Open access • Journal Article • DOI:10.1111/J.1467-937X.2008.00517.X

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Published on: 01 Jan 2009 - The Review of Economic Studies (Oxford University Press)

Topics: Stochastic ordering

The Theory of Assortative Matching Based on Costly Signals

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8.1.2008

Abstract

We study two-sided markets with a finite numbers of agents on each side, and with two-sided incomplete information. Agents are matched assortatively on the basis of costly signals. Asymmetries in signaling activity between the two sides of the market can be explained either by asymmetries in size or in heterogeneity. Our main results identify general conditions under which the potential increase in expected output due to assortative matching (relative to random matching) is offset by the costs of signaling. Finally, we look at the limit model with a continuum of agents, and point out differences and similarities to the finite version. Technically, the paper is based on the elegant theory about stochastic order relations among differences of order statistics, pioneered by R. Barlow and F. Proschan (1966) in the framework of reliability theory.

*We are grateful to Bruno Biais, and to two anonymous referees for numerous comments that greatly improved the quality of the exposition. We also wish to thank Larry Ausubel, Ilan Eshel, John Moore, Georg Nöldeke, Avner Shaked, Moshe Shaked, Xianwen Shi, Lones Smith, Asher Wolinsky and participants at the conference "Matching and Two-Sided Markets", Bonn, 2006, for helpful remarks. Thomas Tröger gave an illuminating discussion during a seminar at the University of Bonn. This research has been partially financed by the Max Planck Research Prize (Moldovanu), and by the German Science Foundation through SFB 15-TR (Hoppe, Moldovanu, Sela). Hoppe: Department of Economics, University of Hannover, Königsworther Platz 1, 30167 Hannover, hoppe@mik.uni-hannover.de; Moldovanu: Department of Economics, University of Bonn, Lennestr. 37, 53113 Bonn, mold@uni-bonn.de; Sela: Department of Economics, Ben Gurion University, 84105 Beer Sheva, Israel, anersela@bgu.ac.il

1 Introduction

Signaling can be observed in virtually every situation where heterogeneous agents form matches, be it in two-sided markets (marriage, labor, education) or in biological settings.[1] By revealing information that is correlated with hidden underlying characteristics, signaling helps determine who gets matched with whom so as to maximize the total output (or utility) generated through matching. On the other hand, this benefit may be offset by the costs of signaling. The precise characterization and various consequences of this trade-off constitute the theme of the present paper.

Pesendorfer (1995) describes how fashion is used to signal unobservable characteristics. Seemingly useless changes in fashion may therefore be very valuable to agents who seek to interact with the 'right' people.[2] According to a Sicilian traveller in 1714, "nothing makes noble persons despise the gilded costume so much as to see it on the bodies of the lowest men in the world".[3] "So", Braudel (1981) concludes, "the upper class had to invent new 'gilded costumes', or new distinctive signs, whatever they might be" (p.324). The need to differentiate oneself from lower ranked agents plays a main role in our analysis.

Pesendorfer also notes that concerns about inefficiencies due to wasteful signaling may explain the numerous attempts to regulate apparel. For example, Sumptuary Laws enacted by the English Parliament in the 14th century controlled the use of cosmetics and the manner of personal dresses in order to preserve class distinctions, and to limit practices which were regarded as harmful in their effects (cf. Baldwin, 1926). Even today, school uniforms are common in many countries.

The idea that conspicuous consumption displays such as luxury cars, yachts, and jewelry might primarily function as a proxy for underlying characteristics valued by potential partners has received much attention in the literature on status goods (e.g., Leibenstein, 1950, Bagwell and Bernheim, 1996). Anecdotal evidence can be found, for instance, in Silverstein and Fiske (2005), Miller (2000) and the literature cited therein. The hypothesis has recently been tested by a group of evolutionary psychologists (Griskevicius et al., 2007).[4] Consistent with the theoretical predictions, the study's

[1] Various instances of financial markets (e.g., the market for underwriting services or for venture capital) also display such features. The American Economic Association has recently established an ad-hoc committee on job market signaling , chaired by Al Roth.

[2] Pesendorfer (1995) develops a model of fashion cycles in order to address the question of why producers spend large amounts of resources on periodic changes in their design. In his model, there are two types of consumers, high and low. Designs are used as signaling devices because each consumer wants to match with a high-type person.

[3] Jean-Paul Marana, Lettre d'un Sicilien à un de ses amis, ed.: V. Dufour, 1883, p.27, quoted in Braudel (1981, p.324).

[4] The participants were divided into two groups, with one group being put into a "romantic mindset" (e.g. by

results indicated that mating motives lead men to spend more on conspicuous products in order to gain status, and women to increase their displays of benevolence in order to demonstrate pro-social attributes. In other words, the individuals were willing to burn resources - whether money or time - once they were exposed to mating-related cues. In the light of these findings, it is interesting to note that substantial excise taxes have often been imposed on luxury goods (e.g., Canada's Jewelry Excise Tax, Australia's Luxury Car Tax, and the U.S. Omnibus Budget Reconciliation Act of 1990).[5]

In student-university relations, universities seek to signal the quality of their education by hiring top faculty and building flashy facilities (library, sports, etc).[6] Students compete by attending prestigious high-schools, taking various tests, and engaging in extracurricular activities.[7] In his famous study of higher education as a filter, Arrow (1973) constructs examples where prohibiting signaling via college education is welfare improving.

While we consider a benchmark model in which agents on both sides engage in wasteful signaling, the aggregate amount of wasteful signaling remains the same when we allow agents on one side to signal their types through payments (e.g., wages, dowries) to the agents on the other side. In her historical analysis of marriage markets, Anderson (2007) reports that dowries were mainly practised in pre-industrial, highly stratified societies in which the role of women in agriculture was limited. In such societies wealthier parents used higher dowries to increase the attractiveness of their daughters to potential grooms. Consistent with the theoretical results in our paper, Anderson reports that the emergence of new wage opportunities for men, resulting in a higher degree of heterogeneity on the men's side as societies became more advanced, was one of the reason behind the transition from brideprices (i.e., a fixed payment from the groom to the bride which tended to be constant across different income levels) to sometimes inflated dowries. Anderson also provides an account

displaying pictures of attractive opposite-sex individuals). The individuals were then asked, firstly, to imagine they possessed $5000 which could be spend on various luxury items as well as inconspicuous goods (such as household medication and basic toiletries), and secondly, to have 60 hours of leisure time which they could devote to volunteer work.

[5] Glazer and Konrad (1996) observe that conspicuous donations to charity may similarly function as signals. For this reason, some economists advocate taxation of charitable donations!

[6] America's universities have complete about $15 billion of building in 2006, a 260% increase over 1997. Standard and Poor described the boom as an "arms race". Particularly amusing is the race among several Texas universities to build the highest climbing wall (see *The Economist*, "Construction on Campus - Just Add Cash", December 1st, 2007)

[7] Universities and scientists also engage in grant acquisition activities in order to demonstrate their relative academic standing as employers and competent faculty, respectively (see, e.g., Arnow, 1983). Not seldom, one hears complaints about the huge amount of resources wasted in the process.

of policy measures that limit dowries and marriage gifts, e.g., the Indian Dowry Prohibition Act of 1961, and the Chinese Marriage Laws of 1950.

In another important application, Acemoglu (1999, 2002) models matching between capital and labor and shows how changes in the composition of jobs (e.g., "middling" jobs offered to both skilled and unskilled workers get replaced by high-quality jobs designed for the skilled, and low-quality jobs targeted at the unskilled) can be used to explain changes in the observed composition of wages.[8]

The present paper combines three main features:

1. We consider a finite number of agents on both sides of the market. More precisely, we "multiply" two tournament models with several heterogenous agents and several heterogenous prizes, as developed by Moldovanu and Sela (2001, 2006). In their one-sided model, bids are submitted and ranked, and then prizes are awarded such that the highest prize goes to the highest bidder, the second-highest prize goes to the second-highest bidder, etc.[9] Here we let "prizes come to life": agents on one side represent the prizes for which the agents on the other side compete. Thus, both sides are active, and the signaling behavior of each agent is affected by features (such as number of agents and distribution of characteristics) of both sides of the market.[10] The analysis of markets with a small, finite number of traders sometimes exhibits phenomena that are different from those in large markets with a continuum of agents.

2. We allow for incomplete information on both sides in a model where the partners' hidden attributes are linearly ordered and complementary.[11] Under complete information, aggregate output is maximized by assortative matching.[12] Since there is a finite number of agents, no

[8] See also the survey of Sattinger (1993). In many labor markets employers also seek to attract workers via non-directed, non-wage components such as various branding activities that make them look "cool" (see e.g. Barrow and Mosley, 2005, and "In the search of the ideal employer", *The Economist*, Aug 18th 2005).

[9] Their focus is on the revenue effects of changes in the number and size of the various prizes, in the bidding costs, and in the tournament's structure (e.g., one-stage or two-stage competition over prizes).

[10] Complete information matching models with one sided offers are analyzed, among others, by Bulow and Levin (2006), and by Felli and Roberts (2002). The latter paper focuses on costly investments, undertaken prior to matching in order to increase the match value. This important variety of complete information models has been pioneered by Cole, Mailath and Postlewaite (1992, 2001a,b).

[11] Many of our results have immediate implications for models with incomplete information on one side, or with complete information, as have been often used in the literature.

[12] The efficiency properties of assortative matching have been emphasized by Becker's (1973) classical study of populations vertically differentiated by an unique, linearly ordered attribute. Becker and many other contributors focused on centralized matching schemes. Shimer and Smith (2000) derive conditions under which a decentralized search equilibrium leads to assortative matching.

agent knows here for sure his/her rank in its own population, nor the quality of a prospective equilibrium partner. This should be contrasted with the situation in models with a continuum of agents, or with complete information, where knowledge of the own attribute and of the distributions of attributes on both sides of the market completely determines own and equilibrium-partner rank, and thus the value of the equilibrium match. In our model these values are interdependent, and agents need to form expectations about the attributes of other agents on both sides of the market.[13]

3. We introduce a new mathematical methodology to the study of two-sided markets. This is based on the elegant work on stochastic orders among normalized mean spacings (i.e., differences) and other linear combinations of order statistics, pioneered by Richard Barlow and Frank Proschan (1966) in the framework of reliability theory.[14] Barlow and Proschan have shown how monotonic failure rates induce monotonicity properties of normalized differences of order statistics. Strategic behavior is determined here by a delicate interplay between probabilities of getting various partners (governed by order statistics on one's own side) and properties of marginal gains from getting stochastically better partners (governed by differences in order statistics on the opposite side).

The seminal contribution on costly signaling in situations with asymmetric information is due to Spence (1973). He shows that investment in education may serve as a signal to prospective employers even if the content of the education is itself negligible.[15] We adopt here Spence's assumption that signals are wasted.[16] Spence and most of the following literature focused on a one-sided activity model (only workers are active), and assumed that firms are homogenous. Therefore matching concerns did not play a role. Several notable exceptions keep the one-sided activity structure, but introduce a role for matching. Chao and Wilson (1987) and Wilson (1989) consider a seller facing

[13] In contrast to the standard case in the double auction literature, our signals (that can be interpreted as bids) only determine who trades with whom, but not the terms of trade. On the other hand, in most of the double auction literature all traded units are identical (so that the optimal matching problem is fairly simple), while they are heterogenous here.

[14] Basic texts on order statistics and stochastic orders are David and Nagaraja, (2003), and Shaked and Shanthikumar (1994), respectively. Boland et al. (2002) is a good survey of the material most relevant for the present study.

[15] Related ideas appear in biology: animals signal their fitness, i.e., their propensity to survive and reproduce, to potential mating partners. According to the *handicap principle (*Zahavi, 1975), signals must be disadvantageous in order to be honest. The handicap principle is widely used to relate the evolution of animal and human traits to *sexual selection*, but we are not aware of a full-fledged signaling-cum matching model in the biological literature (see the survey in Maynard-Smith and Harper, 2003).

[16] But see the discussion below for an interpretation where the signals are payments to a third party.

a continuum of customers who differ in their private valuations for service quality. They show how customers can be "matched" to different service qualities by offering them price menus that fully, or partially induce them to reveal their type. Fernandez and Gali (1999) compare markets to matching tournaments in a model with a continuum of uniformly distributed agents on each side. Only one side is active. Their main result is that, in spite of the wasteful signaling, tournaments may be welfare superior to markets if the active agents have budget constraints. Damiano and Li (2007) allow for two-sided incomplete information in a model of price discrimination with a continuum of types on each side. Their focus is on the intermediary's revenue in such situations.

The paper is organized as follows: In Section 2, we describe the matching model and introduce some useful definitions.

In Section 3, we derive and interpret a side-symmetric signaling equilibrium in strictly monotonic strategies. In this equilibrium, assortative matching based on the ranking of signals is equivalent (in terms of output) to assortative matching based on the ranking of true attributes. We next focus on aggregate measures of signaling and welfare. The amount of signaling is only a fraction of total output - there is no full rent dissipation even if the market gets very large. The reason is that signaling allowing for assortative matching (and thus increased output) creates externalities on both sides of the market. But only the externality on one's own side due to the incentive to win a better partner gets dissipated by competition. The externality interpretation is detailed in Subsection 3.2.

In Section 4 we explain the main intuition and technical apparatus used in the paper in the context of a simple comparative-statics exercises: we illustrate how aggregate measures of output, signaling and net welfare change - on both sides of the market - when we increase the heterogeneity of attributes on one side. For large families of distributions, Barlow and Proschan's results translate changes in the variability of the underlying distribution of abilities into changes in the variability of vectors of order statistics and their spacings.

In Section 5 we dissect asymmetries in signaling behavior between the two sides of the market. In Subsection 5.1, we analyze the effects of increasing the number of agents (i.e., entry) on one side.[17] Entry affects the expected match output, but also the agents' signaling activity since it changes both the amount of competition and the value of perceived prizes. The entry results are also methodologically useful since some of our subsequent proofs proceed by considering a balanced market to which we add agents in order to create a long side. In Subsection 5.2, we use heterogeneity

[17] Our comparative statics results in this and the next section focus on aggregate measures of signaling and welfare. We briefly point out the implications for individual measures - these are governed by the same properties of failure rates.

differentials among the two sides of the market in order to identify circumstances where one side is signaling more than the other. Intuitively, the side who is relatively more homogeneous and who perceives therefore a relatively more dispersed prize structure will signal more. This result has immediate applications for intermediated markets where signals are replaced by payments to a matchmaker, and it provides a new explanation for asymmetries in payments schedules among the two sides of the market.

In Section 6, we compare random matching (without any signaling) to assortative matching (based on costly and wasteful signaling) in terms of total expected net welfare.[18] For distribution functions having a decreasing failure rate (DFR), assortative matching with signaling is welfare-superior, while for distribution functions having an increasing failure rate average (IFR), random matching is superior. In the latter case, we also show in Subsection 6.2 that agents may be trapped: given that all others signal, signaling is individually optimal, even though each type of each agent may be better-off under random matching.[19] Let us note here that, besides the two focal equilibria analyzed here (strictly separating equilibrium and strictly pooling equilibrium yielding random matching) there exist many other intermediate, partially separating equilibria that yield a "coarse" matching of agents.[20]

In Section 7 we analyze the limit version of our model when there is a continuum of agents. Despite the fact that in any given economic situation the number of agents is obviously finite, economists often work with a continuum of agents because it is more convenient, and because this assumption well captures the meaning of "perfect competition".[21] Intuitively, some sort of continuity will ensure that results obtained for large, finite markets will continue to hold in the perfectly competitive limit. It is less natural to "extrapolate backwards" by using insights obtained

[18]While examples where pure pooling is welfare-superior to full separation in the Spence signaling model are known, there are no general results. Rege (2003) studies a model of status consumption with a continuum of agents and uniformly drawn attributes. For certain parameters of a Cobb-Douglas production function, she notes that the increased matching efficiency due to the consumption of status goods (which serves as a signal) may be offset by the needed expenditure.

[19]Charles Darwin once remarked: "The sight of a peacock tail, whenever I gaze at it, makes me sick".

[20]These equilibria combine features of the two focal ones: they involve random matching of agents within corresponding, assortatively matched subsets - see Damiano and Li (2007) for a model with a continuum of agents. Hoppe, Moldovanu and Ozdenoren (2006) estimate an intermediary's revenue loss from coarse matching. McAfee (2002) showed that coarse matching involving only two distinct classes may achieve no less output than the average of assortative matching and random matching.

[21]Gretsky, Ostroy and Zame (1999) formalize the meaning of perfect competition in the assignment game with either a finite number or a continuum of agents. Perfect competition (where agents appropriate their marginal products, and where the core is a singleton) is typical for the continuum version, but rare for the finite version.

for the limit market in order to make predictions about small, finite markets, and in fact such attempts often fail (e.g., as is well known in oligopoly theory).

With a continuum of agents, the welfare comparison between random matching and assortative matching based on signaling hinges on the magnitude of the coefficient of covariation among the two populations. In symmetric settings, assortative matching with signaling is welfare-superior (welfare inferior) to random matching if and only if the coefficient of variation of the common distribution of attributes is larger (smaller) than unity. For *IFR* and *DFR* distributions, we show that this welfare comparison is preserved in markets of *any* finite market size. For distributions that are not *IFR* or *DFR* the comparison crucially depends on market size, and the result for the continuum limit may be completely reversed in small, finite markets.

Section 8 concludes. Appendix A contains several useful results from the statistical literature, while Appendix B contains the proofs of our results.

2 The matching model

There is a finite set $N = \{1, 2, ..., n\}$ of men, and a finite set $K = \{1, 2, ..., k\}$ of women, where $n \geq k$. Each man is characterized by an attribute x, each woman by an attribute y. If a man and a woman are matched, the utility for each of them is the product of their attributes. Thus, total output from a match between agents with types x and y is $2xy$. Note that all our results can be immediately extended to asymmetric production functions having the form $\delta(x)\rho(y)$, where δ and ρ are strictly increasing and differentiable.[22]

Agent i's attribute is private information to i. Attributes are independently distributed over the interval $[0, \tau_F]$, $[0, \tau_G]$, $\tau_{F,}$ $\tau_G \leq \infty$, according to distributions F (men) and G (women), respectively. For all distributions used in the paper we assume, without mentioning it again, that $F(0) = G(0) = 0$, that F and G have continuous densities, $f > 0$ and $g > 0$, respectively, and finite first and second moments. The last requirement ensures that all integrals used below are well defined (e.g., all order statistics have finite expectations).

We study the following matching contest: Each agent sends a costly signal b, and signals are submitted simultaneously. Agents on each side are ranked according to their signals, and are then matched assortatively. That is, the man with the highest signal is matched with the woman with the highest signal, the man with the second-highest signal is matched with woman with the second-highest signal, and so forth. Agents with same signals are randomly matched to the corresponding partners. The net utility of a man with attribute x that is matched to a woman with attribute y

[22] See Subsection 5.2 for an example with a Cobb-Douglas production function.

after sending a signal b is given by $xy - b$ (and similarly for women).[23] Thus, signals are costly. In contrast to standard models, costs differentials are not required here in order to sustain signaling. The reason is that different types of agents expect different marginal gains from signaling.[24]

For the subsequent welfare comparisons we assume that, apart from their function enabling matching, signaling efforts are wasted from the point of view of our men and women. In other variations, not explicitly considered here, these may accrue as rents to a third party. The equilibrium analysis is invariant to such alternative specifications.

For the equilibrium characterization we will need several pieces of notation. Let $X_{(1,n)} \leq X_{(2,n)} \leq \cdots \leq X_{(n,n)}$ and $Y_{(1,k)} \leq Y_{(2,k)} \leq \cdots \leq Y_{(k,k)}$, denote the order statistics of men's and women's characteristics, respectively. We define $X_{(0,n)} \equiv 0$ ($Y_{(0,k)} \equiv 0$).

Let $F_{(i,n)}$ ($G_{(i,k)}$) denote the cumulative distribution of $X_{(i,n)}$ ($Y_{(i,k)}$). The density of $X_{(i,n)}$ is given by:

$$f_{(i,n)}(x) = \frac{n!}{(i-1)!\,(n-i)!} F(x)^{i-1} \left[1 - F(x)\right]^{n-i} f(x),$$

and similarly for $Y_{(i,k)}$.

Let EX (EY) be the expectation of $F(G)$. We denote by $EX_{(i,n)}$ ($EY_{(i,k)}$) the expected value of the order statistic $X_{(i,n)}$ ($Y_{(i,k)}$), and define $EX_{(0,n)} = EY_{(0,k)} = 0$. A useful identity, repeatedly used below, is:

$$\sum_{i=1}^{n} EX_{(i,n)} = nEX$$

3 Equilibrium analysis

In this section, we focus on a symmetric, strictly separating equilibrium where all agents on one side of the market use the same signaling strategy. Obviously, the model has other symmetric equilibria. For example, a strictly pooling equilibrium where no agent ever signals (which yields random matching) can also be sustained. In Section 6 below, we will compare the outcomes of these two focal equilibria.

[23] This is a straightforward generalization of the standard one-sided independent private value model considered in the auction literature. From that literature it is well-known that results beyond the case of risk neutrality are seldom analytically tractable. An advantage of this formulation (which is also used in most of the related matching literature, e.g., McAfee, 2002; Damiano and Li, 2007) is that all our results can be stated solely in terms of properties of the distribution functions.

[24] Notice that b is the amount of money spent on effort, and thus the subsequent equilibrium construction holds for any type-independent, continuous and strictly monotonic effort cost function.

3.1 Existence of signaling equilibrium

Assume then that all men [women] use the same, strictly monotonic and differentiable equilibrium signaling function β $[\gamma]$. A man maximizes his net utility from the expected match minus his signal. Besides the man's own type x, the expected match utility depends on the probabilities of being ranked first, second, etc..., and on the expected qualities of the respective partners for each rank. These are given by the mean order statistics of women. Because a man with attribute x has the option to behave as though his attribute is s, his maximization problem can be written as:

$$\max_s \quad \left\{ \sum_{i=n-k+1}^{n} F_i^n(s)\, xEY_{(k-(n-i),k)} - \beta(s) \right\}$$

where $F_i^n(s)$ denotes the probability that a man with type s meets $n-1$ competitors such that $i-1$ have a lower type and $n-i$ have a higher type.[25] Note that the probability of having *exactly* $i-1$ competitors with a lower type than s can be obtained by subtracting from the the probability that the $i-1$-lowest type is lower than s the probability that the $i-$lowest type is lower than s. These two probabilities are given by the cumulative distribution of the respective order statistics, as described above. For $i = 2, ...n-1$, we have then:

$$F_i^n(s) = F_{(i-1,n-1)}(s) - F_{(i,n-1)}(s) = \frac{(n-1)!}{(i-1)!\,(n-i)!} F(s)^{i-1} \left[1 - F(s)\right]^{n-i},$$

where we let $F_n^n(s) = F_{(n-1,n-1)}(s)$, and $F_1^n(s) = 1 - F_{(1,n-1)}(s)$.

Using the above observation, and the fact that, in equilibrium, the maximum should be attained for $s = x$, we obtain the following differential equation:

$$\begin{aligned} \beta'(x) &= \sum_{i=n-k+1}^{n-1} \left[f_{(i-1,n-1)}(x) - f_{(i,n-1)}(x) \right] xEY_{(k-(n-i),k)} + f_{(n-1,n-1)}(x)\, xEY_{(k,k)} \\ &= \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(x) \left[EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right] x + f_{(n-k,n-1)}(x)\, xEY_{(1,k)} \end{aligned}$$

The equation says that, at the optimum, the marginal cost of signaling must equal the marginal benefit from signaling which is given here by the expected marginal utility from being matched with the next better woman, where the expectation is taken over all possible ranks a man with type x may have. The man with the lowest type either never wins a woman (if $n > k$) or wins for sure the woman with the lowest type (if $n = k$). Hence, the optimal signal of this type is always zero, yielding the boundary condition $\beta(0) = 0$. The solution of the differential equation gives candidate equilibrium effort functions.

[25] Similarly, we denote by $G_i^k(s)$ the probability that a woman with type s meets $k-1$ competitors such that $i-1$ have a lower type and $k-i$ have a higher type.

Proposition 1 *The profile of strategies where each man employs the strictly increasing signaling function*

$$\begin{aligned}\beta(x) &= \int_0^x s\left\{\sum_{i=n-k+1}^{n-1}\left[f_{(i-1,n-1)}(s)-f_{(i,n-1)}(s)\right]EY_{(k-n+i,k)}\right\}ds \\ &+\int_0^x sf_{(n-1,n-1)}(s)\,EY_{(k,k)}ds \qquad (1)\end{aligned}$$

and each woman employs the analogously derived signaling function $\gamma(y)$ constitutes an equilibrium of the matching contest.

Proposition 2 *For any F, G, n, k, it holds in the above equilibrium that:*

1. *Total expected output is given by*

$$O(n,k)=2\sum_{i=n-k+1}^{n}EX_{(i,n)}EY_{(k-(n-i),k)} \qquad (2)$$

2. *Men's total signaling effort and (net) welfare are given by :*

$$\begin{aligned}S_m(n,k) &= n\int_0^{\tau_F}\beta(x)f(x)\,dx \\ &= \sum_{i=n-k+1}^{n}(n-i+1)\left(EY_{(k-n+i,k)}-EY_{(i-(n-k)-1,k)}\right)EX_{(i-1,n)}, \qquad (3)\\ W_m(n,k) &= \sum_{i=n-k+1}^{n}EX_{(i,n)}EY_{(k-(n-i),k)}-S_m(n,k)\\ &= \sum_{i=n-k+1}^{n}(n-i+1)\left(EX_{(i,n)}-EX_{(i-1,n)}\right)EY_{(i-(n-k),k)} \qquad (4)\end{aligned}$$

3. *Women's total signaling effort and (net) welfare are given by :*

$$\begin{aligned}S_w(n,k) &= k\int_0^{\tau_G}\gamma(y)g(x)\,dx \\ &= \sum_{i=n-k+1}^{n}(n-i+1)\left(EX_{(i,n)}-EX_{(i-1,n)}\right)EY_{(i-(n-k)-1,k)} \qquad (5)\\ W_w(n,k) &= \sum_{i=n-k+1}^{n}EX_{(i,n)}EY_{(k-(n-i),k)}-S_w(n,k)\\ &= \sum_{i=n-k+1}^{n}(n-i+1)\left(EY_{(i-(n-k),k)}-EY_{(i-(n-k)-1,k)}\right)EX_{(i,n)} \qquad (6)\end{aligned}$$

4. *Total expected (net) welfare in assortative matching based on costly signaling is at least half the expected output (or, in other words, aggregate signaling efforts are less than half output).*

$$W(n,k) = 2\sum_{i=n-k+1}^{n} EX_{(i,n)}EY_{(k-(n-i),k)} - S_m(n,k) - S_w(n,k) \tag{7}$$

$$\geq \sum_{i=n-k+1}^{n} EX_{(i,n)}EY_{(k-(n-i),k)} \tag{8}$$

3.2 Signaling as externality payments

The spacings of, say, men's mean order statistics, $EX_{(i,n)} - EX_{(i-1,n)}$, represent the expected marginal gains for women from winning a stochastically next better partner. The above proposition reveals that aggregate signaling effort of, say, women is a weighted sum of the *normalized spacings* of these order statistics, $(n - i + 1)(EX_{(i,n)} - EX_{(i-1,n)})$, where the weights (or coefficients), $EY_{(i-(n-k)-1,k)}$, are the mean order statistics of women. Analogous observations hold for the net welfare terms.

In order to understand how these expressions come about, and why the normalization factors of the form $n - i + 1$ do appear, it is very useful to apply an externality argument.[26] Assume for simplicity of notation that $n = k$, and consider, for example, the men's side: one can interpret the interaction among men as a contest for several heterogeneous prizes, where the prizes are represented by the mean orders statistics of women's attributes. Since the above constructed equilibrium implements the assortative matching scheme where, for any realization of types, higher ranked men get higher prizes, the achieved allocation of prizes to men is efficient. The payoff/revenue equivalence principle implies that total expected payments (i.e., signaling here) must be equivalent to the one in a Vickrey mechanism. It is well known that, in such mechanisms, agents must pay for the externality imposed on others.

To compute the externalities, take a realization of ordered men's types $x_1 \leq x_2 \leq .. \leq x_n$, consider ordered prizes (i.e., women's attributes) $y_1 \leq y_2.. \leq y_n$, and set $x_0 = y_0 = 0$. If the man with type x_j is around, the gross welfare of all other men in the efficient allocation (assortative matching) is given by $\sum_{i=1}^{j-1} x_i y_i + \sum_{i=j+1}^{n} x_i y_i$. If that man is not around, the gross welfare of all other men is $\sum_{i=1}^{j-1} x_i y_{i+1} + \sum_{i=j+1}^{n} x_i y_i$. Note that men ranked higher than j are not affected by j's presence, but all lower ranked men "suffer" since they get matched to the next worse woman when j appears. Thus, the (negative) externality of the type x_j is given by:

[26] We are very grateful to Larry Ausubel for suggesting it.

$$
\begin{aligned}
[\sum_{i=1}^{j-1} x_i y_i + \sum_{i=j+1}^{n} x_i y_i] - [\sum_{i=1}^{j-1} x_i y_{i+1} + \sum_{i=j+1}^{n} x_i y_i] &= \sum_{i=1}^{j-1} x_i y_i - \sum_{i=1}^{j-1} x_i y_{i+1} \\
&= \sum_{i=1}^{j-1} x_i (y_i - y_{i+1}) \leq 0
\end{aligned}
$$

Summing up these externalities over all men, we obtain:

$$\sum_{j=1}^{n} \sum_{i=1}^{j-1} x_i (y_{i+1} - y_i) = \sum_{j=1}^{n} (n - j + 1)(y_j - y_{j-1}) x_{j-1},$$

where $-(y_j - y_{j-1})x_{j-1}$ is the externality imposed by *each* of the $n - j + 1$ top men on the man of rank $j - 1$. Taking now expected values in the displayed expression (while recalling that prizes and men's types were ordered), gives the expression for S_m in equation (3). Thus, total men's payments involves the *normalized* spacings of women's mean order statistics because the externality on a certain man needs to be paid by *all* higher ranked men.

A man's presence also creates a positive externality on the women's side. The externality of type x_j is $\sum_{i=1}^{n} x_i y_i - [\sum_{i=1}^{j} x_{i-1} y_i + \sum_{i=j+1}^{n} x_i y_i] = \sum_{i=1}^{j} (x_i - x_{i-1}) y_i \geq 0$. Since output is equally shared, the other half of j's marginal productivity is retained as j's net welfare: $\sum_{i=1}^{j} (x_i - x_{i-1}) y_i = x_j y_j - \sum_{i=1}^{j-1} x_i (y_{i+1} - y_i)$. Summing up over all men yields:

$$\sum_{j=1}^{n} \sum_{i=1}^{j} (x_i - x_{i-1}) y_i = \sum_{j=1}^{n} (n - j + 1) (x_j - x_{j-1}) y_j,$$

where $(x_j - x_{j-1}) y_j$ is half the marginal product of *each* of the $n - j + 1$ top men in the match of the j's woman. Taking expected values in the above expression gives the men's total welfare W_m of equation (4).

The positive externality on the other side of the market explains why total signaling expenditure never raises to total output, even if we let the number of agents go to infinity. This is an important consequence of the two-sided structure of the market, and of the complementarity among the attributes.

4 The main techniques

In this section we explain the main technical tools used in our analysis and the general intuition that can be gleaned from these techniques. We do this by describing how total output, total men's and women's signaling, and welfare are affected when the distribution on one side of the market (say men's) becomes more variable.

4.1 Variability of distributions and their vectors of order statistics

To isolate the role of variability in attributes, assume here that there are equal numbers of men and women, i.e., $n = k$. Recall the expressions for total signaling and welfare displayed in Proposition 2. In order to asses how they change when we increase the variability of the men's distribution, we need to translate the changes in the distribution of attributes into changes in the vector of order statistics $(EX_{(n,n)}, EX_{(n-1,n)}, ..EX_{(1,n)})$ on the one hand, and the vector of normalized spacings $\big(n(EX_{(1,n)}\ ,\ (n-1)(EX_{(2,n)} - EX_{(1,n)})\ , .., \ (EX_{(n,n)} - EX_{(n-1,n)})\big)$ on the other.[27] Such translation results can be found in an elegant paper due to Barlow and Proschan (1966). We now proceed to list these results, starting with appropriate notions of increased variability of distributions and vectors:

Proposition 3 *Barlow and Proschan (1966): Let H, F be two distributions of the men's attributes such that $H(0) = F(0) = 0$. If $H^{-1}F(x)$ is convex, the function $1 - F(x)$ will cross $1 - H(x)$ at most once, and then from above, as x increases from 0 to ∞. As a consequence, if F and H have the same mean, a crossing must occur, and F has a smaller variance than H.*

Note that $H^{-1}F(x)$ convex implies $H^{-1}F(x)$ *star-shaped* (i.e., $H^{-1}F(x)/x$ is increasing), and that $H^{-1}F(x)$ star-shaped implies that F second-order stochastically dominates H if they have the same mean. Variability of vectors is captured by the following famous majorization concepts (see Hardy, Littlewood and Polya, 1934):

Definition 1 *Let $\widehat{a} = (a_1, a_2, ..a_n)$, $\widehat{b} = (b_1, b_2, ..b_n) \in \mathbb{R}^n$ such that $a_1 \geq a_2.. \geq a_n$ and $b_1 \geq b_2.. \geq b_n$. We say that $\widehat{b}$ majorizes $\widehat{a}$, written $\widehat{b} \succ \widehat{a}$, if for all $i = 1, .., n$, $\sum_{j=1}^{i} b_j \geq \sum_{j=1}^{i} a_j$, with equality for $i = n$. A real valued function ϕ defined on a subset of $A \subseteq \mathbb{R}^n$ is said to be Schur-convex (-concave) on A if $\widehat{a}, \widehat{b} \in A$ and $\widehat{b} \succ \widehat{a}$ imply $\phi(\widehat{b}) \geq (\leq)\phi(\widehat{a})$.*

Note that ϕ is Schur-concave if and only if $-\phi$ is Schur-convex. It is well-known, and easy to see that a linear function of the form $\phi(\widehat{b}) = \sum_i c_i b_i$ is Schur-convex (-concave) if and only if $(b_i - b_j)(c_i - c_j) \geq (\leq)0$ holds for any i, j. In words, higher coordinates must have higher (lower) coefficients - or marginal values - in order to get Schur-convexity (-concavity).

Theorem 1 *Barlow and Proschan (1966): Let H, F be two distributions of the men's attributes with the same mean such that $H(0) = F(0) = 0$, and such that $H^{-1}F(x)$ is convex. Let Z (X)*

[27] For both vectors, the sum of all coordinates is equal to n times the mean of the respective distribution.

denote the random variable described by $H(F)$. Then:[28]

1. $(EZ_{(n,n)}, EZ_{(n-1,n)}, ..EZ_{(1,n)}) \succ (EX_{(n,n)}, EX_{(n-1,n)}, ..EX_{(1,n)})$

2. $\sum_{i=1}^{r}(n-i+1)(EX_{(i,n)} - EX_{(i-1,n)}) \geq \sum_{i=1}^{r}(n-i+1)(EZ_{(i,n)} - EZ_{(i-1,n)})$, *for* $1 \leq r \leq n$.

3. $\sum_{i=1}^{n} a_i(n-i+1)(EX_{(i,n)} - EX_{(i-1,n)}) \geq \sum_{i=1}^{n} a_i(n-i+1)(EZ_{(i,n)} - EZ_{(i-1,n)})$ *for* $a_1 \geq a_2.. \geq a_n$.

The meaning of the first statement is clear: increased variability of distributions gets translated, in the sense of majorization, into increased variability of the vector of mean order statistics. Note that vectors of mean order statistics are of course naturally ordered. Recalling that the second statement must hold with equality for $r = n$ (since then both sums are equal to n times the common mean), we see that statements 2 and 3 also have the flavor of majorization: it is tempting to interpret them in the sense of "increased variability in the distribution leads to decreased variability of spacings". But, majorization cannot be meaningfully applied here since the vector of normalized spacing need not be naturally ordered.

Another result due to Barlow and Proschan identifies well-known, large, non-parametric classes of distributions where normalized spacings are naturally ordered. For these classes we obtain then the needed information about the connections between the variability of distributions and the variability of normalized spacings, and we can fully analyze the effects on total signaling and welfare. We start with a well-known definition:

Definition 2 *Let F be a distribution on $[0, \tau_F]$ with density f. The* failure (or hazard) rate *of F is given by the function $\lambda(x) \equiv f(x) / [1 - F(x)]$, $x \in [0, \tau_F)$. F is said to have an increasing (decreasing) failure rate (IFR) (DFR) if $\lambda(x)$ is increasing (decreasing) in x.*

Note that if H is the exponential distribution, then $H^{-1}F(x)$ being convex (concave) is equivalent to F being *IFR (DFR)*.[29] In particular, keeping a constant mean, *IFR (DFR)* distributions are less (more) variable than the exponential distribution. For this distribution, all normalized spacings

[28] Barlow and Proschan prove statements 1-3 for distributions ordered by the weaker star condition. Statement 1 has recently been shown to hold also for distributions ordered by second-order stochastic dominance (Cal and Carcamo, 2006).

[29] The *IFR (DFR)* conditions are equivalent to the *logconcavity (logconvexity)* of the survivor function $1 - F$. A logconcave density is sufficient for the logconcavity of the survivor function. The exponential, uniform, normal, power (for $\alpha \geq 1$), Weibull (for $\alpha \geq 1$), gamma (for $\alpha \geq 1$) distributions are *IFR*. The exponential, Pareto, Weibull (for $0 < \alpha \leq 1$), gamma (for $0 < \alpha \leq 1$) are *DFR*. See Barlow and Proschan (1975).

are independent and identically distributed.[30] Thus, the corresponding vector of expected values is a vector with n equal coordinates, and convex (concave) transformations of the exponential will have a nice, regular effect on normalized spacings:

Theorem 2 *Barlow and Proschan (1966): Assume that F is IFR (DFR). Then the normalized spacing $(n-i+1)\left(X_{(i,n)}-X_{(i-1,n)}\right)$ is stochastically decreasing (increasing) in $i=1,2,...,n$ for fixed n.*

Recall that the spacings of men's order statistics determine the negative externality imposed by all the $n-i-1$ top women on lower ranked women, and also the marginal product of the $n-i-1$ top men in the match of women with the same or lower ranks. Hence, according to Theorem 2, women's expected negative externality imposed on a woman of a given, fixed type is decreasing in her rank if the men's distributions is *IFR*, while the opposite holds for *DFR* distributions. Similarly, men's aggregate marginal product in the match of a given woman with fixed type is decreasing (increasing) in her rank if the men's distributions is *IFR* (*DFR*).

4.2 The comparative static effects of increased heterogeneity

We are now finally ready to describe the comparative statics effects on output, signaling and welfare when increasing the variability of men while keeping their mean attribute constant:

1. Increasing the heterogeneity of men leads to an unambiguous increase in output since $O(n)=\sum_{i=1}^{n}EX_{(i,n)}EY_{(i,n)}$ is a Schur-convex function of the men's mean order statistics. That is, when men become more heterogenous, the expected types of (a number of) relatively highly ranked men tend to increase, while those of (a number of) relatively lowly ranked men tend to decrease (preserving the distribution's mean). This observation is a simple consequence of Theorem 1-1. The result is then obtained by noting that higher ranked men have a higher marginal value (or coefficient) in the production of match output.

2. Increasing the heterogeneity of men leads to a decrease in men's total signaling if the women's distribution is IFR. Here we use Theorem 1-1 and Theorem 2. The result follows because

[30] Economists may know this result from the analysis of inter-arrival times in the Poisson process. Clearly, when the random variable is time, one would expect increasing failure rates in some cases (e.g. life time distributions for a person or a machine) and decreasing failure rates in others (e.g. the duration of tenure at a residence). When the random variable is the attribute of an agent (e.g., skill, human capital, health, income), as in our model, both IFR and DFR are obviously also plausible, albeit without the original failure rate interpretation. Singh and Maddala (1976), for example, take the DFR property as a decisive characteristic of income distributions of US families, while Salem and Mount (1974) advocated to use an IFR distribution.

the function describing men's total signaling is Schur-concave in the vector of men's mean order statistics as long as the coefficients - given here by the normalized mean spacings of women - are decreasing: $S_m(n) = \sum_{i=1}^{n}(n-i+1)\left(EY_{(i,n)} - EY_{(i-1,n)}\right)EX_{(i-1,n)} = \sum_{i=1}^{n}(n-i+1)\left(EY_{(i,n)} - EY_{(i-1,n)}\right)EX_{(i-1,n)} + 0 \cdot EX_{(n,n)}$. In words, if men become less alike, the negative externality imposed on (a number of) relatively highly ranked men, tends to increase, while that of (a number of) relatively lowly ranked men tends to fall (Theorem 1-1). But, when the women's distribution is *IFR*, men's expected negative externality imposed on a man of a given, fixed type is decreasing in his rank. Thus, the externality imposed on men with higher ranks has a lower marginal contribution to men's total externality (Theorem 2). Therefore, men's total signaling (which is equal to men's total externality on men) goes down.

3. An argument similar to point 2 above shows that increasing the heterogeneity of men decreases (increases) women's welfare if the women's distribution is *IFR (DFR)* since the function $W_w(n) = \sum_{i=1}^{n}(n-i+1)\left(EY_{(i,n)} - EY_{(i,n)}\right)EX_{(i,n)}$ is then Schur-concave (convex) in the vector of men's mean order statistics. In contrast to the insight at point 2, we can now obtain a comparative statics result for both *IFR* and *DFR* distributions of women since all n men's mean order statistics appear in the expression for $W_w(n)$, whereas the highest one is missing (or appears with a zero coefficient) in the expression for $S_m(n)$.

4. Increasing the heterogeneity of men leads to an unambiguous increase in women' total signaling, and the same holds for men's total welfare. The result is most easily understood in the case where the men's distributions have naturally ordered vectors of normalized mean spacings. Assume, for example, that the distribution of men's attributes F (random variable X) changes to another distribution H (random variable Z) and that both F, H are *IFR*. Then Theorem 1-2 and Theorem 2 imply together that $[nEX_{(1,n)}\ ,\ (n-1)(EX_{(2,n)} - EX_{(1,n)})\ , ...,$ $(EX_{(n,n)} - EX_{(n-1,n)})] \succ [nEZ_{(1,n)}\ ,\ (n-1)(EZ_{(2,n)} - EZ_{(1,n)})\ \ ,.., (EZ_{(n,n)} - EZ_{(n-1,n)})]$. The result follows by noting that $S_w(n) = \sum_{i=1}^{n}(n-i+1)\left(EX_{(i,n)} - EX_{(i-1,n)}\right)EY_{(i-1,n)}$ is Schur-concave in the vector of men's normalized mean spacings. The general argument uses Theorem 1-3. Intuitively, as men become more heterogenous, women's expected negative externality imposed on relatively highly ranked women (i.e. the reduction in their output share when they get matched to the next worse man) becomes more severe. Since women with higher ranks have a higher marginal contribution to women's total negative externality, women's total signaling becomes larger. The effect on men's total welfare is analogously explained.

It is important to note that points 1-3 above, which use only part 1 of Theorem 1, also hold for increases in heterogeneity according to the weaker, well-known notion of second-order stochastic dominance.

We end this section by mentioning that simpler comparative statics results can be obtained for the case where the distribution of, say, men's attributes F (random variable X) changes to another distribution H (random variable Z) such that $X \leq_{hr} Z$ (Definition 3, Appendix A). This implies, in particular, $EX \leq EZ$, and $EX_{(i,n)} \leq EZ_{(i,n)}$, $i = 1, 2, ...n$. Thus, there is an unambiguous increase in the expected quality of men for every rank. The effects on expected output, men's total signaling, and women's total welfare are also unambiguous: all these measures are higher after the change because women receive better prizes and because competition among men is stronger. The effect on men's total welfare and women's total signaling are subtler: total women's signaling and total men's welfare are higher under H if either F or H are *DFR*. This happens because in that case all normalized spacings of F are stochastically smaller than the corresponding ones under H (see Khaledi and Kochar, 1999). The converse for the *IFR* case is not generally true.

5 Which side signals more ?

As mentioned in the Introduction, many applied studies (e.g., biological studies of sexual selection of various species, development studies of marriage markets in rural societies) have noted that one side of the market engages in much more signaling activity than the other. Another related observation is that in intermediated markets where payments from agents accrue to a third party, prices are often uneven, with one side paying much more than the other.[31]

Besides studying the signaling effects of asymmetries between the sizes of each market side, we identify below another cause for discrepancies in signaling activity (or payments by each side in an intermediated market): relatively higher signaling activity on one side of the market may be due to a relatively smaller degree of heterogeneity on that side.

5.1 Effects of changes in the number of agents

The biological literature has noted that, although the sex-ratio is approximately 50-50 in most species, at any point in time, there are more males ready to mate than females. Thus, males

[31] The recent IO literature on two-sided markets has identified the presence of network externalities as a source of asymmetric payment patterns: attracting customers on one side by lowering the price may be profitable if this side creates externalities on the other side. See Tirole and Rochet (2003) and *The Economist*, "Matchmakers and trustbusters", p.84, Dec 10th, 2005.

perceive more competition, and need to signal more. In a different context and with switched roles, development studies (see the survey of Anderson, 2007) explain the phenomenon of dowry inflation by noting that the modern population explosion in some poor parts of the world has created an imbalance, called a *marriage squeeze*, between the number of men and women active in the marriage market at a given period of time. This may, for example, happen when tradition and economic realities call for men to marry at a relatively older age (e.g., after they acquired sufficient wealth).[32] Since the incoming cohort of younger women is larger, competition among them for the older, economically established men gets stiffer, leading to increased dowries.

We analyze here the effects on both sides of the market caused by a change in the number of agents on one side.[33] If there is entry on the long side (i.e. entry by men), the number of possible pairs remains unchanged, but the expected value of the i'th man is increased. If there is entry on the short side (i.e. entry by women), both the number of possible pairs, and the expected value of the i'th woman gets higher. Thus, adding agents on one side of the market unambiguously increases the expected matching output.

An increase in the number of agents also affects the agents' signaling activity. The next proposition shows that the heuristic argument about imbalances in signaling due to asymmetry in size neatly applies if the distribution of attributes on the men's (long) side is *IFR*: in that case, the addition of more men increases men's signaling while decreasing women's signaling. But, if that distribution is *DFR*, our analysis reveals that the precise effect is subtler since an increase in the number of men also leads to higher women's signaling.

Proposition 4 *Suppose there is entry on the men's side.*[34] *Then, for all* G*,*

1. *men's total signaling* S_m *increases for all* F*.*
2. *women's total signaling* S_w *increases (decreases) if* F *is DFR (IFR).*
3. *men's total welfare* W_m *increases (decreases) if* F *is DFR (IFR).*

[32] Schelling (1978, p. 202) notes that the issue exists also in the US, and (maybe half-jokingly) lists several government policies that could correct the imbalance.

[33] In complete information models, Kelso and Crawford (1982), Mo (1988), and Crawford (1991) studied changes in core payoffs following entry on one side in two-sided markets such as Shapley and Shubik's (1972) assignment game. Entry lowers the payoffs of same-side agents already in the market, while increasing the payoff of all agents on the other side.

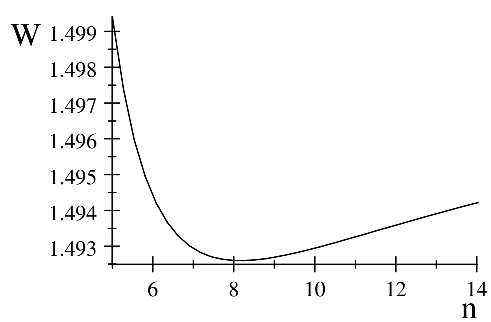
[34] Entry by women (i.e., on the short side) has similar effects to entry by men, except that it leads to a higher number of matches. This increase has, ceteris paribus, a positive effect on the men's total signaling effort. Therefore, even if the distribution of women's types G is *IFR*, men's total signaling may increase due to the presence of additional women.

4. women's total welfare W_w increases for all F.

To understand the intuition, note that entry on the men's side increases all the k highest mean order statistics of men and hence the negative externality (i.e., loss of output) caused by any matched man. There is thus an unambiguous increase in total men's signaling, reflecting stiffer competition on the men's side. The effect on women's signaling is more complex since the externality imposed by a woman on other women depends on the men's mean spacings (determining the loss from getting matched to the next worse partner). Theorem 5 in Appendix A shows that all men's spacings stochastically increase (decrease) jointly in i and n if F is *DFR(IFR)*. Thus, all the k highest *spacings* of men's order statistics increase (decrease) under entry, and hence each women's externality imposed on other women increases (decreases) if the distribution of men's attributes is *DFR(IFR)*, that is, more (less) variable than the exponential distribution, given the same mean (see Subsection 4.1). As a consequence, women's total signaling increases (decreases) if F is *DFR(IFR)*. Analogous effects occur for the welfare terms.[35]

Combining the above observations, we see that entry on the men's side always increases welfare if F is *DFR*, but may reduce welfare if F is *IFR*. Such a welfare loss due to entry is illustrated in the following example. The example will also be used to highlight that the insight can be quite different if there is a continuum of agents. For that model, we find that "entry" on the men's side in this situation always increases total welfare (see Section 7).

Example 1 *Suppose $F = x^{10}$, $G = y$, and $\tau_F = \tau_G = 1$. Fix $k = 3$. Then $W(3,3) \simeq 2.078$. As depicted by the graph below, $W(n,3)$ is decreasing in n for $n \leq 8$ and increasing for $n > 8$. We also have* $\lim_{n\to\infty} W(n,3) = 1.5$.



[35] Clearly, entry on the men's side reduces total and average welfare of the incumbent men for any F due to the entrant's negative externality on same-sided agents (see Subsection 3.2). Since the entrant generates additional output, Proposition 4 implies that the entrant's own share of net welfare is larger than his negative externality if F is DFR, and smaller if F is IFR. It follows that entry reduces average welfare of all men (including the entrant) if F is IFR, while the effect is ambiguous for DFR. Women's average welfare is always increased by additional men.

5.2 The effects of heterogeneity differentials

In their insightful empirical study of marriage in rural Ethiopia, Fafchamps and Quisumbing (2005) write: "if the difference between grooms is large relative to the difference between brides, brides must bring more to fend off competition from lower ranked brides who wish to improve their ranking". A possible cause for such asymmetries was mentioned in the introduction: in traditional societies, modernization leads to new wage opportunities for men, resulting in a higher degree of heterogeneity on the men's side (Anderson, 2007).

We now make the above basic intuition precise. Our result involves the function $F^{-1}G$, which has an important interpretation: it describes which woman attribute is matched to which man attribute under assortative matching in the continuous version of our model (see Section 7). If this function is convex, a marginal increase in a woman's attribute causes a higher marginal increase in her partner's attribute if the woman is higher ranked (and vice-versa for men). In other words, top women perceive higher differences among top men.

Proposition 5 *1. Let $n = k$ and let $F^{-1}G$ be convex.*[36] *Then women's total signaling is higher than men's. Thus, if signals are wasteful, men are better off than women.*

2. Let either F or G be IFR, and let be $G^{-1}F$ be convex. Then, for any $n \geq k$, men's total signaling is higher than women's.[37]

The simplest intuition for the result can be obtained in the case where $n = k$ and where $G = F$, both *IFR.* Obviously, total men's signaling then equals total women's signaling. Consider an increase in men's heterogeneity. As explained in Subsection 4.2 (points 2 and 4), women's signaling will increase, while men's signaling will decrease, and the result follows immediately. The proof of Proposition 5-1 for the general case uses a more refined result about relations among order statistics under the convexity relation (see Theorem 4 in Appendix A). Proposition 5-2 is then obtained by applying the results about the effects of entry.

A simple corollary is that, for any $n \geq k$, men's total signaling effort is higher than women's if F is convex and if G is concave. In this case, the mean spacings (without any normalization) are decreasing on the men's side, and increasing on the women's side (see Boland et al., 2001). In other words, the expected marginal gains of winning a next better partner are larger for highly-ranked men than for highly-ranked women. This yields a stiffer competition among highly ranked agents on the men's side, and total men's effort is higher than women's.

[36] This result can be generalized to distributions ordered in the star sense.

[37] We changed sides here in order to keep the convention whereby men constitute the long side.

For a simple application of the above proposition, consider the Cobb-Douglas production $2\delta(x)\rho(y) = 2x^c y^d$, $c, d > 0$. Let $n = k$, and assume that men's and women's attributes are uniformly distributed on $[0, 1]$. This model is equivalent to the one where the types are $\tilde{x}$, $\tilde{y}$, the production function is $2\tilde{x}\tilde{y}$, and the distributions of attributes are $\tilde{F}(\tilde{x}) = \tilde{x}^{\frac{1}{c}}$, $\tilde{G} = \tilde{y}^{\frac{1}{d}}$. Then, men signal more, and are worse-off if $c \leq d$.

6 Assortative versus random matching

In this section, we compare the equilibrium outcome of assortative matching based on costly signaling to the outcome where agents are randomly matched. Random matching can also be seen as the outcome of a completely pooling equilibrium where nobody signals. While the total matching output (or utility) generated through assortative matching is clearly larger than the one obtainable through random matching, assortative matching involves the cost of signaling efforts. The main questions we address are: 1) Under which conditions is the increase in total expected output achieved by assortative matching completely offset by the increased cost of signaling? 2) Which types prefer random matching, and which types prefer assortative matching with signaling?

6.1 Total welfare

Total welfare in random matching is given by:

$$W^r(n, k) = W_m^r(n, k) + W_w^r(n, k) = 2\min(n, k)EX \cdot EY \tag{9}$$

Comparing assortative and random matching in terms of total welfare, we obtain the following result:

Proposition 6 *1. Suppose that $n = k$. Total men's welfare in random matching is higher (lower) than total men's welfare under assortative matching based on signaling if the distribution of men's attributes F is IFR (DFR).*[38] *An analogous results holds for women. Thus random matching is welfare superior to assortative matching based on signaling if both F and G are IFR (DFR).*

2. For any $n \geq k$, assortative matching based on signaling is welfare superior to random matching if F and G are DFR.

[38] This result can be generalized to distributions that have an increasing (decreasing) *average* failure rate.

To understand the intuition, consider the case of $n = k$ and where the distribution of men's attributes is exponential. Then, men's aggregate marginal product in the match of a woman with a given type is constant in her rank (see Section 4): while $n - i + 1$, the number of men affecting the output of a given woman, is decreasing in her rank i, this effect is exactly offset by the higher expected marginal productivity of higher ranks, $EX_{(i,n)} - EX_{(i-1,n)}$ (see Theorem 2). That is, all men's normalized mean spacings are equal to each other, and hence equal to the mean EX. Thus, men's total welfare under assortative matching equals their welfare in random matching:

$$W_m(n) = \sum_{i=1}^{n} (n - i + 1) \left(EX_{(i,n)} - EX_{(i-1,n)} \right) EY_{(i,n)} = nEX \cdot EY = W_m^r(n).$$

Consider now a decrease in the heterogeneity of men (while keeping the same mean), which yields an *IFR* distribution of men's attributes. While there is no effect on the men's welfare in random matching, men's welfare in assortative matching goes down: The new, decreasing vector of normalized spacings now majorizes the constant vector, and men's total welfare is a Schur concave function of the normalized spacings of men's mean order statistics (see Section 4). Put differently, when men's heterogeneity is reduced, men's aggregate marginal product in the match with a given woman is decreasing in her rank: the higher expected marginal productivity of higher ranks, $EX_{(i,n)} - EX_{(i-1,n)}$, is now outweighed by the number effect. Since higher ranks have a higher marginal contribution to men's total welfare, men's total welfare must be smaller under the *IFR* distribution. An analogous argument yields the result for the *DFR* case.

Part 2 utilizes the observation of Proposition 4 on entry to extend the comparison to the case where $n \geq k$. This is relatively straightforward since an increase in n does not increase the number of pairs, and hence does not affect expected output in random matching.

6.2 Individual welfare

We now compare assortative matching and random matching from each agent's point of view. Obviously, agents with low types prefer random matching since in that case they expect relatively good (i.e., average) partners without incurring any cost, while assortative matching provides then with low-type partners having wasted some resources on signaling.

Lemma 1 *For any distributions F and G, and for any $n \geq k$, there exists at most one cutoff type $\hat{x} \in [0, \tau_F]$ such that all men $x < \hat{x}$ are better-off under random matching, while all men $x \geq \hat{x}$ are better-off under assortative matching based on signaling. An analogous result holds for women.*

From Proposition 6 we know that assortative matching based on signaling yields a higher total welfare than random matching if F is *DFR*. Together with Lemma 1, this implies that, if F is *DFR*, there must exist some types who actually prefer assortative matching with signaling to random matching, i.e., the cutoff point is interior. Suppose that F is not *DFR*. Is it then possible that all agents, including those with high types, are better-off under random matching? The answer is affirmative:

Proposition 7 *Let $n = k$, and assume that $\tau_F < \infty$. For any G, if F stochastically dominates the uniform distribution on $[0, \tau_F]$, then all types of men prefer random matching to assortative matching based on signaling.*[39] *Analogous results hold for women.*

Recall that in the Spence model with two types of workers and homogenous firms, the workers' payoffs in the separating equilibrium are independent of the distribution of worker types.[40] Changes in the distribution only affect the pooling equilibrium wage, which is equal to the workers' expected marginal product. It is known for that model that workers prefer the pooling equilibrium to the separating one if the fraction of high-ability workers is sufficiently high. In contrast, payoffs in both pooling and separating equilibria depend here on the distributions of types, and the comparison between the two is somewhat subtler.

An intriguing situation arises when, say, the distribution of men is *IFR* while the distribution of women is *DFR*. Then, by Proposition 6, at least at the aggregate level there are conflicting interests: men prefer random matching, while women prefer assortative matching based on signaling. If all men prefer random matching (a situation that was shown above to be plausible) then, a concerted measure to "forbid" signaling will be unanimously accepted be men. This will lead to a collapse of the signaling equilibrium, with negative effects on the total welfare of women.

7 Large populations

We now consider a model where there are measures of men and women, distributed according to F and G, respectively. This is the limit case obtained when the number of agents in the finite, discrete model becomes very large. We focus on several similarities and differences between this model and the finite model analyzed so far. The leeway between perfect competition in the continuum version versus less than perfect competition in the finite version plays an important role.

We first normalize both measures of agents to one. Random matching yields now an expected total output (and welfare) of $2EXEY$. Under assortative matching, a man with attribute x is

[39] If F is stochastically dominated by the uniform distribution, then some types of men prefer random matching to assortative matching with signaling.

[40] For a generalization to a continuum of workers' types, see Mailath (1987).

matched with a woman with attribute $y = \psi(x)$, where $\psi(x) = G^{-1}F(x)$. Let $\varphi = \psi^{-1}$. Expected total output under assortative matching is given by $2\int_0^{\tau_F} x\psi(x) f(x)dx$. The signaling equilibrium that enables assortative matching is characterized next:

Proposition 8 *1. In the continuum model, the equilibrium signaling function for men and women are given by* $\beta(x) = \int_0^x z\psi'(z)\,dz$ *and* $\gamma(y) = \int_0^y z\varphi'(z)\,dz$ *, respectively.*

2. In each matched pair, exactly half the output from assortative matching is wasted through signaling in this equilibrium.

3. Men's total signaling is larger (smaller) than women's total signaling if the matching function $\psi = G^{-1}F$ *is convex (concave).*

The proof of Proposition 8 establishes close relations between signaling, net welfare and stable (i.e., core) payoffs for each matched pair.[41] These type of relations are more general, and hold for any fixed sharing rule among partners. Note that half the total output is dissipated through signaling for any level of heterogeneity because each agent shares half of the incremental rents from winning a better partner with his/her matching partner, while completely dissipating the other half in the contest with agents on the same side. Thus, the dissipated rents on both sides add up to half the output. This insight changes accordingly for asymmetric production functions if other fixed sharing rules are used. The last part of the above proposition is the analog to Proposition 5 about heterogeneity differentials in the discrete model.

7.1 Assortative versus random matching

We now turn to the welfare comparison between random matching and assortative matching based on signaling in the continuum model. Applying Proposition 8, we obtain:

Proposition 9 *1. Assortative matching based on signaling is welfare superior (inferior) to random matching if and only if*

$$\frac{Cov(X, \psi(X))}{EX \cdot E\psi(X)} \geq (\leq)\ 1$$

[41] Costrell and Loury (2004) and Suen (2007) study how core payoffs on one side of the market vary with the distribution of attributes on the other side. Hopkins and Kornienko (2005) and Hopkins (2005) study markets with a continuum of workers and firms. While the quality of firms is observable, workers exert effort in order to signal their quality to firms. Their results distinguish the effects on equilibrium behavior of workers caused by changes in the distribution of workers' attributes from those caused by changing the distribution of firms' attributes.

2. *Let $F = G$, and let $CV(X) = \sqrt{Var(X)}/EX$ be the common coefficient of variation. Assortative matching based on signaling is welfare superior (inferior) to random matching if and only if $CV(X) \geq (\leq)\ 1$.*[42] *In particular, assortative matching based on signaling is welfare superior (inferior) to random matching if F is DFR (IFR).*

When the coefficient of (co)variation is greater [smaller] than unity, the increase in total output generated through assortative matching (relative to the random output) outweighs [is outweighed by] the increase in the costs of signaling. Note that the difference in total output between assortative matching and random matching gets smaller as the degree of heterogeneity in the population decreases, while signaling remains proportional to output for any degree of heterogeneity. Hence, as the coefficient of variation gets smaller, total welfare achieved by assortative matching eventually falls below the level achieved by random matching.

The last part of Proposition 9 follows since F being *DFR (IFR)* implies that its coefficient of variation is larger (smaller) than unity (see Barlow and Proschan, 1965).[43]

Notice that the converse relation between coefficients of variation and monotone hazard rates is *not true*: a coefficient of variation larger (smaller) that unity does not necessarily imply that the respective distribution is *DFR (IFR)*. As a consequence, the "if and only if" result for the continuum model applies to a much larger class of distributions than those covered by the weaker "if" result of Proposition 6-1 for the discrete model. This discrepancy illustrates that "backward extrapolation" from the continuum market to small, finite markets may yield wrong insights and predictions if the distributions of attributes are not IFR or DFR. This justifies the need for the precise (if somewhat more tedious) analysis of the finite model.

Example 2 *Let $F = G = x^{\frac{9}{20}}$ and let $\tau_F = \tau_G = 1$. We obtain that $CV \simeq 0.95 < 1$. For the continuum model, random matching is welfare-superior to assortative matching by Proposition 9-2. For the discrete model with an equal finite number of men and women, Proposition 6-1 is not applicable since F is not IFR. It turns out that random matching is welfare superior (inferior) to assortative matching if the number of pairs is at least (strictly below) ten.*

The results of this section have been obtained by direct arguments applied to the continuum model. Note that it is also possible to explicitly take limits in the discrete model: per-capita

[42] Total welfare in random matching equals total welfare in assortative matching for the exponential distribution ($CV(X) = 1$). In a setting where the attributes of one side of the market are known, signaling is one-sided and the waste from signaling is halved. Assortative matching is then welfare superior (inferior) to random matching if $CV(X) \geq (\leq) \sqrt{1/3}$. The alternatives are equivalent for the uniform distribution.

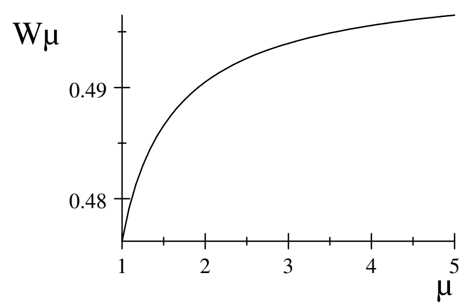
[43] This result holds more generally, for distributions with increasing (decreasing) average failure rates.

output and signaling effort in the discrete model converge to their continuous counterparts when the number of agents goes to infinity.[44] This immediately yields that, in the limit model, assortative matching based on signaling is welfare superior (inferior) to random matching if F *is DFR (IFR),* but it cannot yield the stronger "if and only if" result involving the coefficient of (co)variation.

7.2 Asymmetric market sizes

To deal with asymmetric market sizes in the continuum case, start with F and G, two distributions with normalized mass of one. Consider men's distributions having the form $F_\mu(x) = \mu F(x)$ where $\mu \geq 1$, and let $a_\mu = F_\mu^{-1}(\mu - 1)$. If the populations represented by F_μ and G are assortatively matched, then men with attributes in $[a_\mu, \tau_F]$ obtain women with attributes in $[0, \tau_G]$ according to the matching function $\psi_\mu(x) = G^{-1}(F_\mu(x) - \mu + 1)$. The equilibrium derivations are then analogous to those in Proposition 8.

Example 3 *As in Example 1, let* $F(x) = x^{10}$, $G(y) = y$, $\tau_F = \tau_G = 1$ *. Let* W_μ *denote total welfare as a function of* $\mu \geq 1$*. Recall that in Example 1 we fixed the number of women (and thus of pairs) to be three, and note that* $\lim_{\mu\to\infty} W_\mu = \lim_{n\to\infty} \frac{1}{3} W(n,3) = 0.5$*. Whereas in the discrete setting entry on the men's side initially reduced welfare,* W_μ *is here monotonically increasing in* μ*.*



Note that in a small, finite population agent i faces a considerable uncertainty about the type of the same-side agent ranked just below i: there is a positive probability that this type is much lower than i's own type. In the continuum model, almost perfect substitutes to i are always present. As a consequence, total signaling efforts are lower relative to total expected output in the discrete model. Faced with entry, the amount of signaling may initially rise faster than total output in a small, finite market, while it is very tightly related to output in the perfectly competitive continuum market.

[44] Peters (2004) studies the limit, as the number of agents goes to infinity, of mixed strategy equilibria arising in a complete-information model where a finite number of agents on each side of the market make costly investments prior to the match. In his model, the limit need not correspond to the hedonic equilibrium in the market with a continuum of agents.

8 Conclusion

We have studied two-sided matching models where privately informed agents on each side are matched on the basis of costly signals. Our analysis revealed how output, signaling, and welfare (net of signaling costs) are affected on both sides of the market by changes in primitives of the model such as the number of the agents and the distributions of their attributes, and it yielded a new, heterogeneity based, explanation for the often observed asymmetry in signaling activity. We have also identified conditions under which assortative matching based on wasteful signaling is welfare superior (inferior) to random matching. Thus, the effects of policies that attempt to curb "wasteful" signaling need to be carefully examined in each particular situation.[45]

The analysis of markets with finite numbers of agents has revealed several phenomena that are particular to such markets, and do not occur in very large ones. This analysis been made possible by the application of results and methods from mathematical statistics. We believe that the applications of these methods will be fruitful also in other areas, such as double auctions. There are also many possible extensions of our model. For example, one might introduce productive efforts, adding value to the investor's attribute. While all agents will then choose the same effort under random matching, they will still need to invest excessively in order to credibly signal their types and be matched assortatively, similarly as in our model. Such investments will, however, tend to further increase the expected output from assortative matching (relative to the random output). We leave the precise characterization of this trade-off for future research. Finally, we hope that our model (or some of its variations) will be useful as a sound, theoretical basis around which to organize observations in a variety of empirical studies, e.g., of marriage, labor and education markets.

[45] Alternatively, this holds for policies that attempt to manipulate the rent accruing to a third party, such as an intermediary (see Hoppe, Moldovanu, Ozdenoren, 2006).

9 Appendix A: Order statistics and stochastic orders

Definition 3 *For any two non-negative random variables, X and Z, with distributions F and H and hazard rates λ_x and λ_z, respectively, X is said to be smaller than Z in the hazard rate order (denoted as $X \leq_{hr} Z$) if $\lambda_x(s) \geq \lambda_z(s)$, for all $s \geq 0$. X is said to be smaller than Z in the usual stochastic order (denoted as $X \leq_{st} Z$) if $F(s) \geq H(s)$ for all $s \geq 0$.*

Theorem 3 *(see Shaked and Shanthikumar ,1994):*

1. *If X and Z are two random variables such that $X \leq_{hr} Z$, then $X \leq_{st} Z$.*
2. *Let $X_{(1,n)} \leq X_{(2,n)} .. \leq X_{(n,n)}$ denote the order statistics from n independent random variables identical to X. Then:*
   - $X_{(i,n)} \leq_{hr} X_{(i+1,n)}$ *for* $i = 1, 2, ..., n-1$,
   - $X_{(i-1,n-1)} \leq_{hr} X_{(i,n)}$ *for* $i = 2, 3, ..., n$
   - $X_{(i,n-1)} \geq_{hr} X_{(i,n)}$ *for* $i = 2, 3, ..., n-1$.

An important consequence of the single crossing of random variables ordered by the star-shaped order is:

Theorem 4 *(Barlow and Proschan, 1966) Let X, Z two random variables with distributions F, H respectively, such that $F(0) = H(0) = 0$, and such that $H^{-1}F$ is convex. Then the ratio $EX_{(i,n)}/EZ_{(i,n)}$ is decreasing (increasing) in i (n).*[46]

We also use the following generalization of Barlow and Proschan's results:

Theorem 5 *(Hu and Wei, 2001) Define $U_{(j,i,n)} \equiv X_{(j,n)} - X_{(i,n)}$ for $0 \leq i < j \leq n$. Let F be DFR (IFR). Then $U_{(j-1,i-1,n-1)} \leq_{hr} (\geq_{hr}) U_{(j,i,n)}$.*

10 Appendix B: Proofs

Proof of Proposition 1. We first show that the function β in (1) is strictly monotonically increasing. Note that (1) can be written as

[46] This result holds more generally, for distributions ordered in the star sense.

$$\begin{aligned}\beta(x) &= \int_0^x \left\{ \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s) \left[EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right] \right\} sds \\ &+ \int_0^x f_{(n-k,n-1)}(s) \, EY_{(1,k)} sds\end{aligned}$$

Taking the derivative with respect to x yields

$$\begin{aligned}\beta'(x) &= \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(x) \left[EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)} \right] x \\ &+ f_{(n-k,n-1)}(x) \, EY_{(1,k)} x\end{aligned}$$

which is strictly positive because $Y_{(k-n+i+1,k)} \geq_{st} Y_{(k-n+i,k)}$ by Theorem 3-2 (Appendix A).

Next, we check whether the second-order condition is satisfied. Integrating the RHS of (1) by parts, yields

$$\begin{aligned}\beta(x) &= x \sum_{i=n-k+1}^{n-1} EY_{(k-(n-i),k)} \left[F_{(i-1,n-1)}(x) - F_{(i,n-1)}(x) \right] \\ &+ xF_{(n-1,n-1)}(y) \, EY_{(k,k)} \\ &- \int_0^x \left\{ \sum_{i=n-k+1}^{n-1} EY_{(k-(n-i),k)} \left[F_{(i-1,n-1)}(s) - F_{(i,n-1)}(s) \right] \right\} ds \\ &- \int_0^y F_{(n-1,n-1)}(s) \, EY_{(k,k)} ds \\ &= x \sum_{i=n-k+1}^{n} F_i^n(x) \, EY_{(k-(n-i),k)} - \int_0^x \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s) \, ds\end{aligned}$$

Let $z = \beta^{-1}(b)$ be the type for which the equilibrium effort is b. The expected payoff of a man with type x from exerting effort $\beta(z)$ is thus given by:

$$\begin{aligned}U(b,x) &= \sum_{i=n-k+1}^{n} \left[F_{(i-1,n-1)}(z) - F_{(i,n-1)}(z) \right] xEY_{(k-(n-i),k)} - \beta(z) \\ &= \sum_{i=n-k+1}^{n} F_i^n(z) \, EY_{(k-(n-i),k)} (x-z) \\ &+ \int_0^z \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s) \, ds\end{aligned}$$

Hence, the difference between the expected payoffs of type x when he exerts efforts of $\beta(x)$ and $\beta(z)$ is:

$$U(\beta(x), x) - U(\beta(z), x) = \begin{array}{l} \sum_{i=n-k+1}^{n} F_i^n(z)\, EY_{(k-(n-i),k)}(z-x) \\ -\int_x^z \sum_{i=n-k+1}^{n} EY_{(k-(n-i),k)} F_i^n(s)\, ds \end{array} \tag{10}$$

Since β is strictly increasing, the function $H(s) = \sum_{i=n-k+1}^{n} F_i^n(s)\, EY_{(k-(n-i),k)}$ increases in s and therefore the difference in (10) is always positive. ■

Proof of Proposition 2. **1)** Substituting (1) into (3) yields:

$$\begin{aligned} S_m(n,k) &= n\int_0^{\tau_F}\int_0^x \sum_{i=n-k+1}^{n-1} f_{(i-1,n-1)}(s)\, EY_{(k-(n-i),k)} s ds f(x)\, dx \\ &\quad -n\int_0^{\tau_F}\int_0^x \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s)\, EY_{(k-(n-i),k)} s ds f(x)\, dx \\ &\quad +n\int_0^{\tau_F}\int_0^x f_{(n-1,n-1)}(s)\, EY_{(k,k)} s ds f(x)\, dx \end{aligned} \tag{11}$$

Integrating the first plus the third terms of (11) by parts and rearranging terms, we obtain

$$\begin{aligned} & n\int_0^{\tau_F}\int_0^x \sum_{i=n-k+1}^{n} f_{(i-1,n-1)}(s)\, EY_{(k-(n-i),k)} s ds f(x)\, dx \\ = \; & n\int_0^{\tau_F} [1-F(x)] \sum_{i=n-k+1}^{n} x f_{(i-1,n-1)}(x)\, EY_{(k-(n-i),k)} dx \\ = \; & n\int_0^{\tau_F} \sum_{i=n-k+1}^{n} \frac{n-i+1}{n} x f_{(i-1,n)}(x)\, EY_{(k-(n-i),k)} dx \\ = \; & \sum_{i=n-k+1}^{n} (n-i+1)\, EX_{(i-1,n)} EY_{(k-(n-i),k)} \end{aligned}$$

Similarly, integrating the second term of (11) by parts, we obtain

$$
\begin{aligned}
& -n\int_0^{\tau_F}\int_0^x \sum_{i=n-k+1}^{n-1} f_{(i,n-1)}(s)\, EY_{(k-(n-i),k)} s ds f(x)\, dx \\
= & -n\int_0^{\tau_F} [1-F(x)] \sum_{i=n-k+1}^{n-1} x f_{(i,n-1)}(x)\, EY_{(k-(n-i),k)} dx \\
= & -n\int_0^{\tau_F} \sum_{i=n-k+1}^{n-1} \frac{n-i}{n} x f_{(i,n)}(x)\, EY_{(k-(n-i),k)} dx \\
= & \sum_{i=n-k+1}^{n} (n-i)\, EX_{(i,n)} EY_{(k-(n-i),k)}
\end{aligned}
$$

Collecting terms, yields:

$$
\begin{aligned}
S_m(n,k) = & \sum_{i=n-k+1}^{n} \left[(n-i+1)\, EX_{(i-1,n)} - (n-i)\, EX_{(i,n)}\right] EY_{(k-(n-i),k)} \\
= & \sum_{i=n-k+1}^{n} (n-i+1)\, EX_{(i-1,n)} \left(EY_{(k-n+i+1,k)} - EY_{(k-n+i,k)}\right) \qquad (12)
\end{aligned}
$$

Men's total welfare follows from the definition of gross surplus and (12).

2) Analogous to the above.

3) Note that the only difference in the expressions for $W(n,k)$ on the one hand, and $S_m(n,k) + S_w(n,k)$ on the other, is that the normalized spacings appearing in $W(n,k)$ are multiplied by a higher weight, corresponding to the expectation of a higher order statistic. Thus

$$
\begin{aligned}
S_m(n,k) + S_w(n,k) & \leq W(n,k) \Leftrightarrow \\
W(n,k) + S_m(n,k) + S_w(n,k) & \leq 2W(n,k) \Leftrightarrow \\
2\sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} & \leq 2W(n,k) \Leftrightarrow \\
\sum_{i=n-k+1}^{n} EX_{(i,n)} EY_{(k-(n-i),k)} & \leq W(n,k) \qquad (13)
\end{aligned}
$$

as desired. ■

Proof of Proposition 4. **1)** For $j = i-(n-k)$,we rewrite total men's signaling as:

$$
S_m(n,k) = \sum_{j=1}^{k} (k-j+1)\left(EY_{(j,k)} - EY_{(j-1,k)}\right) EX_{(j+n-k-1,n)} \qquad (14)
$$

The result follows since $EY_{(j,k)} \geq EY_{(j-1,k)}$ and since $EX_{(j+n-k-1,n)}$ is stochastically increasing in n (see Theorem 3, Appendix A).

2) For $j = i - (n-k)$, we rewrite total women's signaling as:

$$S_w(n,k) = \sum_{j=1}^{k} (k-j+1) \left(EX_{(j+n-k,n)} - EX_{(j+n-k-1,n)} \right) EY_{(j-1,k)} \tag{15}$$

The result follows by Theorem 5 (Appendix A).

3) Men's total welfare is given by:

$$W_m(n,k) = \sum_{j=1}^{k} (k-j+1) \left(EX_{(j+n-k,n)} - EX_{(j+n-k-1,n)} \right) EY_{(j,k)} \tag{16}$$

which is is similar to women's signaling (expression (15)). The proof is analogous to that of point 2 above, and we omit it here.

4) Women's total welfare is given by :

$$W_w(n,k) = \sum_{j=1}^{k} (k-j+1) \left(EY_{(j,k)} - EY_{(j-1,k)} \right) EX_{(j+n-k,n)} \tag{17}$$

which is is similar to men's signaling (expression (14)). The proof is analogous to that of point 1 above, and we omit it here. ■.

Proof of Proposition 5. **1)** Using (3) and (5), we get:

$$S_m(n,n) - S_w(n,n) = \sum_{i=1}^{n} (n-i+1) \left(EY_{(i,n)} EX_{(i-1,n)} - EY_{(i-1,n)} EX_{(i,n)} \right) \leq 0$$

The last inequality follows from Theorem 4 (Appendix A) which says that the ratio $EY_{(i,n)}/EX_{(i,n)}$ is decreasing in i .

2) From Proposition 4 we know that:

(i) for any $n \geq k$, and any F, G, $S_m(n,k) \geq S_m(k,k)$,

(ii) for any $n \geq k$, for any G, and for F IFR, $S_w(n,k) \leq S_w(k,k)$

Exchanging the roles of men and women, the result follows directly from **1)** and (i), (ii) if F is IFR. Assume now that G is IFR. This means that $H^{-1}G$ is convex, where H is the exponential distribution. Thus, $H^{-1}GG^{-1}F = H^{-1}F$ is convex (since it is a composition of increasing convex functions), which means that F is IFR. The result follows then as above. ■

Proof of Proposition 6 **1)** Welfare in random matching can be written as:

$$\begin{aligned} W^r(n,n) &= 2nEX \cdot EY = nEX\frac{\sum_{i=1}^n EY_{(i,n)}}{n} + nEY\frac{\sum_{i=1}^n EX_{(i,n)}}{n} \\ &= EX\sum_{i=1}^n EY_{(i,n)} + EY\sum_{i=1}^n EX_{(i,n)} \end{aligned} \tag{18}$$

Welfare in assortative matching is given by (7). Let now $a_1 \geq a_2.. \geq a_n$. *If* F *is IFR (DFR)* then a simple consequence of Theorem 1-3 is :

$$\sum_{i=1}^n a_i[(n-i+1)(EX_{(i,n)} - EX_{(i-1,n)})] \geq (\leq) EX\sum_{i=1}^n a_i$$

Let $a_i = -EY_{(i,n)}$, and note that a_i is decreasing in i. If is IFR (DFR), this yields

$$-\sum_{i=1}^n EY_{(i,n)}[(n-i+1)\left(EX_{(i,n)} - EX_{(i-1,n)}\right)] \geq (\leq) - EX\sum_{i=1}^n EY_{(i,n)} \tag{19}$$

Multiplying by (-1), we obtain: if F is IFR (DFR), then

$$\sum_{i=1}^n EY_{(i,n)}[(n-i+1)\left(EX_{(i,n)} - EX_{(i-1,n)}\right)] \leq (\geq) EX\sum_{i=1}^n EY_{(i,n)} \tag{20}$$

This is the wished result for men's welfare. Similarly, we obtain

$$\sum_{i=1}^n EX_{(i,n)}[(n-i+1)\left(EY_{(i,n)} - EY_{(i-1,n)}\right)] \leq (\geq) EX\sum_{i=1}^n EX_{(i,n)} \tag{21}$$

if G is IFR (DFR). The combination of (20) and (21) completes the proof for total welfare.

2) The result for the general case $n \geq k$ follows by applying the entry results of Proposition 4: Recall that, by Proposition 4, entry by men increases welfare in assortative matching based on signaling if F is DFR. The result follows by noting that entry on the long side does not affect welfare from random matching since the number of matched pairs remains constant. ■

Proof of Lemma 1 Let $U^a(x), U^r(x)$ denote the expected net utility of type x under assortative matching with signaling, and under random matching, respectively.

Note that $U^a(x) = \max_s \left\{\sum_{i=n-k+1}^n F_i^n(s)\, xEY_{(k-n+i,k)} - \beta(s)\right\}$ is an increasing convex function (since it is the maximum of linear increasing functions), while U^r is an increasing linear function with slope EY. Thus, these functions can cross at most once. Note further that the derivative of $U^a(x)$ at $x = 0$ is

$$\left.\frac{dU^a(x)}{dx}\right|_{x=0} = \sum_{i=n-k+1}^n F_i^n(0)\, EY_{(k-n+i,k)} \leq EY_{(1,k)} < EY$$

where the first inequality follows either by $\sum_{i=n-k+1}^{n} F_i^n(0)\, EY_{(k-n+i,k)} = 0$ if $n > k$, (since $F_i^n(0) = 0$ if $i > 1$) or by $\sum_{i=n-k+1}^{n} F_i^n(0)\, EY_{(k-n+i,k)} \leq EY_{(1,n)}$ for $n = k$, (since $F_1^n(0) = \lim_{\varepsilon\to 0} F(\varepsilon)^{\varepsilon} \leq 1$). Thus, $U^a(x) \leq U^r(x)$ in a neighborhood of zero, and the wished result follows. ■

Proof of Proposition 7 By Lemma 1, it is clear that if the man with the highest type prefers random matching, then all other types of men prefer random matching as well (and analogously for women). Under assortative matching based on signaling, the expected utility of the type τ_F man is

$$U^a(\tau_F) = \tau_F EY_{(n,n)} - \sum_{i=1}^{n-1} EX_{(i,n-1)} \left(EY_{(i+1,n)} - EY_{(i,n)}\right)$$

The expected utility of this type under random matching is

$$U^r(\tau) = \tau_F EY = \frac{\tau_F}{n}\left(\sum_{i=1}^{n} EY_{(i,n)}\right)$$

If F stochastically dominates (is stochastically dominated by) the uniform distribution, we obtain that $EX_{(i,n-1)} \geq (\leq) \tau_F \frac{i}{n}$. Then

$$\begin{aligned} U^a(\tau_F) &\leq (\geq)\, \tau_F EY_{(n,n)} - \frac{\tau_F}{n} \sum_{i=1}^{n-1} i \left(EY_{(i+1,n)} - EY_{(i,n)}\right) \\ &= \frac{\tau_F}{n}\left(\sum_{i=1}^{n} EY_{(i,n)}\right) = U^r(\tau_F) \quad \blacksquare \end{aligned}$$

Proof of Proposition 8 **1.** Consider men's types $x, \hat{x}$, $x > \hat{x}$, with equilibrium bids $\beta(x)$, $\beta(\hat{x})$. In equilibrium, type x is assortatively matched with type $\psi(x)$, and $\hat{x}$ is matched with $\psi(\hat{x})$. Type x should not pretend that he is $\hat{x}$ (thus being matched with $\psi(\hat{x})$ and paying $\beta(\hat{x})$), and vice-versa for type $\hat{x}$. This yields:

$$\begin{aligned} x\psi(x) - \beta(x) &\geq x\psi(\hat{x}) - \beta(\hat{x}) \\ \hat{x}\psi(\hat{x}) - \beta(\hat{x}) &\geq \hat{x}\psi(x) - \beta(x) \end{aligned}$$

Combining the above and dividing by $x - \hat{x}$, gives:

$$\frac{\hat{x}\psi(x) - \hat{x}\psi(\hat{x})}{x - \hat{x}} \leq \frac{\beta(x) - \beta(\hat{x})}{x - \hat{x}} \leq \frac{x\psi(x) - x\psi(\hat{x})}{x - \hat{x}}$$

Taking the limit $\hat{x} \to x$ gives $\beta'(x) = x\psi'(x)$. Together with $\beta(0) = 0$, this yields $\beta(x) = \int_0^x z\psi'(z)\, dz$. Letting $\varphi = \psi^{-1}$, we analogously obtain $\gamma(y) = \int_0^y z\varphi'(z)\, dz$.[47]

[47] The results of Mailath (1987) can be applied to show that the second order conditions are satisfied, and that these are indeed equilibrium signaling strategies.

2. It is well known that the unique stable (i.e., core) payoff configuration for our two-sided market with a continuum of agents (and with complete information) is given by:

$$w(x) = 2\int_0^x \psi(t)dt;\ v(y) = 2\int_0^y \varphi(t)dt$$

Since in the core there are no transfers outside matched pairs, it must hold that

$$\forall x,\ w(x) + v(\psi(x)) = 2x\psi(x)$$

By the above calculations, we know that

$$\beta(x) = \int_0^x t\psi'(t)dt = x\psi(x) - \int_0^x \psi(t)dt$$

and similarly for women. This yields:

$$x\psi(x) = \beta(x) + \frac{1}{2}w(x) = \gamma(\psi(x)) + \frac{1}{2}v(\psi(x))$$

Therefore

$$\beta(x) + \gamma(\psi(x)) = 2x\psi(x) - \frac{1}{2}[w(x) + v(\psi(x)] = x\psi(x)$$

as claimed.

3. Comparing total signaling on the two sides of the market (after integrating by parts) yields:

$$\begin{array}{rc} & S_m(\infty) \geq (\leq)\ S_w(\infty) \\ \Leftrightarrow & \int_0^{\tau_F} x\psi'(x)\frac{1-F(x)}{f(x)}dF(x) \leq (\geq) \int_0^{\tau_F} \psi(x)\frac{1-F(x)}{f(x)}dF(x) \\ \Leftrightarrow & \int_0^{\tau_F}\left[\psi(x) - x\psi'(x)\right]\frac{1-F(x)}{f(x)}dF(x) \leq (\geq)\ 0 \end{array}$$

The result follows by noting that $\psi(x) - x\psi'(x) < (>)\, 0$ if $\psi(x)$ is convex (concave). ■

Proof of Proposition 9 By Proposition 8, total net welfare in assortative matching based on signaling is given by $\int_0^{\tau_F} x\psi(x) f(x)dx$. Thus, assortative matching with signaling is welfare superior (inferior) to random matching if:

$$\begin{aligned} \int_0^{\tau_F} x\psi(x) f(x)dx &\geq [\leq]\ 2\int_0^{\tau_F} xf(x)\,dx \int_0^{\tau_F} yg(y)dy \Leftrightarrow \\ E(X\psi(X)) &\geq [\leq]\ 2EX\cdot EY \Leftrightarrow \\ E(X\psi(X)) &\geq [\leq]\ 2EX\cdot E\psi(X) \Leftrightarrow \\ \frac{Cov(X\psi(X))}{EX\cdot E\psi(X)} &\geq [\leq]\ 1 \end{aligned}$$

(Note that $EY = E\psi(X)$; the proof uses the well-known fact that for any random variable Z with cumulative distribution H, $EZ = \int_0^1 H^{-1}(z)\,dz$) ■.

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