.

We now break $\mu''_\sigma(\tau)$ into a sum over set partitions P of $I(\sigma)$. Given a permutation of the symbols appearing in Interpretation B2, define a function

$$\varphi : I(\sigma) \to \{i_j \mid i \in I(\tau), j \text{ even}\} \sqcup \{\text{End}\}$$

recursively by

$$\varphi(i) = \begin{cases} j_{2k} & \text{if } i_{2\sigma_i+1} \text{ for } i \in I(\sigma) \text{ is immediately followed by a symbol} \\ & \qquad \text{of the form } j_{2k} \text{ or } j_{2k+1} \text{ with } j \in I(\tau), \\ \text{End} & \text{if } i_{2\sigma_i+1} \text{ for } i \in I(\sigma) \text{ is immediately followed by } \star \\ & \qquad \text{or at the end of the sequence,} \\ \varphi(j) & \text{if } i_{2\sigma_i+1} \text{ for } i \in I(\sigma) \text{ is immediately followed by a symbol} \\ & \qquad \text{of the form } j_k \text{ with } j \in I(\sigma). \end{cases}$$

Then let P be the set partition of preimages under φ. We will identify the summand of $\mu''_\sigma(\tau)$ corresponding to such a set partition P as $\eta''_{\sigma^P}(\tau)$ times a factor depending only on σ and P.

This factor is equal to

$$\prod_{i \in P} F(i),$$

where

$$F(i) = \frac{(2\sigma_i^P + |i| + 1)!}{\prod_{j \in i}(2\sigma_j + 1)!!}.$$

Here $F(i)$ should be interpreted as the number of permutations of $2\sigma_j + 1$ symbols of kind j for each $j \in i$ and one additional symbol End such that the symbols of each kind appear in comb-like order. If these permutations are interpreted as refinement data, then the permutations counted by the P-summand of $\mu''_\sigma(\tau)$ are the refinements by this data of the permutations counted by $\eta''_{\sigma^P}(\tau)$.

Thus we have the identity

$$\mu''_\sigma = \sum_{P \vdash I(\sigma)} \prod_{i \in P} F(i) \eta''_{\sigma^P}.$$

This is a triangular change of basis with nonzero entries on the diagonal, so the μ'' and η'' span the same subspace in $(\mathbb{Q}^{P(r)})^*$. This completes the proof of the Rank Theorem.

Chapter 4

Constructing relations I: moduli of stable quotients

This chapter presents joint work with Rahul Pandharipande [28].

4.1 Introduction

In this chapter we use the virtual geometry of the moduli space of stable quotients studied by Marian, Oprea, and Pandharipande [24] to construct a large family of kappa relations in the tautological ring of $\mathcal{M}_g$. We analyze these relations and prove the Faber-Zagier relations.

Theorem 4.1. *In $R^r(\mathcal{M}_g)$, the Faber-Zagier relation*

$$\big[\exp(-\gamma^{\mathsf{FZ}})\big]_{t^r\mathbf{p}^\sigma} = 0$$

holds when $g-1+|\sigma| < 3r$ and $g \equiv r+|\sigma|+1 \mod 2$.

In fact, as a corollary of our proof of Theorem 4.1 via the moduli space of stable quotients, we obtain the following stronger boundary result. If $g-1+|\sigma| < 3r$ and $g \equiv r+|\sigma|+1 \mod 2$, then

$$\big[\exp(-\gamma^{\mathsf{FZ}})\big]_{t^r\mathbf{p}^\sigma} \in R^*(\partial\overline{\mathcal{M}}_g) \ . \tag{4.1}$$

Not only is the Faber-Zagier relation 0 on $R^*(\mathcal{M}_g)$, but the relation is equal to a tautological class on the boundary of the moduli space $\overline{\mathcal{M}}_g$.

4.1.1 Plan of the chapter

After reviewing stable quotients on curves in Section 4.2, we derive an explicit set of κ relations from the virtual geometry of the moduli space of stable quotients in Section 4.3. The resulting equations are more tractable than those obtained by classical methods. In a series of steps, the stable quotient relations are transformed to simpler and simpler forms. The first step, Theorem 4.7, comes almost immediately from the virtual localization formula [16] applied to the moduli space of stable quotients. After further analysis in Section 4.4, the simpler form of Proposition 4.11 is found. A change of variables is applied in Section 4.5 that transforms the relations to Proposition 4.16. The proof of Theorem 4.1 is completed in Section 4.6.

4.2 Stable quotients

4.2.1 Stability

Our proof of the Faber-Zagier relations in $R^*(\mathcal{M}_g)$ will be obtained from the virtual geometry of the moduli space of stable quotients. We start by reviewing the basic definitions and results of [24].

Let C be a curve which is reduced and connected and has at worst nodal singularities. We require here only unpointed curves. See [24] for the definitions in the pointed case. Let q be a quotient of the rank N trivial bundle C,

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0.$$

If the quotient subsheaf Q is locally free at the nodes and markings of C, then q is a *quasi-stable quotient*. Quasi-stability of q implies the associated kernel,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0,$$

is a locally free sheaf on C. Let r denote the rank of S.

Let C be a curve equipped with a quasi-stable quotient q. The data (C, q) determine a *stable quotient* if the $\mathbb{Q}$-line bundle

$$\omega_C \otimes (\wedge^r S^*)^{\otimes \epsilon} \tag{4.2}$$

is ample on C for every strictly positive $\epsilon \in \mathbb{Q}$. Quotient stability implies $2g - 2 \geq 0$.

Viewed in concrete terms, no amount of positivity of S^* can stabilize a genus 0 component

$$\mathbf{P}^1 \stackrel{\sim}{=} P \subset C$$

unless P contains at least 2 nodes or markings. If P contains exactly 2 nodes or markings, then S^* *must* have positive degree.

A stable quotient (C, q) yields a rational map from the underlying curve C to the Grassmannian $\mathbb{G}(r, N)$. We will only require the $\mathbb{G}(1, 2) = \mathbf{P}^1$ case for the proof Theorem 4.1.

4.2.2 Isomorphism

Let C be a curve. Two quasi-stable quotients

$$\mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0, \quad \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q'}{\to} Q' \to 0 \tag{4.3}$$

on C are *strongly isomorphic* if the associated kernels

$$S, S' \subset \mathbb{C}^N \otimes \mathcal{O}_C$$

are equal.

An *isomorphism* of quasi-stable quotients

$$\phi : (C, q) \to (C', q')$$

is an isomorphism of curves

$$\phi : C \stackrel{\sim}{\to} C'$$

such that the quotients q and $\phi^*(q')$ are strongly isomorphic. Quasi-stable quotients (4.3) on the same curve C may be isomorphic without being strongly isomorphic.

The following result is proven in [24] by Quot scheme methods from the perspective of geometry relative to a divisor.

Theorem 4.2. *The moduli space of stable quotients $\overline{Q}_g(\mathbb{G}(r, N), d)$ parameterizing the data*

$$(C,\ 0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \stackrel{q}{\to} Q \to 0),$$

with $\operatorname{rank}(S) = r$ *and* $\deg(S) = -d$*, is a separated and proper Deligne-Mumford stack of finite type over* $\mathbb{C}$*.*

4.2.3 Structures

Over the moduli space of stable quotients, there is a universal curve

$$\pi : U \to \overline{Q}_g(\mathbb{G}(r, N), d) \tag{4.4}$$

with a universal quotient

$$0 \to S_U \to \mathbb{C}^N \otimes \mathcal{O}_U \stackrel{q_U}{\to} Q_U \to 0.$$

The subsheaf S_U is locally free on U because of the stability condition.

The moduli space $\overline{Q}_g(\mathbb{G}(r, N), d)$ is equipped with two basic types of maps. If $2g - 2 > 0$, then the stabilization of C determines a map

$$\nu : \overline{Q}_g(\mathbb{G}(r, N), d) \to \overline{\mathcal{M}}_g$$

by forgetting the quotient.

The general linear group $\mathbf{GL}_N(\mathbb{C})$ acts on $\overline{Q}_g(\mathbb{G}(r, N), d)$ via the standard action on $\mathbb{C}^N \otimes \mathcal{O}_C$. The structures π, q_U, ν and the evaluations maps are all $\mathbf{GL}_N(\mathbb{C})$-equivariant.

4.2.4 Obstruction theory

The moduli of stable quotients maps to the Artin stack of pointed domain curves

$$\nu^A : \overline{Q}_g(\mathbb{G}(r, N), d) \to \mathcal{M}_g.$$

The moduli of stable quotients with fixed underlying curve $[C] \in \mathcal{M}_g$ is simply an open set of the Quot scheme of C. The following result of [24, Section 3.2] is obtained from the standard deformation theory of the Quot scheme.

Theorem 4.3. *The deformation theory of the Quot scheme determines a 2-term obstruction theory on the moduli space* $\overline{Q}_g(\mathbb{G}(r, N), d)$ *relative to* ν^A *given by* $RHom(S, Q)$*.*

More concretely, for the stable quotient,

$$0 \to S \to \mathbb{C}^N \otimes \mathcal{O}_C \xrightarrow{q} Q \to 0,$$

the deformation and obstruction spaces relative to ν^A are $\mathrm{Hom}(S, Q)$ and $\mathrm{Ext}^1(S, Q)$ respectively. Since S is locally free, the higher obstructions

$$\mathrm{Ext}^k(S, Q) = H^k(C, S^* \otimes Q) = 0, \quad k > 1$$

vanish since C is a curve. An absolute 2-term obstruction theory on the moduli space $\overline{Q}_g(\mathbb{G}(r, N), d)$ is obtained from Theorem 4.3 and the smoothness of $\mathcal{M}_g$, see [2, 3, 13]. The analogue of Theorem 4.3 for the Quot scheme of a *fixed* nonsingular curve was observed in [23].

The $\mathbf{GL}_N(\mathbb{C})$-action lifts to the obstruction theory, and the resulting virtual class is defined in $\mathbf{GL}_N(\mathbb{C})$-equivariant cycle theory,

$$[\overline{Q}_g(\mathbb{G}(r, N), d)]^{vir} \in A_*^{\mathbf{GL}_N(\mathbb{C})}(\overline{Q}_g(\mathbb{G}(r, N), d)).$$

For the construction of the Faber-Zagier relations, we are mainly interested in the open stable quotient space

$$\nu : Q_g(\mathbf{P}^1, d) \longrightarrow \mathcal{M}_g$$

which is simply the corresponding relative Hilbert scheme. However, we will require the full stable quotient space $\overline{Q}_g(\mathbf{P}^1, d)$ to prove the Faber-Zagier relations can be completed over $\mathcal{M}_g$ with tautological boundary terms.

4.3 Stable quotients relations

4.3.1 First statement

Our relations in the tautological ring $R^*(\mathcal{M}_g)$ obtained from the moduli of stable quotients are based on the function

$$\Phi(t, x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1 - it} \, \frac{(-1)^d}{d!} \frac{x^d}{t^d} \ . \tag{4.5}$$

Define the coefficients $\widetilde{C}^d_r$ by the logarithm,

$$\log(\Phi) = \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}^d_r \ t^r \frac{x^d}{d!} \ .$$

By an application of Wick's formula in Section 4.3.3, the t dependence has at most a simple pole. Let

$$\widetilde{\gamma} = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} + \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}^d_r \kappa_r t^r \frac{x^d}{d!} \ . \tag{4.6}$$

Denote the $t^r x^d$ coefficient of $\exp(-\widetilde{\gamma})$ by

$$\Big[\exp(-\widetilde{\gamma})\Big]_{t^r x^d} \in \mathbb{Q}[\kappa_{-1}, \kappa_0, \kappa_1, \kappa_2, \ldots] \ .$$

In fact, $[\exp(-\widetilde{\gamma})]_{t^r x^d}$ is homogeneous of degree r in the κ classes.

The first form of the tautological relations obtained from the moduli of stable quotients is given by the following result.

Proposition 4.4. *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[\exp(-\widetilde{\gamma})\Big]_{t^r x^d} = 0$$

holds when $g - 2d - 1 < r$ and $g \equiv r + 1 \bmod 2$.

For fixed r and d, if Proposition 4.4 applies in genus g, then Proposition 4.4 applies in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$. The genus shifting mod 2 property is present also in the Faber-Zagier relations.

4.3.2 K-theory class $\mathbb{F}_d$

For genus $g \geq 2$, we consider as before

$$\pi^d : \mathcal{C}^d_g \to \mathcal{M}_g \ ,$$

the d-fold product of the universal curve over M_g. Given an element

$$[C, p_1, \ldots, p_d] \in \mathcal{C}^d_g \ ,$$

there is a canonically associated stable quotient

$$0 \to \mathcal{O}_C(-\sum_{j=1}^{d} p_j) \to \mathcal{O}_C \to Q \to 0. \tag{4.7}$$

Consider the universal curve

$$\epsilon : U \to \mathcal{C}_g^d$$

with universal quotient sequence

$$0 \to S_U \to \mathcal{O}_U \to Q_U \to 0$$

obtained from (4.7). Let

$$\mathbb{F}_d = -R\epsilon_*(S_U^*) \in K(\mathcal{C}_g^d)$$

be the class in K-theory. For example,

$$\mathbb{F}_0 = \mathbb{E}^* - \mathbb{C}$$

is the dual of the Hodge bundle minus a rank 1 trivial bundle.

By Riemann-Roch, the rank of $\mathbb{F}_d$ is

$$r_g(d) = g - d - 1.$$

However, $\mathbb{F}_d$ is not always represented by a bundle. By the derivation of [24, Section 4.6],

$$\mathbb{F}_d = \mathbb{E}^* - \mathbb{B}_d - \mathbb{C}, \tag{4.8}$$

where $\mathbb{B}_d$ has fiber $H^0(C, \mathcal{O}_C(\sum_{j=1}^d p_j)|_{\sum_{j=1}^d p_j})$ over $[C, p_1, \ldots, p_d]$.

The Chern classes of $\mathbb{F}_d$ can be easily computed. Recall the divisor $D_{i,j}$ where the markings p_i and p_j coincide. Set

$$\Delta_i = D_{1,i} + \ldots + D_{i-1,i},$$

with the convention $\Delta_1 = 0$. Over $[C, p_1, \ldots, p_d]$, the virtual bundle $\mathbb{F}_d$ is the formal difference

$$H^1(\mathcal{O}_C(p_1 + \ldots + p_d)) - H^0(\mathcal{O}_C(p_1 + \ldots + p_d)).$$

Taking the cohomology of the exact sequence

$$0 \to \mathcal{O}_C(p_1 + \ldots + p_{d-1}) \to \mathcal{O}_C(p_1 + \ldots + p_d) \to \mathcal{O}_C(p_1 + \ldots + p_d)|_{\widehat{p}_d} \to 0,$$

we find

$$c(\mathbb{F}_d) = \frac{c(\mathbb{F}_{d-1})}{1 + \Delta_d - \psi_d}.$$

Inductively, we obtain

$$c(\mathbb{F}_d) = \frac{c(\mathbb{E}^*)}{(1 + \Delta_1 - \psi_1) \cdots (1 + \Delta_d - \psi_d)}.$$

Equivalently, we have

$$c(-\mathbb{B}_d) = \frac{1}{(1 + \Delta_1 - \psi_1) \cdots (1 + \Delta_d - \psi_d)}. \tag{4.9}$$

4.3.3 Proof of Proposition 4.4

Consider the proper morphism

$$\nu : Q_g(\mathbf{P}^1, d) \to \mathcal{M}_g.$$

Certainly the class

$$\nu_* \left(0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) \in A^*(\mathcal{M}_g, \mathbb{Q}), \tag{4.10}$$

where 0 is the first Chern class of the trivial bundle, vanishes if $c > 0$. Proposition 4.4 is proven by calculating (4.10) by localization. We will find Proposition 4.4 is a subset of the much richer family of relations of Theorem 4.7 of Section 4.3.4.

Let the torus $\mathbb{C}^*$ act on a 2-dimensional vector space $V \stackrel{\sim}{=} \mathbb{C}^2$ with diagonal weights $[0, 1]$. The $\mathbb{C}^*$-action lifts canonically to $\mathbf{P}(V)$ and $Q_g(\mathbf{P}(V), d)$. We lift the $\mathbb{C}^*$-action to a rank 1 trivial bundle on $Q_g(\mathbf{P}(V), d)$ by specifying fiber weight 1. The choices determine a $\mathbb{C}^*$-lift of the class

$$0^c \cap [Q_g(\mathbf{P}(V), d)]^{vir} \in A_{2d+2g-2-c}(Q_g(\mathbf{P}(V), d), \mathbb{Q}).$$

The pushforward (4.10) is determined by the virtual localization formula [16]. There are only two $\mathbb{C}^*$-fixed loci. The first corresponds to a vertex lying over $0 \in \mathbf{P}(V)$. The locus is isomorphic to

$$\mathcal{C}_g^d \ / \ \mathbb{S}_d$$

and the associated subsheaf (4.7) lies in the first factor of $V \otimes \mathcal{O}_C$ when considered as a stable quotient in the moduli space $Q_g(\mathbf{P}(V), d)$. Similarly, the second fixed locus corresponds to a vertex lying over $\infty \in \mathbf{P}(V)$.

The localization contribution of the first locus to (4.10) is

$$\frac{1}{d!}\pi^d_* \left(c_{g-d-1+c}(\mathbb{F}_d)\right) \quad \text{where} \quad \pi^d : \mathcal{C}^d_g \to \mathcal{M}_g \ .$$

Let $c_-(\mathbb{F}_d)$ denote the total Chern class of $\mathbb{F}_d$ evaluated at -1. The localization contribution of the second locus is

$$\frac{(-1)^{g-d-1}}{d!}\pi^d_*\Big[c_-(\mathbb{F}_d)\Big]^{g-d-1+c}$$

where $[\gamma]^k$ is the part of γ in $A^k(\mathcal{C}^d_g, \mathbb{Q})$.

Both localization contributions are found by straightforward expansion of the vertex formulas of [24, Section 7.4.2]. Summing the contributions yields

$$\pi^d_*\Big(c_{g-d-1+c}(\mathbb{F}_d) + (-1)^{g-d-1}\Big[c_-(\mathbb{F}_d)\Big]^{g-d-1+c}\Big) = 0 \quad \text{in} \quad R^*(\mathcal{M}_g)$$

for $c > 0$. We obtain the following result.

Lemma 4.5. *For $c > 0$ and $c \equiv 0 \mod 2$,*

$$\pi^d_*\Big(c_{g-d-1+c}(\mathbb{F}_d)\Big) = 0 \quad \textit{in} \quad R^*(\mathcal{M}_g) \ .$$

For $c > 0$, the relation of Lemma 4.5 lies in $R^r(\mathcal{M}_g)$ where

$$r = g - 2d - 1 + c \ .$$

Moreover, the relation is trivial unless

$$g - d - 1 \equiv g - d - 1 + c = r - d \mod 2 \ . \tag{4.11}$$

We may expand the right side of (4.9) fully. The resulting expression is a polynomial in the $d + \binom{d}{2}$ variables.

$$\psi_1, \ldots, \psi_d, -D_{12}, -D_{13}, \ldots, -D_{d-1,d} \ .$$

Let $\widetilde{M}_r^d$ denote the coefficient in degree r,

$$c_t(-\mathbb{B}_d) = \sum_{r=0}^{\infty} \widetilde{M}_r^d(\psi_i, -D_{ij})\ t^r.$$

Let $\widetilde{S}_r^d$ be the summand of the evaluation $\widetilde{M}_r^d(\psi_i = 1, -D_{ij} = 1)$ consisting of the contributions of only the connected monomials.

Lemma 4.6. *We have*

$$\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \widetilde{S}_r^d\ t^r \frac{x^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty}\prod_{i=1}^{d} \frac{1}{1-it}\ \frac{x^d}{d!}\right)\ .$$

Proof. By Wick's formula, the connected and disconnected counts are related by exponentiation,

$$\exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \widetilde{S}_r^d\ t^r \frac{x^d}{d!}\right) = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \widetilde{M}_r^d(\widehat{\psi}_i = 1, -D_{ij} = 1)\ t^r \frac{x^d}{d!}\ .$$

$\square$

Since a connected monomial in the variables ψ_i and $-D_{ij}$ must have at least $d-1$ factors of the variables $-D_{ij}$, we see $\widetilde{S}_r^d = 0$ if $r < d-1$. Using the self-intersection formulas, we obtain

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi_*^d\big(c_r(-\mathbb{B}_d)\big)\ t^r \frac{x^d}{d!} = \exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \widetilde{S}_r^d (-1)^{d-1}\kappa_{r-d}\ t^r \frac{x^d}{d!}\right)\ . \tag{4.12}$$

To account for the alternating factor $(-1)^{d-1}$ and the κ subscript, we define the coefficients $\widetilde{C}_r^d$ by

$$\sum_{d=1}^{\infty}\sum_{r\geq -1} \widetilde{C}_r^d\ t^r \frac{x^d}{d!} = \log\left(1 + \sum_{d=1}^{\infty}\prod_{i=1}^{d} \frac{1}{1-it}\ \frac{(-1)^d}{t^d}\frac{x^d}{d!}\right)\ .$$

The vanishing $\widetilde{S}_{r<d-1}^d = 0$ implies the vanishing $\widetilde{C}_{r<-1}^d = 0$.

Again using Mumford's Grothendieck-Riemann-Roch calculation [26],

$$c_t(\mathbb{E}^*) = -\sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1}\ .$$

Putting the above results together yields the following formula:

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi_*^d\big(c_r(\mathbb{F}_d)\big)\ t^{r-d}\frac{x^d}{d!} = \exp\left(-\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1} - \sum_{d=1}^{\infty}\sum_{r\geq -1}\widetilde{C}_r^d\kappa_r t^r\frac{x^d}{d!}\right)\ .$$

The restrictions on g, d, and r in the statement of Proposition 4.4 are obtained from (4.11). $\square$

4.3.4 Extended relations

The universal curve

$$\epsilon : U \to Q_g(\mathbf{P}^1, d)$$

carries the basic divisor classes

$$s = c_1(S_U^*), \quad \omega = c_1(\omega_\pi)$$

obtained from the universal subsheaf S_U of the moduli of stable quotients and the ϵ-relative dualizing sheaf. Following [24, Proposition 5], we can obtain a much larger set of relations in the tautological ring of $\mathcal{M}_g$ by including factors of $\epsilon_*(s^{a_i}\omega^{b_i})$ in the integrand:

$$\nu_* \left(\prod_{i=1}^{n} \epsilon_*(s^{a_i}\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) = 0 \ \text{ in } A^*(\mathcal{M}_g, \mathbb{Q})$$

when $c > 0$. We will study the associated relations where the a_i are always 1. The b_i then form the parts of a partition σ.

To state the relations we obtain, we start by extending the function $\widetilde{\gamma}$ of Section 4.3.1,

$$\begin{aligned} \gamma^{\mathsf{SQ}} &= \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} \\ &\quad + \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \kappa_{r+|\sigma|} \ t^r \frac{x^d}{d!} \ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \ . \end{aligned}$$

Let $\overline{\gamma}^{\mathsf{SQ}}$ be defined by a similar formula,

$$\begin{aligned} \overline{\gamma}^{\mathsf{SQ}} &= \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} (-t)^{2i-1} \\ &\quad + \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \kappa_{r+|\sigma|} \ (-t)^r \frac{x^d}{d!} \ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \ . \end{aligned}$$

The sign of t in $t^{|\sigma|}$ does not change in $\overline{\gamma}^{\mathsf{SQ}}$. The κ_{-1} terms which appear will later be set to 0.

The full system of relations are obtained from the coefficients of the functions

$$\exp(-\gamma^{\mathsf{SQ}}), \quad \exp(-\sum_{r=0}^{\infty} \kappa_r t^r p_{r+1}) \cdot \exp(-\overline{\gamma}^{\mathsf{SQ}})$$

Theorem 4.7. *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[\exp(-\gamma^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^\sigma} = (-1)^g \Big[\exp(-\sum_{r=0}^{\infty} \kappa_r t^r p_{r+1}) \cdot \exp(-\overline{\gamma}^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^\sigma}$$

holds when $g - 2d - 1 + |\sigma| < r$.

Again, we see the genus shifting mod 2 property. If the relation holds in genus g, then the *same* relation holds in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$.

In case $\sigma = \emptyset$, Theorem 4.7 specializes to the relation

$$\begin{aligned} \Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} &= (-1)^g \Big[\exp(-\widetilde{\gamma}(-t,x))\Big]_{t^r x^d} \\ &= (-1)^{g+r} \Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} , \end{aligned}$$

nontrivial only if $g \equiv r + 1 \bmod 2$. If the mod 2 condition holds, then we obtain the relations of Proposition 4.4.

Consider the case $\sigma = (1)$. The left side of the relation is then

$$\Big[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(-\sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} \widetilde{C}_s^d \;\kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} .$$

The right side is

$$(-1)^g \Big[\exp(-\widetilde{\gamma}(-t,x)) \cdot \left(-\kappa_0 t^0 + \sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} \widetilde{C}_s^d \;\kappa_{s+1} (-t)^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} .$$

If $g \equiv r + 1 \bmod 2$, then the large terms cancel and we obtain

$$-\kappa_0 \cdot \Big[\exp(-\widetilde{\gamma}(t,x))\Big]_{t^r x^d} = 0 .$$

Since $\kappa_0 = 2g - 2$ and

$$(g - 2d - 1 + 1 < r) \quad \Longrightarrow \quad (g - 2d - 1 < r),$$

we recover most (but not all) of the $\sigma = \emptyset$ equations.

If $g \equiv r \bmod 2$, then the resulting equation is

$$\Big[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(\kappa_0 - 2\sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} \widetilde{C}_s^d \;\kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} = 0$$

when $g - 2d < r$.

4.3.5 Proof of Theorem 4.7

Partitions, differential operators, and logs.

We will write partitions σ as $(1^{n_1}2^{n_2}3^{n_3}\ldots)$ with

$$\ell(\sigma) = \sum_i n_i \quad \text{and} \quad |\sigma| = \sum_i in_i \ .$$

The empty partition $\emptyset$ corresponding to $(1^0 2^0 3^0 \ldots)$ is permitted. In all cases, we have

$$|\mathrm{Aut}(\sigma)| = n_1! n_2! n_3! \cdots \ .$$

In the infinite set of variables $\{p_1, p_2, p_3, \ldots\}$, let

$$\Phi^{\mathbf{p}}(t,x) = \sum_\sigma \sum_{d=0}^\infty \prod_{i=1}^d \frac{1}{1-it} \frac{(-1)^d}{d!} \frac{x^d}{t^d} \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|} \ ,$$

where the first sum is over all partitions σ. The summand corresponding to the empty partition equals $\Phi(t,x)$ defined in (4.5).

The function $\Phi^{\mathbf{p}}$ is easily obtained from Φ,

$$\Phi^{\mathbf{p}}(t,x) = \exp\left(\sum_{i=1}^\infty p_i t^i x \frac{d}{dx}\right) \ \Phi(t,x) \ .$$

Let D denote the differential operator

$$D = \sum_{i=1}^\infty p_i t^i x \frac{d}{dx} \ .$$

Expanding the exponential of D, we obtain

$$\begin{aligned} \Phi^{\mathbf{p}} &= \Phi + D\Phi + \frac{1}{2} D^2 \Phi + \frac{1}{6} D^3 \Phi + \ldots \qquad (4.13) \\ &= \Phi \left(1 + \frac{D\Phi}{\Phi} + \frac{1}{2}\frac{D^2\Phi}{\Phi} + \frac{1}{6}\frac{D^3\Phi}{\Phi} + \ldots\right) \ . \end{aligned}$$

Let $\gamma^* = \log(\Phi)$ be the logarithm,

$$D\gamma^* = \frac{D\Phi}{\Phi} \ .$$

After applying the logarithm to (4.13), we see

$$\begin{aligned}
\log(\Phi^{\mathbf{P}}) &= \gamma^* + \log\left(1 + D\gamma^* + \frac{1}{2}(D^2\gamma^* + (D\gamma^*)^2) + \ \ldots\right) \\
&= \gamma^* + D\gamma^* + \frac{1}{2}D^2\gamma^* + \ldots
\end{aligned}$$

where the dots stand for a universal expression in the $D^k\gamma^*$. In fact, a remarkable simplification occurs,

$$\log(\Phi^{\mathbf{P}}) = \exp\left(\sum_{i=1}^{\infty} p_i t^i x \frac{d}{dx}\right) \ \gamma^* \ .$$

The result follows from a general identity.

Proposition 4.8. *If f is a function of x, then*

$$\log\left(\exp\left(\lambda x \frac{d}{dx}\right) \ f\right) = \exp\left(\lambda x \frac{d}{dx}\right) \ \log(f) \ .$$

Proof. A simple computation for monomials in x shows

$$\exp\left(\lambda x \frac{d}{dx}\right) \ x^k = (e^{\lambda} x)^k \ .$$

Hence, since the differential operator is additive,

$$\exp\left(\lambda x \frac{d}{dx}\right) \ f(x) = f(e^{\lambda} x) \ .$$

The Proposition follows immediately. □

We apply Proposition 4.8 to $\log(\Phi^{\mathbf{P}})$. The coefficients of the logarithm may be written as

$$\begin{aligned}
\log(\Phi^{\mathbf{P}}) &= \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d(\sigma) \ t^r \frac{x^d}{d!} \mathbf{p}^{\sigma} \\
&= \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \ t^r \frac{x^d}{d!} \exp\left(\sum_{i=1}^{\infty} d p_i t^i\right) \\
&= \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d \ t^r \frac{x^d}{d!} \ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|} \ .
\end{aligned}$$

We have expressed the coefficients $\widetilde{C}_r^d(\sigma)$ of $\log(\Phi^{\mathbf{P}})$ solely in terms of the coefficients $\widetilde{C}_r^d$ of $\log(\Phi)$.

Cutting classes

Let $\theta_i \in A^1(U,\mathbb{Q})$ be the class of the i^{th} section of the universal curve

$$\epsilon : U \to \mathcal{C}_g^d \tag{4.14}$$

The class $s = c_1(S_U^*)$ on the universal curve over $Q_g(\mathbf{P}^1,d)$ restricted to the $\mathbb{C}^*$-fixed locus $\mathcal{C}_g^d/\mathbb{S}_d$ and pulled-back to (4.14) yields

$$s = \theta_1 + \ldots + \theta_d \ \in A^1(U,\mathbb{Q}).$$

We calculate

$$\epsilon_*(s\ \omega^b) = \psi_1^b + \ldots + \psi_d^b \quad \in A^b(\mathcal{C}_g^d,\mathbb{Q})\ . \tag{4.15}$$

Wick form

We repeat the Wick analysis of Section 4.3.3 for the vanishings

$$\nu_*\left(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1,d)]^{vir}\right) = 0 \ \text{ in } A^*(\mathcal{M}_g,\mathbb{Q})$$

when $c > 0$. We start by writing a formula for

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi_*^d\left(\exp\Big(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i)\Big) \cdot c_r(\mathbb{F}_d) t^r\right)\ \frac{1}{t^d}\frac{x^d}{d!}\ .$$

Applying the Wick formula to equation (4.15) for the cutting classes, we see

$$\sum_{d=1}^{\infty}\sum_{r\geq 0} \pi_*^d\left(\exp\Big(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i)\Big) \cdot c_r(\mathbb{F}_d) t^r\right)\ \frac{1}{t^d}\frac{x^d}{d!} = \exp(-\widetilde{\gamma}^{\mathsf{SQ}}) \tag{4.16}$$

where $\widetilde{\gamma}^{\mathsf{SQ}}$ is defined by

$$\widetilde{\gamma}^{\mathsf{SQ}} = \sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1} + \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty}\widetilde{C}_r^d(\sigma)\kappa_r\ t^r\frac{x^d}{d!}\ \mathbf{p}^{\sigma}\ .$$

We follow here the notation of Section 4.3.5,

$$\Phi^{\mathbf{P}}(t,x) = \sum_{\sigma}\sum_{d=0}^{\infty}\prod_{i=1}^{d}\frac{1}{1-it}\ \frac{(-1)^d}{d!}\frac{x^d}{t^d}\ \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ ,$$

$$\log(\Phi^{\mathbf{P}}) = \sum_{\sigma} \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} \widetilde{C}_r^d(\sigma)\ t^r \frac{x^d}{d!} \mathbf{p}^{\sigma}\ .$$

In the Wick analysis, the class $\epsilon_*(s\omega^b)$ simply acts as dt^b.

Using the expression for the coefficents $\widetilde{C}_r^d(\sigma)$ in terms of $\widetilde{C}_r^d$ derived in Section 4.3.5, we obtain the following result from (4.16).

Proposition 4.9. *We have*

$$\sum_{d=1}^{\infty} \sum_{r\geq 0} \pi_*^d \left(\exp\Big(\sum_{i=1}^{\infty} p_i t^i \epsilon_*(s\omega^i) \Big) \cdot c_r(\mathbb{F}_d) t^r \right)\ \frac{1}{t^d} \frac{x^d}{d!} = \exp(-\gamma^{\mathsf{SQ}})\ .$$

Geometric construction

We apply $\mathbb{C}^*$-localization on $Q_g(\mathbf{P}^1, d)$ to the geometric vanishing

$$\nu_* \left(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir} \right) = 0 \ \text{ in } A^*(\mathcal{M}_g, \mathbb{Q}) \tag{4.17}$$

when $c > 0$. The result is the relation

$$\pi_* \Big(\prod_{i=1}^{\ell} \epsilon_*(s\omega^{b_i}) \cdot c_{g-d-1+c}(\mathbb{F}_d) + \\ (-1)^{g-d-1} \Big[\prod_{i=1}^{\ell} \epsilon_* \left((s-1)\omega^{b_i} \right) \cdot c_-(\mathbb{F}_d) \Big]^{g-d-1+\sum_i b_i + c} \Big) = 0 \tag{4.18}$$

in $R^*(\mathcal{M}_g)$. After applying the Wick formula of Proposition 4.9, we immediately obtain Theorem 4.7.

The first summand in (4.18) yields the left side

$$\Big[\exp(-\gamma^{\mathsf{SQ}}) \Big]_{t^r x^d \mathbf{p}^{\sigma}}$$

of the relation of Theorem 4.7. The second summand produces the right side

$$(-1)^g \Big[\exp(-\sum_{r=0}^{\infty} \kappa_r t^r p_{r+1}) \cdot \exp(-\widehat{\gamma}^{\mathsf{SQ}}) \Big]_{t^r x^d \mathbf{p}^{\sigma}}\ . \tag{4.19}$$

Recall the localization of the virtual class over $\infty \in \mathbf{P}^1$ is

$$\frac{(-1)^{g-d-1}}{d!} \pi_*^d \Big[c_-(\mathbb{F}_d) \Big]^{g-d-1+c}\ .$$

Of the sign prefactor $(-1)^{g-d-1}$,

- $(-1)^{-1}$ is used to move the term to the right side,
- $(-1)^{-d}$ is absorbed in the $(-t)$ of the definition of $\widehat{\gamma}^{\mathrm{SQ}}$,
- $(-1)^{g}$ remains in (4.19).

The -1 of $s-1$ produces the the factor $\exp(-\sum_{r=0}^{\infty}\kappa_r t^r p_{r+1})$.

Finally, a simple dimension calculation (remembering $c>0$) implies the validity of the relation when $g-2d-1+|\sigma|<r$. $\square$

4.4 Analysis of the relations

4.4.1 Expanded form

Let $\sigma=(1^{a_1}2^{a_2}3^{a_3}\ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We can directly write the corresponding tautological relation in $R^r(\mathcal{M}_g)$ obtained from Theorem 4.7.

A *subpartition* $\sigma'\subset\sigma$ is obtained by selecting a nontrivial subset of the parts of σ. A *division* of σ is a disjoint union

$$\sigma=\sigma^{(1)}\cup\sigma^{(2)}\cup\sigma^{(3)}\ldots \tag{4.20}$$

of subpartitions which exhausts σ. The subpartitions in (4.20) are unordered. Let $\mathcal{S}(\sigma)$ be the set of divisions of σ. For example,

$$\begin{aligned}\mathcal{S}(1^12^1) &= \{\ (1^12^1),\ (1^1)\cup(2^1)\ \}\ ,\\ \mathcal{S}(1^3) &= \{\ (1^3),\ (1^2)\cup(1^1)\ \}\ .\end{aligned}$$

We will use the notation $\sigma^\bullet$ to denote a division of σ with subpartitions $\sigma^{(i)}$. Let

$$m(\sigma^\bullet)=\frac{1}{|\mathrm{Aut}(\sigma^\bullet)|}\frac{|\mathrm{Aut}(\sigma)|}{\prod_{i=1}^{\ell(\sigma^\bullet)}|\mathrm{Aut}(\sigma^{(i)})|}.$$

Here, $\mathrm{Aut}(\sigma^\bullet)$ is the group permuting equal subpartitions. The factor $m(\sigma^\bullet)$ may be interpreted as counting the number of different ways the disjoint union can be made.

To write explicitly the $\mathbf{p}^\sigma$ coefficient of $\exp(\gamma^{\mathrm{SQ}})$, we introduce the functions

$$F_{n,m}(t,x)=-\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty}\widetilde{C}_s^d\ \kappa_{s+m}t^{s+m}\frac{d^nx^d}{d!}$$

for $n, m \geq 1$. Then,

$$|\mathrm{Aut}(\sigma)| \cdot \Big[\exp(-\gamma^{\mathsf{SQ}})\Big]_{t^r x^d \mathbf{p}^\sigma} = \Big[\exp(-\widetilde{\gamma}(t,x)) \cdot \left(\sum_{\sigma^\bullet \in \mathcal{S}(\sigma)} m(\sigma^\bullet) \prod_{i=1}^{\ell(\sigma^\bullet)} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \Big]_{t^r x^d}.$$

Let $\sigma^{*,\bullet}$ be a division of σ with a marked subpartition,

$$\sigma = \sigma^* \cup \sigma^{(1)} \cup \sigma^{(2)} \cup \sigma^{(3)} \ldots, \tag{4.21}$$

labelled by the superscript $*$. The marked subpartition is permitted to be empty. Let $\mathcal{S}^*(\sigma)$ denote the set of marked divisions of σ. Let

$$m(\sigma^{*,\bullet}) = \frac{1}{|\mathrm{Aut}(\sigma^\bullet)|} \frac{|\mathrm{Aut}(\sigma)|}{|\mathrm{Aut}(\sigma^*)| \prod_{i=1}^{\ell(\sigma^{*,\bullet})} |\mathrm{Aut}(\sigma^{(i)})|}.$$

The length $\ell(\sigma^{*,\bullet})$ is the number of unmarked subpartitions.

Then, $|\mathrm{Aut}(\sigma)|$ times the right side of Theorem 4.7 may be written as

$$(-1)^{g+|\sigma|} |\mathrm{Aut}(\sigma)| \cdot \Big[\exp(-\widetilde{\gamma}(-t,x)) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1} (-t)^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}(-t,x) \right) \Big]_{t^r x^d}$$

To write Theorem 4.7 in the simplest form, the following definition using the Kronecker δ is useful,

$$m^\pm(\sigma^{*,\bullet}) = (1 \pm \delta_{0,|\sigma^*|}) \cdot m(\sigma^{*,\bullet}).$$

There are two cases. If $g \equiv r + |\sigma| \bmod 2$, then Theorem 3 is equivalent to the vanishing of

$$|\mathrm{Aut}(\sigma)| \Big[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m^-(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1} t^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \Big]_{t^r x^d}.$$

If $g \equiv r + |\sigma| + 1 \bmod 2$, then Theorem 4.7 is equivalent to the vanishing of

$$|\mathrm{Aut}(\sigma)| \Big[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} m^+(\sigma^{*,\bullet}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^*-1} t^{\sigma_j^*-1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \Big]_{t^r x^d}.$$

In either case, the relations are valid in the ring $R^*(\mathcal{M}_g)$ only if the condition $g - 2d - 1 + |\sigma| < r$ holds.

We denote the above relation corresponding to g, r, d, and σ (and depending upon the parity of $g-r-|\sigma|$) by

$$\mathsf{R}(g,r,d,\sigma)=0$$

The $|\mathrm{Aut}(\sigma)|$ prefactor is included in $\mathsf{R}(g,r,d,\sigma)$, but is only relevant when σ has repeated parts. In case of repeated parts, the automorphism scaled normalization is more convenient.

4.4.2 Further examples

If $\sigma=(k)$ has a single part, then the two cases of Theorem 4.7 are the following. If $g\equiv r+k$ mod 2, we have

$$\Big[\exp(-\widetilde{\gamma})\cdot\kappa_{k-1}t^{k-1}\Big]_{t^r x^d}=0$$

which is a consequence of the $\sigma=\emptyset$ case. If $g\equiv r+k+1$ mod 2, we have

$$\Big[\exp(-\widetilde{\gamma})\cdot\big(\kappa_{k-1}t^{k-1}+2F_{1,k}\big)\Big]_{t^r x^d}=0$$

If $\sigma=(k_1k_2)$ has two distinct parts, then the two cases of Theorem 4.7 are as follows. If $g\equiv r+k_1+k_2$ mod 2, we have

$$\Big[\exp(-\widetilde{\gamma})\cdot\big(\kappa_{k_1-1}\kappa_{k_2-1}t^{k_1+k_2-2}+\kappa_{k_1-1}t^{k_1-1}F_{1,k_2}+\kappa_{k_2-1}t^{k_2-1}F_{1,k_1}\big)\Big]_{t^r x^d}=0\ .$$

If $g\equiv r+k_1+k_2+1$ mod 2, we have

$$\Big[\exp(-\widetilde{\gamma})\cdot\big(\kappa_{k_1-1}\kappa_{k_2-1}t^{k_1+k_2-2}+\kappa_{k_1-1}t^{k_1-1}F_{1,k_2}$$
$$+\kappa_{k_2-1}t^{k_2-1}F_{1,k_1}+2F_{2,k_1+k_2}+2F_{1,k_1}F_{1,k_2}\big)\Big]_{t^r x^d}=0\ .$$

In fact, the $g\equiv r+k_1+k_2$ mod 2 equation above is not new. The genus g and codimension $r_1=r-k_2+1$ case of partition (k_1) yields

$$\Big[\exp(-\widetilde{\gamma})\cdot\big(\kappa_{k_1-1}t^{k_1-1}+2F_{1,k_1}\big)\Big]_{t^{r_1} x^d}=0\ .$$

After multiplication with $\kappa_{k_2-1}t^{k_2-1}$, we obtain

$$\Big[\exp(-\widetilde{\gamma})\cdot\big(\kappa_{k_1-1}\kappa_{k_2-1}t^{k_1+k_2-2}+2\kappa_{k_2-1}t^{k_2-1}F_{1,k_1}\big)\Big]_{t^r x^d}=0\ .$$

Summed with the corresponding equation with k_1 and k_2 interchanged yields the above $g \equiv r+k_1+k_2 \mod 2$ case.

4.4.3 Expanded form revisited

Consider the partition $\sigma = (k_1 k_2 \cdots k_\ell)$ with distinct parts. Relation $\mathsf{R}(g,r,d,\sigma)$, in the $g \equiv r + |\sigma| \mod 2$ case, is the vanishing of

$$\left[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1 - \delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^* - 1} t^{\sigma_j^* - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

since all the factors $m(\sigma^{*,\bullet})$ are 1. In the $g \equiv r + |\sigma| + 1 \mod 2$ case, $\mathsf{R}(g,r,d,\sigma)$ is the vanishing of

$$\left[\exp(-\widetilde{\gamma}) \cdot \left(\sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1 + \delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma_j^* - 1} t^{\sigma_j^* - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

for the same reason.

If σ has repeated parts, the relation $\mathsf{R}(g,r,d,\sigma)$ is obtained by viewing the parts as distinct and specializing the indicies afterwards. For example, the two cases for $\sigma = (k^2)$ are as follows. If $g \equiv r + 2k \mod 2$, we have

$$\left[\exp(-\widetilde{\gamma}) \cdot \left(\kappa_{k-1}\kappa_{k-1} t^{2k-2} + 2\kappa_{k-1} t^{k-1} F_{1,k}\right)\right]_{t^r x^d} = 0 \ .$$

If $g \equiv r + 2k + 1 \mod 2$, we have

$$\left[\exp(-\widetilde{\gamma}) \cdot \left(\kappa_{k-1}\kappa_{k-1} t^{2k-2} + 2\kappa_{k-1} t^{k-1} F_{1,k} + 2F_{2,2k} + 2F_{1,k}F_{1,k}\right)\right]_{t^r x^d} = 0 \ .$$

The factors occur via repetition of terms in the formulas for distinct parts.

Proposition 4.10. *The relation $\mathsf{R}(g,r,d,\sigma)$ in the $g \equiv r + |\sigma| \mod 2$ case is a consequence of the relations in $\mathsf{R}(g,r',d,\sigma')$ where $g \equiv r' + |\sigma'| + 1 \mod 2$ and $\sigma' \subset \sigma$ is a strictly smaller partition.*

Proof. The strategy follows the example of the phenonenon already discussed in Section 4.4.2.

If $g \equiv r + |\sigma| \mod 2$, then for every subpartition $\tau \subset \sigma$ of odd length, we have

$$g \equiv r - |\tau| + \ell(\tau) + |\sigma/\tau| + 1 \mod 2$$

where σ/τ is the complement of τ. The relation

$$\prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}(g, r - |\tau| + \ell(\tau), d, \sigma/\tau)$$

is of codimension r.

Let $g \equiv r + |\sigma| \bmod 2$, and let σ have distinct parts. The formula

$$\mathsf{R}(g,r,d,\sigma) = \sum_{\tau \subset \sigma} \left(\frac{2^{\ell(\tau)+2} - 2}{\ell(\tau)+1} \right) B_{\ell(\tau)+1} \cdot \prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}\Big(g, r - |\tau| + \ell(\tau), d, \sigma/\tau\Big) \tag{4.22}$$

follows easily by grouping like terms and the Bernoulli identity

$$\sum_{k \geq 1} \binom{n}{2k-1} \left(\frac{2^{2k+1} - 2}{2k} \right) B_{2k} = -\left(\frac{2^{n+2} - 2}{n+1} \right) B_{n+1} \tag{4.23}$$

for $n > 0$. The sum in (4.22) is over all subpartitions $\tau \subset \sigma$ of odd length.

The proof of the Bernoulli identity (4.23) is straightforward. Let

$$a_i = \left(\frac{2^{i+2} - 2}{i+1} \right) B_{i+1} \ , \quad A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Using the definition of the Bernoulli numbers as

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} \ ,$$

we see

$$A(x) = \frac{2}{x} \sum_{i=0}^{\infty} (2^i - 1) B_r \frac{x^r}{r!} = \frac{2}{x} \left(\frac{2x}{e^{2x} - 1} - \frac{x}{e^x - 1} \right) = -\left(\frac{2}{1 + e^x} \right) \ .$$

The identity (4.23) follows from the series relation

$$e^x A(x) = -A(x) - 2 \ .$$

Formula (4.22) is valid for $\mathsf{R}(g,r,d,\sigma)$ even when σ has repeated parts: the sum should be interpreted as running over all odd subsets $\tau \subset \sigma$ (viewing the parts of σ as distinct). $\square$

4.4.4 Recasting

We will recast the relations $\mathsf{R}(g,r,d,\sigma)$ in case $g \equiv r + |\sigma| + 1 \bmod 2$ in a more convenient form. The result will be crucial to the further analysis in Section 4.5.

Let $g \equiv r + |\sigma| + 1 \mod 2$, and let $\mathsf{S}(g, r, d, \sigma)$ denote the κ polynomial

$$|\mathrm{Aut}|\Big[\exp\left(-\widetilde{\gamma}(t,x) + \sum_{\sigma \neq \emptyset}\Big(F_{\ell(\sigma),|\sigma|} + \frac{\delta_{\ell(\sigma),1}}{2}\kappa_{|\sigma|-1}\Big)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{t^r x^d \mathbf{p}^\sigma} .$$

We can write $\mathsf{S}(g, r, d, \sigma)$ in terms of our previous relations $\mathsf{R}(g, r', d, \sigma')$ satisfying $g \equiv r' + |\sigma'| + 1 \mod 2$ and $\sigma' \subset \sigma$:

If $g \equiv r + |\sigma| + 1 \mod 2$, then for every subpartition $\tau \subset \sigma$ of even length (including the case $\tau = \emptyset$), we have

$$g \equiv r - |\tau| + \ell(\tau) + |\sigma/\tau| + 1 \mod 2$$

where σ/τ is the complement of τ. The relation

$$\prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}(g, r - |\tau| + \ell(\tau), d, \sigma/\tau)$$

is of codimension r.

In order to express S in terms of R, we define $z_i \in \mathbb{Q}$ by

$$\frac{2}{e^x + e^{-x}} = \sum_{i=0}^{\infty} z_i \frac{x^i}{i!} .$$

Let $g \equiv r + |\sigma| + 1 \mod 2$, and let σ have distinct parts. The formula

$$\mathsf{S}(g, r, d, \sigma) = \sum_{\tau \subset \sigma} \frac{z_{\ell(\tau)}}{2^{\ell(\tau)+1}} \cdot \prod_i \kappa_{\tau_i - 1} \cdot \mathsf{R}\Big(g, r - |\tau| + \ell(\tau), d, \sigma/\tau\Big) \tag{4.24}$$

follows again grouping like terms and the combinatorial identity

$$\sum_{i \geq 0} \binom{n}{i} \frac{z_i}{2^i + 1} = -\frac{z_n}{2^{n+1}} - \frac{1}{2^n} \tag{4.25}$$

for $n > 0$. The sum in (4.24) is over all subpartitions $\tau \subset \sigma$ of even length.

As before, there the identity (4.25) is straightforward to prove. We see

$$Z(x) = \sum_{i=0}^{\infty} \frac{z_i}{2^{i+1}} \frac{x^i}{i!} = \frac{1}{e^{x/2} + e^{-x/2}} .$$

The identity (4.25) follows from the series relation

$$e^x Z(x) = e^{x/2} - Z(x).$$

Formula (4.22) is valid for $\mathsf{S}(g,r,d,\sigma)$ even when σ has repeated parts: the sum should be interpreted as running over all even subsets $\tau \subset \sigma$ (viewing the parts of σ as distinct). We have proved the following result.

Proposition 4.11. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp\left(-\widetilde{\gamma}(t,x) + \sum_{\sigma\neq\emptyset}\left(F_{\ell(\sigma),|\sigma|} + \frac{\delta_{\ell(\sigma),1}}{2}\kappa_{|\sigma|-1}\right)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{t^r x^d \mathbf{p}^\sigma} = 0$$

holds when $g-2d-1+|\sigma| < r$ and $g \equiv r+|\sigma|+1$ mod 2.

4.5 Transformation

4.5.1 Differential equations

The function Φ satisfies a basic differential equation obtained from the series definition:

$$\frac{d}{dx}(\Phi - tx\frac{d}{dx}\Phi) = -\frac{1}{t}\Phi\ .$$

After expanding and dividing by Φ, we find

$$-tx\frac{\Phi_{xx}}{\Phi} - t\frac{\Phi_x}{\Phi} + \frac{\Phi_x}{\Phi} = -\frac{1}{t}$$

which can be written as

$$-t^2x\gamma^*_{xx} = t^2x(\gamma^*_x)^2 + t^2\gamma^*_x - t\gamma^*_x - 1 \tag{4.26}$$

where, as before, $\gamma^* = \log(\Phi)$. Equation (4.26) has been studied by Ionel in *Relations in the tautological ring* [18]. We present here results of hers which will be useful for us.

To kill the pole and match the required constant term, we will consider the function

$$\Gamma = -t\left(\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}t^{2i-1} + \gamma^*\right)\ . \tag{4.27}$$

The differential equation (4.26) becomes

$$tx\Gamma_{xx} = x(\Gamma_x)^2 + (1-t)\Gamma_x - 1\ .$$

The differential equation is easily seen to uniquely determine Γ once the initial conditions

$$\Gamma(t,0) = -\sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)} t^{2i}$$

are specified. By Ionel's first result,

$$\Gamma_x = \frac{-1+\sqrt{1+4x}}{2x} + \frac{t}{1+4x} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1} q_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}-1}$$

where the postive integers $q_{k,j}$ (defined to vanish unless $k \geq j \geq 0$) are defined via the recursion

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l}$$

from the initial value $q_{0,0} = 1$.

Ionel's second result is obtained by integrating Γ_x with respect to x. She finds

$$\Gamma = \Gamma(0,x) + \frac{t}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1} c_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}}$$

where the coefficients $c_{k,j}$ are determined by

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}$$

for $k \geq 1$ and $k \geq j \geq 0$.

While the derivation of the formula for Γ_x is straightforward, the formula for Γ is quite subtle as the intial conditions (given by the Bernoulli numbers) are used to show the vanishing of constants of integration. Said differently, the recursions for $q_{k,j}$ and $c_{k,j}$ must be shown to imply the formula

$$c_{k,0} = \frac{B_{k+1}}{k(k+1)} \ .$$

A third result of Ionel's is the determination of the extremal $c_{k,k}$,

$$\sum_{k=1}^{\infty} c_{k,k} z^k = \log\left(\sum_{k=1}^{\infty} \frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right) \ .$$

The formula for Γ becomes simpler after the following very natural change of variables,

$$u = \frac{t}{\sqrt{1+4x}} \quad \text{and} \quad y = \frac{-x}{1+4x} \ . \tag{4.28}$$

The change of variables defines a new function

$$\widehat{\Gamma}(u,y) = \Gamma(t,x) \ .$$

The formula for Γ implies

$$\frac{1}{t}\widehat{\Gamma}(u,y) = \frac{1}{t}\widehat{\Gamma}(0,y) - \frac{1}{4}\log(1+4y) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j} u^k y^j \ . \tag{4.29}$$

Ionel's fourth result relates coefficients of series after the change of variables (4.28). Given any series

$$P(t,x) \in \mathbb{Q}[[t,x]],$$

let $\widehat{P}(u,y)$ be the series obtained from the change of variables (4.28). Ionel proves the coefficient relation

$$\big[P(t,x)\big]_{t^r x^d} = (-1)^d \big[(1+4y)^{\frac{r+2d-2}{2}} \cdot \widehat{P}(u,y)\big]_{u^r y^d} \ .$$

4.5.2 Analysis of the relations of Proposition 4.4

We now study in detail the simple relations of Proposition 4.4,

$$\big[\exp(-\widetilde{\gamma})\big]_{t^r x^d} = 0 \ \in R^r(\mathcal{M}_g)$$

when $g - 2d - 1 < r$ and $g \equiv r + 1 \bmod 2$. Let

$$\widehat{\gamma}(u,y) = \widetilde{\gamma}(t,x)$$

be obtained from the variable change (4.28). Equations (4.6), (4.27), and (4.29) together imply

$$\widehat{\gamma}(u,y) = \frac{\kappa_0}{4}\log(1+4y) + \sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j$$

modulo κ_{-1} terms which we set to 0.

Applying Ionel's coefficient result,

$$\begin{aligned}
\left[\exp(-\widetilde{\gamma})\right]_{t^r x^d} &= \left[(1+4y)^{\frac{r+2d-2}{2}} \cdot \exp(-\widehat{\gamma})\right]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r+2d-2}{2}-\frac{\kappa_0}{4}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} .
\end{aligned}$$

In the last line, the substitution $\kappa_0 = 2g-2$ has been made.

Consider first the exponent of $1+4y$. By the assumptions on g and r in Proposition 4.4,

$$\frac{r-g+2d-1}{2} \geq 0$$

and the fraction is integral. Hence, the y degree of the prefactor

$$(1+4y)^{\frac{r-g+2d-1}{2}}$$

is exactly $\frac{r-g+2d-1}{2}$. The y degree of the exponential factor is bounded from above by the u degree. We conclude

$$\left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} = 0$$

is the *trivial* relation unless

$$r \geq d - \frac{r-g+2d-1}{2} = -\frac{r}{2} + \frac{g+1}{2} .$$

Rewriting the inequality, we obtain $3r \geq g+1$ which is equivalent to $r > \lfloor \frac{g}{3} \rfloor$. The conclusion is in agreement with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

A similar connection between Proposition 4.4 and Ionel's relations in [18] has also been found by Shengmao Zhu [39].

4.5.3 Analysis of the relations of Theorem 4.7

For the relations of Theorem 4.7, we will require additional notation. To start, let

$$\gamma^c(u,y) = \sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j .$$

By Ionel's second result,

$$\frac{1}{t}\Gamma = \frac{1}{t}\Gamma(0,x) + \frac{1}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^k c_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}} \ . \tag{4.30}$$

Let $c^0_{k,j} = c_{k,j}$. We define the constants $c^n_{k,j}$ for $n \geq 1$ by

$$\left(x\frac{d}{dx}\right)^n \frac{1}{t}\Gamma = \left(x\frac{d}{dx}\right)^{n-1}\left(\frac{-1}{2t} + \frac{1}{2t}\sqrt{1+4x}\right) - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n} t^k c^n_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}} \ .$$

Lemma 4.12. *For $n > 0$, there are constants b^n_j satisfying*

$$\left(x\frac{d}{dx}\right)^{n-1}\left(\frac{1}{2t}\sqrt{1+4x}\right) = \sum_{j=0}^{n-1} b^n_j u^{-1} y^j \ .$$

Moreover, $b^n_{n-1} = -2^{n-2}\cdot(2n-5)!!$ where $(-1)!! = 1$ and $(-3)!! = -1$.

Proof. The result is obtained by simple induction. The negative evaluations $(-1)!! = 1$ and $(-3)!! = -1$ arise from the Γ-regularization. □

Lemma 4.13. *For $n > 0$, we have $c^n_{0,n} = 4^{n-1}(n-1)!$.*

Proof. The coefficients $c^n_{0,n}$ are obtained directly from the t^0 summand $\frac{1}{4}\log(1+4x)$ of (4.30). □

Lemma 4.14. *For $n > 0$ and $k > 0$, we have*

$$c^n_{k,k+n} = (6k)(6k+4)\cdots(6k+4(n-1))\ c_{k,k}.$$

Proof. The coefficients $c^n_{k,k+n}$ are extremal. The differential operators $x\frac{d}{dx}$ must always attack the $(1+4x)^{-j-\frac{k}{2}}$ in order to contribute $c^n_{k,k+n}$. The formula follows by inspection. □

Consider next the full set of equations given by Theorem 4.7 in the expanded form of Section 4.4. The function $F_{n,m}$ may be rewritten as

$$\begin{aligned} F_{n,m}(t,x) &= -\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}^d_s\ \kappa_{s+m} t^{s+m}\frac{d^n x^d}{d!} \\ &= -t^m\left(x\frac{d}{dx}\right)^n \sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} \widetilde{C}^d_s\ \kappa_{s+m} t^s \frac{x^d}{d!}. \end{aligned}$$

We may write the result in terms of the constants b_j^n and $c_{k,j}^n$,

$$t^{-(m-n)}F_{n,m} = -\delta_{n,1}\frac{\kappa_{m-1}}{2} + (1+4y)^{-\frac{n}{2}}\Big(\sum_{j=0}^{n-1}\kappa_{m-1}b_j^n u^{n-1}y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n}\kappa_{k+m}c_{k,j}^n u^{k+n}y^j\Big)$$

Define the functions $G_{n,m}(u,y)$ by

$$G_{n,m}(u,y) = \sum_{j=0}^{n-1}\kappa_{m-1}b_j^n u^{n-1}y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n}\kappa_{k+m}c_{k,j}^n u^{k+n}y^j \ .$$

Let $\sigma = (1^{a_1}2^{a_2}3^{a_3}\ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We assume the parity condition

$$g \equiv r + |\sigma| + 1 \ . \tag{4.31}$$

Let $G_\sigma^{\pm}(u,y)$ be the following function associated to σ,

$$G_\sigma^{\pm}(u,y) = \sum_{\sigma^\bullet \in \mathcal{S}(\sigma)}\prod_{i=1}^{\ell(\sigma^\bullet)}\left(G_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \pm \frac{\delta_{\ell(\sigma^{(i)}),1}}{2}\sqrt{1+4y}\ \kappa_{|\sigma^{(i)}|-1}\right) \ .$$

The relations of Theorem 4.7 in the the expanded form of Section 4.4.1 written in the variables u and y are

$$\left[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}}\exp(-\gamma^c)\left(G_\sigma^+ + G_\sigma^-\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}y^d} = 0$$

In fact, the relations of Proposition 4.11 here take a much more efficient form. We obtain the following result.

Proposition 4.15. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}}\exp\left(-\gamma^c - \sum_{\sigma\neq\emptyset}G_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}y^d\mathbf{p}^\sigma} = 0$$

holds when $g - 2d - 1 + |\sigma| < r$ and $g \equiv r + |\sigma| + 1$ mod 2.

Consider the exponent of $1+4y$. By the inequality and the parity condition (4.31),

$$\frac{r-|\sigma|-g+2d-1}{2} \geq 0$$

and the fraction is integral. Hence, the y degree of the prefactor

$$(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}} \tag{4.32}$$

is exactly $\frac{r-|\sigma|-g+2d-1}{2}$. The y degree of the exponential factor is bounded from above by the u degree. We conclude the relation of Theorem 4 is *trivial* unless

$$r-|\sigma|+\ell(\sigma) \geq d - \frac{r-|\sigma|-g+2d-1}{2} = -\frac{r-|\sigma|}{2} + \frac{g+1}{2} \ .$$

Rewriting the inequality, we obtain

$$3r \geq g+1+3|\sigma|-2\ell(\sigma)$$

which is consistent with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

4.5.4 Another form

A subset of the equations of Proposition 4.15 admits an especially simple description. Consider the function

$$H_{n,m}(u) = 2^{n-2}(2n-5)!! \ \kappa_{m-1}u^{n-1} + 4^{n-1}(n-1)! \ \kappa_m u^n \\ + \sum_{k=1}^{\infty}(6k)(6k+4)\cdots(6k+4(n-1))c_{k,k} \ \kappa_{k+m}u^{k+n} \ .$$

Proposition 4.16. *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} = 0$$

holds when $3r \geq g+1+3|\sigma|-2\ell(\sigma)$ and $g \equiv r+|\sigma|+1$ mod 2.

Proof. Let $g \equiv r+|\sigma|+1$, and let

$$\frac{3}{2}r - \frac{1}{2}g - \frac{1}{2} - \frac{3}{2}|\sigma| + \ell(\sigma) = \Delta > 0 \ .$$

By the parity condition, δ is an integer. For $0 \le \delta \le \Delta$, let

$$\mathsf{E}_\delta(g,r,\sigma) = \Big[\exp\left(-\gamma^c + \sum_{\sigma\neq\emptyset} G_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{u^{r-|\sigma|+\ell(\sigma)}y^{r-|\sigma|+\ell(\sigma)-\delta}\mathbf{p}^\sigma} .$$

The $\delta = 0$ case is special. Only the monomials of $G_{n,m}$ of equal u and y degree contribute to the relations of Proposition 4.15. By Lemmas 4.12 - 4.14, $H_{u,m}(uy)$ is exactly the subsum of $G_{n,m}$ consisting of such monomials. Similarly,

$$\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k y^k$$

is the subsum of γ^c of monomials of equal u and y degree. Hence,

$$\begin{aligned}\mathsf{E}_0(g,r,\sigma) = \\ &\Big[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k y^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}(uy)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{(uy)^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} = \\ &\Big[\exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k u^k - \sum_{\sigma\neq\emptyset} H_{\ell(\sigma),|\sigma|}(u)\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{u^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} .\end{aligned}$$

We consider the relations of Proposition 4.15 for fixed g, r, and σ as d varies. In order to satisfy the inequalty $g - 2d - 1 + |\sigma| < r$, let

$$d(\widehat{\delta}) = \frac{-r+g+1+|\sigma|}{2} + \widehat{\delta}\ , \quad \text{for} \ \ \widehat{\delta} \ge 0.$$

For $0 \le \widehat{\delta} \le \Delta$, relation of Proposition 4.15 for g, r, σ, and $d(\widehat{\delta})$ is

$$\sum_{i=0}^{\widehat{\delta}} 4^i \binom{\widehat{\delta}}{i} \cdot \mathsf{E}_{\Delta-\widehat{\delta}+i}(g,r,\sigma) = 0\ .$$

As $\widehat{\delta}$ varies, we therefore obtain all the relations

$$\mathsf{E}_\delta(g,r,\sigma) = 0 \tag{4.33}$$

for $0 \le \delta \le \Delta$. The relations of Proposition 4.16 are obtained when $\delta = 0$ in (4.33). $\square$

The main advantage of Proposition 4.16 is the dependence on only the function

$$\sum_{k=1}^{\infty} c_{k,k} z^k = \log\left(\sum_{k=1}^{\infty} \frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right) . \tag{4.34}$$

Proposition 4.16 only provides finitely many relations for fixed g and r. In Section 4.6, we show Proposition 4.16 is equivalent to the Faber-Zagier conjecture.

4.5.5 Relations left behind

In our analysis of relations obtained from the virtual geometry of the moduli space of stable quotients, twice we have discarded large sets of relations. In Section 4.3.4, instead of analyzing all of the geometric possibilities

$$\nu_*\left(\prod_{i=1}^{n} \epsilon_*(s^{a_i}\omega^{b_i}) \cdot 0^c \cap [Q_g(\mathbf{P}^1, d)]^{vir}\right) = 0 \text{ in } A^*(\mathcal{M}_g, \mathbb{Q}) ,$$

we restricted ourselves to the case where $a_i = 1$ for all i. And just now, instead of keeping all the relations (4.33), we restricted ourselves to the $\delta = 0$ cases.

In both instances, the restricted set was chosen to allow further analysis. In spite of the discarding, we will arrive at the Faber-Zagier relations. We expect the discarded relations are all redundant, but we do not have a proof.

4.6 Equivalence

4.6.1 Notation

The relations in Proposition 4.16 have a similar flavor to the Faber-Zagier relations. We start with formal series related to

$$A(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!}\left(\frac{z}{72}\right)^i ,$$

we insert classes κ_r, we exponentiate, and we extract coefficients to obtain relations among the κ classes. In order to make the similarities clearer, we will need a little notation.

First, if F is a formal power series in z,

$$F = \sum_{r=0}^{\infty} c_r z^r$$

with coefficients in a ring, let

$$\{F\}_\kappa = \sum_{r=0}^{\infty} c_r \kappa_r z^r$$

be the series with κ-classes inserted (as in the previous chapter except that our power series are in z now instead of t).

Let A be as above, and let

$$B(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left(\frac{z}{72}\right)^i$$

be the second power series appearing in the Faber-Zagier relations. Note that we have scaled the definitions of A and B by $z \mapsto \frac{z}{72}$ to match them more closely with the series appearing in Ionel's work. Let

$$C = \frac{B}{A} \in \mathbb{Q}[[z]] \ ,$$

and let

$$E = \exp(-\{\log(A)\}_\kappa) = \exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k z^k\right) \in \mathbb{Q}[\kappa_1, \kappa_2, \ldots][[z]].$$

We will rewrite the Faber-Zagier relations and the relations of Proposition 4.16 in terms of C and E. The equivalence between the two will rely on the principal differential equation satisfied by C,

$$12z^2 \frac{dC}{dz} = 1 + 4zC - C^2. \tag{4.35}$$

4.6.2 Rewriting the relations

The Faber-Zagier relations are straightforward to rewrite using the above notation:

$$\left[E \cdot \exp\left(-\left\{\log\left(1 + p_3 z + p_6 z^2 + \cdots + C(p_1 + p_4 z + p_7 z^2 + \cdots)\right)\right\}_\kappa\right)\right]_{z^r p^\sigma} = 0 \tag{4.36}$$

for $3r \geq g + |\sigma| + 1$ and $3r \equiv g + |\sigma| + 1 \bmod 2$. The above relation (4.36) will be denoted $\mathsf{FZ}(r, \sigma)$.

The stable quotient relations of Proposition 4.16 are more complicated to rewrite in terms of C and E. Define a sequence of power series $(C_n)_{n\geq 1}$ by

$$2^{-n}C_n = 2^{n-2}(2n-5)!!z^{n-1} + 4^{n-1}(n-1)!z^n + \sum_{k=1}^{\infty}(6k)(6k+4)\cdots(6k+4(n-1))c_{k,k}z^{k+n}.$$

We see

$$H_{n,m}(z) = 2^{-n}z^{n-m}\{z^{m-n}C_n\}_\kappa.$$

The series C_n satisfy

$$C_1 = C, \quad C_{i+1} = \left(12z^2 \frac{d}{dz} - 4iz\right) C_i. \tag{4.37}$$

Using the differential equation (4.35), each C_n can be expressed as a polynomial in C and z:

$$C_1 = C, \quad C_2 = 1 - C^2, \quad C_3 = -8z - 2C + 2C^3, \ldots, .$$

Proposition 4.16 can then be rewritten as follows (after an appropriate change of variables):

$$\left[E \cdot \exp\left(-\sum_{\sigma \neq \emptyset} \{z^{|\sigma| - \ell(\sigma)} C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|}\right)\right]_{z^r p^\sigma} = 0 \tag{4.38}$$

for $3r \geq g + 3|\sigma| - 2\ell(\sigma) + 1$ and $3r \equiv g + 3|\sigma| - 2\ell(\sigma) + 1 \bmod 2$. The above relation (4.38) will be denoted $\mathsf{SQ}(r, \sigma)$.

The FZ and SQ relations now look much more similar, but the relations in (4.36) are indexed by partitions with no parts of size 2 mod 3 and satisfy a slightly different inequality. The indexing differences can be erased by removing the variables p_{3k} (these relations still generate the same *ideal*, as discussed in Section 2.4) and reindexing the others by replacing p_{3k+1} with p_{k+1}. The result is the following equivalent form of the FZ relations:

$$\left[E \cdot \exp\left(-\left\{\log(1 + C(p_1 + p_2 z + p_3 z^2 + \cdots))\right\}_\kappa\right)\right]_{z^r p^\sigma} = 0 \tag{4.39}$$

for $3r \geq g + 3|\sigma| - 2\ell(\sigma) + 1$ and $3r \equiv g + 3|\sigma| - 2\ell(\sigma) + 1 \bmod 2$.

4.6.3 Comparing the relations

We now explain how to write the SQ relations (4.38) as linear combinations of the FZ relations (4.39) with coefficients in $\mathbb{Q}[\kappa_0, \kappa_1, \kappa_2, \ldots]$. In fact, the associated matrix will be triangular with diagonal entries equal to 1.

We start with further notation. For a partition σ, let

$$\mathsf{FZ}_\sigma = \left[\exp\left(-\left\{\log(1 + C(p_1 + p_2 z + p_3 z^2 + \cdots))\right\}_\kappa\right)\right]_{p^\sigma}$$

and

$$\mathsf{SQ}_\sigma = \left[\exp\left(-\sum_{\sigma \neq \emptyset} \{z^{|\sigma| - \ell(\sigma)} C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|}\right)\right]_{p^\sigma}$$

be power series in z with coefficients that are polynomials in the κ classes. The relations themselves are given by

$$\mathsf{FZ}(r,\sigma) = [E \cdot \mathsf{FZ}_\sigma]_{z^r} \ , \quad \mathsf{SQ}(r,\sigma) = [E \cdot \mathsf{SQ}_\sigma]_{z^r} \ .$$

It is straightforward to expand FZ_σ and SQ_σ as linear combinations of products of factors $\{z^a C^b\}$ for $a \geq 0$ and $b \geq 1$, with coefficients that are polynomials in the kappa classes. When expanded, FZ_σ always contains exactly one term of the form

$$\{z^{a_1}C\}_\kappa\{z^{a_2}C\}_\kappa \cdots \{z^{a_m}C\}_\kappa \ . \tag{4.40}$$

All the other terms involve higher powers of C. If we expand SQ_σ, we can look at the terms of the form (4.40) to determine what the coefficients must be when writing the SQ_σ as linear combinations of the FZ_σ. For example,

$$\begin{aligned}
\mathsf{SQ}_{(111)} &= -\frac{1}{6}\{C_3\}_\kappa + \frac{1}{2}\{C_2\}_\kappa\{C_1\}_\kappa - \frac{1}{6}\{C_1\}_\kappa^3 \\
&= \frac{4}{3}\kappa_1 z + \frac{1}{3}\{C\}_\kappa - \frac{1}{3}\{C^3\}_\kappa + \frac{1}{2}(\kappa_0 - \{C^2\}_\kappa)\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3 \\
&= \left(\frac{4}{3}\kappa_1 z\right) + \left(\left(\frac{1}{3} + \frac{\kappa_0}{2}\right)\{C\}_\kappa\right) \\
&\qquad + \left(-\frac{1}{3}\{C^3\}_\kappa - \frac{1}{2}\{C^2\}_\kappa\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3\right) \\
&= \frac{4}{3}\kappa_1 z\,\mathsf{FZ}_\emptyset + \left(-\frac{1}{3} - \frac{\kappa_0}{2}\right)\mathsf{FZ}_{(1)} + \mathsf{FZ}_{(111)} \, .
\end{aligned}$$

In general we must check that the terms involving higher powers of C also match up. The matching will require an identity between the coefficients of C_i when expressed as polynomials in C. Define polynomials $f_{ij} \in \mathbb{Z}[z]$ by

$$C_i = \sum_{j=0}^{i} f_{ij} C^j.$$

It will also be convenient to write $f_{ij} = \sum_k f_{ijk} z^k$, so

$$C_i = \sum_{\substack{j,k \geq 0 \\ j+3k \leq i}} f_{ijk} z^k C^j.$$

If we define

$$F = 1 + \sum_{i,j \geq 1} \frac{(-1)^{j-1} f_{ij}}{i!(j-1)!} x^i y^j \in \mathbb{Q}[z][[x,y]],$$

then we will need a single property of the power series F.

Lemma 4.17. *There exists a power series $G \in \mathbb{Q}[z][[x]]$ such that $F = e^{yG}$.*

Proof. The recurrence (4.37) for the C_i together with the differential equation (4.35) satisfied by C yield a recurrence relation for the polynomials f_{ij}:

$$f_{i+1,j} = (j+1)f_{i,j+1} + 4(j-i)zf_{ij} - (j-1)f_{i,j-1}.$$

This recurrence relation for the coefficients of F is equivalent to a differential equation:

$$F_x = -yF_{yy} + 4zyF_y - 4zxF_x + yF.$$

Now, let $G \in \mathbb{Q}[z][[x, y]]$ be $\frac{1}{y}$ times the logarithm of f (as a formal power series). The differential equation for F can be rewritten in terms of G:

$$G_x = -2G_y - yG_{yy} - (G + yG_y)^2 + 4z(G + yG_y) - 4zxG_x + 1.$$

We now claim that the coefficient of $x^k y^l$ in G is zero for all $k \geq 0, l \geq 1$, as desired. For $k = 0$ this is a consequence of the fact that $F = 1 + O(xy)$ and thus $G = O(x)$, and higher values of k follow from induction using the differential equation above. $\square$

We can now write the SQ_σ as linear combinations of the FZ_σ.

Theorem 4.18. *Let σ be a partition. Then $\mathsf{SQ}_\sigma - \mathsf{FZ}_\sigma$ is a $\mathbb{Q}$-linear combination of terms of the form*

$$\kappa_\mu z^{|\mu|}\, \mathsf{FZ}_\tau,$$

where μ and τ are partitions (μ possibly containing parts of size 0) satisfying $\ell(\tau) < \ell(\sigma)$, $3|\mu| + 3|\tau| - 2\ell(\tau) \leq 3|\sigma| - 2\ell(\sigma)$, and

$$3|\mu| + 3|\tau| - 2\ell(\tau) \equiv 3|\sigma| - 2\ell(\sigma) \mod 2\ .$$

Proof. We will need some additional notation for subpartitions. If σ is a partition of length $\ell(\sigma)$ with parts $\sigma_1, \sigma_2, \ldots$ (ordered by size) and S is a subset of $\{1, 2, \ldots, \ell(\sigma)\}$, then let $\sigma_S \subset \sigma$ denote the subpartition consisting of the parts $(\sigma_i)_{i \in S}$.

Using this notation, we explicitly expand SQ_σ and FZ_σ as sums over set partitions of $\{1,\dots,\ell(\sigma)\}$:

$$\mathsf{SQ}_\sigma = \frac{1}{|\operatorname{Aut}(\sigma)|}\sum_{P\vdash\{1,\dots,\ell(\sigma)\}}\prod_{S\in P}\left(\sum_{j,k} -f_{|S|,j,k}\{z^{|\sigma_S|-|S|+k}C^j\}_\kappa\right),$$

$$\mathsf{FZ}_\sigma = \frac{1}{|\operatorname{Aut}(\sigma)|}\sum_{P\vdash\{1,\dots,\ell(\sigma)\}}\prod_{S\in P}\left((-1)^{|S|}(|S|-1)!\{z^{|\sigma_S|-|S|}C^{|S|}\}_\kappa\right).$$

Matching coefficients for terms of the form (4.40) tells us what the linear combination must be. We claim

$$\mathsf{SQ}_\sigma = \sum_{\substack{R\vdash\{1,\dots,\ell(\sigma)\}\\ P\sqcup Q=R\\ k:R\to\mathbb{Z}_{\geq 0}}}\frac{|\operatorname{Aut}(\sigma')|}{|\operatorname{Aut}(\sigma)|}\times \prod_{S\in P}\left(-f_{|S|,0,k(S)}\kappa_{|\sigma_S|-|S|+k(S)}z^{|\sigma_S|-|S|+k(S)}\right)\prod_{S\in Q}\left(f_{|S|,1,k(S)}\right)\mathsf{FZ}_{\sigma'}, \tag{4.41}$$

where σ' is the partition with parts $|\sigma_S|-|S|+1+k(S)$ for $S\in Q$. Using the vanishing $f_{i,j,k}=0$ unless $j+3k\leq i$ and $j+3k\equiv i \mod 2$, we easily check the above expression for SQ_σ is of the desired type.

Expanding SQ_σ and $\mathsf{FZ}_{\sigma'}$ in (4.41) and canceling out the terms involving the $f_{i,0,k}$ coefficients, it remains to prove

$$\sum_{\substack{Q\vdash\{1,\dots,\ell(\sigma)\}\\ k:Q\to\mathbb{Z}_{\geq 0}\\ j:Q\to\mathbb{N}}}\prod_{S\in Q}\left(-f_{|S|,j(S),k(S)}\{z^{|\sigma_S|-|S|+k(S)}C^{j(S)}\}_\kappa\right)$$
$$= \sum_{\substack{Q\vdash\{1,\dots,\ell(\sigma)\}\\ k:Q\to\mathbb{Z}_{\geq 0}}}\prod_{S\in Q}\left(f_{|S|,1,k(S)}\right)\sum_{P\vdash\{1,\dots,\ell(\sigma')\}}\prod_{S\in P}\left((-1)^{|S|}(|S|-1)!\{z^{|(\sigma')_S|-|S|}C^{|S|}\}_\kappa\right).$$

A single term on the left side of the above equation is determined by choosing a set partition Q_{left} of $\{1,\dots,\ell(\sigma)\}$ and then for each part S of Q_{left} choosing a positive integer $j(S)$ and a nonnegative integer $k_{\text{left}}(S)$. We claim that this term is the sum of the terms of the right side given by choices Q_{right}, k_{right}, P such that Q_{right} is a refinement of Q_{left} that breaks each part S in Q_{left} into exactly $j(S)$ parts in Q_{right}, P is the associated grouping of the parts of Q_{right}, and the $k_{\text{right}}(S)$ satisfy

$$k_{\text{left}}(S) = \sum_{T\subseteq S} k_{\text{right}}(T)\ .$$

These terms all are integer multiples of the same product of $\{z^aC^b\}_\kappa$ factors, so we are left with the identity

$$\frac{(-1)^{j_0-1}}{(j_0-1)!} f_{i_0,j_0,k_0} = \sum_{\substack{P \vdash \{1,\ldots,i_0\} \\ |P|=j_0 \\ k:P\to\mathbb{Z}_{\geq 0} \\ |k|=k_0}} \prod_{S\in P} f_{|S|,1,k(S)}. \tag{4.42}$$

to prove.

But by the exponential formula, identity (4.42) is simply a restatement of Lemma 4.17. □

The conditions on the linear combination in Theorem 4.18 are precisely those needed so that multiplying by E and taking the coefficient of z^r allows us to write any SQ relation as a linear combination of FZ relations. The associated matrix is triangular with respect to the partial ordering of partitions by size, and the diagonal entries are equal to 1. Hence, the matrix is invertible. We conclude the SQ relations are equivalent to the FZ relations.

Chapter 5

Tautological relations on $\overline{\mathcal{M}}_{g,n}$

The ideas in this chapter were originally presented in the notes [32], which were based on informal talks given by the author at the workshop at KTH Stockholm on “The moduli space of curves and its intersection theory” in April 2012.

5.1 Introduction

In the previous chapter, we proved that the Faber-Zagier (FZ) relations on $\mathcal{M}_g$ are not only true but extend to tautological relations on the compactification $\overline{\mathcal{M}}_g$. What is this extension? In theory, the methods of the previous chapter give an explicit answer to this, but the combinatorics of the stable quotients on the boundary is significantly more complicated[1].

In this chapter we give a conjectural description of this extension and also explain how to add marked points to the relations. The result is a very large class of conjectural relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

The motivation for guessing this extension was as follows. First, there is a very natural guess for FZ-type relations on $\mathcal{M}_{g,n}$: in the notation of Section 2.4, the relations in degree d are parametrized by a partition σ with no parts congruent to 2 mod 3 as usual, along with a nonnegative integer a_i not congruent to 2 mod 3 for each marking $i = 1, 2, \ldots, n$. These parameters should satisfy $3d \geq g + 1 + |\sigma| + \sum_i a_i$ and $3d \equiv g + 1 + |\sigma| + \sum_i a_i \bmod 2$. Then the relations are

[1] In private communication, F. Janda has announced that he has worked out these details and obtained precisely the relations conjectured here on $\overline{\mathcal{M}}_g$.

$$\Big[\overbrace{\exp(-\{1-C_0(t)\}_\kappa)\{C_1(t)\}_\kappa^{\sigma_1}\{C_3(t)\}_\kappa^{\sigma_3}\{C_4(t)\}_\kappa^{\sigma_4}\{C_6(t)\}_\kappa^{\sigma_6}\cdots}C_{a_1}(\psi_1 t)\cdots C_{a_n}(\psi_n)t\Big]_{t^d}=0, \quad (5.1)$$

where $C_{3k}(t) = t^k A(t)$ and $C_{3k+1}(t) = t^k B(t)$. It is easily checked when $n = 1$ that these relations hold in the Gorenstein quotient $\mathrm{Gor}^*(\mathcal{M}_{g,1})$.

If we assume that *all* relations on $\mathcal{M}_{g,n}$ are linear combinations of these relations, then this gives many constraints on relations on $\overline{\mathcal{M}}_{g,n}$. For any boundary stratum, we may restrict a relation to the interior of the boundary stratum and then it must be expressable in terms of the relations above. These constraints lead naturally to the form of the relations conjectured in this chapter.

5.2 The strata algebra

We begin by defining the *strata algebra*, a commutative algebra formed based on the standard generating set (2.2) that surjects onto the tautological ring of $\overline{\mathcal{M}}_{g,n}$. The strata algebra $\mathcal{S}_{g,n}$ plays the same role for $\overline{\mathcal{M}}_{g,n}$ that the ring of formal polynomials in the kappa classes did for $\mathcal{M}_g$.

Let Γ be a stable graph. A *basic class* on $\overline{\mathcal{M}}_\Gamma$ is defined to be a product of monomials in κ classes at each vertex of the graph and powers of ψ classes at each half-edge (including the legs),

$$\gamma = \prod_{v\in \mathrm{V}}\prod_{i>0}\kappa_i[v]^{x_i[v]} \ \cdot \ \prod_{h\in \mathrm{H}}\psi_h^{y[h]} \ \in H^*(\overline{\mathcal{M}}_\Gamma,\mathbb{Q}) \ ,$$

where $\kappa_i[v]$ is the i^{th} kappa class on $\overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$. We impose the condition

$$\sum_{i>0} i x_i[v] + \sum_{h\in \mathrm{H}[v]} y[h] \le \dim_{\mathbb{C}} \overline{\mathcal{M}}_{\mathrm{g}(v),\mathrm{n}(v)} = 3\mathrm{g}(v)-3+\mathrm{n}(v)$$

at each vertex to avoid the trivial vanishing of γ. Here, $\mathrm{H}[v]\subset \mathrm{H}$ is the set of half-edges (including the legs) incident to v.

Consider the $\mathbb{Q}$-vector space $\mathcal{S}_{g,n}$ whose basis is given by the isomorphism classes of pairs $[\Gamma,\gamma]$, where Γ is a stable graph of genus g with n legs and γ is a basic class on $\overline{\mathcal{M}}_\Gamma$. Since there are only finitely many pairs $[\Gamma,\gamma]$ up to isomorphism, $\mathcal{S}_{g,n}$ is finite dimensional.

A product on $\mathcal{S}_{g,n}$ is defined by the intersection theory of $\overline{\mathcal{M}}_{g,n}$. Let

$$[\Gamma_1,\gamma_1], \ [\Gamma_2,\gamma_2]\in \mathcal{S}_{g,n}$$

be two basis elements. The fiber product of ξ_{Γ_1} and ξ_{Γ_2} over $\overline{\mathcal{M}}_{g,n}$ is canonically described as a disjoint union of ξ_Γ for stable graphs Γ endowed with contractions[2] onto Γ_1 and Γ_2. More precisely, the set of edges E of Γ should be represented as a union of two (not necessarily disjoint) subsets,

$$E = E_1 \cup E_2,$$

in such a way that Γ_1 is obtained by contracting all the edges outside E_1 and Γ_2 is obtained by contracting all edges outside E_2 (see Proposition 9 in the Appendix of [17]). The intersection of ξ_{Γ_1} and ξ_{Γ_2} in $\overline{\mathcal{M}}_{g,n}$ is then canonically given by Fulton's excess theory as a sum of elements in $\mathcal{S}_{g,n}$. We define

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2] = \sum_{\Gamma} [\Gamma, \gamma_1 \gamma_2 \varepsilon_\Gamma]$$

where

$$\varepsilon_\Gamma = \prod_{e \in E_1 \cap E_2} -(\psi_e' + \psi_e'')$$

is the excess class. Here, ψ_e' and ψ_e'' are the two cotangent line classes corresponding to the two half-edges of the edge e.

Via the above intersection product, $\mathcal{S}_{g,n}$ is a finite dimensional $\mathbb{Q}$-algebra, called the *strata algebra*. It is straightforward to verify that $\mathcal{S}_{g,n}$ is associative by defining (A, B, C)-graphs analogously to the (A, B)-graphs in [17]. Pushforward along ξ_Γ defines a canonical surjective ring homomorphism

$$q : \mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n}), \quad q([\Gamma, \gamma]) = \xi_{\Gamma *}(\gamma)$$

from the strata algebra to the tautological ring. An element of the kernel of q is called a *tautological relation*.

Each basis element $[\Gamma, \gamma]$ has a degree grading given by the number of edges of Γ plus the usual (complex) degree of γ,

$$\deg[\Gamma, \gamma] = |\mathrm{E}| + \deg_{\mathbb{C}}(\gamma) \ .$$

Hence, $\mathcal{S}_{g,n}$ is graded,

$$\mathcal{S}_{g,n} = \bigoplus_{d=0}^{3g-3+n} \mathcal{S}_{g,n}^d \ .$$

[2]If there are several different pairs of contractions from a given Γ, the corresponding ξ_Γ appears with multiplicity.

Since the product respects the grading, $\mathcal{S}_{g,n}$ is a graded algebra. Of course, we have

$$q : \mathcal{S}_{g,n}^d \to R^d(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

In addition, the rules given in [17] and [1] for taking the pushforward or pullback of one of the generators along a forgetful or gluing map allow us to define $\mathbb{Q}$-linear maps between the strata algebras lifting the pushforward and pullback maps between the tautological rings. It is again straightforward to check that these pushforward and pullback maps on strata algebras satisfy the basic ring-theoretic properties one would expect (i.e. pullbacks are ring homomorphisms and the projection formula holds). The collection of ideals of tautological relations is clearly closed under these maps.

5.3 The relations

We now begin to construct conjectural relations in $R^d(\overline{\mathcal{M}}_{g,n})$. When $n = 0$, it will be clear that the restriction of these relations to the interior are simply the usual FZ relations described by (2.7). In fact, for all n the restriction of these relations to $\mathcal{M}_{g,n}$ will be the relations (5.1) described in Section 5.1. The relations are parametrized by the same data as described there: a partition σ with no parts of size 2 mod 3 together with nonnegative integers $a_1, \ldots, a_n$ not 2 mod 3, subject to the conditions $3d \geq g + 1 + |\sigma| + \sum_i a_i$ and $3d \equiv g + 1 + |\sigma| + \sum_i a_i$ mod 2.

We denote the relation coming from this data by

$$\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n) \in \mathcal{S}_{g,n},$$

and we write

$$\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n) = \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} \big[\Gamma, \mathcal{R}_\Gamma(g, n, d; \sigma, a_1, \ldots, a_n)\big],$$

where the sum is over isomorphism classes of dual graphs Γ. Here $\mathcal{R}_\Gamma(g, n, d; \sigma, a_1, \ldots, a_n)$ is a polynomial of degree $d - |E(\Gamma)|$ in the κ and ψ classes on the components of $\overline{\mathcal{M}}_\Gamma$.

In order to describe this polynomial, we need to modify the definitions used in the FZ relations to include extra parity information. This will involve augmenting the polynomials and power series with extra commuting variables ζ satisfying $\zeta^2 = 1$.

First, we define series $\overline{C}_i(T,\zeta) \in \left(\mathbb{Q}[\zeta]/(\zeta^2-1)\right)[[T]]$ by

$$\overline{C}_{3i}(T,\zeta) = T^i A(\zeta T)$$

and

$$\overline{C}_{3i+1}(T,\zeta) = \zeta T^i B(\zeta T).$$

Next, if F is a power series in T and ζ then we let $[F]_{T^n\zeta^a}$ denote the coefficient of $T^n\zeta^a$ and define

$$\{F\}_K = \sum_{n\in\mathbb{Z}, a\in\mathbb{Z}/2} [F]_{T^n\zeta^a} K_{n,a} T^n,$$

where the $K_{n,a}$ are formal indeterminates.

Finally, we redefine the $\overbrace{\cdot}$ kappa basis change operator defined in (2.3) as a linear operator converting polynomials in the $K_{n,a}$ into polynomials in the kappa variables $\kappa_i[v]$ along with an additional variable ζ_v (satisfying $\zeta_v^2 = 1$) for each vertex v, defined by

$$\overbrace{K_{e_1,a_1}\cdots K_{e_l,a_l}} = \sum_{\tau\in S_l} \prod_{c \text{ cycle in } \tau} \left(\sum_{v\in V(\Gamma)} \kappa_{e_c}[v]\zeta_v^{a_c}\right)$$

where e_c and a_c are the sums of the e_i and a_i respectively appearing in the cycle c.

Then we can write

$$\begin{aligned}\mathcal{R}_\Gamma(g,n,d;\sigma,a_1,\ldots,a_n) = \Bigg[&\frac{1}{2^{h_1(\Gamma)}} \overbrace{\exp\left(\{1-\overline{C}_0\}\right)\{\overline{C}_{\sigma_1}\}\cdots\{\overline{C}_{\sigma_l}\}} \\ &\cdot \prod_{i=1}^n \overline{C}_{a_i}(\psi_{h_i}T,\zeta_{v_i}) \prod_{e\in E(\Gamma)} \Delta_e\Bigg]_{T^{d-|E(\Gamma)|}} \prod_{v\in V(\Gamma)} \zeta_v^{g_v+1},\end{aligned} \tag{5.2}$$

where $h_1(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ is the cycle number of Γ, marking i corresponds to half-edge h_i on vertex v_i, and g_v is the genus of vertex v.

Also, for each edge $e \in E(\Gamma)$, let e_1 and e_2 be the two halves, attached to vertices v_1 and v_2 respectively. The edge contribution Δ_e appearing in the above formula is a power series in T with coefficients that are polynomials in $\psi_1 := \psi_{e_1}, \psi_2 := \psi_{e_2}, \zeta_1 := \zeta_{v_1}$, and $\zeta_2 := \zeta_{v_2}$:

$$\Delta_e = \frac{A(\zeta_1\psi_1T)\zeta_2B(\zeta_2\psi_2T) + \zeta_1B(\zeta_1\psi_1T)A(\zeta_2\psi_2T) + \zeta_1 + \zeta_2}{(\psi_1+\psi_2)T}.$$

The fact that $\psi_1 + \psi_2$ divides the numerator in this formula is a consequence of the identity

$$A(T)B(-T) + A(-T)B(T) + 2 = 0.$$

This completes the definition of $\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n)$.

Conjecture 5.1. *$\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n)$ maps to $0 \in R^*(\overline{\mathcal{M}}_{g,n})$ if $3d \geq g + 1 + |\sigma| + \sum_i a_i$.*

In the next chapter, we will prove that these relations are at least true in cohomology (see Corollary 6.2).

We also conjecture that this construction produces *all* relations in the tautological ring. To make the statement of this precise, let $\mathcal{R}_{g,n}$ be the linear span of all elements of the strata algebra $\mathcal{S}_{g,n}$ produced as follows: choose a dual graph Γ for a boundary stratum of $\overline{\mathcal{M}}_{g,n}$, pick one of the components $\mathcal{S}_{g',n'}$ in $\mathcal{S}_\Gamma = \mathcal{S}_{g_1,n_1} \times \cdots \times \mathcal{S}_{g_m,n_m}$, take the product of a relation $\mathcal{R}(g', n', d; \sigma, a_1, \ldots, a_n)$ on the chosen component together with arbitrary classes on the other components, and push forward along the gluing map $\mathcal{S}_\Gamma \to \mathcal{S}_{g,n}$.

Conjecture 5.2. *$\mathcal{R}_{g,n}$ is the kernel of the natural surjection $q : \mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n})$.*

Of course, relations on $\overline{\mathcal{M}}_{g,n}$ can be restricted to relations on $\mathcal{M}^c_{g,n}, \mathcal{M}^{rt}_{g,n}$, or $\mathcal{M}_{g,n}$. Strata algebras for these moduli spaces can be defined in the same way as for $\mathcal{M}_{g,n}$ except with restrictions on the graphs Γ, and then there is a well-defined notion of giving all tautological relations there as well.

Conjecture 5.3. *Restricting $\mathcal{R}_{g,n}$ gives all tautological relations on $\mathcal{M}^c_{g,n}, \mathcal{M}^{rt}_{g,n}$, and $\mathcal{M}_{g,n}$.*

In the case of $\mathcal{M}_g$, this conjecture repeats the existing conjecture that the FZ relations give all relations in the tautological ring of $\mathcal{M}_g$, which is well known to contradict for $g \geq 24$ Faber's conjecture [9] that this ring is Gorenstein. The conjecture similarly contradicts the Gorenstein conjecture in various cases for $\mathcal{M}^{rt}_{g,n}$ and $\mathcal{M}^c_{g,n}$; see Appendix A for more details.

The motivation for believing Conjectures 5.2 and 5.3 is mainly just that there are no known counterexamples. The known "exotic" tautological relations, such as Getzler's relation [12] in $R^2(\overline{\mathcal{M}}_{1,4})$ and the Belorousski-Pandharipande relation [4] in $R^2(\overline{\mathcal{M}}_{2,3})$, are contained in $\mathcal{R}_{g,n}$.

With the aid of a computer, we have calculated the ideal $\mathcal{R}_{g,n}$ and the betti numbers of the quotients $\mathcal{S}_{g,n}/\mathcal{R}_{g,n}$ for many small values of g and n, and also the compact type and rational tails versions of these quotients. For small values of g and n, these betti numbers agree with the Gorenstein ranks, many of which were computed by Yang [38]. In such cases, we know that $\mathcal{R}_{g,n}$

must contain all relations. For moduli spaces near the boundaries of computability, the relations $\mathcal{R}_{g,n}$ become insufficient to make the quotient Gorenstein, so there are possible missing relations: this is the same phenomenon that happens with $g \geq 24$ for $\mathcal{M}_g$. In Appendix A, we present some of the data from these computations.

5.4 Properties

In this section we list a couple of properties satisfied by the relations $\mathcal{R}(g, n, d; \sigma, a_1, \dots, a_n)$.

First, there are no tautological ways of enlarging the space of conjectured relations $\mathcal{R}_{g,n}$.

Proposition 5.4. *The vector subspace $\mathcal{R}_{g,n}$ of the strata algebra $\mathcal{S}_{g,n}$ is an ideal. Moreover, this collection of ideals of the stata algebra is closed under pushforward and pullback by the gluing and forgetful maps.*

Proof. We need to check that the span of our relations is closed under five operations. We will implicitly use the projection formula and facts about composing pullbacks and pushforwards along tautological morphisms in order to reduce the number of things that have to be checked. In particular, we only ever have to apply these operations directly to the basic relations $R(g, n, d; \sigma, a_1, \dots, a_n)$ and not to the pushforwards of these from the boundary.

1. Multiplication: First, multiplication by a boundary stratum is the same thing as pulling back to the boundary stratum and then pushing forward, so it will be handled by the later cases. Next, multiplication by a psi class is given by the very simple formula

$$\psi_i \mathcal{R}(g, n, d; \sigma, a_1, \dots, a_n) = \mathcal{R}(g, n, d + 1; \sigma, a_1, \dots, a_i + 3, \dots, a_n).$$

   Finally, multiplication by a kappa class is given by the usual formula

$$\kappa_a \mathcal{R}(g, n, d; \sigma, a_1, \dots, a_n) = \kappa_a \mathcal{R}(g, d+a, \sigma \sqcup \{3a\}, a_1, \dots, a_n) - \sum_{\tau} \mathcal{R}(g, n, d+a; \tau, a_1, \dots, a_n),$$

   where the sum runs over partitions τ formed by increasing one part of σ by $3a$.

2. Pullback by a gluing map: We consider the gluing map

$$\phi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2};$$

the case of gluing two points on the same curve together follows similarly. We can compute[3]

$$\phi^*\mathcal{R}(g_1+g_2, n_1+n_2, d; \sigma, a_1, \ldots, a_{n_1+n_2}) =$$
$$\sum_{\substack{d_1+d_2=d\\ \sigma_1\sqcup\sigma_2=\sigma}} \mathcal{R}(g_1, n_1+1, d_1; \sigma_1, a_1, \ldots, a_{n_1}, e)\otimes\mathcal{R}(g_2, n_2+1, d_2; \sigma_2, a_{n_1+1}, \ldots, a_{n_1+n_2}, 1-e),$$

where $e\in\{0,1\}$ is chosen for each term to meet the parity constraint on the parameters. At least one of the inequalities

$$3d_1 \geq g_1 + |\sigma_1| + a_1 + \cdots + a_{n_1} + e + 1,$$

$$3d_2 \geq g_2 + |\sigma_2| + a_{n_1+1} + \cdots + a_{n_1+n_2} + (1-e) + 1$$

must hold for each term, so each term belongs to the ideal generated by $\mathcal{R}_{g_1,n_1+1}$ and $\mathcal{R}_{g_2,n_2+1}$ inside $\mathcal{S}_{g_1,n_1+1}\otimes\mathcal{S}_{g_2,n_2+1}$.

3. Pushforward by a gluing map: The ideals $\mathcal{R}_{g,n}$ are closed under this by definition.

4. Pullback by a forgetful map: The basic pullback formula for $\pi : \overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ is

$$\pi^*\left(\widehat{\kappa_\alpha}\prod_{1\leq i\leq n}\psi_i^{e_i}\right) =$$
$$\widehat{\kappa_\alpha}\prod_{1\leq i\leq n}\psi_i^{e_i} - \sum_{x\in\alpha}\widehat{\kappa_{\alpha\setminus\{x\}}}\left(\prod_{1\leq i\leq n}\psi_i^{e_i}\right)\psi_{n+1}^x - \sum_{\substack{1\leq j\leq n\\ e_j>0}}\iota_{j*}\widehat{\kappa_\alpha}\prod_{\substack{1\leq i\leq n\\ i\neq j}}\psi_i^{e_i}\psi_\bullet^{e_j-1},$$

where $\iota_j : \overline{\mathcal{M}}_{g,n}\times\overline{\mathcal{M}}_{0,3}\to\overline{\mathcal{M}}_{g,n+1}$ is the gluing map corresponding to the boundary stratum in which markings j and $n+1$ come together and $\psi_\bullet$ is the psi class at the half-edge leading to this rational tail.

Applying this, we get

$$\begin{aligned}\pi^*\mathcal{R}(g,n,d;\sigma,a_1,\ldots,a_n) = {}& \mathcal{R}(g,n+1,d;\sigma,a_1,\ldots,a_n,0)\\ &-\sum_{i\in\sigma}R(g,n+1,d;\sigma\setminus\{i\},a_1,\ldots,a_n,i)\\ &-\sum_{\substack{1\leq j\leq n\\ a_j\geq 3}}\iota_{j*}\mathcal{R}(g,n,d-1;\sigma,a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_n,a_j-3).\end{aligned} \tag{5.3}$$

[3]This is basically the splitting axiom for a cohomological field theory: see Chapter 6.

5. Pushforward by a forgetful map: When $a_{n+1} \geq 4$, we have the very simple formula

$$\pi_* \mathcal{R}(g, n+1, d; \sigma, a_1, \ldots, a_{n+1}) = \mathcal{R}(g, n, d-1; \sigma \sqcup \{a_{n+1} - 3\}, a_1, \ldots, a_n).$$

When $a_{n+1} = 3$, we can use the simple combinatorial identity

$$\overbrace{\exp(\{1-X\}_\kappa)\{X\}_\kappa\{F_1\}_\kappa \cdots \{F_m\}_\kappa} = (\kappa_0 + m)\overbrace{\exp(\{1-X\}_\kappa)\{F_1\}_\kappa \cdots \{F_m\}_\kappa}$$

to obtain

$$\pi_* \mathcal{R}(g, n+1, d; \sigma, a_1, \ldots, a_n, 3) = (2g - 2 + n + \ell(\sigma))\mathcal{R}(g, n, d-1; \sigma, a_1, \ldots, a_n).$$

The pushforward formula when $a_{n+1} = 0$ or 1 is much more complicated. At first sight, it isn't clear how to handle the terms that involve factors $\{\frac{A+B}{T}\}_\kappa$ that appear. However, these terms can be canceled out by using the differential equations

$$864T\frac{dA}{dT} = -144A + \frac{A+B}{T} \text{ and } 864T\frac{dB}{dT} = 144B + \frac{A+B}{T}$$

and applying $T\frac{d}{dT}$ to the vertex and leg factors and $1 + T\frac{d}{dT}$ to the edge factors appearing in (5.2) to get an alternative expression for $d \cdot \mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n)$.

□

We can also ask how many of these relations are actually necessary to generate them all using multiplication, pushforward, and pullback. To make this question precise, let $\mathcal{R}_{g,n}^{\text{old}}$ be the sub-ideal of $\mathcal{R}_{g,n}$ generated by $\mathcal{S}_{g,n}^{>0}\mathcal{R}_{g,n}$ together with the images of other $\mathcal{R}_{g',n'}$ under pushforwards via gluing maps or pullbacks via forgetful maps; these are the relations that come from some simpler moduli space or from lower codimension.

Proposition 5.5. *If $g > 0$ then $\mathcal{R}_{g,n}/\mathcal{R}_{g,n}^{old}$ is generated by the relations $\mathcal{R}(g, n, d; \sigma, 1, \ldots, 1)$ with all parts of σ congruent to 1 mod 3.*

Proof. Let M be the subspace of $\mathcal{R}_{g,n}/\mathcal{R}_{g,n}^{\text{old}}$ generated by the relations $\mathcal{R}(g, n, d; \sigma, 1, \ldots, 1)$ with all parts of σ congruent to 1 mod 3. We need to show that all relations $\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n)$ are contained in M.

Multiplying by kappa and psi classes gives that $\mathcal{R}(g, n, d; \sigma, a_1, \ldots, a_n) \in M$ whenever all the a_i

are nonzero. Now suppose that some $a_j = 0$. By (5.3), we have

$$\mathcal{R}(g,n,d;\sigma,a_1,\ldots,a_n) = \sum_{i\in\sigma} R(g,n,d;\sigma\setminus\{i\},a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_n,i)$$

in $\mathcal{R}_{g,n}/\mathcal{R}_{g,n}^{\mathrm{old}}$. Each term on the left hand side has one fewer zero in σ and the a_i combined, so we may inductively use this equation to write any $\mathcal{R}(g,n,d;\sigma,a_1,\ldots,a_n)$ in terms of relations in which all the a_i are nonzero. This completes the proof. □

In particular, all the new relations in positive genus in our set of conjectural relations are S_n-invariant. This explains why fundamental relations such as Getzler's relation [12] in $R^2(\overline{\mathcal{M}}_{1,4})$ are symmetric in the marked points.

Chapter 6

Constructing relations II: moduli of r-spin curves

This chapter presents joint work with Rahul Pandharipande and Dimitri Zvonkine [30, 29].

6.1 Introduction

We prove that the conjectured relations $\mathcal{R}_{g,n}$ of the previous chapter hold in the cohomology ring $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. By restriction, we obtain a second proof of the Faber-Zagier conjecture in cohomology.

We study here the geometry of 3-spin curves. Witten's class on the moduli space of 3-spin curves defines a non-semisimple cohomological field theory. After a canonical modification (obtained by moving to a semisimple point of the associated Frobenius manifold), we construct a semisimple CohFT with a non-trivial vanishing property obtained from the homogeneity of Witten's class. Using the classification of semisimple CohFTs by Givental-Teleman [14, 36], we derive an explicit formula in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ for Witten's 3-spin class *and* use the vanishing property to establish the relations $\mathcal{R}_{g,n}$.

6.1.1 The tautological relations T

We define a subset $\mathbf{T}$ of the tautological relations described in Chapter 5 consisting of the elements

$$\mathcal{R}^d_{g,A} := (-1)^{d+a_1+\cdots+a_n}\mathcal{R}(g,n,d;\sigma=\emptyset,a_1,\ldots,a_n) \in \mathcal{S}^d_{g,n} \tag{6.1}$$

associated to the data

- $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range $2g - 2 + n > 0$,
- $A = (a_1, \ldots, a_n), \quad a_i \in \{0, 1\}$,
- $d \in \mathbb{Z}_{\geq 0}$ satisfying $d > \frac{g-1+\sum_{i=1}^{n} a_i}{3}$.

(When $d \not\equiv g - 1 + \sum_{g-1}^{n} a_i \mod 2$, we set $\mathcal{R}_{g,A}^{d} = 0$.)

The sign factors appearing in this definition can be interpreted as replacing the standard hypergeometric functions $A(T)$ and $B(T)$ by $A(-T)$ and $-B(-T)$; these are how these series will appear in the homogeneous calibration of the Frobenius manifold associated to A_2.

Theorem 6.1. *Every element $\mathcal{R}_{g,A}^{d} \in \mathbf{T}$ lies in the kernel of the homomorphism*

$$q : \mathcal{S}_{g,n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \ .$$

By Proposition 5.5, the relations $\mathbf{T}$ imply the more complete ideal of relations $\mathcal{R}_{g,n}$.

Corollary 6.2. *The ideal $\mathcal{R}_{g,n}$ of relations conjectured in Conjecture 5.1 holds in cohomology.*

Furthermore, we will identify $\mathcal{R}_{g,(a_1,\ldots,a_n)}^{d}$ as a simple multiple of Witten's class for $r = 3$ when $d = \frac{g-1+\sum a_i}{3}$ and a simple multiple of a pushforward of Witten's class under a forgetful map when $d < \frac{g-1+\sum a_i}{3}$.

6.1.2 Cohomological field theories

We recall here the basic definitions of a cohomological field theory by Kontsevich and Manin [20].

Let V be a finite dimensional $\mathbb{Q}$-vector space with a non-degenerate symmetric 2-form η and a distinguished element $\mathbf{1} \in V$. The data $(V, \eta, \mathbf{1})$ is the starting point for defining a cohomological field theory. Given a basis $\{e_i\}$ of V, we write the symmetric form as a matrix

$$\eta_{jk} = \eta(e_j, e_k) \ .$$

The inverse matrix is denoted by η^{jk} as usual.

A cohomological field theory consists of a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

We view $\Omega_{g,n}$ as associating a cohomology class on $\overline{\mathcal{M}}_{g,n}$ to elements of V assigned to the n markings. The CohFT axioms imposed on Ω are:

(i) Each $\Omega_{g,n}$ is S_n-invariant, where the action of the symmetric group S_n permutes both the marked points of $\overline{\mathcal{M}}_{g,n}$ and the copies of V^*.

(ii) Denote the basic gluing maps by

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n} \ ,$$

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n} \ .$$

The pullbacks $q^*(\Omega_{g,n})$ and $r^*(\Omega_{g,n})$ are equal to the contractions of $\Omega_{g-1,n+2}$ and $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$ by the bi-vector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points.

(iii) Let $v_1, \ldots, v_n \in V$ be any vectors and let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. We require

$$\Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbf{1}) = p^*\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \ ,$$

$$\Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2) \ .$$

Definition 6.3. A system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements

$$\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying properties (i) and (ii) is a *cohomological field theory* or a *CohFT*. If (iii) is also satisfied, Ω is a *CohFT with unit*.

A CohFT Ω yields a *quantum product* $\bullet$ on V via

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3) \ .$$

Associativity of $\bullet$ follows from (ii). The element $\mathbf{1} \in V$ is the identity for $\bullet$ by (iii).

A CohFT ω composed only of degree 0 classes,

$$\omega_{g,n} \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n} ,$$

is called a *topological field theory.* Via property (ii), $\omega_{g,n}(v_1, \ldots, v_n)$ is determined by considering stable curves with a maximal number of nodes. Such a curve is obtained by identifying several rational curves with three marked points. The value of $\omega_{g,n}(v_1 \otimes \cdots \otimes v_n)$ is thus uniquely specified by the values of $\omega_{0,3}$ and by the quadratic form η. In other words, given V and η, a topological field theory is uniquely determined by the associated quantum product.

6.1.3 Witten's r-spin class

For every integer $r \geq 2$, there is a beautiful CohFT obtained from Witten's r-spin class. We review here the basic properties of the construction. The integer r is fixed once and for all.

Let V be an $(r-1)$-dimensional $\mathbb{Q}$-vector space with basis $e_0, \ldots, e_{r-2}$, bilinear form

$$\eta_{ab} = \langle e_a, e_b \rangle = \delta_{a+b,r-2} ,$$

and unit vector $\mathbf{1} = e_0$. Witten's r-spin theory provides a family of classes

$$W_{g,n}(a_1, \ldots, a_n) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

for $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$. These define a CohFT by

$$\mathsf{W}_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}), \quad \mathsf{W}_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) = W_{g,n}(a_1, \ldots, a_n) .$$

To emphasize r, we will often refer to V as V_r.

Witten's class $W_{g,n}(a_1, \ldots, a_n)$ has (complex) degree given by the formula

$$\begin{aligned} \deg_{\mathbb{C}} W_{g,n}(a_1, \ldots, a_n) &= \mathsf{D}_{g,n}(a_1, \ldots, a_n) \\ &= \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r} . \end{aligned} \tag{6.2}$$

If $\mathsf{D}_{g,n}(a_1, \ldots, a_n)$ is not an integer, the corresponding Witten class vanishes.

In genus 0, the construction was first carried out by Witten [37] using r-spin structures (r^{th} roots of the canonical bundle) and satisfies the following initial conditions:

$$W_{0,3}(a_1, a_2, a_3) = \begin{cases} 1 & \text{if } a_1 + a_2 + a_3 = r - 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

$$W_{0,4}(1, 1, r-2, r-2) = \frac{1}{r}[\text{point}] \ \in H^2(\overline{\mathcal{M}}_{0,4}, \mathbb{Q}) \ .$$

Uniqueness of Witten's r-spin theory in genus 0 follows easily from the initial conditions (6.3) and the axioms of a CohFT with unit.

The genus 0 sector defines a quantum product $\bullet$ on V with unit e_0,

$$\langle e_a \bullet e_b, e_c \rangle = W_{0,3}(a, b, c) \ .$$

The resulting algebra, even after extension to $\mathbb{C}$, is not semisimple.

The existence of Witten's class in higher genus is both remarkable and highly non-trivial. An algebraic construction was first obtained by Polishchuk and Vaintrob [33] defining

$$W_{g,n}(a_1, \ldots, a_n) \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

as an algebraic cycle class. The algebraic approach was later simplified by Chiodo [5]. Analytic constructions have been given by Mochizuki [25] and later by Fan, Jarvis, and Ruan [11]. The equivalence between the above analytic and algebraic constructions was heretofore unknown.

Theorem 6.4. *For every $r \geq 2$, there is a unique CohFT which extends Witten's r-spin theory in genus 0 and has pure dimension* (6.2). *The unique extension takes values in the tautological cohomology ring*

$$RH^*(\mathcal{M}_{g,n}) \subset H^*(\mathcal{M}_{g,n}, \mathbb{Q}).$$

As a consequence of Theorem 6.4, the analytic and algebraic approaches coincide and yield tautological classes in cohomology. Our proof of Theorem 6.4 is not valid for Chow field theories as topological results play an essential role.

6.1.4 Strategy of proof

Theorems 6.1 and 6.4 are proven together. Let $\mathsf{W}_{g,n}$ be any CohFT with unit which extends Witten's r-spin theory in genus 0 and has pure dimension (6.2).

We use a canonical procedure (a shift on the Frobenius manifold) to define a new CohFT $\widetilde{\mathsf{W}}_{g,n}$ satisfying the following three properties:

(i) $\widetilde{\mathsf{W}}_{0,n}$ in genus 0 is constructed canonically from the genus 0 sector of $\mathsf{W}_{0,n}$ with the associated quantum product defining a semisimple algebra on V_r,

(ii) the component of $\widetilde{\mathsf{W}}_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n})$ in complex degree $\mathsf{D}_{g,n}(a_1, \ldots, a_n)$ equals $W_{g,n}(a_1, \ldots, a_n)$,

(iii) the class $\widetilde{\mathsf{W}}_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n})$ has no components in degrees higher than $\mathsf{D}_{g,n}(a_1, \ldots, a_n)$.

In other words, $\widetilde{\mathsf{W}}$ is constructed from W by adding only *lower* degree terms.

By the results of Givental and Teleman, $\widetilde{\mathsf{W}}$ is determined via a universal formula in the tautological ring by the semisimple genus 0 sector. By property (ii), we deduce a formula for $\widetilde{\mathsf{W}}$ in the tautological ring depending only upon Witten's r-spin theory in genus 0 and obtain Theorem 6.4.

To prove Theorem 6.1, we write explicitly Givental's formula for the modified CohFT $\widetilde{\mathsf{W}}$ in the 3-spin case. The series A and B appear in the associated Frobenius structure. By property (iii), we obtain vanishings in the tautological ring in degrees

$$d > \mathsf{D}_{g,n}(a_1, \ldots, a_n) = \frac{g - 1 + \sum_{i=1}^{n} a_i}{3} \quad \text{for } r = 3.$$

The outcome is exactly the relations $\mathbf{T}$.

As a further outcome of the above investigation, we obtain the following formula for Witten's 3-spin class.

Theorem 6.5. *Let $r = 3$. Then, for $g, n \in \mathbb{Z}_{\geq 0}$ in the stable range, we have*

$$W_{g,n}(a_1, \ldots, a_n) = 2^g \, 1728^d \, q\left(\mathcal{R}^d_{g,(a_1,\ldots,a_n)}\right) \in H^{2d}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

when $d = \frac{g-1+\sum_{i=1}^{n} a_i}{3}$ is integral (and $W_{g,n}(a_1, \ldots, a_n)$ is 0 otherwise).

6.1.5 Plan of the chapter

In Section 6.2, we define the shifted Witten class for the r-spin theory. Theorem 6.4 is proven as a consequence of semisimplicity and Teleman's uniqueness result. A short review of the R-matrix action on CohFTs is presented in Section 6.3. In Section 6.4, we compute the R-matrix for the 3-spin case and prove Theorems 6.1 and 6.5. We conclude by saying a little bit about tautological relations coming from the r-spin case for $r > 3$ in Section 6.5.

6.2 A_{r-1} and the shifted Witten class

6.2.1 Potentials

Frobenius manifolds were introduced and studied in detail in Dubrovin's monograph [6]. For a concise summary see [15, Section 1].

As for every CohFT, the genus 0 part of Witten's r-spin class determines a Frobenius manifold structure on the underlying vector space V_r. For Witten's class, the Frobenius manifold coincides with the canonical Frobenius structure on the versal deformation of the A_{r-1} singularity [7] up to a coordinate change. We will denote by $t^0, \dots, t^{r-2}$ the coordinates in the basis $e_0, \dots, e_{r-2}$ of V_r.

The structure of a Frobenius manifold is governed by the Gromov-Witten potential. The genus 0 Gromov-Witten potential of Witten's r-spin class (without descendants) is

$$\mathsf{F}(t^0, \dots, t^{r-2}) = \sum_{n \geq 3} \sum_{a_1, \dots, a_n} \int_{\overline{\mathcal{M}}_{0,n}} W_{0,n}(a_1, \dots, a_n) \frac{t^{a_1} \cdots t^{a_n}}{n!}.$$

Example 6.6. For $r = 3$, the genus 0 potential obtained from Witten's class equals

$$\mathsf{F}(x, y) = \frac{1}{2}x^2 y + \frac{1}{72}y^4,$$

where $x = t^0$ and $y = t^1$.

For $r = 4$, the potential is

$$\mathsf{F}(x, y, z) = \frac{1}{2}x^2 z + \frac{1}{2}xy^2 + \frac{1}{16}y^2 z^2 + \frac{1}{960}z^5,$$

where $x = t^0$, $y = t^1$, and $z = t^2$.

The third derivatives of F determine an associative algebra structure (the *quantum product*) in each tangent space to the Frobenius manifold. Let ∂_i denote the vector field on V_r associated to differentiation by t^i. Then,

$$\partial_i \bullet \partial_j = \sum_{k,l} \frac{\partial^3 \mathsf{F}}{\partial t^i \partial t^j \partial t^k} \eta^{kl} \partial_l \ .$$

The algebra on tangent spaces is semisimple outside the discriminant of A_{r-1}. For instance, for $r = 3$, the discriminant is $\{y = 0\}$.

Definition 6.7. Let $\tau \in V_r$. We define the *shifted Witten class* by

$$\mathsf{W}^{\tau}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 0} \frac{1}{m!} \, (p_m)_* \mathsf{W}_{g,n+m}(v_n \otimes \cdots \otimes v_n \otimes \tau \otimes \cdots \otimes \tau),$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

Remark 6.8. We have the following degree bound:

$$\begin{aligned} \deg\Big[(p_m)_* \mathsf{W}_{g,n+m}(e_{a_1} \otimes \cdots \otimes e_{a_n} \otimes \tau \otimes \cdots \otimes \tau)\Big] &\leq \frac{(g-1)(r-2) + \sum a_i + m(r-2)}{r} - m \\ &= \mathsf{D}_{g,n}(a_1, \ldots, a_n) - \frac{2m}{r} \ . \end{aligned}$$

The sum in Definition 6.7 is thus finite for any given g and $a_1, \ldots, a_n$. The shifted Witten class is therefore well-defined. Moreover, the highest degree term of the shifted Witten class is equal to the Witten class itself – all the other terms are of smaller degrees.

Remark 6.9. Let $\mathsf{F}(t)$ and $\mathsf{F}^{\tau}(\widehat{t}\,)$ be the genus 0 potentials of W and W^{τ} respectively. By elementary verification,

$$\mathsf{F}^{\tau}(\widehat{t}\,) = \mathsf{F}(\tau + \widehat{t}\,) - (\text{terms of degree} < 3).$$

The following proposition is a straightforward check:

Proposition 6.10. *The shifted Witten class W^{τ} is a CohFT with unit.*

6.2.2 The Euler field

A Frobenius manifold is called *conformal* if carries an affine Euler field E, a vector field satisfying the following properties:

(i) in flat coordinates t^i, the field has the form

$$E = \sum_i (\alpha_i t^i + \beta_i) \frac{\partial}{\partial t^i},$$

(ii) the quantum product $\bullet$, the unit $\mathbf{1}$, and the metric η are eigenfunctions of the Lie derivative L_E with weights 0, -1, and $2 - \delta$ respectively.

The rational number δ is called the *conformal dimension* of the Frobenius manifold.

For instance, on the Frobenius manifold A_{r-1}, an Euler field is given by

$$E = \sum_{a=0}^{r-2} \left(1 - \frac{a}{r}\right) t^a \frac{\partial}{\partial t^a},$$

$$\delta = \frac{r-2}{r}.$$

Remark 6.11. We follow here Givental's conventions for the Euler field. In Teleman's conventions, the Euler vector field and hence the eigenvalues of L_E have the opposite sign.

Let Ω be a CohFT and V the corresponding Frobenius manifold. Given an Euler field E on V, a natural action of E on Ω is defined as follows. Let

$$\deg : H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

be the operator which acts on H^{2k} by multiplication by k. As usual, ∂_i is the vector field[1] on V associated to differentiation by the coordinate t^i. Then

$$(E.\Omega)_{g,n}(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n}) =$$

$$\left(\deg + \sum_{l=1}^{n} \alpha_{i_l}\right) \Omega_{g,n}(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n}) + p_* \Omega_{g,n+1}\left(\partial_{i_1} \otimes \cdots \otimes \partial_{i_n} \otimes \sum \beta_i \partial_i\right),$$

where $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

Definition 6.12. A CohFT Ω is *homogeneous* if

$$(E.\Omega)_{g,n} = [(g-1)\delta + n]\, \Omega_{g,n}$$

for all g and n.

Witten's r-spin class is easily seen to be homogeneous. Indeed, we have

$$\mathsf{D}_{g,n}(a_1, \ldots, a_n) + \sum_{i=1}^{n} \left(1 - \frac{a_i}{r}\right) =$$

$$\frac{(r-2)(g-1) + \sum a_i}{r} + n - \frac{\sum a_i}{r} = (g-1)\frac{r-2}{r} + n = (g-1)\delta + n.$$

The underlying vector space V_r and basis $e_0, \ldots, e_{r-2}$ are the same for the CohFT obtained from the shifted Witten class. We denote the coordinates on V_r in the basis $e_0, \ldots, e_{r-2}$ for the shifted

[1]We will often use the canonical identification of V with the tangent space of $0 \in V$.

r-spin Witten theory by $\widehat{t}^{\,0}, \dots, \widehat{t}^{\,r-2}$.

Proposition 6.13. *The shifted Witten class is a homogeneous CohFT with Euler field*

$$E = \sum_{a=0}^{r-2} \left(1 - \frac{a}{r}\right) (\tau^a + \widehat{t}^{\,a}) \frac{\partial}{\partial \widehat{t}^{\,a}},$$

of conformal dimension $\delta = \frac{r-2}{r}$.

Proof. Assume for simplicity $\tau = u\partial_b$ for some fixed $b \in \{0, \dots, r-2\}$. Denote

$$\mathsf{W}_{g,n+m}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n} \otimes \partial_b \otimes \cdots \otimes \partial_b)$$

here by just $\mathsf{W}_{g,n+m}$. Then we have

$$(E.\mathsf{W}^\tau)_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) =$$

$$\sum_{m \geq 0} \frac{u^m}{m!} \left\{ \left[\frac{(r-2)(g-1) + \sum a_i + mb}{r} - m \right] + \sum_{i=0}^{r-2} \left(1 - \frac{a_i}{r}\right) \right\} (p_m)_* \mathsf{W}_{g,n+m}$$
$$+ \sum_{m \geq 0} \frac{u^m}{m!} u \left(1 - \frac{b}{r}\right) (p_{m+1})_* \mathsf{W}_{g,n+m+1}$$

After simplifying, the above equals

$$[(g-1)\delta + n] \sum_{m \geq 0} \frac{u^m}{m!} (p_m)_* \mathsf{W}_{g,n+m}$$
$$- \sum_{m \geq 1} \frac{u^m}{(m-1)!} \left(1 - \frac{b}{r}\right) (p_m)_* \mathsf{W}_{g,n+m}$$
$$+ \sum_{m \geq 0} \frac{u^{m+1}}{m!} \left(1 - \frac{b}{r}\right) (p_{m+1})_* \mathsf{W}_{g,n+m+1}.$$

The last two sums cancel each other, so we obtain

$$[(g-1)\delta + n] \sum_{m \geq 0} \frac{u^m}{m!} (p_m)_* \mathsf{W}_{g,n+m} = [(g-1)\delta + n]\ \mathsf{W}^\tau_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}).$$

The general case is similar. ☐

Since the shifted r-spin Witten class is a homogeneous semisimple CohFT, we can apply the following theorem by C. Teleman [36, Theorem 1].

Theorem 6.14 (Teleman). *Let $\Omega_{0,n}$ be a genus 0 homogeneous semisimple CohFT with unit. The following results hold:*

(i) There exists a unique homogeneous CohFT with unit $\Omega_{g,n}$ extending $\Omega_{0,n}$ to higher genus.

(ii) The extended CohFT $\Omega_{g,n}$ is obtained by an R-matrix action on the topological (degree 0) sector of $\Omega_{0,n}$ determined by $\Omega_{0,3}$.

(iii) The R-matrix is uniquely specified by $\Omega_{0,3}$ and the Euler field.

The unit-preserving R-matrix action in part (ii) of Theorem 6.14 will be described in Section 6.3, see Definition 6.21.

We will compute the R-matrix for the 3-spin Witten class in Section 6.4. Since the shifted 3-spin Witten class (considered for all genera) is a homogeneous CohFT with unit, the expressions obtained by the R-matrix action coincide with the shifted 3-spin Witten class. In particular, if we split the expression of the R-matrix action into pure degree parts, the parts of degree

$$d > \mathsf{D}_{g,n}(a_1, \dots, a_n)$$

vanish while the part of degree $\mathsf{D}_{g,n}(a_1, \dots, a_n)$ coincides with Witten's class, which proves Theorem 6.4.

6.3 The R-matrix action

We describe the R-matrix action on CohFTs. The action was first defined on Gromov-Witten potentials by Givental [14]. Its lifting to CohFTs was independently discovered by several authors: the papers by Teleman [36] and Shadrin [34] give an abbreviated treatment of the subject and refer to unpublished notes by Kazarian and by Katzarkov, Kontsevich, and Pantev. For a detailed and self-contained exposition of the subject, see Section 2 of [30].

6.3.1 The R-matrix action on CohFTs

Let V be a vector space with basis $\{e_i\}$ and a symmetric bilinear form η. Consider the group of $\mathrm{End}(V)$-valued power series

$$R(z) = 1 + R_1 z + R_2 z^2 + \cdots \tag{6.4}$$

satisfying the *symplectic condition*,

$$R(z)R^*(-z) = 1 \ ,$$

where R^* is the adjoint with respect to η.

Remark 6.15. Let R_j^k be the matrix form of an endomorphism R in the given basis,

$$R\left(t^j e_j\right) = \sum_{j,k} R_j^k t^j \, e_k \ .$$

The symplectic condition in coordinates is

$$R_l^j(z)\eta^{ls}R_s^k(-z)\eta_{ku} = \delta_u^j.$$

After multiplying by η^{-1} on the right, we obtain equivalent condition in bi-vector form

$$R_l^j(z)\eta^{ls}R_s^k(-z) = \eta^{jk} \ .$$

We conclude that the expression

$$\frac{\eta^{jk} - R_l^j(z)\eta^{ls}R_s^k(w)}{z+w}$$

is a well-defined *power series* in z and w.

Associated to $R(z)$ is the power series $R^{-1}(z) = \frac{1}{R(z)}$ which also satisfies the symplectic condition. Hence,

$$\frac{\eta^{jk} - (R^{-1})_l^j(z)\eta^{ls}(R^{-1})_s^k(w)}{z+w} \in V^{\otimes 2}[[z,w]]. \tag{6.5}$$

We will denote the series (6.5) by

$$\frac{\eta^{-1} - R^{-1}(z)\eta^{-1}R^{-1}(w)^{\mathsf{t}}}{z+w}$$

for short (where the superscript t denotes matrix transpose).

Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a CohFT on V, and let R be an element of the group (6.4). The CohFT $R\Omega$ is defined as follows.

Definition 6.16. Let $\mathsf{G}_{g,n}$ be the finite set of stable graphs of genus g with n legs. For each $\Gamma \in \mathsf{G}_{g,n}$, define a contribution

$$\mathrm{Cont}_\Gamma \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

by the following construction:

(i) place $\Omega_{\mathsf{g}(v),\mathsf{n}(v)}$ at each vertex v of Γ,

(ii) place $R^{-1}(\psi_l)$ at every leg l of Γ,

(iii) at every edge e of Γ, place
$$\frac{\eta^{-1} - R^{-1}(\psi_e')\eta^{-1}R^{-1}(\psi_e'')^{\mathsf{t}}}{\psi_e' + \psi_e''} \,.$$

Define $(R\Omega)_{g,n}$ to be the sum of contributions of all stable graphs,

$$(R\Omega)_{g,n} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \operatorname{Cont}_\Gamma \,.$$

We use the inverse of the R-matrix in all of our formulas in Definition 6.16. There are two reasons for the seemingly peculiar choice. First, the result will be a left group action rather than a right group action on CohFTs. Second, the same convention is used by Givental and Teleman in their papers.

A few remarks about Definition 6.16 are needed for clarification. By the symmetry property of CohFTs, the placement of $\Omega_{\mathsf{g}(v),\mathsf{n}(v)}$ does not depend upon an ordering of the half-edges at v. At a leg l attached to a vertex v, we have

$$R^{-1}(\psi_l) \in H^*(\overline{\mathcal{M}}_{\mathsf{g}(v),\mathsf{n}(v)}, \mathbb{Q}) \otimes \operatorname{End}(V).$$

The first factor acts on the cohomology of the moduli space $\overline{\mathcal{M}}_{\mathsf{g}_v,\mathsf{n}_v}$ by multiplication. The endomorphism factor acts on the vectors which are "fed" to $\Omega_{\mathsf{g}(v),\mathsf{n}(v)}$ at the legs.

For an edge e attached to vertices v' and v'' (possibly the same vertex), denote by $\overline{\mathcal{M}}_{g',n'}$ and $\overline{\mathcal{M}}_{g'',n''}$ the corresponding moduli spaces. The insertion on e is an element of

$$H^*(\overline{\mathcal{M}}_{g',n'}, \mathbb{Q}) \otimes H^*(\overline{\mathcal{M}}_{g'',n''}, \mathbb{Q}) \otimes V^{\otimes 2}$$

obtained by substituting $z = \psi_e'$ and $w = \psi_e''$ in (6.5). Once again, the cohomology factors act on the corresponding cohomology spaces by multiplication. The bivector part is used to contract the two covectors sitting on the half-edges e' and e'' in the corresponding CohFT elements at v' and v. In the expression $R^{-1}(\psi_e')\eta^{-1}R^{-1}(\psi_e'')^{\mathsf{t}}$, the bivector η^{-1} sits in the middle of the edge, while the action of R^{-1} is directed from the middle of the edge towards the vertices.

The similarity of Definition 6.16 with the form of the relations $\mathcal{R}^d_{g,A}$ inspired the work presented in this chapter.

Proposition 6.17. *The R-matrix action is a left group action on the set of CohFTs.*

6.3.2 Action by translations

Let Ω be a CohFT based on the vector space V, and let

$$T(z) = T_2 z^2 + T_3 z^3 + \cdots$$

be a V-valued power series with vanishing coefficients in degrees 0 and 1.

Definition 6.18. The *translation of* Ω *by* T is the CohFT $T\Omega$ defined by

$$(T\Omega)_{g,n}(v_1 \otimes \cdots \otimes v_n)$$

$$= \sum_{m \geq 0} \frac{1}{m!} \, (p_m)_* \Omega_{g,n+m}\Big(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m})\Big),$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map.

The use of $T(\psi_i)$ as an argument in a CohFT is an abuse of notation. The result should be understood as

$$\Omega_{g,n}(\cdots T(\psi_i) \cdots) = \sum_{k \geq 2} \psi_i^k \Omega_{g,n}(\cdots T_k \cdots).$$

Remark 6.19. The action by translations is very close to the shift of Definition 6.7. However, unlike shifts, the translation action is always well-defined for degree reasons: the degree of the m^{th} summand of the definition is at least m, so the sum is actually finite for any given g, n.

The action by translations can be described in terms of stable graphs. It is a summation over stable graphs with a single vertex and $n + m$ legs for $m \geq 0$. The first n legs carry the vectors $v_1, \ldots, v_n$, and the last m legs carry the series $T(\psi_i)$. The latter legs are then suppressed by a forgetful map. We will call the first n legs *main legs* and the last m legs κ*-legs*, since the push-forward of powers of ψ-classes gives rise to κ-classes.

Proposition 6.20. *Translations form an abelian group action on CohFTs.*

6.3.3 CohFTs with unit

Let Ω be a CohFT with unit $\mathbf{1} \in V$ satisfying

$$\Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbf{1}) = p^* \Omega_{g,n}(v_1 \otimes \cdots \otimes v_n),$$

$$\Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2) ,$$

where $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. The R-matrix action and the translation action defined in the previous sections do not preserve the property of being a CohFT with unit. However, the two actions can be combined in a unique way so as to preserve the unit.

Definition 6.21. Let Ω be a CohFT with unit $\mathbf{1} \in V$. Let $R(z)$ be an R-matrix satisfying the symplectic condition, and let

$$T(z) = z \cdot [\mathbf{1} - R^{-1}(\mathbf{1})](z) \in z^2V[[z]].$$

The *unit-preserving R-matrix action* on Ω is

$$R.\Omega = RT\Omega .$$

Proposition 6.22. *The unit-preserving R-matrix action is a left group action on the set of CohFTs with unit.*

6.4 The R-matrix for A_2

We compute the R-matrix for the Frobenius manifold of the A_2 singularity and deduce an expression for the shifted 3-spin Witten class in terms of stable graphs. The outcome is a proof of Theorems 6.1 and 6.5.

6.4.1 The Frobenius manifold A_2

We compute all the differential geometric data associated with the Frobenius manifold A_2 for use in the following calculations.

The Frobenius manifold A_2 is based on the 2-dimensional vector space[2] V with coordinates $x = t^0$ and $y = t^1$ corresponding to the remainders 0 and 1 modulo 3 respectively. The *unit vector field* is $\partial_x = \frac{\partial}{\partial x}$. The *metric* is

$$\eta = dx \otimes dy + dy \otimes dx \qquad \text{or} \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

[2]In the notation of Section 6.1.3, V is V_3.

Since the only nonzero values of Witten's 3-spin class in genus 0 are

$$W_{0,3}(0,0,1) = 1, \qquad W_{0,4}(1,1,1,1) = \frac{1}{3} \ ,$$

the genus 0 Gromov-Witten potential is

$$\mathsf{F}(x,y) = \frac{1}{2}x^2 y + \frac{1}{72}y^4 \ .$$

The *Euler field* is

$$E = x\frac{\partial}{\partial x} + \frac{2}{3}y\frac{\partial}{\partial y} \ .$$

The Lie derivatives of E on the basis vectors fields are easily calculated:

$$L_E(\partial_x) = [E, \partial_x] = -\partial_x \ ,$$
$$L_E(\partial_y) = [E, \partial_y] = -\frac{2}{3}\partial_y \ .$$

By Proposition 6.13, the *conformal dimension* equals

$$\delta = \frac{r-2}{r} = \frac{1}{3} \ .$$

Let v be a tangent vector at a point of the Frobenius manifold. We define the *shifted degree operator* $\mu(v)$, also called the *Hodge grading operator*, by

$$\mu(v) = [E, v] + (1 - \delta/2)v \ .$$

Here, the vector v is extended to a flat tangent vector field in order to compute the commutator. We have

$$\mu(\partial_x) = -\frac{1}{6}\partial_x \ ,$$
$$\mu(\partial_y) = \frac{1}{6}\partial_y \ .$$

Definition 6.23. To simplify the formulas, we will use the following notation:

$$\phi = \frac{y}{3} \ , \quad \widehat{\partial}_x = \phi^{1/4}\partial_x \ , \quad \widehat{\partial}_y = \phi^{-1/4}\partial_y \ .$$

The frame $(\widehat{\partial}_x, \widehat{\partial}_y)$ in the tangent space of V at (x, y) is the most practical for the computations. The dual frame of the cotangent space is denoted by

$$\widehat{dx} = \phi^{-1/4} dx \ , \quad \widehat{dy} = \phi^{1/4} dy \ .$$

The *quantum multiplication* of vector fields on the Frobenius manifold is given by

$$\begin{aligned} \widehat{\partial}_x \bullet \widehat{\partial}_x &= \phi^{1/4} \widehat{\partial}_x \ , \\ \widehat{\partial}_x \bullet \widehat{\partial}_y &= \phi^{1/4} \widehat{\partial}_y \ , \\ \widehat{\partial}_y \bullet \widehat{\partial}_y &= \phi^{1/4} \widehat{\partial}_x \ . \end{aligned}$$

Whether in basis (∂_x, ∂_y) or in frame $(\widehat{\partial}_x, \widehat{\partial}_y)$, the shifted degree operator is expressed by the matrix

$$\frac{1}{6} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Unlike ∂_x, the vector field $\widehat{\partial}_x$ is not flat. However, in the definition of μ, we use the flat extension of $\widehat{\partial}_x$ at a given point, which only differs from ∂_x by a multiplicative constant.

We will also need the operator ξ of quantum multiplication by E. In the frame $(\widehat{\partial}_x, \widehat{\partial}_y)$, ξ is given by

$$\xi = \begin{pmatrix} x & 2\phi^{3/2} \\ 2\phi^{3/2} & x \end{pmatrix} .$$

Remark 6.24. The computations not involving the Euler vector field apply more generally to 2-dimensional Frobenius manifolds whose Gromov-Witten potential has the form

$$\mathsf{F}(x, y) = \frac{1}{2} x^2 y + \Phi(y)$$

with the convention $\phi = \phi(y) = \Phi'''(y)$. For instance, the Gromov-Witten potential of $\mathbb{C}\mathrm{P}^1$ has the above form with

$$\Phi = \phi = Q e^y .$$

6.4.2 The topological field theory

A topological field theory $\omega_{g,n}$ is a CohFT of degree 0, as discussed in Section 6.1.2. Teleman's reconstruction, used to prove Theorem 6.14, expresses every semisimple CohFT Ω as a unit-preserving R-matrix action (see Definition 6.21) on the topological field theory $\omega_{g,n}$ with unit where

$$\omega_{0,3} = \Omega_{0,3}.$$

Let us start by determining the topological field theory $\omega_{g,n}$ for Witten's 3-spin class.

Lemma 6.25. *For the topological (degree 0) part of Witten's* 3*-spin theory, we have*

$$\omega_{g,n}(\widehat{\partial}_x^{\otimes n_0} \otimes \widehat{\partial}_y^{\otimes n_1}) = 2^g \phi^{\frac{2g-2+n}{4}} \cdot \delta^{\text{odd}}_{g+n_1},$$

where $n = n_0 + n_1$. *Here,*

$$\delta^{\text{odd}}_{g+n_1} = \begin{cases} 1 & \text{if } g + n_1 \text{ is odd,} \\ 0 & \text{if } g + n_1 \text{ is even.} \end{cases}$$

Proof. The values of $\omega_{0,3}$ are prescribed by the quantum product:

$$\begin{aligned} \omega_0(\widehat{\partial}_x \otimes \widehat{\partial}_x \otimes \widehat{\partial}_x) &= \omega_0(\widehat{\partial}_x \otimes \widehat{\partial}_y \otimes \widehat{\partial}_y) &= 0 \\ \omega_0(\widehat{\partial}_x \otimes \widehat{\partial}_x \otimes \widehat{\partial}_y) &= \omega_0(\widehat{\partial}_y \otimes \widehat{\partial}_y \otimes \widehat{\partial}_y) &= \phi^{1/4}\ . \end{aligned}$$

For other g and n, we consider a stable curve with a maximal possible number of nodes (each component is rational with 3 special points). The vectors $\widehat{\partial}_x$ and $\widehat{\partial}_y$ are placed in some way on the marked points, and we must place either $\widehat{\partial}_x \otimes \widehat{\partial}_y$ or $\widehat{\partial}_y \otimes \widehat{\partial}_x$ at each node in such a way that the number of $\widehat{\partial}_y$'s is odd on each component of the curve. If $g + n_1$ is even, such a placement is impossible. If $g+n_1$ is odd, the placement can be done in 2^g ways, since the dual graph of the curve has g independent cycles.

By the factorization rules for CohFTs, the contribution of each successful placement of the $\widehat{\partial}_x$'s and $\widehat{\partial}_y$'s equals $\phi^{\frac{2g-2+n}{4}}$, where $2g - 2 + n$ is the number of rational components of the curve. □

6.4.3 The R-matrix

Givental [15, pages 4-5] gives a general method for computing the R-matrix of a Frobenius manifold without using an Euler field. The method is ambiguous: the R-matrix depends on the choice of certain integration constants. In the presence of an Euler field E, there is a unique choice of constants such that

$$L_E R_m = -mR_m$$

for every m. In the conformal case, Givental's method can be simplified by substituting i_E into his recursive equation. The simplified method for computing the R-matrix of a conformal Frobenius manifold is given, for instance, by Teleman [36, Theorem 8.15]. Since the 3-spin theory yields a conformal Frobenius manifold, the simplified method is suitable for us.

Let ξ be the operator of quantum multiplication by the tangent vector E. The matrices R_m then satisfy the following recursive equation[3]:

$$[R_{m+1}, \xi] = (m + \mu)R_m. \tag{6.6}$$

At a semisimple point of a conformal Frobenius manifold, the above equation determines the matrices R_m uniquely starting from $R_0 = 1$. Let

$$R_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}.$$

Using the formulas of Section 6.4.1 for ξ and μ, we rewrite (6.6) as

$$\left[\begin{pmatrix} a_{m+1} & b_{m+1} \\ c_{m+1} & d_{m+1} \end{pmatrix}, \begin{pmatrix} x & 2\phi^{3/2} \\ 2\phi^{3/2} & x \end{pmatrix} \right] = \frac{1}{6} \begin{pmatrix} 6m-1 & 0 \\ 0 & 6m+1 \end{pmatrix} \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix},$$

or in other words

$$2\phi^{3/2} \begin{pmatrix} b_{m+1} - c_{m+1} & a_{m+1} - d_{m+1} \\ d_{m+1} - a_{m+1} & c_{m+1} - b_{m+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} (6m-1)a_m & (6m-1)b_m \\ (6m+1)c_m & (6m+1)d_m \end{pmatrix}.$$

[3]In Teleman's paper, the commutator has the opposite sign, since his Euler field is the opposite of ours.

The following formulas are easily checked to be the unique solutions:

$$\begin{aligned}
a_m &= \frac{1}{1728^m\,\phi^{3m/2}}\,\frac{1+6m}{1-6m}\,\frac{(6m)!}{(3m)!\,(2m)!}\,\delta_m^{\text{even}},\\
b_m &= \frac{1}{1728^m\,\phi^{3m/2}}\,\frac{1+6m}{1-6m}\,\frac{(6m)!}{(3m)!\,(2m)!}\,\delta_m^{\text{odd}},\\
c_m &= \frac{1}{1728^m\,\phi^{3m/2}}\,\frac{(6m)!}{(3m)!\,(2m)!}\,\delta_m^{\text{odd}},\\
d_m &= \frac{1}{1728^m\,\phi^{3m/2}}\,\frac{(6m)!}{(3m)!\,(2m)!}\,\delta_m^{\text{even}}.
\end{aligned}$$

We now make explicit the connection with the A and B power series discovered by Faber and Zagier, which we alter slightly:

$$\boldsymbol{B}_0(T) := A(-T) = \sum_{m\geq 0}\frac{(6m)!}{(2m)!(3m)!}(-T)^m, \quad \boldsymbol{B}_1(T) := -B(-T) = \sum_{m\geq 0}\frac{1+6m}{1-6m}\frac{(6m)!}{(2m)!(3m)!}(-T)^m.$$

Denote by $\boldsymbol{B}_0^{\text{even}}$, $\boldsymbol{B}_0^{\text{odd}}$, $\boldsymbol{B}_1^{\text{even}}$, and $\boldsymbol{B}_1^{\text{odd}}$ the respective even and odd degree parts. The final expression for the R-matrix is:

$$R(z) = \begin{pmatrix} \boldsymbol{B}_1^{\text{even}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) & -\boldsymbol{B}_1^{\text{odd}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) \\ \\ -\boldsymbol{B}_0^{\text{odd}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) & \boldsymbol{B}_0^{\text{even}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right)\end{pmatrix}. \tag{6.7}$$

The symplectic condition for the R-matrix follows from the identity

$$\boldsymbol{B}_0(T)\boldsymbol{B}_1(-T) + \boldsymbol{B}_0(-T)\boldsymbol{B}_1(T) = 2,$$

or, equivalently,

$$\boldsymbol{B}_0^{\text{even}}(T)\boldsymbol{B}_1^{\text{even}}(T) - \boldsymbol{B}_0^{\text{odd}}(T)\boldsymbol{B}_1^{\text{odd}}(T) = 1.$$

Using the identity, we find

$$R^{-1}(z) = \begin{pmatrix} \boldsymbol{B}_0^{\text{even}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) & \boldsymbol{B}_1^{\text{odd}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) \\ \\ \boldsymbol{B}_0^{\text{odd}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right) & \boldsymbol{B}_1^{\text{even}}\left(\dfrac{z}{1728\,\phi^{3/2}}\right)\end{pmatrix}. \tag{6.8}$$

6.4.4 An expression for the shifted 3-spin Witten class

We combine here the expression for the topological field theory from Section 6.4.2 with the R-matrix action from Definition 6.21 using the explicit formulas for the R-matrix of Section 6.4.3.

Let $\tau = (x, y)$, $y \neq 0$, be a point of the Frobenius manifold A_2. Let $a_1, \ldots, a_n \in \{0, 1\}$ and let

$$D = \frac{g - 1 + \sum_{i=1}^n a_i}{3}$$

be the degree of Witten's 3-spin class. By convention, $\phi = y/3$. Recall the expressions $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$ of (6.1).

Theorem 6.26. *Witten's class for the shifted* 3*-spin theory equals*

$$\mathsf{W}^\tau_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = 2^g \sum_{d \geq 0} \frac{\phi^{\frac{3}{2}(D-d)}}{1728^d}\, q\left(\mathcal{R}^d_{g,(a_1,\ldots,a_n)}\right),$$

where $\partial_0 = \partial_x$*,* $\partial_1 = \partial_y$*.*

The following Corollary is an immediate consequence of Theorem 6.26 and the equation

$$\mathsf{W}^\tau_{g,n}(\partial_{a_1} \otimes \cdots \otimes \partial_{a_n}) = W_{g,n}(a_1, \ldots, a_n) + \text{ lower degree terms.}$$

explained in Section 6.2. Theorems 6.1 and 6.5 are implied by the Corollary.

Corollary 6.27. *We have the evaluations:*

$$\begin{aligned} q\left(\mathcal{R}^d_{g,(a_1,\ldots,a_n)}\right) &= 2^g\, 1728^D\, W_{g,n}(a_1, \ldots, a_n) && \text{for} \quad d = D, \\ q\left(\mathcal{R}^d_{g,(a_1,\ldots,a_n)}\right) &= 0 && \text{for} \quad d > D. \end{aligned}$$

Proof of Theorem 6.26. By Teleman's reconstruction result in the conformal semisimple case, Witten's shifted 3-spin class is given by $R.\omega$ where

- R is given by (6.7),
- ω is the topological part of the shifted 3-spin theory.

The proof now just amounts to a systematic matching of all factors in the sums over stable graphs which occur in Definition 6.21 for the R-matrix action and Section 5.3 for $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$.

Consider first the expression for the CohFT $R.\omega$ applied to a tensor product of n vectors ∂_x and ∂_y. As before, we denote by n_0 and n_1 the number of 0s and 1s among $a_1, \ldots, a_n$ so that $n_0 + n_1 = n$.

Powers of ϕ.

Since we wrote the R-matrix in frame $(\widehat{\partial}_x, \widehat{\partial}_y)$, we must substitute

$$\partial_x \mapsto \phi^{-1/4}\widehat{\partial}_x \ , \quad \partial_y \mapsto \phi^{1/4}\widehat{\partial}_y$$

in the tensor product argument for $R.\omega$. The result of the substitution is a factor of $\phi^{\frac{n_1-n_0}{4}}$.

By formula (6.8), all coefficients of R_m^{-1} contain a factor of $\phi^{-3m/2}$. Tracing through the definitions of the actions

$$R.\omega = RT\omega \ , \quad T(z) = z \cdot [\partial_x - R^{-1}(\partial_x)](z), \tag{6.9}$$

the R-matrix contributes a factor of $\phi^{-3d/2}$, where d is the degree of the class.

By Lemma 6.25, the topological field theory ω contributes (subject to parity condition accounted for later) a factor of $\phi^{\frac{2g_v-2+n_v}{4}}$ for every vertex v. These factors combine to yield $\phi^{\frac{2g-2+n}{4}}$.

Finally, each κ-leg created by the translation action contributes in two ways. First, since we must substitute

$$\partial_x \mapsto \phi^{-1/4}\widehat{\partial}_x$$

in formula (6.9) for $T(z)$, each κ-leg contributes $\phi^{-1/4}$. Second, because the κ-leg increases the valence of the vertex by 1, a factor of $\phi^{1/4}$ is contributed via the topological field theory. Thus, the contributions of each κ-leg to the power of ϕ cancel.

Collecting all of the above factors, we obtain a final calculation of the exponent of ϕ:

$$\frac{n_1-n_0}{4} - \frac{3d}{2} + \frac{2g-2+n}{4} = \frac{g-1+n_1-3d}{2} = \frac{3D-3d}{2} = \frac{3}{2}(D-d).$$

Powers of 1728.

All coefficients of R_m^{-1} contain a factor of $1/1728$. Hence, as above, we obtain a factor of 1728^{-d} from the R-matrix action.

Powers of 2.

At each vertex the topological field theory contributes a factor of 2^{g_v}. These combine into

$$\prod_{v\in V(\Gamma)} 2^{g_v} = \frac{2^g}{2^{h^1(\Gamma)}}.$$

The factor $2^{-h^1(\Gamma)}$ is present in the definition of $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$, and the remaining 2^g is included in the statement of Theorem 6.26.

Parity conditions at the vertices.

The topological field theory ω provides a nonzero contribution at a vertex if and only if $g_v + n_1(v)$ is odd. We must prove the parity condition which occurs in the definition of $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$ exactly matches.

The parity condition is imposed on $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$ by extracting the coefficient of $\zeta_v^{g_v-1}$, at each vertex v: see (5.2). We may view the factors of ζ_v as having the following sources. A leg carrying the assignment $a_l = 1$ (corresponding to ∂_y) contributes a ζ_v, while a leg carrying the assignment $a_l = 0$ (corresponding to ∂_x) does not. The terms of $\boldsymbol{B}_0^{\text{odd}}$ (including the effect of the κ-legs) and the terms of $\boldsymbol{B}_1^{\text{odd}}$ contribute a ζ_v. The terms of $\boldsymbol{B}_0^{\text{even}}$ and $\boldsymbol{B}_1^{\text{even}}$ do not contribute anything (because they leave the parity invariant). Finally, every edge insertion Δ_e contributes a factor if e is adjacent to v. The edge term can be expanded via

$$\boldsymbol{B}_0 = \boldsymbol{B}_0^{\text{even}} + \boldsymbol{B}_0^{\text{odd}} \ , \quad \boldsymbol{B}_1 = \boldsymbol{B}_1^{\text{even}} + \boldsymbol{B}_1^{\text{odd}}$$

and matched with the edge term of the CohFT $R.\omega$ using (6.5) and (6.8). Then the contributing factor is ζ_v if the bi-vector includes a factor $\widehat{\partial}_y$ on the side of the vertex v and 1 otherwise. Hence, the power of the variable ζ_v correctly counts the parity of entries $\widehat{\partial}_y$ submitted to the topological field theory ω at the vertex v.

Coefficients of the series $\boldsymbol{B}_0, \boldsymbol{B}_1$.

These coefficients simply coincide in the expression for $\mathcal{R}^d_{g,(a_1,\ldots,a_n)}$ and the formulas of the unit-preserving R-matrix action in all instances (legs, κ-legs, and edges). $\square$

6.5 Higher r

The methods used in this chapter can also be applied to study the shifted r-spin Witten class when $r > 3$. The result will be the construction of more families of tautological relations on $\overline{\mathcal{M}}_{g,n}$, beginning in codimension $\frac{r-2}{r}g$ now instead of $\frac{g}{3}$. This section outlines some preliminary results on this subject from ongoing work [29].

The most notable change that occurs with higher r is that there are multiple distinct choices of shift $\tau = (x_0, \ldots, x_{r-2})$. In the case $r = 3$, only the value of x_1 matters, and that value just scales the resulting theories. Now we must choose a point $[x_1 : \cdots : x_{r-2}] \in \mathsf{P}^{r-3}$.

6.5.1 Quantum product

The computations in Section 6.4 are for the most part straightforward to modify for the Frobenius manifold A_{r-1}. The exception is the structure of the quantum multiplication at the shift τ. Computing this requires knowing the dimension 0 part of the r-spin Witten classes

$$W_{0,n}(a, b, c, \tau, \tau, \dots, \tau).$$

It turns out that there is a very nice combinatorial formula for these classes.

For $a_1, \dots, a_n \in \{0, \dots, r-2\}$, let $W^{(0)}(a_1, \dots, a_n) \in \mathbb{Q}$ be equal to x if $W_{0,n}(a_1, \dots, a_n)$ is x times the class of a point and 0 otherwise. By the dimension formula for the Witten class, this means that $W^{(0)}(a_1, \dots, a_n) = 0$ unless $\sum_{i=1}^n a_i = (n-2)r - 2$.

Denote by ρ_k the k-th symmetric power of the standard 2-dimensional representation of sl_2. It is well-known that $\dim \rho_k = k+1$ and the ρ_k, $k \geq 0$, form the complete list of irreducible representations of sl_2.

Theorem 6.28. *Let $a_1, \dots, a_n \in \{0, \dots, r-2\}$ such that $\sum_{i=1}^n a_i = (n-2)r - 2$. Then we have*

$$W^{(0)}(a_1, \dots, a_n) = \frac{(n-3)!}{r^{n-3}} \dim \Big[\rho_{r-2-a_1} \otimes \cdots \otimes \rho_{r-2-a_n}\Big]^{\mathsf{sl}_2},$$

where the superscript sl_2 denotes the sl_2-invariant subspace.

Proof. Let $a, b, c, d, e_1, \dots, e_k \in \{0, \dots, r-2\}$ satisfy $a + b + c + d + \sum e_i = (k+1)r - 2$. Then $W_{0,k+4}(a, b, c, d, e_1, \dots, e_k)$ is a 1-dimensional class, and we may get a relation between the $W^{(0)}$ by cutting this with a relation between boundary divisors on $\overline{\mathcal{M}}_{0,k+4}$ and using the CohFT splitting axiom for W. If we use a pullback of the WDVV relation on $\overline{\mathcal{M}}_{0,4}$, we get

$$\sum_{\substack{I \sqcup J = \{1, \dots, k\} \\ * + \widehat{*} = r-2}} W^{(0)}(e_I, a, c, *) W^{(0)}(e_J, b, d, \widehat{*})$$
$$= \sum_{\substack{I \sqcup J = \{1, \dots, k\} \\ * + \widehat{*} = r-2}} W^{(0)}(e_I, a, d, *) W^{(0)}(e_J, b, c, \widehat{*}),$$

where e_I and e_J are the inputs $(e_i : i \in I)$ and $(e_j : j \in J)$ respectively, and $*$ and $\widehat{*}$ are nonnegative integers with sum $r-2$.

It is straightforward to check that the theorem statement is true for $n \leq 4$, and then the WDVV equation above uniquely determines $W^{(0)}(a_1, \dots, a_n)$ for $n \geq 5$. So it suffices to check that the sl_2

formula also satisfies this WDVV equation. Replacing each of $a, b, c, d, e_1, \ldots, e_k$ by $r-2$ minus itself, the desired identity is

$$\begin{aligned}
&\sum_{I \sqcup J = \{1,\ldots,k\}} |I|!|J|! \langle e_I, a, c, 2r-4-a-c-2|I| - \sum_{i \in I} e_i \rangle \times \\
&\qquad \langle e_J, b, d, 2r-4-b-d-2|J| - \sum_{j \in J} e_j \rangle \\
= &\sum_{I \sqcup J = \{1,\ldots,k\}} |I|!|J|! \langle e_I, a, d, 2r-4-a-d-2|I| - \sum_{i \in I} e_i \rangle \times \\
&\qquad \langle e_J, b, c, 2r-4-b-c-2|J| - \sum_{j \in J} e_j \rangle,
\end{aligned}$$

where

$$\langle a_1, \ldots, a_k \rangle := \dim \Big[\rho_{a_1} \otimes \cdots \otimes \rho_{a_n} \Big]^{\mathsf{sl}_2}$$

and

$$a + b + c + d + \sum_{i=1}^{k} e_i = 3r - 6 - 2k.$$

The following formula for the dimensions of the sl_2-invariants follows easily from the multiplication rule for the ρ_a:

$$\langle a_1, \ldots, a_k, 2s - a_1 - \cdots - a_k \rangle = \left[(1 - t^{-1}) \prod_{i=1}^{k} \frac{1 - t^{a_i+1}}{1-t} \right]_{t^s}.$$

Applying this formula, we can rewrite the identity we want to prove as saying that

$$\sum_{I \sqcup J = \{1,\ldots,k\}} |I|!|J|! \left[\frac{(1-t^{a+1})(1-t^{c+1})(1-u^{b+1})(1-u^{d+1})}{(1-t)(1-u)} \prod_{i \in I} \frac{t - t^{e_i+2}}{1-t} \prod_{j \in J} \frac{u - u^{e_j+2}}{1-u} \right]_{t^{r-1}u^{r-1}}$$

is equal to the same thing with c and d swapped.

Subtracting one side from the other and moving factors outside the sum, we have that it is equivalent to show that the coefficient of $t^{r-1}u^{r-1}$ in

$$\begin{aligned}
&(1-t^{a+1})(1-u^{b+1}) \frac{(1-t^{c+1})(1-u^{d+1}) - (1-t^{d+1})(1-u^{c+1})}{(1-t)(1-u)} \times \\
&\qquad \sum_{I \sqcup J = \{1,\ldots,k\}} |I|!|J|! \prod_{i \in I} \frac{t - t^{e_i+2}}{1-t} \prod_{j \in J} \frac{u - u^{e_j+2}}{1-u}
\end{aligned}$$

vanishes. Interchanging t and u and adding, we can replace this polynomial by a symmetric one:

$$((1-t^{a+1})(1-u^{b+1})-(1-t^{b+1})(1-u^{a+1}))\frac{(1-t^{c+1})(1-u^{d+1})-(1-t^{d+1})(1-u^{c+1})}{(1-t)(1-u)}\times$$
$$\sum_{I\sqcup J=\{1,\ldots,k\}}|I|!|J|!\prod_{i\in I}\frac{t-t^{e_i+2}}{1-t}\prod_{j\in J}\frac{u-u^{e_j+2}}{1-u}.$$

Now replace t by t/v and u by u/v and multiply by v^m for $m=a+b+c+d+3+\sum_{i=1}^k(e_i+2)=3r-3$. Also replace a,b,c,d by $a-1,b-1,c-1,d-1$ and e_i by e_i-2. The resulting identity to be proven is that the coefficient of $t^{r-1}u^{r-1}v^{r-1}$ vanishes in the polynomial

$$\frac{(t^au^b-u^at^b-t^av^b+v^at^b+u^av^b-v^au^b)(t^cu^d-u^dt^c-t^cv^d+v^ct^d+u^cv^d-v^cu^d)}{t^2u-u^2t-t^2v+v^2u+u^2v-v^2u}\times$$
$$(t-u)v\sum_{I\sqcup J=\{1,\ldots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{e_i}v-v^{e_i}t}{t-v}\prod_{j\in J}\frac{u^{e_j}v-v^{e_j}u}{u-v}.$$

The initial factor is invariant under cyclically permuting t,u,v, so this is equivalent to the coefficient of $t^{r-1}u^{r-1}v^{r-1}$ in the polynomial

$$\frac{(t^au^b-u^at^b-t^av^b+v^at^b+u^av^b-v^au^b)(t^cu^d-u^dt^c-t^cv^d+v^ct^d+u^cv^d-v^cu^d)}{t^2u-u^2t-t^2v+v^2u+u^2v-v^2u}\times$$
$$\sum_{\text{cyc}}(t-u)v\sum_{I\sqcup J=\{1,\ldots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{e_i}v-v^{e_i}t}{t-v}\prod_{j\in J}\frac{u^{e_j}v-v^{e_j}u}{u-v},$$

being zero, where the sum runs over the three cyclic permutations of t,u,v.

In fact, we claim that

$$\sum_{\text{cyc}}(t-u)v\sum_{I\sqcup J=\{1,\ldots,k\}}|I|!|J|!\prod_{i\in I}\frac{t^{e_i}v-v^{e_i}t}{t-v}\prod_{j\in J}\frac{u^{e_j}v-v^{e_j}u}{u-v}=0 \tag{6.10}$$

for any $e_i\in\mathbb{Z}$. It remains to prove (6.10) to complete the proof of the theorem.

Let $A\sqcup B\sqcup C=\{1,\ldots,k\}$ give an partition of the e_i into three sets, of sizes $a=|A|,b=|B|,c=|C|$. We can compute the coefficient of $\prod_{i\in A}t^{e_i}\prod_{i\in B}u^{e_i}\prod_{i\in C}v^{e_i}$ in (6.10):

$$\sum_{\text{cyc}}(t-u)v\sum_{i+j=c}\binom{c}{i}(i+a)!(j+b)!\left(\frac{v}{t-v}\right)^a\left(\frac{v}{u-v}\right)^b\left(\frac{-t}{t-v}\right)^i\left(\frac{-u}{u-v}\right)^j=0,$$

where now a,b,c are also cyclically permuted in correspondence with t,u,v.

Multiplying by $(-1)^{a+b+c}(t-u)^{a+b}(u-v)^{b+c}(v-t)^{c+a}$ and dividing by $a!b!c!$ gives

$$\sum_{\text{cyc}}\sum_{i+j=c}(-1)^{b+i}\binom{a+i}{i}\binom{b+j}{j}t^iu^jv^{a+b+1}(t-u)^{a+b+1}(u-v)^i(v-t)^j=0$$

Now set $x_1=t(u-v), x_2=u(v-t), x_3=v(t-u)$, multiply by $y_1^ay_2^by_3^c$, and sum the left hand side over all nonnegative integers a,b,c. The result is

$$\sum_{\text{cyc}}\frac{x_3}{(1+x_1y_3-x_3y_1)(1+x_3y_2-x_2y_3)},$$

which expands to

$$\frac{x_1+x_2+x_3}{(1+x_1y_3-x_3y_1)(1+x_2y_1-x_1y_2)(1+x_3y_2-x_2y_3)}.$$

But $x_1+x_2+x_3=tu-tv+uv-ut+vt-vu=0$, so this is zero, as desired. □

6.5.2 The shift by $\tau=(0,\ldots,0,r\phi)$

Let $a+b+c=r-2+2k$. By Theorem 6.28, we have

$$W(a,b,c,\underbrace{r-2,\ldots,r-2}_{k})=\begin{cases}\frac{k!}{r^k} & \text{if}\quad \min(a,b,c)\geq k,\\ 0 & \text{otherwise.}\end{cases}$$

This implies that the quantum multiplication at $\tau=(0,\ldots,0,r\phi)$ is given by

$$\partial_a\bullet\partial_b=\sum_{k=\max(0,a+b-r+2)}^{\min(a,b)}\phi^k\partial_{a+b-2k}.$$

Using this, it is straightforward to compute the R-matrix and then a family of tautological relations on $\overline{\mathcal{M}}_{g,n}$. When restricted to $\mathcal{M}_g$, these relations are precisely the generalized FZ relations of Theorem 3.2.

The other shifts $\tau=(0,\ldots,x_i,\ldots,0)$ can also be studied for $i=1,2,\ldots,r-3$. In each case, we obtain a formula for the r-spin Witten class and a family of tautological relations, true in cohomology.

Appendix A

Computations

Sage [35] code written by the author to compute the relations of Chapter 5 can be found at `http://www.math.princeton.edu/~apixton/programs/`. There is also some code to compute the Gorenstein ranks of tautological rings.

The code works by making a list of all possible dual graphs for a given moduli space of curves ($\mathcal{M}^{rt}, \mathcal{M}^c$, or $\overline{\mathcal{M}}$) and then decorating the dual graphs with kappa and psi classes to construct a list of generators for the strata algebra. Then given parameters for a relation, it computes the coefficient of each generator in the relation. The program can then compute the betti numbers of the quotient of the strata algebra by the span of all of the conjectured relations. Conjecture 5.3 says that these are actually the betti numbers of the tautological ring. Non-conjecturally, this procedure gives upper bounds on the betti numbers of the tautological cohomology by Theorem 6.1.

If these bounds agree with the Gorenstein ranks (which are computed by implementing the multiplication rule described in [17]), then the result is that the tautological cohomology ring is Gorenstein and all relations are given by our conjectured list of relations. If there is a discrepancy between the ranks given by the relations and the Gorenstein ranks, then we predict that the tautological ring is not Gorenstein. Figuring out a way of resolving whether the tautological ring is Gorenstein in these situations seems to be a difficult problem. The case of $\mathcal{M}_6^c$ is especially intriguing.

We now give several tables presenting some of the data that was obtained with this code. Computationally, progressing much beyond what is presented would require either significant code optimization or use of a dedicated computing cluster. For ease of computation, we always restrict to the S_n-invariant part when there are n marked points. Also, instead of doing computations in $\mathcal{M}_{g,n}^{rt}$, we work with $\mathcal{C}_g^n$, the nth power of the universal curve over $\mathcal{M}_g$[1]. The tautological rings of these

[1]Our conjectured set of relations on $\mathcal{M}_{g,n}^{rt}$ can be pushed to $\mathcal{C}_g^n$ via the canonical map $\pi : \mathcal{M}_{g,n}^{rt} \to \mathcal{C}_g^n$.

two spaces are very closely related, and it seems likely that the Gorenstein discrepancies are always equal in these two cases.

In the following tables, an asterisk indicates a Gorenstein discrepancy. All middle-dimensional ranks listed agree with the Gorenstein rank, including $R^6(\overline{\mathcal{M}}_5)$. Blank entries do not usually mean that the ring is zero; for computational reasons, we often did not compute the ranks for $d > \frac{D}{2} + 1$, where D is the socle degree. Question marks indicate that $d \leq \frac{D}{2} + 1$ but that the computation was too demanding to finish.

The minimal locations of Gorenstein discrepancies among those that were found are

$$\mathcal{M}_{24}, \mathcal{C}_{20}, \mathcal{C}^2_{17}, \mathcal{C}^3_{14}, \mathcal{C}^4_{11}, \mathcal{C}^5_{10}, \mathcal{C}^6_9, \mathcal{M}^c_6, \mathcal{M}^c_{5,2}. \tag{A.1}$$

In each case, there is a unique S_n-invariant missing relation in degree $d = \lfloor \frac{D}{2} \rfloor + 1$.

Conjecture A.1. *The tautological rings of the moduli spaces listed in* (A.1) *are not Gorenstein.*

Ranks given by relations for $R^d(\mathcal{C}_g^n)^{S_n}$												
	d:	0	1	2	3	4	5	6	7	8	9	10
n=1:	g=18:	1	2	4	7	12	19	30	42	53	53	42
	g=19:	1	2	4	7	12	19	30	43	57	64	57
	g=20:	1	2	4	7	12	19	30	44	61	75	76*
n=2:	g=15:	1	3	7	13	24	40	62	81	81	62	
	g=16:	1	3	7	13	24	40	64	88	103	88	
	g=17:	1	3	7	13	24	40	65	95	122	123*	
n=3:	g=12:	1	3	9	19	37	62	87	87	62		
	g=13:	1	3	9	19	37	64	97	114	97		
	g=14:	1	3	9	19	37	65	105	140	141*		
n=4:	g=9:	1	3	10	24	47	69	69	47			
	g=10:	1	3	10	24	49	79	98	79			
	g=11:	1	3	10	24	50	87	124	125*			
n=5:	g=8:	1	3	10	25	52	77	77	52			
	g=9:	1	3	10	26	57	95	117	95			
	g=10:	1	3	10	26	59	106	152	153*			
n=6:	g=7:	1	3	10	25	51	76	76	51			
	g=8:	1	3	10	26	58	98	121	98			
	g=9:	1	3	10	27	63	117	168	169*			
n=7:	g=6:	1	3	10	23	44	62	62	?			
	g=7:	1	3	10	25	53	87	106	87			
	g=8:	1	3	10	26	60	110	157	157			

Ranks given by relations for $R^d(\mathcal{M}^c_{g,n})^{S_n}$							
	d:	0	1	2	3	4	5
n=0:	g=2:	1	1				
	g=3:	1	2	2	1		
	g=4:	1	3	6	6	3	
	g=5:	1	3	10	19	19	10
	g=6:	1	4	15	42	71	72*
n=1:	g=2:	1	2	1			
	g=3:	1	4	7	4	1	
	g=4:	1	5	17	25	17	
	g=5:	1	6	28	75	107	75
n=2:	g=2:	1	4	4	1		
	g=3:	1	6	17	17	6	
	g=4:	1	8	36	81	81	
	g=5:	1	9	57	205	405	406*
n=3:	g=2:	1	5	10	5		
	g=3:	1	8	32	52	32	
	g=4:	1	10	62	190	285	190
n=4:	g=2:	1	7	20	20		
	g=3:	1	10	53	125	125	
	g=4:	1	13	96	387	799	799

Ranks given by relations for $R^d(\overline{\mathcal{M}}_{g,n})^{S_n}$									
	d:	0	1	2	3	4	5	6	7
n=0:	g=2:	1	2	1					
	g=3:	1	3	7	10	7			
	g=4:	1	4	13	32	50	50		
	g=5:	1	4	19	65	165	291	355	?
n=1:	g=2:	1	3	5	3				
	g=3:	1	5	16	29	29			
	g=4:	1	6	30	93	191	240	?	
n=2:	g=2:	1	5	11	11				
	g=3:	1	7	31	74	100	74		

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