The Teichmüller Space of an Entire Function

Núria Fagella*
Dept. de Mat. Aplicada i Anàlisi
Univ. de Barcelona, Gran Via 585
08007 Barcelona, Spain
fagella@maia.ub.es

Christian Henriksen†
The Tech. Univ. of Denmark
Building 303
DK-2800 Lyngby, Denmark
christian.henriksen@mat.dtu.dk

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Abstract

We consider the Teichmüller space of a general entire transcendental function $f : \mathbb{C} \to \mathbb{C}$ regardless of the nature of the set of singular values of f (critical values and asymptotic values). We prove that, as in the known case of periodic points and critical values, asymptotic values are also fixed points of any quasiconformal automorphism that commutes with f and which is homotopic to the identity, rel. the ideal boundary of the domain. As a consequence, the general framework of McMullen and Sullivan [McMullen & Sullivan 1998] for rational functions applies also to entire functions and we can apply it to study the Teichmüller space of f, analyzing each type of Fatou component separately. Baker domains were already considered in citefh, but wandering domains are new. We provide different examples of wandering domains, each of them adding a different quantity to the dimension of the Teichmüller space. In particular we give examples of rigid wandering domains.

1 Introduction

Given a holomorphic endomorphism $f : S \to S$ on a Riemann surface S, we consider the dynamical system generated by the iterates of f denoted by $f^n = f \circ \overset{n}{\cdots} \circ f$. There is a dynamically natural partition of the phase space S into the *Fatou set* $\mathcal{F}(f)$, where the iterates of f form a normal family, and the *Julia set*, $\mathcal{J}(f) = S - \mathcal{F}(f)$ which is the complement.

If $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then f is a rational map. If $S = \mathbb{C}$ and f does not extend to $\widehat{\mathbb{C}}$ then f is an entire transcendental mapping, i.e., infinity is an essential singularity. Entire transcendental functions present fundamental differences with respect to rational maps.

One of them concerns the set of singularities of the inverse function. For a rational map f, all branches of the inverse function are locally well defined except on the set of *critical values*, i.e., points $v = f(c)$ where $f'(c) = 0$. If f is transcendental, there is another obstruction to inverse branches being well defined: Some inverse branches are not well defined in any

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neighborhood of an *asymptotic value*. A point $a \in \mathbb{C}$ is called an asymptotic value if there exists a path $\gamma(t) \underset{t\to\infty}{\longrightarrow} \infty$ such that $f(\gamma(t)) \underset{t\to\infty}{\longrightarrow} a$. We call a point $z \in \mathbb{C}$ *regular* if there exists a neighborhood U of z such that $f : V \to U$ is a conformal isomorphism for every component V of $f^{-1}(U)$. The complement of the set of regular values is called the set of *singular values* and it can be shown to be equal to the closure of the union of the set of critical values and the set of asymptotic values. The set of critical values, even when infinite, always forms a discrete set. This is in contrast to the fact that there are virtually no restrictions on the set of asymptotic values: there even exists an entire map for which every value is an asymptotic value [Gross 1918]. However, it follows from a theorem of Denjoy, Carleman and Ahlfors that entire functions of finite order may have only a finite number of asymptotic values (see e.g. [Nevanlinna 1970] or Theorem 4.11 in [Hua & Yang 1998]).

This fact motivated the definition and study of special classes of entire transcendental maps like, for example, the class $\mathcal{S}$ of functions of *finite type*, which are those with a finite number of singular values (see e.g. [Eremenko & Lyubich 1992] and [Devaney & Tangerman 1986]). The space of entire functions with a given finite number of singular values form a finite dimensional space and therefore share many properties with rational maps, like for example the fact that every component of $\mathcal{F}(f)$ is eventually periodic [Eremenko & Lyubich 1992, Goldberg & Keen 1986]. There is a classification of the periodic components of the Fatou set of a rational map or a map in class $\mathcal{S}$: Such a component can either be a cycle of rotation domains or the basin of attraction of an attracting, superattracting or indifferent periodic point.

If we allow f to have infinitely many singular values then there are more possibilities. For example a component of $\mathcal{F}(f)$ may be *wandering*, that is, it will never be iterated to a periodic component. The first example of a wandering domain was given by Baker [Baker 1976] and elementary examples were given by Herman in [Herman 1981]. It is easy to check, for instance, that the function $f(z) = z + 2\pi + \sin(z)$ has a sequence of wandering domains, each of them containing a critical point (see example 2). In general however, wandering domains need not have any singular value inside. It was shown in [Baker 1975] that wandering domains are the only Fatou components of an entire transcendental function that may be multiply connected.

For maps of finite type, there is a classification of the eventually periodic components of the Fatou set (see [Bergweiler 1993]) and entire maps with infinitely many singular values allow for one more possibility: An invariant connected component of the Fatou set, U, is called a Baker domain, if for all $z \in U$ we have $f^n(z) \to \infty$, as $n \to \infty$. We call U a period p Baker domain if $f^k(U)$ is an invariant Baker domain under f^p, for some $k \leq p$. The first example of an entire function with a Baker domain was given by Fatou in [Fatou 1920], who considered the function $f(z) = z + 1 + \mathrm{e}^{-z}$ and showed that the right half-plane is contained in an invariant Baker domain. Since then, many other examples have been considered, showing various properties that are possible for this type of Fatou components (see for example [Eremenko & Lyubich 1987], [Bergweiler 1995], [Baker & Domínguez 1999], [Rippon & Stallard 1999(1)], [Rippon & Stallard 1999(2)], [König 1999] and [Baranski & Fagella 2000]). It follows from [Baker 1975] that a Baker domain of an entire function is simply connected.

As it is the case with basins of attraction and rotation domains, there is also a relation between Baker domains and the singularities of the inverse map. In particular, it is shown in [Eremenko & Lyubich 1992] that Baker domains do not exist for a map such that the set $\mathrm{Sing}(f^{-1})$ is bounded, where $\mathrm{Sing}(f^{-1})$ denotes the closure in $\mathbb{C}$ of the set of singular values. The actual relationship between this set and a Baker domain U is related to the distance of the singular orbits to the boundary of U (see [Bergweiler 2001] for a precise statement). We remark that it is not necessary, however, that any of the singular values be inside the Baker domain. Indeed, there are examples of Baker domains with an arbitrary finite number of singular values (including none) inside.

We consider two holomorphic endomorphisms $f, g : S \to S$ to be equivalent if they are conjugate via a conformal isomorphism of S, i.e., if there exists a holomorphic bijective map $c : S \to S$ such that $g = c^{-1} \circ f \circ c$. In the case of entire maps, the conjugacy needs to be affine, so our maps live in the space of entire functions modulo affine conjugations. From now on we will mostly consider the case $S = \mathbb{C}$.

Our goal in this paper is to study the possible quasiconformal deformations of an entire transcendental map f, i.e., those entire maps $g : \mathbb{C} \to \mathbb{C}$ such that $g = h^{-1} \circ f \circ h$, for some quasiconformal homeomorphism h, but where h cannot be chosen to be conformal. This analysis was started in [Harada & Taniguchi 1997] for entire maps whose set of singular values is a countable set. Later on in [Fagella & Henriksen 2006], the case of entire maps possessing an invariant Baker domain was studied, with no condition required for the set of singular values. In this paper, we extend the study to all entire maps, also without restrictions on the set of singularities.

More precisely, we will consider the Teichmüller space $\mathcal{T}(\mathbb{C}, f)$ of an entire function f using the very general framework given by [McMullen & Sullivan 1998] (see Section 2.2 for an introduction). Also, another approach can be found in [Epstein 1995], applied to entire functions of finite type. Morally, the Teichmüller space of f is precisely the space of quasiconformal deformations of f provided with a nice manifold structure. Under some conditions, it is shown in [McMullen & Sullivan 1998] that it can be split into several smaller parts, in correspondence with completely invariant subsets of the dynamical plane. In section 3 we will see how these splitting theorems can be applied to our setting. In order to do that we need to show the following technical result, which is the main difficulty we overcome to prove our results for *all* entire functions. By the *grand orbit* of a point $z \in \mathbb{C}$, we mean all points $w \in \mathbb{C}$ such that $f^n(z) = f^m(w)$ for some $n, m \in \mathbb{N}$. Alternatively, if we define the grand orbit relation as the finest equivalence relation on $\mathbb{C}$ with $z \simeq f(z)$ then the grand orbit of z is the equivalence class of z. Let $\mathrm{QC(U, f)}$ be the group of quasiconformal automorphisms of U that commute with f, and let $\mathrm{QC}_0(U, f) \subset \mathrm{QC(U, f)}$ the subgroup of automorphisms which are homotopic to the identity rel the ideal boundary of U, through a uniformly K-qc subset of $\mathrm{QC(U, f)}$ (see Section 2.2).

Theorem. *Let f be an entire transcendental function and $\widehat{\mathcal{J}}$ the closure of the grand orbits of all* marked points *of f (periodic points and singular values). Let ω be an element of $\mathrm{QC}_0(\mathbb{C}, f)$. Then ω restricts to the identity on $\widehat{\mathcal{J}}$.*

This is well known in the case of rational maps (critical values, as periodic points of a given period always form a discrete set) and also for entire maps of finite type (for the same reason), and it extends to the case of functions with a countable set of singularities [Harada & Taniguchi 1997]. However in our case, we deal with the possibility of having asymptotic values forming a continuum, or even a set of positive measure.

The main consequence of the theorem above is that marked points play a special role when studying the Teichmüller space of a given function (see Section 3). In other words we can split the dynamical plane into two sets, namely $\widehat{\mathcal{J}}$ and its complement, and study the Teichmüller space separately in each of them. Likewise, we can then split $\mathbb{C} \setminus \widehat{\mathcal{J}}$ into several completely invariant subsets, namely the different grand orbits of Fatou components (or subsets of it).

For attracting basins of attracting, superattracting or parabolic periodic points, and for rotation domains, the results are practically the same as those for rational maps: if U is a Fatou component of f and $\mathcal{U}$ is its grand orbit then $\mathcal{U}$ contributes with a certain finite quantity to the dimension of Teichmüller space, plus the number of distinct singular grand orbits that belong to it, which in this case may be infinite. In the case of rotation domains this is not a precise statement although the moral is the same.

New situations arise with the presence of Baker and wandering domains, since these are not present in the dynamical plane of rational maps. For Baker domains, we recall the results obtained in [Fagella & Henriksen 2006] (see Proposition 4.2), where we classify the different types of Baker domains and we show that the Teichmüller space can have 0, finite nonzero or infinite dimension depending on the class they fall into and the number of singular grand orbits they contain.

For wandering domains we show that mainly all possibilities can occur. We give particular examples for all of them, some of which were not known to exist up to our knowledge.

Finally in Section 5, we address the problem of studying the Teichmüller space supported on the Julia set, which of course can give rise to several different possibilities.

2 Preliminaries

In this section we recall shortly the relevant definitions and results related to quasiconformal mappings and to Teichmüller spaces. For the first part, the standard references are [Ahlfors 1966] and [Lehto & Virtanen 1973]. In the second part, we will use the general framework of [McMullen & Sullivan 1998]. We also refer to [Hubbard] as a reference for Riemann surfaces and Teichmüller theory.

2.1 Quasiconformal mappings

Let $V, V' \subset \mathbb{C}$ be open subsets of the complex plane or more generally, one dimensional complex manifolds.

Definition 2.1. Given a measurable function $\mu : V \to \mathbb{C}$, we say that μ is a $k-$*Beltrami coefficient* of V if $||\mu(z)||_\infty \leq k < 1$ almost everywhere in V. Two Beltrami coefficients of V are equivalent if they coincide almost everywhere in V. We say that μ is a Beltrami coefficient if μ is a $k-$Beltrami coefficient for some $k < 1$.

Definition 2.2. A homeomorphism $\phi : V \to V'$ is said to be *quasiconformal* if it has locally square integrable generalized derivatives and

$$\mu_\phi(z) = \frac{\frac{\partial \phi}{\partial \bar{z}}(z)}{\frac{\partial \phi}{\partial z}(z)} = \frac{\bar{\partial}\phi(z)}{\partial \phi(z)}$$

is a $k-$Beltrami coefficient for some $k < 1$. In this case, we say that ϕ is $K-$quasiconformal where $K = \frac{1+k}{1-k} < \infty$, and μ_ϕ is the *complex dilatation* or the *Beltrami coefficient* of ϕ.

With the same definition, but skipping the hypothesis on ϕ to be a homeomorphism, ϕ is called a $K-$*quasiregular map*.

Definition 2.3. Given a Beltrami coefficient μ of V and a quasiregular map $f : V \to V'$, we define the *pull-back* of μ by f as the Beltrami coefficient of V defined by:

$$f^*\mu = \frac{\frac{\partial f}{\partial \bar{z}} + (\mu \circ f)\overline{\frac{\partial f}{\partial z}}}{\frac{\partial f}{\partial z} + (\mu \circ f)\overline{\frac{\partial f}{\partial \bar{z}}}}.$$

We say that μ is f–invariant if $f^*\mu = \mu$. If $\mu = \mu_g$ for some quasiregular map g, then $f^* = \mu_{g \circ f}$.

It follows from Weyl's Lemma that a quasiregular map f is holomorphic if and only if $f^*\mu_0 = \mu_0$, where $\mu_0 \equiv 0$.

Definition 2.4. Given a Beltrami coefficient μ, the partial differential equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z)\frac{\partial \phi}{\partial z} \tag{1}$$

is called *the Beltrami equation.* By *integration* of μ we mean the construction of a quasiconformal map ϕ solving this equation almost everywhere, or equivalently, such that $\mu_\phi = \mu$ almost everywhere.

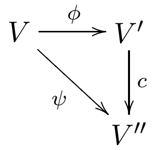
The famous *Measurable Riemann Mapping Theorem* by Morrey, Bojarski, Ahlfors and Bers states that every k-Beltrami coefficient is integrable.

Theorem 2.5 (Measurable Riemann Mapping Theorem)**.** *Let μ be a $k-$Beltrami coefficient, for some $k < 1$, of V where $V = \mathbb{C}$ or is isomorphic to $\mathbb{D}$. Then, there exists a quasiconformal map $\phi : V \to \mathbb{C}$ such that $\mu_\phi = \mu$ almost everywhere. Moreover, ϕ is unique up to post-composition with conformal isomorphisms of V.*

2.2 Teichmüller Space

In this section we set up the machinery that is necessary for defining the Teichmüller space of a dynamical system, using the general framework of McMullen and Sullivan. The classical concept of the Teichmüller space of a Riemann surface is a special case of the general definition.

Let V be an open subset of the complex plane or more generally a one dimensional complex manifold and f a holomorphic endomorphism of V. Define an equivalence relation $\sim$ on the set of quasiconformal homeomorphisms on V by identifying $\phi : V \to V'$ with $\psi : V \to V''$ if there exists a conformal isomorphism $c : V' \to V''$ such that $c \circ \phi = \psi$, i.e. the following diagram commutes.



It then follows that $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate (although the converse is not true in general). Then the deformation space of f on V is

$$\mathrm{Def}(V, f) = \{\phi : V \to V' \text{ quasi conformal } \mid \mu_\phi \text{ is } f\text{-invariant } \}/\sim .$$

As a consequence of the Measurable Riemann Mapping Theorem (see [Ahlfors 1966] or Theorem 2.5) one obtains a bijection between $\mathrm{Def}(V, f)$ and

$$\mathcal{B}_1(V, f) = \{f\text{-invariant Beltrami forms } \mu \in L^\infty \text{ with } ||\mu||_\infty < 1\},$$

and this is used to endow $\mathrm{Def}(V, f)$ with the structure of a complex manifold. Indeed, $\mathcal{B}_1(V, f)$ is the unit ball in the Banach space of f-invariant Beltrami forms equipped with the infinity norm.

We denote by $\mathrm{QC}(V, f)$ the group of quasiconformal automorphisms of V that commute with f. Given $K < \infty$, a family of q.c. mappings is called uniformly K-q.c. if each element of the family is K-q.c.

A hyperbolic Riemann surface V is covered by the unit disk; in fact V is isomorphic to $\mathbb{D}/\Gamma$ where Γ is a Fuchsian group. Let $\Omega \subseteq \mathbb{S}^1$ denote the complement of the limit set of Γ. Then $(\mathbb{D} \cup \Omega)/\Gamma$ is a bordered surface and Ω/Γ is called the ideal boundary of V. A homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ is called *rel ideal boundary* if there exists a lift $\hat{\omega}_t : \mathbb{D} \to \mathbb{D}$ that extends continuously to Ω as the identity. If V is not hyperbolic then the ideal boundary is defined to be the empty set.

We denote by $\mathrm{QC}_0(V, f) \subseteq \mathrm{QC}(V, f)$ the subgroup of automorphisms which are homotopic to the identity rel the ideal boundary of V through a uniformly K-q.c. subset of $\mathrm{QC}(V, f)$, for some $K < \infty$.

Earle and McMullen [Earle & McMullen 1988] prove the following result for hyperbolic subdomains of the Riemann sphere.

Theorem 2.6. *Suppose $V \subseteq \widehat{\mathbb{C}}$ is a hyperbolic subdomain of the Riemann sphere. Then a uniformly quasiconformal homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ rel ideal boundary can be extended to a uniformly quasiconformal homotopy of $\widehat{\mathbb{C}}$ by letting $\omega_t = \mathrm{Id}$ on the complement of V. Conversely, a uniformly quasiconformal homotopy $\omega_t : V \to V$ such that each ω_t extends continuously as the identity to the topological boundary $\partial V \subseteq \widehat{\mathbb{C}}$ is a homotopy rel the ideal boundary.*

Proof. The proof can be found in [Earle & McMullen 1988]: Proposition 2.3 and the proof of Corollary 2.4 imply the first statement. Theorem 2.2 implies the second. □

The group $\mathrm{QC}(V, f)$ acts on $\mathrm{Def}(V, f)$ by $\omega_* \phi = \phi \circ \omega^{-1}$. Indeed if ϕ and ψ represent the same element in $\mathrm{Def}(V, f)$ then $\omega_* \phi = \omega_* \psi$ as elements of $\mathrm{Def}(V, f)$.

Definition 2.7. The Teichmüller space $\mathcal{T}(V, f)$ is the deformation space $\mathrm{Def}(V, f)$ modulo the action of $\mathrm{QC}_0(V, f)$, i.e. $\mathcal{T}(V, f) = \mathrm{Def}(V, f)/\mathrm{QC}_0(V, f)$. If V is a one dimensional complex manifold we denote by $\mathcal{T}(V)$ the Teichmüller space $\mathcal{T}(V, \mathrm{Id})$.

Teichmüller space can be equipped with the structure of a complex manifold and a (pre)-metric (we refer to [McMullen & Sullivan 1998]).

Let us give a rough idea of Teichmüller space and the motivation for studying it. In holomorphic dynamics one is often interested in studying the set $\mathbf{F}$ of holomorphic mappings that are quasiconformally conjugate to a given holomorphic map $f : V \to V$ modulo conjugacy by conformal isomorphisms. Such a mapping can be written as $\phi \circ f \circ \phi^{-1}$ for a $\phi \in \mathrm{Def}(V, f)$. Now $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate exactly when they represent the same element in $\mathrm{Def}(V, f)/\mathrm{QC}(V, f)$. So we can study $\mathbf{F}$ by looking at $\mathrm{Def}(V, f)/\mathrm{QC}(V, f)$.

Clearly the Teichmüller space is related to this space, and it can be shown to be, at least morally, a covering of it. Because of the nice properties it has, it is often more convenient to study Teichmüller space than **F**.

As we will see in the next section, it is often useful to split the Teichmüller space into several smaller parts that are easier to study separately. In this direction Sullivan and McMullen prove stronger versions of the following two theorems. To state them, we need the definition of a restricted product of Teichmüller spaces.

Definition 2.8. Let f be an entire function, and suppose that $\{U_\alpha\}$ is a family of pairwise disjoint completely invariant open subsets of $\mathbb{C}$.

For an element $u \in \mathcal{T}(U_\alpha, f) = \mathcal{B}_1(U_\alpha, f)/\mathrm{QC}_0(U_\alpha, f)$ we set

$$|u| = \inf\{||\mu||_\infty \mid \mu \in \mathcal{B}_1(U_\alpha, f) \text{ is a representative of } u.\}$$

Then $0 \leq |u| < 1$, and we define the *restricted product of Teichmüller spaces*

$$\prod_\alpha^* \mathcal{T}(U_\alpha, f)$$

as the set of sequences $\{u_\alpha \mid u_\alpha \in \mathcal{T}(U_\alpha, f)\}$ that satisfy $\sup_\alpha |u_\alpha| < 1$. In particular, if the family $\{U_\alpha\}$ is finite, the restricted product coincides with the usual product of spaces.

The motivation for this definition is that we could have a collection of invariant Beltrami forms μ_α defined on each U_α and each with complex dilatation $k_\alpha < 1$. However, if we tried to put them all together to form a global invariant Beltrami form μ on $\cup_\alpha U_\alpha$, with $\mu\,|_{U_\alpha} = \mu_\alpha$, we would obtain a complex dilatation for μ whose infinity norm would be bounded by $k = \sup_\alpha k_\alpha$ which, in the case of infinitely many $\alpha' s$, could be equal to 1. In such a case, μ would not belong to $\mathcal{B}_1(\bigcup_\alpha U_\alpha)$. By considering the restricted product, we are allowing exactly for the sequences where this problem does not arise.

With this concept we are now ready to state the following splitting theorem.

Theorem 2.9. *Let f be an entire function, and suppose that U_α is a family of pairwise disjoint completely invariant open subsets of $\mathbb{C}$. Then*

$$\mathcal{T}(\cup U_\alpha, f) \simeq \prod^* \mathcal{T}(U_\alpha, f).$$

Proof. This follows from Theorem 5.5 in [McMullen & Sullivan 1998]. □

The next Theorem characterizes completely the Teichmüller space of an open, invariant, hyperbolic subset of the complex plane, as long as it has no singular values and it is in some sense "minimal". For the statement we need the concept of the grand orbit relation and the notion of discreteness.

Definitions 2.10. Let U be a subset of $\mathbb{C}$ and $f : U \to U$ be a holomorphic endomorphism. The *grand orbit* of a point $z \in U$ is the set of $w \in U$ such that $f^n(z) = f^m(w)$ for some $n, m \geq 0$. We denote the grand orbit equivalence relation by $z \sim w$. We denote by U/f the quotient of U by the grand orbit equivalence relation. We call this relation *discrete* if all grand orbits are discrete; otherwise we call it *indiscrete*.

As an example, if U is a basin of attraction, the grand orbit relation is discrete. On the other hand, Siegel disks or basins of superattraction give rise to indiscrete grand orbit relations. See section also 4.

Theorem 2.11 ([McMullen & Sullivan 1998] Thm. 6.1)**.** *Let U be an open subset of $\mathbb{C}$ and suppose every connected component of U is hyperbolic. Assume $f : U \to U$ is a holomorphic covering map and U/f is connected.*

(a) *If the grand orbit equivalence relation is discrete, then U/f is a Riemann surface or orbifold and*
$$\mathcal{T}(U, f) \simeq \mathcal{T}(U/f).$$

(b) *If the grand orbit equivalence relation is indiscrete, and some component A of U is an annulus of finite modulus, then*
$$\mathcal{T}(U, f) \simeq \mathcal{T}(A, \mathrm{Aut}_0(\mathrm{A})) \simeq \mathbb{H}.$$

(c) *Otherwise, $\mathcal{T}(U, f)$ is a point.*

3 The structure of Teichmüller space

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. By using the decomposition results at the end of the previous section, we partition the complex plane into dynamically meaningful subsets in such a way that the Teichmüller space $\mathcal{T}(\mathbb{C}, f)$ can be studied in each of them separately. First we notice the following.

Proposition 3.1. *Suppose $U \subset \mathbb{C}$ is a completely invariant open set and $K \subset U$ is a closed completely invariant subset with the property that each element of $\mathrm{QC}_0(U, f)$ fixes K pointwise. Then*
$$\mathcal{T}(U, f) = \mathcal{B}_1(K, f) \times \mathcal{T}(U - K, f).$$

Proof. Since K is measurable we have $\mathcal{B}_1(U, f) \simeq \mathcal{B}_1(K, f) \times \mathcal{B}_1(U - K, f)$. If we write $\mathrm{QC}_0(U, f)|_{U-K}$ for the set formed by the elements of $\mathrm{QC}_0(U, f)$ restricted to $U - K$, it follows that $\mathrm{QC}_0(U, f)|_{U-K} = \mathrm{QC}_0(U - K, f)$ by Theorem 2.6. Hence, we get that
$$\begin{aligned} \mathcal{T}(U, f) &\simeq (\mathcal{B}_1(K, f) \times \mathcal{B}_1(U - K, f)) / \mathrm{QC}_0(U, f) \\ &\simeq \mathcal{B}_1(K, f) \times (\mathcal{B}_1(U - K, f) / \mathrm{QC}_0(U - K, f)) \\ &\simeq \mathcal{B}_1(K, f) \times \mathcal{T}(U - K, f). \end{aligned}$$

□

The completely invariant set we can use as K is the set of *marked points*, which consists of all periodic points of f together with the singular values of the map (i.e., critical and asymptotic values). More precisely, we define the set $\widehat{\mathcal{J}}$ as the closure of the Grand Orbit of the set of marked points of f, that is
$$\widehat{\mathcal{J}} = \mathrm{Cl}\,(\mathrm{GO}\{\text{ marked points of } f\})\,.$$

Observe that $\widehat{\mathcal{J}}$ always contains the Julia set of f. We will see that points in the grand orbit of the set of marked points are dynamically distinguished, in the sense that every automorphism of $\mathbb{C}$ isotopic to the identity that commutes with f must have them as fixed points.

Theorem 3.2. *Let f be an entire transcendental function and $\widehat{\mathcal{J}}$ the closure of the grand orbits of all marked points of f (periodic points and singular values). Let ω be an element of $\mathrm{QC}_0(\mathbb{C}, f)$. Then ω restricts to the identity on $\widehat{\mathcal{J}}$.*

The proof of this theorem is involved mainly because we do not know what the set of asymptotic values looks like. For instance, it can have interior. Recall from the introduction that every point in the plane may be asymptotic value. For the sake of exposition we leave the proof for the end of the section.

Combining this theorem with Proposition 3.1 we obtain

$$\mathcal{T}(\mathbb{C}, f) = \mathcal{T}(\widehat{\mathcal{J}}, f) \times \mathcal{T}(\mathbb{C} - \widehat{\mathcal{J}}, f) = \mathcal{B}_1(\widehat{\mathcal{J}}, f) \times \mathcal{T}(\mathbb{C} - \widehat{\mathcal{J}}, f).$$

We now would like to further decompose the space $\mathcal{T}(\mathbb{C} - \widehat{\mathcal{J}}, f)$ in order to apply Theorem 2.11 to each of its parts. Let S denote the set of singular values and periodic points of f that lie in the Fatou set $\mathcal{F}$, and $\widehat{S}$ the grand orbit of the elements of S. Clearly $\widehat{S}$ is a subset of $\mathcal{F}$ (in fact $\widehat{\mathcal{J}}$ equals the disjoint union $\mathcal{J} \coprod \widehat{S}$) and therefore $\mathbb{C} - \widehat{\mathcal{J}} = \mathcal{F} - \widehat{S} := \widehat{\mathcal{F}}$ which is an open hyperbolic set. This set splits naturally into completely invariant subsets. Indeed let us define an equivalence relation in the set of connected components of $\widehat{\mathcal{F}}$, by identifying all those that have the same grand orbit. Since they are all open, we may choose one representative of each class and obtain a countable collection of connected open sets $\{U_i\}$. We denote by $V_i = \mathrm{GO}(U_i)$ and then, $\widehat{\mathcal{F}} = \bigcup_i V_i$ where the V_i are pairwise disjoint, completely invariant, hyperbolic and open. It then follows from Theorem 2.9 that

$$\mathcal{T}(\mathbb{C} - \widehat{\mathcal{J}}, f) = \prod_i^{*} \mathcal{T}(V_i, f).$$

In summary, we have proved the following theorem.

Theorem 3.3 (First Structure Theorem). *Let f be an entire transcendental function. Let $\widehat{S}$ denote the closure of the grand orbits of the marked points in the Fatou set and $\widehat{\mathcal{J}} = \mathcal{J} \cup \widehat{S}$. Let $\{V_i\}_i$ denote the collection of pairwise disjoint grand orbits of the connected components of $\mathbb{C} - \widehat{\mathcal{J}}$. Then,*

$$\mathcal{T}(\mathbb{C}, f) = \mathcal{B}_1(\widehat{\mathcal{J}}, f) \times \prod_i^{*} \mathcal{T}(V_i, f).$$

We now would like to determine the Teichmüller space of each of the completely invariant sets V_i using Theorem 2.11. Recall that if X is an open subset of the plane which is completely invariant by f, we defined X/f to be the quotient of X by the grand orbit equivalence relation. We will split up the collection of $V_i's$ into two different ones. Given a fixed i, we will say that V_i belongs to $\widehat{\mathcal{F}}^{dis}$ if all grand orbits in V_i are discrete. Otherwise, the grand orbit relation in V_i is indiscrete and we say that V_i belongs to $\widehat{\mathcal{F}}^{ind}$. In this last case we rename it $V_i' = V_i$ and modify the indices as necessary so that

$$\widehat{\mathcal{F}}^{dis} = \bigcup_i V_i; \qquad \widehat{\mathcal{F}}^{ind} = \bigcup_j V_j'; \qquad \widehat{\mathcal{F}} = \widehat{\mathcal{F}}^{dis} \cup \widehat{\mathcal{F}}^{ind}. \tag{2}$$

In the same fashion, we distinguish between U_i and U_j' so that $V_i = \mathrm{GO}(U_i)$ and $V_j' = \mathrm{GO}(U_j')$.

Theorem 3.4 (Second Structure Theorem). *Let f be an entire transcendental function of $\mathbb{C}$ and define $\widehat{\mathcal{J}}, \widehat{\mathcal{F}}^{dis}, \widehat{\mathcal{F}}^{ind}, V_i, U_i, V_j'$ and U_j' as above. Then, for every i, the quotient $U_i/f \simeq V_i/f$ is a connected Riemann surface or orbifold and*

$$\mathcal{T}(V_i, f) = \mathcal{T}(U_i/f).$$

Consequently, we can write

$$\mathcal{T}(\mathbb{C}, f) = \mathcal{B}_1(\widehat{\mathcal{J}}, f) \times \prod_i^* \mathcal{T}(U_i/f) \times \prod_j^* \mathcal{T}(V_j', f).$$

Moreover, for every j such that a component of V_j' is an annulus of finite modulus we have

$$\mathcal{T}(V_j', f) \simeq \mathbb{H},$$

and otherwise $\mathcal{T}(V_j', f)$ is trivial.

Proof. This theorem is in fact Theorem 2.11 adapted to our setting. One thing we must check though is that indeed the quotients V_i/f and V_j'/f are connected sets, so we can apply the mentioned theorem. We first check that they coincide with U_i/f and U_j'/f respectively and then see that the latter are connected. Let U be U_i or U_j' and V be V_i or V_j', for some i or j.

By definition V is the grand orbit of U, so any point in V has an image or a preimage in U. In other words, all grand orbits must have at least one representative in U. This implies that $V/f \subset U/f$, while the other inclusion follows trivially from $U \subset V$. To see that U/f is connected we first recall that U is connected (by definition). Let $\pi : U \to U/f$ denote the natural projection and endow U/f with the quotient topology. Then if U/f could be split inside two disjoint open sets, so could U.

We are now ready to apply Theorem 2.11 and obtain the first and last parts of the Theorem. The middle decomposition comes from (2) and Theorem 2.9.

□

The remainder of this section will be dedicated to the proof of Theorem 3.2. The proof is divided in two different parts. The first one concerns the set of periodic points and critical values, and the proof is relatively simple, given the fact that critical values form a discrete set and so do periodic points of a given period. The second part concerns asymptotic values and the proof is much more delicate since, a priori, we do not know what this set looks like (we remind the reader that it could be all of $\mathbb{C}$). For the proof of the second part we need some preliminary results which we include in the following subsection.

3.1 Preliminary results

The first ingredient is due to Lindelöf.

Lindelöf's theorem (e.g. Theorem 5.4 in [Conway 1987]). *Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and $\Gamma : [0,1] \to \mathbb{D}$ a curve such that $\Gamma(1) = 1$. If there exists $\lim_{t\to 1} g(\Gamma(t))$ and equals L, then, the map g has radial limit L at 1.*

What we will actually use is the following immediate corollary.

Corollary 3.5. *Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic map and $\Gamma_1, \Gamma_2 : [0,1] \to \mathbb{D}$ be two curves such that $\Gamma_1(1) = \Gamma_2(1) = 1$. If the curves $g(\Gamma_1(t))$ and $g(\Gamma_2(t))$ have a limit when $t \to 1$, then they must land at the same point.*

Proof. If any image curve lands, then there exists a radial limit L and the landing must occur at L. Hence any other landing image curve must do so at the same point L. □

The second ingredient is the following. The statement can be found in Theorem 1.1 of [Earle & McMullen 1988] but we include a proof for completeness.

Proposition 3.6. *Let $V \subset \mathbb{C}$ be an open hyperbolic subset of the plane and $\omega : V \to V$ an element of $\mathrm{QC}_0(V, \mathrm{Id})$. Then, there exists a lift $\widetilde{\omega} : \mathbb{D} \to \mathbb{D}$ which extends as the identity to $\partial\mathbb{D}$.*

Proof. By hypothesis we know that there exists a lift $\widetilde{\omega}$ which extends as the identity to a subset of $\partial\mathbb{D}$, namely, the complement Ω of the limit set of the Fuchsian group Γ. We must see that the extension also exists at the remaining points and equals the identity.

Let μ be the Beltrami form on $\mathbb{D}$ induced by the quasiconformal map $\widetilde{\omega}$. We can extend μ to $\mathbb{C}$ by setting $\mu = 0$ on the complement of the unit disk.

Let φ be the quasiconformal map that integrates μ, given by the Measurable Riemann Mapping Theorem. Then $\varphi(\partial\mathbb{D})$ is a quasicircle We define the composition $\psi = \widetilde{\omega} \circ \varphi^{-1}$ on the $\varphi(\mathbb{D})$ as shown in the following diagram.

$$\begin{array}{ccc} (\mathbb{D}, \mu) & \xrightarrow{\widetilde{\omega}} & (\mathbb{D}, \mu_0) \\ \varphi \big\downarrow & \nearrow_{\psi} & \\ (\varphi(\mathbb{D}), \mu_0) & & \end{array}$$

Then ψ is conformal since it transports the standard complex structure $\mu_0 = 0$ to itself. Hence it extends continuously to the boundary of $\varphi(\mathbb{D})$. Let us denote by $\widehat{\psi}$ this extension.

Now define

$$\widehat{\omega} = \begin{cases} \widetilde{\omega} & \text{on } \mathbb{D} \\ \widehat{\psi} \circ \varphi & \text{on } \partial\mathbb{D} \end{cases}$$

and this is a continuous extension of $\widetilde{\omega}$ to the boundary of the unit disk. By hypothesis we know that $\widetilde{\omega}$ extends as the identity to $\Omega \subset \mathbb{S}^1$, the complement of the limit set of γ. It follows that $\widehat{\omega}$ is the identity on Ω and, since Ω is dense in the circle, we conclude that $\widehat{\omega}$ is the identity on the whole unit circle. □

3.2 Proof of Theorem 3.2

We first show that ω restricts to the identity on the set of periodic points and on the Julia set, as well as on the set of critical points of f.

Let ω_t be a homotopy, i.e., a path in $\mathrm{QC}(\mathbb{C}, f)$ that connects $\omega_0 = \mathrm{Id}$ to $\omega_1 = \omega$. Since ω_t commutes with f, the set of periodic points of a given period is ω_t–invariant. So, if $p \in \mathbb{C}$ is a periodic point of period N say, the path $t \mapsto \omega_t(p)$ is a subset of the periodic points of period N. Since this set is discrete $\omega_t(p) = p$ for all t. Since ω_t commutes with f we immediately get that ω_t fixes all periodic points for all t.

By continuity we obtain that every ω_t fixes any point in the closure of the set of periodic points, and in particular, any point in the Julia set.

The same argument proves that ω_t fixes each of the critical points, since these also form a discrete set.

Since any automorphism ω in $\mathrm{QC}_0(\mathbb{C}, f)$ restricts to the identity on the Julia set, it follows that ω restricts to an automorphism of the Fatou set, which is a completely invariant open set whose connected components are hyperbolic. It is for this reason that the remaining cases (singular values and their grand orbits, and the closure of such) follow from the proposition below.

Proposition 3.7. *Let f be an entire function and $\mathcal{U}$ a totally invariant open set whose connected components are hyperbolic. Denote by S the set of singular values of f in $\mathcal{U}$. Then, any ω in $\mathrm{QC}_0(U, f)$ restricts to the identity on the closure of the grand orbit of S in U.*

We remark that this result is very similar to Proposition 3 in [Fagella & Henriksen 2006] except that here we do not require $\mathcal{U}$ to be simply connected. The proof, however, is simplified.

Proof. The case of critical points was dealt with above. Now assume we know also that ω fixes every asymptotic value of f and let us see how to conclude the proof.

Since every singular value is in the closure of the set of asymptotic and critical values (see Section 1), we get by continuity that ω_t fixes the singular values of f in $\mathcal{U}$. Since ω_t commutes with f we get that ω_t restricts to the identity on the forward orbit of this set. Now suppose $\omega_t(y) = y$ for all t and that $f^n(x) = y$. Then $\omega_t(x)$ must map into $f^{-n}\{y\}$. Since this set is discrete we get that $\omega_t(x) = x$ for all x. It follows that ω_t restricts to the identity on the grand orbit of S for all t and by continuity this is also true on the closure.

So it remains to prove that ω fixes every asymptotic value of f. Let $z_0 \in \mathcal{U}$ be an asymptotic value of f and let $\gamma = \gamma_0 \subset U$ be an associated asymptotic path, that is $\gamma : [0, \infty) \to \mathcal{U}$ is a curve such that $\gamma(s) \to \infty$ and $f(\gamma(s)) \to z_0$ as $s \to \infty$. Our goal is to show that $\omega_t(f(\gamma(s)))$ also tends to z_0 as $s \to \infty$ which will imply that z_0 is fixed by ω_t.

Let U and U' be the connected components of $\mathcal{U}$ containing z_0 and γ respectively (we could have $U = U'$) and denote by $\pi' : \mathbb{D} \to U'$ the universal covering of U'. Let $\widetilde{\omega}_t : \mathbb{D} \to \mathbb{D}$ be the lift of $\omega_t : U' \to U'$ that extends as the identity on the boundary of $\mathbb{D}$ (see Proposition 3.6) and depends continuously on t. Then, $\gamma_t = \omega_t(\gamma)$ is a family of curves in U', all of them tending to infinity, since ω_t extend as the identity on the boundary of U' (see Theorem 2.6).

We now lift this family of curves $\{\gamma_t\}$ to a family $\{\widetilde{\gamma}_t\}$ in the unit disk in the following way: we choose any lift $\widetilde{\gamma}_0$ of γ_0 and then define $\widetilde{\gamma}_t = \widetilde{\omega}_t(\widetilde{\gamma}_0)$ (see Figure 1). Observe that these are lifts of $\widetilde{\gamma}_t$ since

$$\pi'(\widetilde{\gamma}_t) = \pi' \circ \widetilde{\omega}_t(\widetilde{\gamma}_0) = \omega_t \circ \pi'(\widetilde{\gamma}_0) = \omega_t(\gamma_0) = \gamma_t.$$

Notice that all curves $\widetilde{\gamma}_t$ must land at the same point, say u, in the boundary of $\mathbb{D}$, given that $\widetilde{\omega}_t$ extends to the identity to $\partial\mathbb{D}$ and $\widetilde{\gamma}_t = \widetilde{\omega}_t(\widetilde{\gamma}_0)$.

Also observe that the image curves, $f(\pi'(\widetilde{\gamma}_t))$ also land at some point $z_t = \omega_t(z_0) \in U$, since

$$f(\pi'(\widetilde{\gamma}_t)) = f(\omega_t(\gamma_0)) = \omega_t(f(\gamma_0))$$

and the ω_t are continuous maps. We would like to show that $z_t = z_0$ for all t.

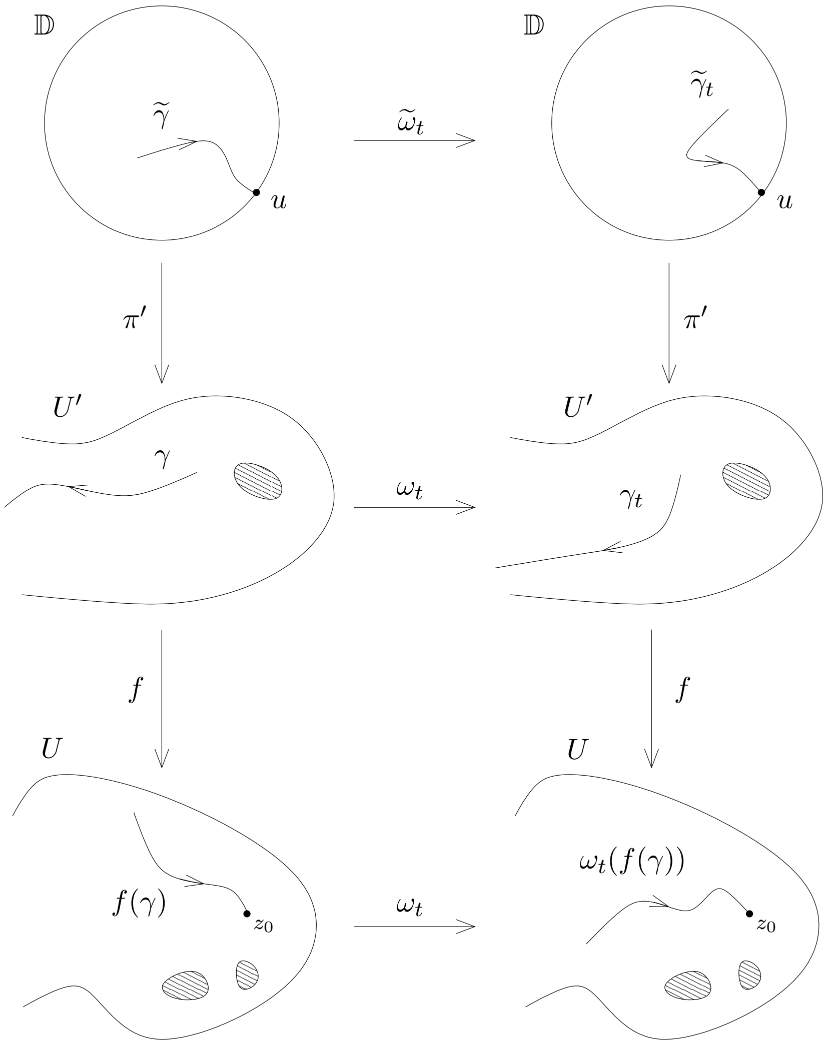
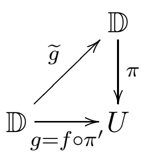


Figure 1: Commutative diagram showing the construction in the proof of Proposition 3.7.

To that end, we observe that we are under the hypotheses of Corollary 3.5, except that U is not necessarily a disk. To fix this problem, let $\pi : \mathbb{D} \to U$ be the universal covering of U such that, for example, $\pi(0) = z_0$. Denote by $g : \mathbb{D} \to U$ the composition $g = f \circ \pi'$ and by $\widetilde{g} : \mathbb{D} \to \mathbb{D}$ one of its lift to the disk, as shown in the following diagram.



Then $\widetilde{g}$ is holomorphic and the curves $\widetilde{\gamma}_t$ are mapped to curves in $\mathbb{D}$ that land at some interior points. Indeed if this were not the case, their images by π could not be the γ_t landing at interior points of U. Hence we are under the hypotheses of Lindelöf's theorem and we conclude that all the curves $\widetilde{g}(\widetilde{\gamma}_t)$ land at the same point. Hence their images by π must also land at the same point and we conclude that $z_t = z_0$ for all t. $\square$

4 Teichmüller Space supported on the Fatou set

In this section we treat each of the possible types of Fatou components separately to study how they might contribute to the Teichmüller space of f, in view of the structure theorems in the previous section. We will put more emphasis in the types of Fatou components that cannot occur for rational maps (Baker and wandering domains) and which, consequently, were not treated in [McMullen & Sullivan 1998]. In general we will use caligraphic letters to denote Fatou components and their grand orbits and regular letters to denote these same objects after removing the marked points from them.

4.1 Fatou components with discrete grand orbit relations

4.1.1 Attracting basins

Suppose f has a periodic attracting basin of period p. Let $\mathcal{U}$ be a connected component of the immediate basin, and $\mathcal{V}$ the grand orbit of $\mathcal{U}$. In such a Fatou component each grand orbit is a discrete set (points are isolated in the linearizing domain and then one can just pull back). It is well known that we can choose (using linearizing coordinates) a simple closed curve γ in the linearizable domain of the periodic orbit such that after p iterations γ is mapped to a curve γ' strictly inside γ. It follows that all orbits in $\mathcal{V}$ (except the periodic points and its preimages) pass exactly once through the annulus bounded by γ and γ' (including only one of the two curves). Identifying these two boundaries we obtain a fundamental domain T isomorphic to a torus.

Now we must remove the points of $\widehat{S}$ from T (we already did not consider the grand orbit of the periodic orbit which is part of $\widehat{S}$). Suppose for a moment that the periodic basin contains only one singular value v, which is the minimum amount possible. Then the grand orbit of v has exactly one element in T. Thus if we denote $V = \mathcal{V} - \widehat{S}$ we have that $V/f = (\mathcal{V}/f) - (\widehat{S}/f)$ is isomorphic to a torus with one marked point. It follows that the complex dimension of $\mathcal{T}(V/f)$ is 1.

In practice, the basin of attraction could contain many singular values (even a set with interior). The important number to take in account is the number of *singular grand orbits* i.e., the number of distinct grand orbits that contain a singular value. Indeed, if this number is finite, each of the singular grand orbits will add a marked point to the torus T and increase the dimension of $\mathcal{T}(V/f)$ by one.

In the case where the number of distinct singular grand orbits is infinite the dimension of $\mathcal{T}(\mathcal{V}, f)$ is infinite. If V is not the empty set , V/f has a component with infinitely many punctures or ideal boundary (or both). If $V = \emptyset$ then $\mathcal{T}(\mathcal{V}, f) = B_1(\mathcal{V}/f, f)$ again infinite dimensional.

Summarizing we have the following statement.

Proposition 4.1. *Let $\mathcal{V}$ denote the grand orbit of a periodic attracting basin and $V = \mathcal{V} - \widehat{S}$, where $\widehat{S}$ is the closure of the grand orbits of the singular values. Let $1 \leq N \leq \infty$ equal the number of distinct grand orbits in $\widehat{S} \cap \mathcal{V}$ (other than the attracting periodic point). Then, if N is finite, V/f is a torus with N punctures and*

$$\dim \mathcal{T}(\mathcal{V}, f) = \dim_{\mathbb{C}} (\mathcal{T}(V/f)) = N.$$

If N is infinite, $\mathcal{T}(\mathcal{V}, f)$ has infinite dimension.

Clearly it is easy to produce examples of functions with periodic attracting basins adding any finite or infinite number to the dimension of the Teichmüller space.

4.1.2 Parabolic basins

Parabolic basins can be treated in the same way as attracting basins. Passing to Fatou coordinates we find that V/f is an infinite cylinder with as many punctures as distinct singular grand orbits lying in $\mathcal{V}$, which must be at least one. The conclusions are then the same as those in Proposition 4.1.

4.1.3 Baker Domains

An extensive treatment of Baker domains was done in [Fagella & Henriksen 2006], and we refer the reader to there for details. Here we summarize the main ideas and results.

Suppose that f has an invariant Baker domain $\mathcal{U}$ (again the periodic case can be handled the same way). Then $\mathcal{U}$ is simply connected [Baker 1975] and we can choose a Riemann map $\varphi : \mathcal{U} \to \mathbb{D}$ that maps $\mathcal{U}$ conformally onto the unit disk. Such a map conjugates f to a self-mapping of $\mathbb{D}$ that we denote by $B_{\mathcal{U}}$ and we call the *inner function* associated to $\mathcal{U}$. It follows from the Denjoy-Wolf Theorem that there exists a point $z_0 \in \partial\mathbb{D}$ such that $B_{\mathcal{U}}^n$ converges towards the constant mapping z_0 locally uniformly in $\mathbb{D}$ as $n \to \infty$.

It can be deduced from the work of Cowen in [Cowen 1981] that the grand orbit relation in $\mathcal{U}$ is always discrete. Hence, by Theorem 3.4, we have to analyze the set $(\mathcal{U} - \widehat{S})/f$ or, equivalently, $(\mathcal{U}/f)-(\widehat{S}/f)$. In [Fagella & Henriksen 2006] we have the following classification following Cowen.

Proposition 4.2 ([Fagella & Henriksen 2006], Prop. 1)**.** *Let f be entire and $\mathcal{U}$ an invariant Baker domain. Then $\mathcal{U}/f$ is a Riemann surface conformally isomorphic to one of the following cylinders:*

(1) $\{-s < \mathrm{Im}(z) < s\}/\mathbb{Z}$ *for some* $s > 0$ *and we call* $\mathcal{U}$ hyperbolic*;*

(2) $\{\mathrm{Im}(z) > 0\}/\mathbb{Z}$ *and we call* $\mathcal{U}$ simply parabolic*;*

(3) $\mathbb{C}/\mathbb{Z}$ *and we call* $\mathcal{U}$ doubly parabolic*. In this case $f : \mathcal{U} \to \mathcal{U}$ is not proper or has degree at least* 2*.*

Once we know what kind of Riemann surface $\mathcal{U}/f$ is we now need to remove the elements of $\widehat{S}$. Observe that in the hyperbolic and simply parabolic cases, the cylinders we obtain have an ideal boundary. This means that their Teichmüller space is infinite dimensional, even before adding any of the punctures given by $\widehat{S}$. In the doubly parabolic case, we have a doubly infinite cylinder and hence the Teichmüller space will depend on the number of punctures created when removing $\widehat{S}$. Summarizing we have the following theorem.

Theorem 4.3 ([Fagella & Henriksen 2006], Main Theorem)**.** *Let $\mathcal{U}$ be a fixed Baker domain of the entire function f and $V = \mathrm{GO}(\mathcal{U}) - \widehat{S}$. Then $\mathcal{T}(V, f) = \mathcal{T}(V/f)$ which is infinite dimensional except if $\mathcal{U}$ is doubly parabolic and the cardinality of $\widehat{\mathcal{S}}/f$ is finite. In that case the dimension of $\mathcal{T}(V, f)$ equals $\#\widehat{\mathcal{S}}/f - 1$.*

Examples of all 3 types can be found in [Fagella & Henriksen 2006] including an example of a Baker domain whose Teichmüller dimension is trivial, and another one of dimension one. By introducing more critical points in a doubly parabolic Baker domain one can construct examples of Baker domains adding any finite number to the dimension of the global Teichmüller space.

4.2 Fatou components with indiscrete grand orbit relations

In the case of indiscrete grand orbit relations it is not important how many distinct singular grand orbits we have but how many of them have distinct closures. It is in this spirit that we need to introduce the following concept (following [McMullen & Sullivan 1998]).

Definition 4.4. A singular value is *acyclic* if its forward orbit is infinite. Two points z and w in the Fatou set belong to the same *foliated equivalence class* if the closures of their grand orbits agree.

Two singular values whose grand orbits are not discrete may belong to the same foliated equivalence class although their grand orbits might be disjoint. This is the case for example if their grand orbits intersect the same equipotential in a superattracting basin or the same invariant curve in a Siegel disk. If their orbits are discrete then they belong to the same foliated equivalence class if and only if their grand orbits coincide.

4.2.1 Superattracting basins

Suppose f has a superattracting periodic basin $\mathcal{V}'$. For such a Fatou component the grand orbit equivalence relation is indiscrete (except for the grand orbit of the periodic point). To see this, let us consider the map $z \mapsto z^2$ and chose $z_0 = re^{i\theta}$ for some fixed r and θ. It is easy to check that all the points of the form $\{r^{2^s} \exp(i(2^s\theta + \frac{n}{2^k}) \mid s \in \mathbb{Z},\, 1 \leq k < \infty,\, 0 \leq n < 2^k\}$ belong to the same grand orbit. Clearly the closure of these points is an infinite set of concentric circles of radii r^{2^s} for all $s \in \mathbb{Z}$. So the grand orbit relation is indiscrete. This is also true in the general case and it can be seen easily using Böttcher coordinates from the dynamical plane of $z \mapsto z^d$.

In order to apply Theorem 3.4 we need to remove the points of $\widehat{S}$ from the superattracting basin $\mathcal{V}'$. Suppose first that there are no singular values in the basin. Then, the set $V' = \mathcal{V}' - \widehat{S}$ consists of a collection of punctured disks (the punctures correspond to the grand orbit of the periodic point), all belonging to the same grand orbit. No component of his set is an annulus of finite modulus (for it to be mapped to a disk it would have to contain a critical point and we are assuming there are none) and therefore it follows from Theorem 3.4 that $T(V', f)$ is trivial.

Now suppose that $\mathcal{V}'$ contains singular values. We need to remove from $\mathcal{V}'$ the closure of their grand orbits. Assume that $\mathcal{V}'$ contains a finite number of singular grand orbits each belonging to different foliated equivalence class. Denote their closures by $\widehat{S}_1, \ldots, \widehat{S}_N$. Let $\mathcal{U}'$ be a connected component of the immediate basin which we assume for simplicity to be fixed (if not, one only needs to consider f^p). We start by removing $\widehat{S}_1$. Close enough to the fixed (and critical) point we can assume that there are no critical points, so $\widehat{S}_1$ consists of infinitely many concentric simple closed curves (equipotentials). After removal of those, we are left (close enough to the fixed point) with infinitely many annuli, all belonging to the same grand orbit. Let U' be one of these annuli. Then, U' is a fundamental domain for $V' = \mathcal{V}' - \widehat{S}_1$. Let us now consider the remaining $N-1$ singular orbits. Since U' is a fundamental domain, each of these grand orbits will intersect U' in one single equipotential. Therefore, removing $\widehat{S}_2, \ldots, \widehat{S}_N$ will partition U' into exactly N disjoint open annuli, $U'_1, \ldots, U'_n$ each one giving rise to a distinct grand orbit. Denote these grand orbits by $V'_1, \ldots, V'_n$, which form a partition of V' into completely invariant open subsets.

To each V'_i we can apply Theorem 3.4. Since each of them has a connected component which is an annulus of finite modulus (exactly U'_i) we conclude that $\mathcal{T}(V'_i, f) = \mathbb{H}$. It follows that

$$\mathcal{T}(V', f) = \prod_{i=1}^{N} \mathcal{T}(V'_i, f) = \mathbb{H}^N$$

and hence the superattracting basin adds N dimensions to the global Teichmüller space.

Finally we consider the case were there are infinitely many singular grand orbits in $\mathcal{V}'$, each of them in a distinct foliated equivalence class. Then either the set $\widehat{S} \cap \mathcal{V}'$ has interior or $\mathcal{V}' \setminus \widehat{S}$ contains infinitely many distinct grand orbits of annuli. In both cases $\mathcal{T}(\mathcal{V}', f)$ has infinite dimension.

We can state a summary as follows.

Proposition 4.5. *Let $\mathcal{V}'$ denote the grand orbit of a superattracting periodic attracting basin and $V' = \mathcal{V}' - \widehat{S}$, where $\widehat{S}$ is the closure of the grand orbits of the singular values and periodic points. Let $0 \leq N \leq \infty$ equal the number of grand orbits in $\widehat{S} \cap \mathcal{V}'$ other than the superattracting periodic point, that belong to distinct foliated equivalence classes. Then, the grand orbit relation is indiscrete and,*

1. *if $N = 0$, the set V' is a collection of punctured disks and therefore $\mathcal{T}(V', f)$ is trivial.*
2. *if $N > 0$ is finite, V' splits into N completely invariant subsets $V'_1, \ldots, V'_N$, and each of them has a connected component which is an annulus of finite modulus. Then,*

$$\dim_{\mathbb{C}} \left(\mathcal{T}(V', f)\right) = N.$$

3. *Finally, if $N = \infty$ either V' is empty or $\mathcal{T}(V', f)$ has infinite dimension. In any case, $\dim \mathcal{T}(\mathcal{V}', f) = \infty$.*

4.2.2 Siegel disks

Suppose that f has a Siegel fixed point z_0 with an invariant Siegel disk Δ (the periodic case is analogous) and let $\mathcal{V}'$ denote the grand orbit of Δ. Observe that the grand orbit relation is indiscrete since, given any point in the Siegel disk (except for the fixed point), the accumulation of its grand orbit is exactly one invariant simple closed curve in Δ, and all its preimages in the other components of $\mathcal{V}'$.

As before, let us first assume that $\mathcal{V}'$ contains no singular grand orbits so that $V' = \mathcal{V}' - \widehat{S} = \mathcal{V}' - \mathrm{GO}(z_0)$. Then V'/f is isomorphic to $\Delta - \{z_0\}$ and therefore connected (or equivalently, V' has no completely invariant proper subsets). Hence we can apply Theorem 2.11 or 3.4 to V' directly. Since no component of V' is an annulus of finite modulus (for it to be mapped to Δ it would have to contain a critical point and we are assuming there are none) it follows that the Teichmüller space $\mathcal{T}(V', f)$ is trivial.

Let us now suppose we have a finite number, say N, of singular grand orbits (other than the Siegel point), each of them belonging to a distinct foliated equivalence class. Then the set $\widehat{S} \cap \Delta$ consists of the fixed Siegel point together with N disjoint invariant closed curves around z_0. Hence, $\Delta - \widehat{S}$ consists of N annuli of finite modulus, $U'_1, \ldots, U'_N$ together with a punctured disk U'_{N+1}. For $1 \leq i \leq N+1$ define $V'_i = \mathrm{GO}(U'_i)$. Then the sets $V'_1, \ldots, V'_{N+1}$ are open and completely invariant and form a partition of V'. Applying Theorem 3.4 to each of them we obtain that $\mathcal{T}(V'_i, f) = \mathbb{H}$ for $1 \leq i \leq N$ while $\mathcal{T}(V'_{N+1}, f)$ is trivial.

If we have infinitely many foliated equivalence classes in $\mathcal{V}'$ then the same considerations of the other cases apply.

Summarizing we can state the following.

Proposition 4.6. *Let $\mathcal{V}'$ denote the grand orbit of a periodic cycle of Siegel disks and $V' = \mathcal{V}' - \widehat{S}$, where $\widehat{S}$ is the closure of the grand orbits of the singular values and periodic points. Let $0 \leq N \leq \infty$ equal the number of grand orbits in $\widehat{S} \cap \mathcal{V}'$ that belong to distinct foliated equivalence classes. Then, the grand orbit relation is indiscrete and,*

1. *if $N = 0$, the set V' is a collection of punctured disks and therefore $\mathcal{T}(V', f)$ is trivial.*
2. *if $N > 0$ is finite, V' splits into $N+1$ completely invariant subsets $V'_1, \ldots, V'_{N+1}$, and each of them, except one, has a connected component which is an annulus of finite modulus. Then,*
$$\dim_{\mathbb{C}} \left(\mathcal{T}(V', f)\right) = N.$$
3. *Finally, if $N = \infty$ either V' is empty or $\mathcal{T}(V', f)$ has infinite dimension. In any case* $\dim_{\mathbb{C}} (\mathcal{T}(\mathcal{V}', f)) = \infty$.

4.3 Wandering Domains: discrete and indiscrete grand orbit relations

We will treat the case of wandering domains by giving several examples that illustrate the different situations that can occur. In particular some wandering domains have discrete grand orbit relations while others do not. At the same time they may contribute to the dimension of global Teichmüller space either with a finite amount, infinite or zero. To our knowledge, no example was previously known of a wandering domain having a Teichmüller space of finite nonzero dimension. We provide such example (see Example 3) using surgery, with a wandering domain with an indiscrete grand orbit relation. We have not been able to construct a wandering domain with a discrete grand orbit relation, and with a finite dimensional Teichmüller space. The summary of examples are shown in the following table.

Example	GO relation	V/f	$\dim(\mathcal{T}(V,f)$
1. Lift of a Siegel disk	discrete	$\mathbb{D}$	∞
2. Lift of a superattracting basin	indiscrete	–	0
3. Surgery on 2	indiscrete	–	$1 \leq N < \infty$
4. Lift of a basin of attraction	discrete	$\mathbb{C}^+ * \setminus \{a_i\}_{i\in\mathbb{N}}$	∞
5. Lift of a superattracting basin with an additional critical point	indiscrete	–	∞
Unknown	discrete	–	$0 \leq N < \infty$

Example 1 (discrete relation, $\mathbf{V/f} \simeq \mathbb{D}$, $\mathbf{dim}(\mathcal{T}(\mathcal{V}, f)) = \infty$).
Let $f(w) = aw^2e^{-w}$ and set $a = e^{2-\lambda}/(2-\lambda)$, with $\lambda = \exp(2\pi i(1-\sqrt{5})/2) = -0.737369... + i0.67549....$ It is easy to check that f has a superattracting fixed point at $w = 0$ (for all

values of a) and that for this particular value of the parameter, f has an indifferent fixed point at $w_0 = 2 - \lambda$. The multiplier of w_0 is precisely λ and thus w_0 is linearizable and has a Siegel disk Δ around it. Given the presence of a superattracting basin for $w = 0$ we know that Δ does not intersect a neighborhood of this point. Just to make the picture complete, notice that f has two critical points: the origin and $c = 2$, whose orbit accumulates on the boundary of Δ. See Figure 2.

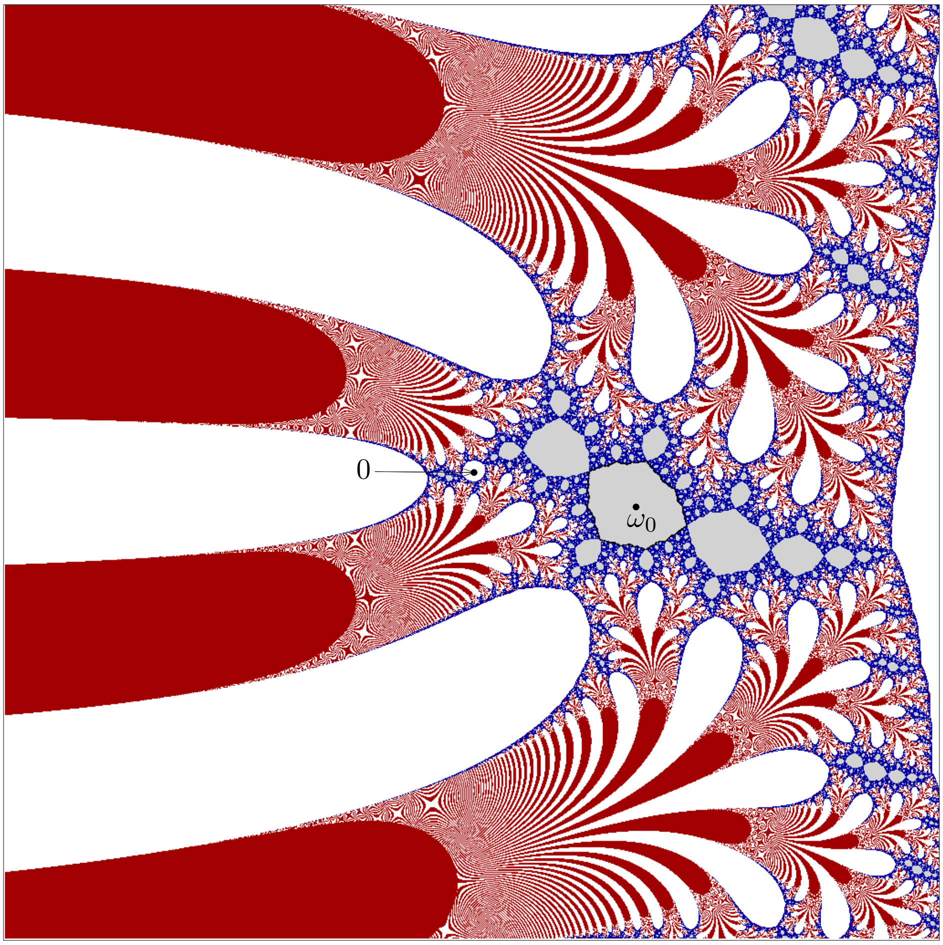


Figure 2: Dynamical plane of $f(w) = aw^2e^{-w}$ for $a = e^{2-\lambda}/(2-\lambda)$, with $\lambda = \exp(2\pi i(1-\sqrt{5})/2) = -0.737369... + i0.67549...$. In white we see the basin of superattracting of $w = 0$ (the superattracting fixed point is at the center of the image). In gray the Siegel disk around w_0 and its preimages.

We are now going to take a logarithmic lift of f in such a way that the Siegel disk lifts to a wandering domain. More precisely let

$$F(z) = A + 2z - e^z; \qquad A = \log(a) = 2 - \lambda - \log(2-\lambda),$$

and observe that the function $w = e^z$ is a semiconjugation between f and F. Thus it follows that the superattracting basin of f lifts to a (hyperbolic) Baker domain of F, as studied originally in [Bergweiler 1995] and later in [Baranski & Fagella 2000] and in [Fagella & Henriksen 2006] for some other values of the parameter. Let us see what happens to the Siegel disk after lifting. See Figure 3.

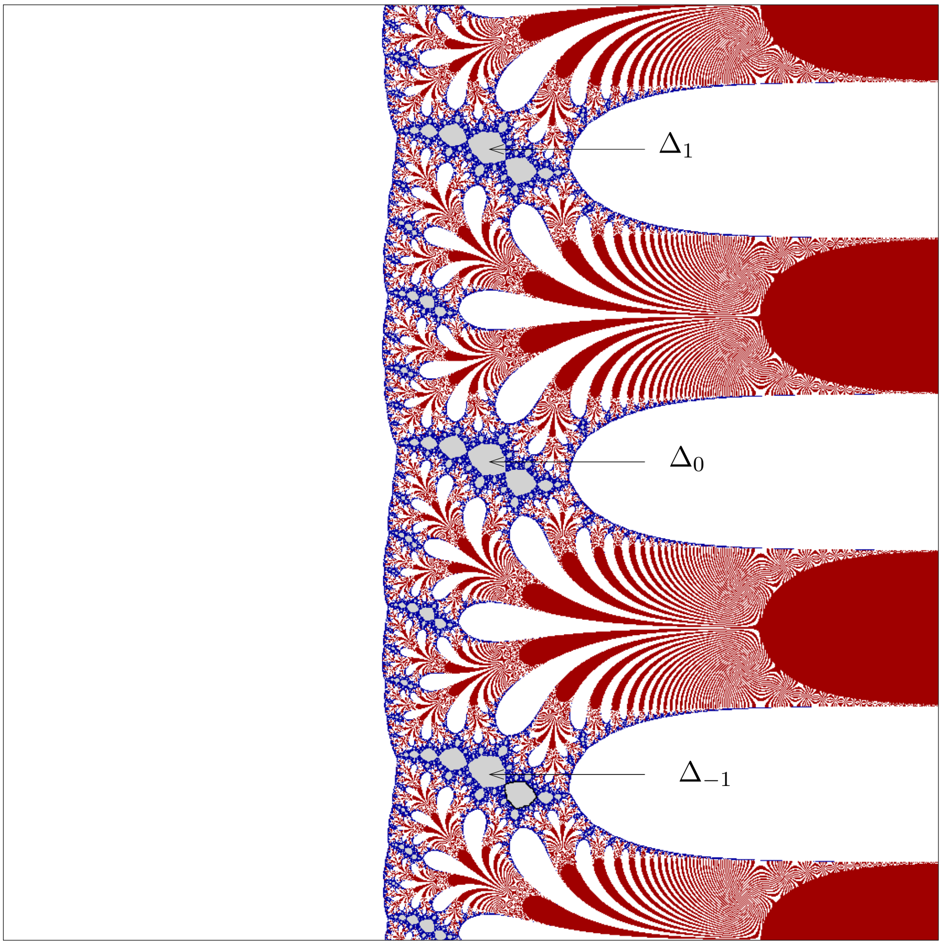


Figure 3: Dynamical plane of $F(z) = A + 2z - e^z$ as in Example 1. In white we see the Baker domain which is the lift of the superattracting immediate basin of f, and all its preimages. In gray we see the domains Δ_k and their preimages, which correspond to lifting the Siegel disk in Figure 2.

An easy computation shows that all points of the form

$$z_k = \log(2 - \lambda) + 2k\pi i$$

project down to the fixed Siegel point. However these are not all fixed points of f but instead they satisfy that $F(z_k) = z_{2k}$. Since the Siegel disk Δ omits a neighborhood of zero and is simply connected, it must lift to infinitely many disjoint domains Δ_k each surrounding the point z_k. Since the points z_k map to z_{2k} it follows that the domains Δ_k map to Δ_{2k} and hence we obtain a Siegel disk Δ_0 and infinitely many distinct grand orbits of wandering domains. Since the original Δ was a Siegel disk it follows that F is one to one on each of the domains. In fact, F is morally still acting like a rotation, with the only difference that it moves to a different fiber of Siegel disk every time it is applied.

Let us choose one of these sequences, say $\Delta_1, \Delta_2, \Delta_4, \ldots$ and denote by V its grand orbit. Notice that V does not contain any singular value nor any periodic point or, in other words, no element of $\widehat{S}$. Since F is one to one in each of the domains the grand orbit relation is clearly discrete. In view of Theorem 3.4 we then must look at V/f which is isomorphic to, say, Δ_1, since every grand orbit in V passes once and only once through Δ_1. Hence V/f

is a disk and its Teichmüller space has infinite dimension because of the presence of ideal boundary.

Example 2 (indiscrete relation, $\dim(\mathcal{T}(\mathcal{V}', f)) = 0$**).**
Consider the map $F(z) = z+2\pi+\sin(z)$. Observe that F has infinitely many (double) critical points on the real line located at $c_k = (2k+1)\pi$ which map as $F(c_k) = c_{k+1}$. Simultaneously, consider the vertical lines whose real part is an even multiple of π which we denote by $l_k = 2k\pi + it$, $t \in \mathbb{R}$, and observe that they map as $F(l_k) = l_{k+1}$. Moreover these lines belong to the Julia set since all their points (with nonzero imaginary part) tend to infinity exponentially fast under iteration.

It follows that there is a sequence of distinct Fatou domains W_k containing the critical point c_k in its interior such that $F(W_k) = W_{k+1}$. These are wandering domains mapping to each other with degree 3 because of the presence of the double critical point inside. See figure 4.

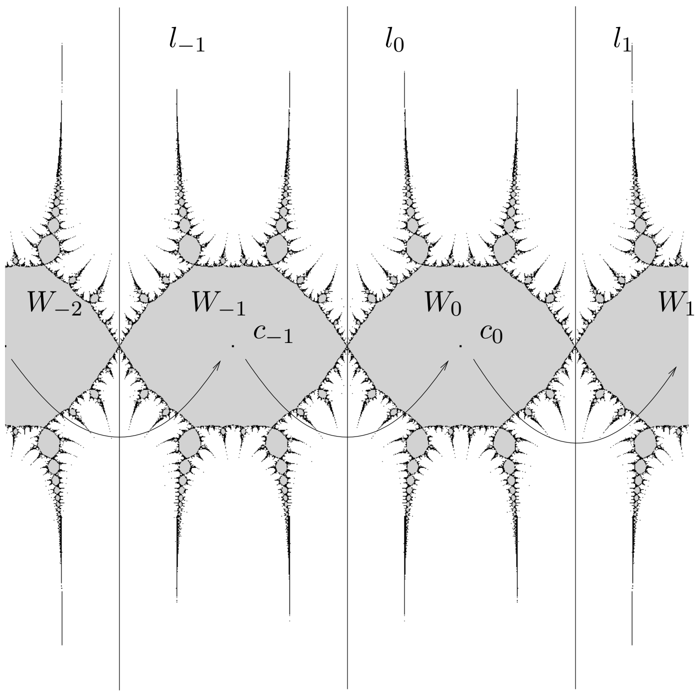


Figure 4: The dynamics of the mapping $F : z \to z+2\pi+\sin z$ used in example 2 The vertical lines are part of the Julia set and cut the plane into vertical strips each of which contains a wandering domain W_i.

Just as an observation, the map F projects under $w = e^{iz}$ to $f(w) = w\exp(\frac{1}{2}(w - \frac{1}{w}))$ which has a superattracting fixed point at $w = -1$ and a repelling fixed point at $w = 1$. The wandering domains W_k are in fact the logarithmic lifts of the superattracting basin around $w = -1$. Abusing language, we will call *equipotential curves* those simple closed curves in W_k that are lifts of the equipotentials in the superattracting basin.

Returning to F, observe that the grand orbit relation is indiscrete. Just as it occurred in the superattracting basins, the grand orbit of any point accumulates in one of the equipotential curves, and a different one in each W_k. This is easy to check by picking any point, map it forward several times and then start considering *all* its preimages in each of the preceding domains. Taking this procedure to the limit we see that we fill up an equipotential curve in each of the $W_k' s$.

Let $\mathcal{V}'$ denote the grand orbit of W_k (for any k) and $V' = \mathcal{V}' - \widehat{S}$, which in this case it is a collection of punctured disks (the only elements in $\widehat{S}$ are the critical points c_k and their preimages). Since the orbit relation is indiscrete and no component of V' is an annulus of finite modulus, we conclude from Theorem 3.4 that $\mathcal{T}(V', f)$ is trivial. Now since $\widehat{S}$ has measure zero, we have from Proposition 3.1 that $\mathcal{T}(\mathcal{V}', f) = \mathcal{T}(V', f)$ and thus has dimension 0.

Example 3 (indiscrete relation, $\dim(\mathcal{T}(\mathcal{V}, f)) = 1$**).**
We modify the previous example in such a way that instead of one grand orbit of singular values we get two. So let F and W_k, $k \in \mathbb{Z}$ be defined as before. Notice that each W_k is simply connected and that $F : W_k \to W_{k+1}$ is a degree 3 proper map. Let U be an open disk neighborhood of π that is compactly contained in W_0. Let U' be the component of $F^{-1}(U)$ that contains $-\pi$. Then $U' = F^{-1}(U) \cap W_{-1}$, and more importantly $F : U' \to U$ is a proper degree 3 mapping.

Construct $h : \overline{U} \to \overline{U}$ such that h satisfies $h|_{\partial U} = \mathrm{Id}_{\partial U}$, $h(\pi) \neq \pi$ and $h|_U$ is quasiconformal. Set

$$\tilde{F} = \begin{cases} h \circ F & \text{on } \overline{U'} \\ F & \text{on } \mathbb{C} \setminus U'. \end{cases}$$

Then F is a quasiregular mapping. Since, as for F, $\tilde{F}$ maps W_k onto W_{k+1} for all k, we have that a grand orbit contains at most one point in U'.

We now define a almost complex structure with respect to which $\tilde{F}$ is holomorphic. Let μ_0 denote the standard complex structure. Define μ by letting it equal the pullback $(\tilde{F}^n)^*\mu_0$ on $\tilde{F}^{-n}(U)$, $n = 1, 2, \ldots$ and letting it equal μ_0 on the complement, i.e. the set $\{z \mid \tilde{F}^n(z) \notin U\}$.

Then $||\mu||_\infty = ||h^*(\mu_0)||_\infty < 1$. So by the measurable Riemann mapping Theorem there exists $\phi : \mathbb{C} \to \mathbb{C}$ that satisfies $\phi^*\mu_0 = \mu$. By Weyl's lemma $G = \phi \circ \tilde{F} \circ \phi^{-1}$ is a holomorphic mapping.

Notice that ϕ conjugates F to G everywhere except on U', and that $G(\phi(W_{-1})) = \phi(W_0)$. It follows that ϕ maps the Julia set of F to the Julia set of G, and the Fatou set of F to that of G. By construction, $\phi(W_0)$ is a simply connected wandering domain. The critical points of G are $p_k = \phi((2k+1)\pi), k \in \mathbb{Z}$. Again by the construction of G we see that G maps p_k to p_{k+1} for all $k \in \mathbb{Z} \setminus \{-1\}$, and that $F(p_{-1}) \neq p_0$.

We conclude that we have at most two distinct grand orbits of singular values, which are $\mathrm{GO}(p_{-1})$ and $\mathrm{GO}(p_0)$. That these two grand orbits are in fact distinct follows from an easy induction using that the only point in W_k that is mapped to p_{k+1} is p_k, $k = 0, 1, \ldots$.

Let $S_i = \overline{\mathrm{GO}(p_i)}, i = -1, 0$, denote the closure of the two distinct grand orbits of singular values. We can argue as in the previous example that $\phi(W_0) \cap S_{-1}$ is a simple closed analytic curve encircling $\{p_0\} = \phi(W_0) \cap S_0$. So $\phi(W_0) \setminus \tilde{S}$ is composed of an annulus X_A and a pointed disk X_B. A connected component of the preimage of a pointed disk by an entire function cannot be an annulus of finite modulus. It follows that no component of $\mathrm{GO}(X_B)$ is an annulus of finite modulus. We can therefore apply Theorem 2.11 and Proposition 3.1 and get $\mathcal{T}(\mathrm{GO}(X_A), G) \simeq \mathbb{H}$ whereas $\mathcal{T}(\mathrm{GO}(X_B), G)$ is trivial. Since $\mathrm{GO}(\phi(W_0)) \cap \widehat{S}$ has measure zero, we conclude that $\mathcal{T}(\mathrm{GO}(\phi(W_0)), G)$ has dimension 1.

Remark. We could repeat the surgery and modify G such that $G(p_0) \neq p_1$. In this way we would obtain a mapping with three distinct grand orbits of singular values and add one to the dimension of the Teichmüller space. In fact given any n we could do this a finite number of times and obtain a map with a wandering domain on which the grand orbit relation is indiscrete and where the corresponding Teichmüller Space has dimension n.

Example 4 (discrete relation, $V/f \simeq \mathbb{C} - \{a_i\}_{i\in\mathbb{Z}}$, $\dim(\mathcal{T}(\mathcal{V}, f)) = \infty$).
Let $f(w) = \lambda w e^w$ where we set $\lambda = e^{\frac{1}{2}}$. Then, f has a repelling fixed point at the asymptotic value $w = 0$, and an attracting fixed point at $w = -1/2$. By plotting the map on the real line we can see that the semi-axis $[0, \infty)$ belongs to the Julia set (all its points except the origin escape to ∞), while $(-\infty, 0)$ is contained in the basin of attraction Ω of the attracting fixed point. The map has only one critical point at $w_0 = -1$ which is of course contained in Ω.

We now choose a lift of f by $w = e^z$ and obtain a new function

$$F(z) = z + \frac{1}{2} + 2\pi i + e^z.$$

The basin of attraction Ω lifts to infinitely many band-like Fatou domains $\{\Omega_k\}_{k\in\mathbb{Z}}$, each of them containing the horizontal line $\{\mathrm{Im(z)} = (2\mathrm{k} + 1)\pi\}$, and a critical point at $c_k = (2k + 1)\pi i$. They are separated by the lifts of the positive real line, i.e., by the horizontal lines $\{\mathrm{Im(z)} = 2\mathrm{k}\pi\}$, which belong to the Julia set. Because of the choice of the lift, the domains are mapped in such a way that $F(\Omega_k) = \Omega_{k+1}$ with degree 2. Hence they are all wandering domains and belong to the same grand orbit which we denote by $\mathcal{V}$. Observe that these sets are morally basins of attraction with one critical point inside, except that as the orbits move closer to a fiber of the fixed point, they also jump to the next domain.

Since the grand orbit relation of f in $\Omega \setminus \widehat{S}$ is discrete, so is the grand orbit relation of F in $\mathcal{V}$. We now would like to know how the set $V/f = (\mathcal{V} - \widehat{S})/f$ looks like. To that end, observe that the grand orbit of every point must have at least one element in each of the Ω_k. Hence $\mathcal{V}/f = \Omega_k/f$ for any k. Let us take for example $k = 0$.

Proposition 4.7. *(a) Let $z_1, z_2 \in \Omega_0$, not belonging to any singular grand orbit. Then z_1 and z_2 belong to the same grand orbit if and only if $F^n(z_1) = F^n(z_2)$ for some $n \geq 0$.*

(b) There exists a holomorphic bijection $\phi : \Omega_0/f \to \mathbb{C}$.

(c) There are infinitely many distinct singular grand orbits in $\mathcal{V}$, and

$$V/f \simeq (\Omega_0 - \widehat{S})/f \simeq \mathbb{C}^* - \{a_i\}_{i\in\mathbb{Z}}.$$

where $a_i \in \mathbb{C}$ for all $i \in \mathbb{Z}$. We conclude that $\dim(\mathcal{T}(\mathcal{V}, f)) = \dim(\mathcal{T}(V/f)) = \infty$.

Proof. (a) The if part is clear by definition. To see the only if, note that if $z_1, z_2 \in \Omega_0$ belongs to the same grand orbit, there exist $n, m \geq 0$ so that $f^n(z_1) = f^m(z_2)$. Since $f^n(z_1) \in \Omega_n$ and $f^m(z_2) \in \Omega_m$ we must have $n = m$.

(b) Recall that F was obtained as the logarithmic lift of a function f having an attracting basin Ω. Let $\varphi : \Omega \to \mathbb{C}$ denote the extended linearizing coordinates conjugating f in Ω to $M_\rho(w) = \rho w$ where $\rho = 1/2$ is the multiplier of the fixed point. Define the maps

$$\begin{array}{rcccccc} \phi_k : & \Omega_k & \xrightarrow{\exp} & \Omega & \xrightarrow{\varphi} & \mathbb{C} \\ & z & \longmapsto & \exp(z) & \longmapsto & \varphi(\exp(z)), \end{array}$$

which are holomorphic. Then, for any $n \geq 0$, the following diagram commutes:

$$\begin{array}{ccc} \Omega_0 & \xrightarrow{F^n} & \Omega_n \\ \exp \big\downarrow & & \big\downarrow \exp \\ \Omega & \xrightarrow{f^n} & \Omega \\ \varphi \big\downarrow & & \big\downarrow \varphi \\ \mathbb{C} & \xrightarrow{M_\rho^n} & \mathbb{C} \end{array}$$

and therefore $M_\rho^{-n} \circ \phi_n \circ F^n = \phi_0$ on Ω_0. It follows that if $F^n(z_1) = F^n(z_2)$ then $\phi_0(z_1) = \phi_0(z_2)$, so the map $\phi : \Omega_0/f \to \mathbb{C}$ defined by $\phi[z] = \phi_0(z)$ is well defined. To see that it is a bijection observe that the linearizing coordinate satisfies $\varphi(w_1) = \varphi(w_2)$ if and only if $f^n(w_1) = f^n(w_2)$ for some $n \geq 0$. Now suppose that z_1 and z_2 are in different classes in Ω_0/f but $\phi(z_1) = \phi(z_2)$. Since the exponential is bijective on Ω_0 we have that $\varphi(\exp(z_1)) = \varphi(\exp(z_2))$ and therefore $f^n(\exp(z_1)) = f^n(\exp(z_2))$ for some $n \geq 0$. Lifting to Ω_n this implies that $F^n(z_1) = F^n(z_2)$ and thus z_1 and z_1 belong to the same class. This proves the injectivity of ϕ. The surjectivity follows immediately from the surjectivity of φ.

(c) There is one critical point c_k in each wandering domain Ω_k and each of them belongs to a distinct grand orbit, since they all project to the same point $c \in \Omega$ which is not fixed nor periodic. Given the absence of asymptotic values, this implies that $\phi(\widehat{S}/f)$ consists of a countable sequence of points $\{a_i\} \cup \{0\}$ which, more precisely, are the points in the full orbit of $\phi(c)$ under M_ρ, together with the point at 0, which is the image of the former fixed point in Ω_0 under ϕ. In other words, $\phi(\widehat{S}/f) = \{\rho^i \varphi(c)\}_{i\in\mathbb{Z}} \cup \{0\}$ and the statement follows. ☐

Example 5 (indiscrete relation, $\dim(\mathcal{T}(\mathcal{V}, f)) = \infty$**).**
As in other examples we get the mapping by lifting. First consider the entire map f given by $f_{a,b}(z) = az^2 \exp(bz^2 + z), a \neq 0$. We claim there exists real values of a and b such that $f_{a,b}$ has three real critical points $\omega_2 < \omega_1 < \omega_0 = 0$, and such that ω_1 is a fixed point having ω_2 in its immediate basin of attraction. Indeed, $\omega_0 = 0$ is a critical point, and for $0 < b \leq 1/16$, the two other critical points are given by $\omega_1 = \frac{-1+\sqrt{1-16b}}{4b}$ and $\omega_2 = \frac{-1-\sqrt{1-16b}}{4b}$. Still for $0 < b \leq 1/16$, there exists a function $a(b)$ such that $f_{a(b),b}(\omega_1) = \omega_1$. Abusing notation we write $f_b = f_{a(b),b}$. For $b = \frac{1}{16}$, ω_1 and ω_2 coincide and for $b < \frac{1}{16}$ they are distinct. By semicontinuity of the basin of the immediate attraction of ω_1 it follows that for values of b slightly less than $1/16$, ω_2 stays in the immediate basin of ω_1. This proves the claim.

Fix a, b as in the claim, and write f for $f_{a,b}$. Let W be the immediate basin of ω_1. The equipotential going through ω_2 is a figure eight that cuts the sphere into three simply connected regions and we let U denote the one that contains ω_1. Now lift f by the exp to obtain a mapping F given by $F(w) = \log a + 2w + b\exp(2w) + exp(w)$. Let $\tilde{W}, \tilde{U}$ denote the preimage of W, U by the exponential. Since the positive real axis does not meet W, the preimage $\tilde{W}$ contains countably many components W_k which we can number so that $F(\tilde{W}_k) = \tilde{W}_{2k}, k \in \mathbb{Z}$. Similarly define $\tilde{\omega}_i(k) \in W_k$ as the fibers of $\omega_i, i = 1, 2$.

Let $\mathcal{V}$ denote the grand orbit of $\tilde{W}_1$. We leave it to the reader to verify that the grand orbit relation is indiscrete, and we sketch a proof that $\mathcal{T}(\mathcal{V}, f)$ is infinite dimensional.

Number the components $\tilde{U}_k$ of $\log(U)$ so that $\tilde{U}_k \subset \tilde{W}_k$. Notice that W_1 is a wandering component. Now, the closure of $\mathrm{GO}(\omega_1)$ cuts U into an infinite and countable number of annuli each delimited by equipotential curves of $f^k(\omega_2)$ and $f^{k+1}(\omega_2)$. For F, the grand orbits of the singular values $\widehat{S}$ is the grand orbit of the set $O = \cup_k \omega_1(k) \bigcup \cup_k \omega_2(k)$. It follows that $\widehat{S}$ cuts $\widetilde{U}_1$ into infinitely many nested annuli, and we conclude that on the wandering domain $\mathcal{V}$ the grand orbit relation is indiscrete and that the dimension of $\mathcal{T}(\mathcal{V}, f)$ is infinite.

5 Teichmüller Space supported on the Julia set

Whereas the dynamics of an entire function on the Fatou set is well understood, the dynamics on the Julia set is generally not so well understood. In this we survey a few well known facts.

The Teichüller space supported on the Julia set is isomorphic to $\mathcal{B}_1(J, f)$, because the Julia set is formed by dynamically distinguished points. In particular, to have a non trivial space, the Julia set has to have positive Lebesgue measure. Thus in many examples $\mathcal{B}_1(J, f)$ can be understood by pointing out that J has zero measure. A very general result to show that J has zero measure is the following theorem by Eremenko and Lyubich.

Theorem 5.1 ([Eremenko & Lyubich 1992], Prop. 4 and Thm. 8)**.** *Let f be an entire transcendental function of finite order and finite type such that f^{-1} has a logarithmic singularity $a \in \mathbb{C}$. Assume that the orbit of every singular point of f^{-1} is either absorbed by a cycle or converges to an attracting or parabolic cycle. Then, either $J(f) = \mathbb{C}$ or meas($J(f)$=0.*

Just because the Julia set have positive measure does not mean that $\mathcal{B}_1(J, f)$ is non-trivial. For instance, in the case of quadratic polynomials it is known that there exist polynomials whose Julia set has positive measure (see [Buff & Chéritat 2006]), but even if it is not known, it's generally believed that every quadratic polynomial P has $\mathcal{B}_1(J, P)$ trivial.

However, there are examples of entire transcendental mappings which have $\mathcal{B}_1(J, f)$ non-trivial. Such example was given in [Eremenko & Lyubich 1987], stated as follows.

> *There is an entire transcendental map f whose Julia set $J(f)$ is nowhere dense, has positive measure and supports an invariant line field.*

By an invariant line field, they mean a measurable function μ supported on the Julia set such that $||\mu||_\infty = 1$ and $f^*\mu = \mu$. Notice that the existence of an invariant line field implies that the measurable functions $\mu_t = t\mu$ are $f-$invariant, $|t|-$Beltrami forms on the Julia set for all $t \in \mathbb{D}$. It follows that $\mathcal{B}_1(J, f)$ has dimension at least 1.

In fact, Eremenko and Lyubich in [Eremenko & Lyubich 1987] indicate how to modify this example so that it could support an infinite-dimensional family of measurable invariant line fields.

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