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THE THEORY OF FIGURAL CONCEPTS

ABSTRACT. The main thesis of the present paper is that geometry deals with mental entities (the so-called geometrical figures) which possess simultaneously conceptual and figural characters. A geometrical sphere, for instance, is an abstract ideal, formally determinable entity, like every genuine concept. At the same time, it possesses figural properties, first of all a certain shape. The ideality, the absolute perfection of a geometrical sphere cannot be found in reality. In this symbiosis between concept and figure, as it is revealed in geometrical entities, it is the image component which stimulates new directions of thought, but there are the logical, conceptual constraints which control the formal rigour of the process. We have called the geometrical figures *figural concepts* because of their double nature. The paper analyzes the internal tensions which may appear in figural concepts because of this double nature, development aspects and didactical implications.

The Notion of Figural Concept

Concepts and mental images are usually distinguished in current psychological theories. Piéron, in his "Vocabulaire de la Psychologie", defines a concept in the following way: "Symbolic representation (almost always verbal) used in the process of abstract thinking and possessing a general significance corresponding to an ensemble of concrete representations with regard to what they have in common" (Piéron, 1957, p. 72). What then characterizes a concept is the fact that it expresses *an idea*, a general, ideal representation of a class of objects, based on their common features.

In contrast, an image (we refer here to mental images) is a *sensorial* representation of an object or phenomenon. The concept of metal is the *general idea* of a class of substances having in common a number of properties: electrically conductive, etc. The *image* of a metallic object is the *sensorial* representation of the respective object (including color, magnitude, etc.).

In all the actual cognitive theories, concepts and images are considered two basically distinct categories of mental entities. Even the propositional theory – according to which both types of information are finally encoded in the same propositional format – refers to images and concepts as two distinct types of mental entities.

But let us consider the following example: consider the isosceles triangle ABC with $AB = AC$ (Figure 1). We want to prove that $\angle B = \angle C$. We may imagine the following proof: let us consider that one detaches the triangle from itself, one reverses it such that AC is on the left side and AB on the right side, and one superposes the reversed triangle on the original one. The angle A remaining the same and AB and AC having the same length, AC will coincide perfectly

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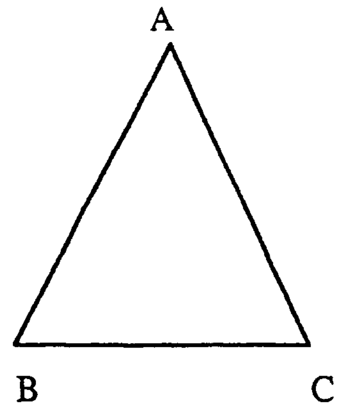

Fig. 1.

with AB on the left side and AB and AC will coincide perfectly on the right side. Then the reversed and the original triangle will coincide perfectly. As a consequence, the angles $\angle B$ and $\angle C$ must be equal. Q.E.D.

In this proof one has used a certain amount of knowledge expressed conceptually: the two sides AB and AC have been declared to be equal. One has used the concepts of point, side, angle and triangle. One has mentioned verbally the process of reversion. But, at the same time, one has used figural information and figurally represented operations – mainly the idea of *detaching* the triangle ABC from itself, reversing it and superposing it upon the original one.

Are we dealing here with a mixture of two independent, defined entities, that is abstract ideas (concepts), on one hand, and sensorial representations reflecting some concrete operations, on the other?

Let us consider the core of the proof, that is the operation of detaching the triangle ABC from itself and of reversing it. Concepts cannot be detached, reversed and matched. We deal here with descriptions of apparently practical operations. But in *reality*, is it possible to detach an object from itself? Certainly not. Such an operation has no concrete meaning. We deal with an ideal world, with ideal meanings. The objects to which we refer – points, sides, angles and the operations with them – have only an ideal existence. They are of a conceptual nature. *At the same time*, they have an *intrinsic* figural nature: only while referring to images one may consider operations like detaching, reversing or superposing.

As a matter of fact, the triangle to which we refer and its elements cannot be considered either pure concepts or mere common images. The operations mentioned above could not have been performed either with pure concepts or with real objects. Nevertheless, these entities and operations participated in a formal, logical proof, mathematically valid and, at the same time, the conclusion, the equality of the angles $\angle B$ and $\angle C$, may be checked practically.

The entities to which we have referred above – points, sides (line segments), angles, the triangle itself, and the operations with them – possess conceptual qualities. In mathematical reasoning one does not refer to them as material objects or as drawings. The material objects – solids or drawings – are only materialized models of the mental entities with which the mathematician deals. Secondly, only in a conceptual sense one may consider the absolute perfection of geometrical entities: straight lines, circles, squares, cubes, etc.

Thirdly, these geometrical entities do not have genuine material correspondents. Points (zero-dimensional objects), lines (uni-dimensional objects), planes (bi-dimensional objects) do not exist, cannot exist in reality. The real objects of our practical experience are necessarily tri-dimensional. But even the cube or the sphere to which the mathematician refers do not exist in reality, though they are tri-dimensional. These also are mere mental constructs which are not supposed to possess any substantial reality whatsoever.

Fourth, all these constructs are *general* representations, like every concept, and never mental copies of particular, concrete objects. When you draw a *certain* triangle ABC on a sheet of paper in order to check some of its properties (for instance, the property of its heights to be concurrent) you do not refer to the respective particular drawing but to a certain *shape* which may be the shape of an infinite class of objects. Even the particular shape drawn by you with its given sides and angles may be the shape of an infinity of objects. As a matter of fact, we deal with a hierarchy of shapes, from an apparently particular one – but in fact corresponding to an infinity of possible objects – to the universal category of triangles. Ideality, abstractness, absolute perfection, universality are properties which make sense in the domain of concepts.

But there is a fifth property which characterizes the geometrical figures and which, also, is related to their conceptual nature. The properties of geometrical figures are imposed by, or derived from definitions in the realm of a certain axiomatic system. From this point of view, also, a geometrical figure has a conceptual nature. A square is not an image drawn on a sheet of paper. It is a shape controlled by its definition (though it may be inspired by a real object). A square is a rectangle having equal sides. Starting from these properties one may go on for discovering other properties of the square (the equality of angles which are all right angles, the equality of diagonals, etc.).

A geometrical figure may, then, be described as having *intrinsically* conceptual properties. Nevertheless, a geometrical figure is *not* a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property.

When conceptualizing, for instance, a wheel in order to describe its roundness, we may get not only the *idea* of roundness, not only the *image* of the wheel associated with it, but also a third type of construct which is the geometrical figure called circle. If one has to solve a problem in which one has to calculate, for instance, the distance covered by a vehicle, knowing the radius of the wheels, the number of rotations per time unit and the time spent, the computation is made considering an abstract model of the wheel which is neither a pure image nor a pure concept. Concepts do not turn, do not move, and images, as such, do not possess the perfection, the generalization, the abstractness, the purity which are

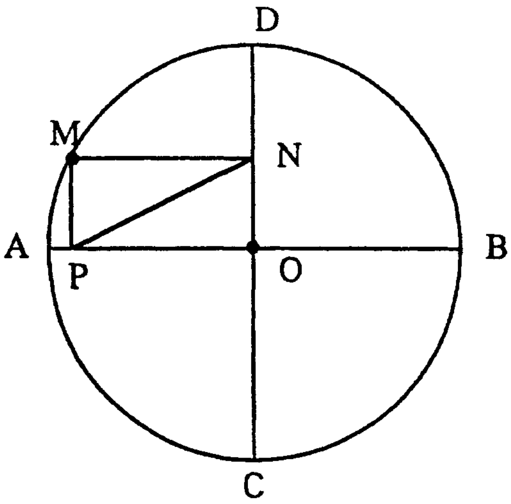

Fig. 2.

supposed when performing the calculations.

The triangle, the circle, the square, the point, the line, the plane, mentioned in the above examples and, in general, all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties.

Certainly, when we *imagine* a circle, we imagine a drawn circle (including, for instance, the color of the ink) and not the ideal, perfect circle. But the mathematical circle, which is the object of our mathematical reasoning, has no color, no material substance, no mass, etc. and it is supposedly ideally perfect. It has all the properties of a concept, it may participate, as it is, in a mathematical reasoning and this despite the fact that it still includes the representation of spatial properties.

Let us consider the following example: "In a circle with its center in C we draw two perpendicular diameters AB and CD. We chose arbitrarily a point M and we draw the perpendiculars MN and MP on the two diameters. What is the length of PN?"

At a first glance, it seems that the problem cannot be solved because the lengths of the segments MP and MN depend on the position of the point M. But, suddenly, one remarks that $MPON$ is a rectangle and that the segment MO is a diagonal of that rectangle. Consequently $PN = MO$ and MO is the radius of the circle. The equality of the diagonals is not questioned, the equality of the radiuses is not questioned. These relationships do not depend on the d:awing itself. They are imposed by definitions and theorems. The essential aspect we want to stress is that the conclusion is not drawn by considering separately the image and the formal constraints but by a unique process in which a distilled figure is considered, revealing logical relationships. We do not have to make any effort in order to "polish" the figure, to purify it – mentally – from its irregularities and impurities. The process of idealizing the figure takes place automatically so as to become an integral, active component of a strict logical reasoning.

The fact that we jump to the conclusion suddenly – $PN = MO$ = radius = constant – at the very moment when we have grasped the rectangle $PONM$, without an intervening investigation, supports the idea that the considered figure is, *from the beginning*, not an ordinary image but an already logically controlled structure. The *fusion between concept and figure tends to be, in this case, complete.*

The objects of investigation and manipulation in geometrical reasoning are then mental entities, called by us *figural concepts*, which reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities –like ideality, abstractness, generality, perfection.

I do not intend to affirm that the representation we have in mind, when imagining a geometrical figure, is devoid of any sensorial quality (like color) except space properties. But I affirm that, while operating with a geometrical figure, we act *as if no other quality counts.*

I ask myself: which shape will I get as a result of sectioning a cube with a plane through the diagonals of two opposite faces? The operation is easy to imagine. But *two* distinct mental realities have to be considered. One is the representation of a real cube (something like a wooden cube) and the operation of cutting it. It is a sensorial image like so many images which come into mind as an effect of our practical experience: the house in which I live, the room in which I use to work, representations of relatives, friends, students, etc.

Beyond that image there is another image not sensorially perceived but *thought*, the genuine object of our geometrical reasoning. This is the image to which we refer when performing a mathematical operation.

We are so used to distinguishing between images, as "pictures in the head", and concepts, i.e., general, non-sensorial ideas, that it is very difficult to accept a construct which would have, simultaneously, conceptual and imaginative spatial qualities.

It should be clear that the fusion between concept and figure in geometrical reasoning expresses only an ideal, extreme situation usually not reached absolutely because of psychological constraints. The history of mathematics is witnessing the complex dynamics of the process of conceptualizing and axiomatizing the figural information. Many of the axioms used by Euclid in his Elements, have never been stated explicitly by him. "As Gauss noted, Euclid spoke of points lying *between* other points and lines lying between other lines, but never treated the notion of betweenness and its properties" (cf. Kline, 1982, p. 102). It has been Moritz Pasch who, in the nineteenth century conferred a formal status to "betweenness" which previously was accepted as figurally based information.

In the following pages, we will encounter examples of conflictual phenomena taking place in the genesis of figural concepts in the individual.

THE INTERACTION BETWEEN IMAGES AND CONCEPTS

As a matter of fact, it is common to accept that, in the course of a productive reasoning process, images and concepts interact intimately. Shepard has quoted many introspective reports of scientists who describe the ways in which the discovery of a new idea has been based on imagery triggered by a theoretical investigation (Shepard, 1978). For instance, referring to Einstein's work, he writes: "Throughout, Einstein's work in theoretical physics was marked by an interplay between concrete perceptual visualization, on the one hand, and a relentless drive toward abstract aesthetic principles of symmetry or invariance, on the other. This interplay seems to have been mediated, not by verbal deductions, logical bridges or mathematical formalisms, but by soaring leaps of spatial and physical intuition" (Shepard, 1978, p. 135).

Shepard reminds of the famous mental experience of Kékulé which led him to the discovery of the hexagonal ringlike structure of the molecule of benzene. While dozing before the fire one afternoon (1865) he found that "the atoms were juggling before my eyes ... my mind's eyes, sharpened by repeated sights of similar kind, could now distinguish larger structures of different forms and in long chains, many of them close together: everything was moving in a snake-like and twisting manner. Suddenly, what was this? One of the snakes got hold of its own tail and the whole structure was mockingly twisting in front of my eyes. As if struck by lightning, I awake." (Shepard, 1978, p. 147). The reader may find tens of examples of the same kind in Shepard's paper.

The essential idea, repeatedly mentioned in the recent literature, is, then, that productive reasoning in both, every day life and scientific situations, includes a permanent interplay between conceptual and imaginative dynamics. Is the course of the reasoning process determined essentially by conceptual constructions (symbolized or mediated by imaginary means) or vice versa: is it the play of images which pushes forward the reasoning process in its creative attempts? The phenomena are so complex that it is not possible to get a definitive answer. The most plausible hypothesis seems to be that we deal in fact with *one game* in which active conceptual networks interact with imaginative sources. Moreover, we have reasons to admit that, in the course of that interplay, meanings shift from one category to the other, images getting more generalized significance and concepts largely enriching their connotations and their combinational power.

There is extensive experimental evidence concerning the reciprocal role played by images and concepts in learning and solving activities (see for reviews and general theories, Rohwer, 1970; Paivio, 1970; Paivio, 1971, Blanc-Garin, 1974; Denis and Dubois, 1976; Anderson, 1978; Kosslyn, 1980; Shepard, 1982; Kosslyn, 1983; Anderson, 1990).

But in this interplay, images and concepts are considered distinct categories of mental entities. *What we assume is that, in the special case of geometrical reasoning, one has to do with a third type of mental objects which simultaneously possess both conceptual and figural properties.*

The reason for this profound symbiosis between symbolic, analytical constraints and figural properties in geometrical reasoning is that we deal in fact with *axiomatic systems*. We have then, to distinguish between formal, mathematical validity and empirical validity. As far as a geometrical figure is considered in the realm of a certain axiomatic structure, its properties and the corresponding theorems are dictated directly or indirectly by implicit or explicit definitions. The investigation of these properties is confined only to an intellectual endeavor, and we deal with a formal type of validity. If we are interested in the empirical validity of the properties or theorems, things change fundamentally and one has to confront the respective mathematical assertions with empirical facts.

As we have already mentioned, the total conceptualization of spatial images in geometrical reasoning represents, in fact, an ideal phenomenon. The figural component is usually influenced by figural-Gestalt forces and the conceptual components may be affected by logical fallacies. With age, and as an effect of instruction – as we will see – the fusion between the figural and the conceptual facets tends to improve.

In the studies concerning space representations performed so far, their particular status has not been taken into account. Even Shepard who has devoted a large amount of research to the mental manipulation of geometrical figures (like rotations and unfoldings), does not emphasize this aspect of the problem (see Shepard and Cooper, 1982). Only Piaget and Inhelder mention the particular status of spatial images but they too do not draw all the consequences concerning the relationships between the figural and the logical constraints in this domain (Piaget and Inhelder, 1966, pp. 373–412).

DEVELOPMENTAL ASPECTS

Are figural concepts a natural product of the human mind as concepts and images are, or do they develop only as an effect of systematic training?

The problem is difficult to be answered because, in many situations, the material embodiment and the genuine conceptual interpretation yield the same answer. If one asks a subject to unfold a cube (mentally represented or a material embodiment of it), the drawing obtained is the same, no matter if the subject thinks in terms of conceptual figures or he is referring to a concrete cube. Special experimental situations are required in order to be able to make a distinction.

In an earlier work, we have devised such experimental situations. Let us give an example: subjects of various ages (grades 2 to 6) were confronted with the following question (Figure 3).

> In 3a there are four lines which intersect (point 1). In 3b, there are two lines which intersect (point 2). Compare the two points 1 and 2. Are these two points different? Is one of them bigger? If yes, which one? Is one of them heavier? If yes, which one? Have the two points the same shape?

The question was deliberately ambiguous. It may be considered either from a geometrical or from a material (graphical) point of view. It was our intention

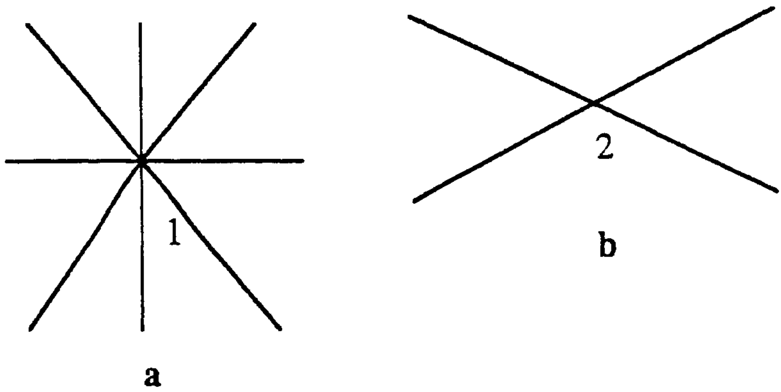

Fig. 3.

to detect the evolution with age of the subjects' interpretation, and the possible emergence of figural concepts (point, line).

The findings show a relatively systematic evolution of the answers from a concrete representation to an abstract-conceptual one. In grade 2, 68% of the children do not answer at all to the question referring to the magnitude of the points. Thirteen percent answered that point 1 is bigger, 6% answered that point 2 is bigger and only 7% answered that the points are the same. Two percent answered, generally, that the points are different.

In grade 3, 40% did not answer, but 45.7% claimed that point 1 is bigger. Only 2% affirmed that the points are the same. In grade 4 takes place a phenomenon of polarization of answers: 50.9% argue that point 1 is bigger, 27.3% claim that the points are the same, and only 12.3% do not answer. In grade 5, the proportion of concretist answers starts to decrease: 40% claim that point 1 is bigger, 28.8% claim that the points are the same and 20% do not answer. In grade 6 the image is different. Twenty percent only find still that point 1 is bigger, while 45.4% of the subjects answer that the points are the same.

Let us quote some explanations given by children, starting from grade 4, which reveal their contradictory interpretations:

S. A.: "Point 1 is bigger because it is the intersection of more lines. The points have no weight. They have the same shape" (Fischbein, 1963, p. 222).

M. N.: "Point 1 is bigger because more lines are intersecting. Because point 1 has a bigger volume it is also greater. The points have the same shape because they are the same thing" (Fischbein, 1963, p. 223).

The child is not able to organize his information in a coherent structure. On one hand, point 1 is considered bigger because, *perceptively*, it represents the intersection of four lines. But, the same point is on its way to become an autonomous entity detached from the context, thus preparing the geometrical concept of point.

M. N.: "The points are almost the same. They have the same weight. The points have the same shape because both are triangles" (cf. Fischbein, 1963, p. 224).

One can see the uncoordinated mixture of perceptive and geometrical interpretations. The figural concept (the abstract zero-dimensional point generated by abstract uni-dimensional lines) does not yet exist.

F. S.: "The points have the same magnitude and weight. They have different shapes. The points have the same magnitude no matter how many lines intersect" (cf. Fischbein, ibid, p. 225).

On one hand, the correct, formal interpretation seems to be present: the number of intersected lines does not influence the magnitude of the points of intersection (implicitly, this means that the lines are uni-dimensional and the points have zero dimension). But, nevertheless, the shapes are considered different: the perceptive factor, disclosing a tacit, concretist orientation, is still active.

Grade 5. There are no significant differences in comparison with grade 4. One may observe the same mixture of perceptive and formal-geometrical (learned) interpretations.

A. P.: "Point 1 is bigger because it has a bigger area. Point 1 is also heavier because it has a bigger area" (cf. Fischbein, ibid, p. 228).

C. V.: "The two points do not have the same magnitude. One is bigger, the other is smaller. The two points have the same shape, because both are round" (cf. Fischbein, ibid, p. 228).

On one hand, the points differ in magnitude. This means that they depend on the lines (supposed, tacitly, different in width) which generate them. On the other hand, the points are similar, circular, because they are considered independent graphical entities (the graphical points are approximately circular).

In grade 6 the picture is totally changed. The abstract lines and points are manifest. Genuine figural concepts seem to be present.

D. N.: "The intersection point of the lines does not possess any weight, magnitude or shape, they do not have any dimension" (Fischbein, ibid, p. 230).

R. R.: "Points have no dimension. Through a point pass an infinity of lines. Points have no shape" (Fischbein, ibid, p. 230).

L. C.: "Points have no magnitude, no weight, but they are represented by small round traces" (Fischbein, ibid, p. 230).

The subjects have also been asked to compare a point on the blackboard with a point on the copybook. Generally, the same type of evolution could be found. But it is worth mentioning that, in some cases, some interesting contradictions appear.

Let me quote a couple of examples from grade 6. The comparison between a point on the blackboard and one on the copy book:

R. A.: "No one of the points is heavier, no one is bigger, because a point has no dimensions" (Fischbein, ibid, p. 232).

The same student (R. A.) (the question referring to the intersecting lines): "Point 1 is bigger because more lines intersect. Point 2 is smaller because less lines intersect. The points have the same weight" (Fischbein, ibid, p. 232).

The student (R. A.) affirms that points have no dimension and therefore no one is bigger and no one is heavier. The same student referring to the intersecting lines claims that point 1 is bigger because it is the intersection of more lines. There is certainly a conflict here generated by the fact that the two systems, the figural and the conceptual, did not yet blend in genuine figural concepts. The child knows that points have no dimension, and he uses this knowledge when referring to the point on the blackboard and the point marked in his copybook. At the same time, when referring to the points generated by intersecting lines, the figural effect is too strong and it seems to cancel the conceptual constraints.

Let me quote a second example of the same type.

J. M.: "The point on the blackboard and the point on the copy book are the same because we know that the point has no dimensions" (Fischbein, ibid, p. 232).

The same student (J. M.): "The two points (generated by the intersecting lines) do not have the same magnitude. The weight of point 1 is greater, and so is the magnitude and they do not have the same shape."

How is it possible that the same subject – who affirms that the point on the blackboard and the point on the copy book are the same because both have no dimensions – claims that the two points generated by intersections are different?

The eleven year old child (grade 6) is aware of the fact that the two graphical signs *represent* geometrical, non-dimensional entities. The fact that one is made by chalk and the other by a pencil, contributes to neutralize the significance of the material embodiment. But in the case of the intersecting lines, the child has to do only with graphical representations. It seems that, in this case, the influence of the figural representation is much more subtle and succeeds to capture *by itself* the entire meaning of the concepts of point and line.

The above examples show the complexity of relationships between the figural and the conceptual aspects in the organization of figural concepts and the fragility of that organization in the students' minds.

The Definition, the Image, and the Figural Concept

One has, then, to consider three categories of mental entities when referring to geometrical figures: the definition, the image (based on the perceptive-sensorial experience, like the image of a drawing) and the figural concept. The figural concept is a mental reality, it is the construct handled by mathematical reasoning in the domain of geometry. It is devoid of any concrete-sensorial properties (like color, weight, density, etc.) but displays figural properties. This figural construct is controlled and manipulated, in principle without residuals, by logical rules and procedures in the realm of a certain axiomatic system. The difficulty to accept the existence of this third type of mental entities is determined by the fact that we are *directly* aware of only the mental representation (including various *sensorial* properties like color) and the corresponding concept. *We need an intellectual effort in order to understand that mathematical-logical operations manipulate only a purified version of the image, the spatial-figural content of the image.*

When we manipulate words in a verbal activity, the sounds (heard or expressed) are the external, material representatives of meaning. *The meaning lies beyond the materiality of the expressed word*: the meaning is an idea fixed by a complex of relationships. *The figural concept is also meaning. The particularity of this type of meaning is that it includes figure as an intrinsic property.* The genuine meaning of the word *circle* in geometry, as it is manipulated by our reasoning process, is not reducible to a purely formal definition. *It is an image entirely controlled by a definition. Without this type of spatial images, geometry would not exist as a branch of mathematics.*

The term "figure" is ambiguous and may denote a large variety of meanings. In the present text, "figure" refers only to spatial images. Usually a figure possesses a certain structure, a shape or "Gestalt". Geometrical figures correspond to this description, but some specifications have to be added: (a) a geometrical figure is a *mental* image, the properties of which are completely controlled by a definition; (b) a drawing is not the geometrical figure itself, but a graphical or a concrete, material embodiment of it; (c) the mental image of a geometrical figure is, usually, the representation of the materialized model of it. The geometrical figure itself is only the corresponding *idea* that is the abstract, idealized, purified figural entity, strictly determined by its definition.

As already mentioned, the geometrical figures are not the only images controlled by corresponding concepts. As a matter of fact, this is the common situation especially in scientific reasoning. For a biologist, for instance, terms like vertebrate, batrachian, mammal, etc. indicate classes of animals which, on one hand, have their meanings synthesized by concepts and, on the other hand, are related, in the scientist's mind, with certain images. When thinking about these categories of animals, the scientist manipulates the images according to the respective concepts.

The difference between empirical sciences and geometry, in this respect, is that in geometry the images may be exhaustively controlled by concepts while in empirical sciences they are not.

In empirical sciences *the concept tends to approximate* the corresponding *existing* reality, while in mathematics it is the concept, through its definition, which dictates the properties of the corresponding figures.

This leads to a fundamental consequence. *The entire investigative process of the mathematician may be performed mentally, in accordance with a certain axiomatic system, while the empirical scientist must, sooner or later, return to empirical sources.* For a mathematician, reality may be a source of inspiration but never an object of research leading to mathematical truths, and certainly not a final instance for proving a mathematical truth (as far as mathematics is concerned).

The mathematician, like the physicist or the biologist, uses observation, experimentation, induction, comparisons, generalizations, but the objects of his investigation are purely mental. His laboratory is, in principle, confined to his mind. His proofs are never of an empirical nature but only of a logical one.

CONFLICTS

As we mentioned above, *figural concepts constitute only the ideal limit of a process of fusion and integration between the logical and the figural facets.*

A similar idea has been expressed by Tall and Vinner who have distinguished between "concept image" and "concept definition". While the term "concept definition" applies to the mathematical meaning, as it is formally defined, the term *concept image* describes "the total cognitive structure that is associated with the concept which includes all mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (cf. Tall, 1991, p. 7). In geometry the ideal figural concept corresponds with the concept definition, while its mental reflection with all its connotations and ambiguities corresponds with what Tall and Vinner have called "concept image". "Image", in their terminology, does not mean "picture" in the sensorial sense, but rather a mental, subjective reconstruction of a formally given mathematical entity.

Let us come back to the figural concept. In usual psychological conditions the figural and the conceptual features of a figural concept remain relatively dependent on the two systems with their specific constraints. This basic fact very often leads to contradictions, conflicts, internal tensions, up to a total dissolution of the figural concept into its two basic components.

Let me give some examples. In an experiment, carried out some years ago, we have presented the following theorem: "$ABCD$ is a quadrilateral and $PQRS$ the midpoints of its sides. One should prove that $PQRS$ is a parallelogram" (Figure 4).

The subjects were presented with the proof of the theorem and asked if they agree with the correctness of the proof. In order to check whether the subjects understand that the proof guarantees the universal validity of the theorem, several additional questions were asked. One of these questions was: "V is a doubter. He thinks that we have to check at least a hundred quadrilaterals in order to be sure that $PQRS$ is a parallelogram. What is your opinion? Explain your answer."

It has been found that about 40% of the subjects (N = 396) agreed with the proof, but only about 10% rejected any need for further empirical checks (Fischbein and Kedem, 1982, pp. 128–131).

Some of the students gave the following type of explanation: one has to check for various categories of quadrilaterals (parallelograms, rectangles, squares, etc.).

We claimed above that a figural concept is a mental construct characterized by all the properties of concepts (generality, essentiality, abstraction, ideality), but which at the same time preserves figural properties (shape, distances, positions). In principle, the fusion between figure and concept should be absolute, but it is the conceptual organization which should dictate, completely, the figural properties and relationships. As a matter of fact, as we have already mentioned, this is an ideal situation which usually may be accomplished in the trained mind of the mathematician.

What happens is that the conceptual and the figural properties remain under

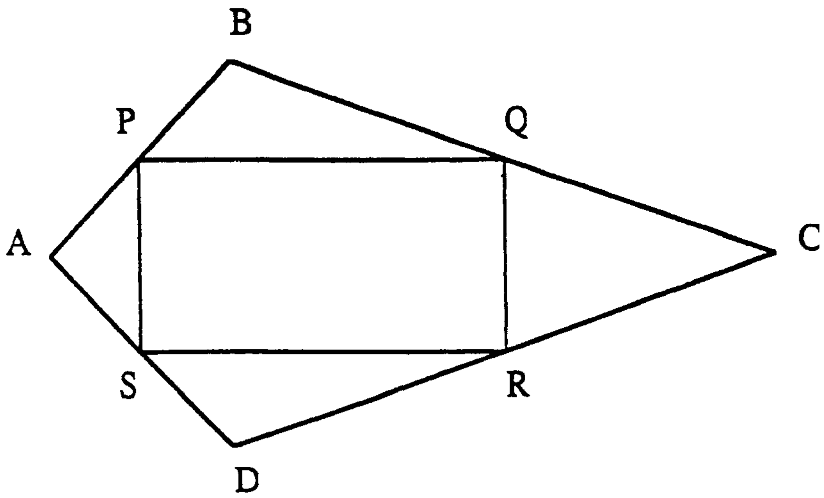

Fig. 4.

the influence of the respective systems, the conceptual and the figural ones. Very often the figural constraints – usually following the laws of Gestalt – may escape the conceptual control and impose, to the line of thought, interpretations which are figurally consistent but which are not subject any more to the conceptual constraints.

Though the student knows the definition of the parallelogram (a quadrilateral the opposite sides of which are parallel) it may become difficult for him to see in the various shapes corresponding to that definition, the same *Gestalt, the same* category of *figures*. An oblique parallelogram, a rectangle, a square are *figurally* so different that the unifying effect of the common concept simply vanishes. The same subject who accepts the correctness of the given proof for supporting the validity of the theorem, may claim that more checks are necessary, *for every category of quadrilaterals*, in order to reach certitude.

Alessandra Mariotti mentions the following example: Alessia (16 years old, 11th grader) has been addressed the following problem: how many angles do you see in the figures a and b? (see Figure 5).

Alessia: "Whenever I see two lines which intersect, I know that the space between the two lines is an angle. I think that, in both figures, there is only one angle, even if at first I thought that in the second figure there were two angles. I can explain my supposition. First, I thought that, in this representation, line 1 and line 2 form one angle, and line 2 and line 3 form a second angle. However, now I think that there is only one angle formed by the crossing lines (1, 3) and that line 2 is the bisector of this angle" (Mariotti, 1992, p. 875).

Alessia's difficulty is generated by the fact that the concept is unable to control the figure. And this, not because she does not possess the concept correctly but

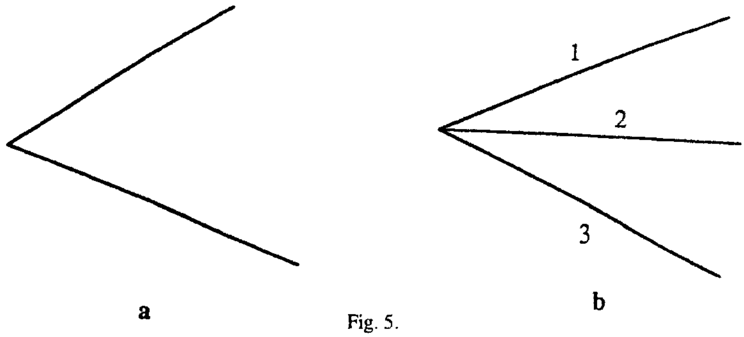

Fig. 5.

because the figure still carries with it Gestalt features inspired by practice. As a matter of fact, the complete symbiosis about which we talked above does not yet exist; if you cut into two halves a piece of cake, you get two pieces of cake not three (Alessia's first interpretation). If line 2 is the bisector of the angle, it cannot belong, at the same time, to two other angles (the second interpretation).

The *concept* of an angle does not control, totally, the figure. The interpretation of the figure still depends, partially, on non-formal constraints.

Theoretical Implications

One may distinguish two main theories referring to the relationships between concepts and images. These two points of view have been emphasized in the realm of the information processing approach.

(a) The dual code theory

Paivio has devoted a number of research projects to the role of mental images, especially in the learning process. Similarly to the theory of Piaget and Inhelder (1966), he emphasizes the symbolic nature of images, and retaining the same line of thinking, he distinguishes them from verbal processes.

"The empirical approach involving the different sets of convergent operations just described, can be linked to a theoretical framework in which the functions attributed to images in learning and memory are clearly distinguished from those attributed to verbal processes. Without such a differentiation, it would be redundant to retain both imaginal and verbal processes as theoretical constructs. One such distinction is that imagery is functionally linked to stimulus or task concreteness, whereas verbal processes are more independent of this dimension. That is, images presumably are more useful in dealing with concrete situations than with abstract. Another theoretical distinction is that imagery is specialized for the processing of spatial information, whereas the verbal system is characterized more by its capacity for sequential processing" (Paivio, 1970, pp. 386–387).

(b) The propositional theory

This dual code theory has been contested. Various authors have argued that imagery, like verbal information, is encoded in an abstract propositional format (see, for a review, Anderson, 1978). Anderson describes three features that define a proposition: it is abstract, it has a truth value and has its rules of formation (Anderson, 1978, p. 250). A proposition is not a mere sentence. The notion of abstractness is related to a concept of invariance under paraphrase, says Anderson (ibid, p. 250). That is, various linguistic paraphrases and cross-language translations would be assigned the same propositional representation.

According to the supporters of the propositional theory the processes related to mental images and verbal memory cannot be explained by the dual-code theory.

Pylyshyn (1973) refers to the fact that people can describe pictures in words or create pictures to illustrate verbally expressed ideas. "The abstract propositional code would serve as a mental format into which and out of which, pictorial and verbal information could be translated. It serves as a 'half-way house' for the process of translating between the two peripheral codes" (Anderson, ibid, p. 256). Anderson and Bower (1973) and Pylyshyn (1973) affirm that a propositional code is needed to represent meaning.

Anderson claims that the theory of a common, abstract propositional code is flawed by the following consideration: if for translating from code 1 to code 2 one needs to translate first from code 1 to code 3, this would imply that for translating from code 1 to code 3 one needs a new code 4 and this would lead to an infinite regress (Anderson, 1978, p. 256).

Briefly speaking, it is difficult, based on the present data, to decide which one of the two theories – the dual code theory or the propositional theory – represent a more adequate explanation for the storage and dynamics of verbal and image representations. For both, there are strong arguments in favor and against (see Anderson, 1978, for a comprehensive discussion).

(c) The figural concepts and the propositional theory

The existence of figural concepts – in addition to pure concepts and images – represents a strong argument in favor of a central, unifying, relatively autonomous mental level which not only facilitates the communication between verbal and pictorial information but also creates the possibility for mental constructs characterized, simultaneously, by conceptual properties (generality, ideality, essentiality) and by pictorial (basically spatial) properties.

To manipulate an image, a spatial representation under the strict but also intrinsic control of a definition would not be possible if only two independent processing codes would exist. *When solving a geometrical problem we manipulate geometrical figures as if they were homogeneous mental entities, not combinations of two categories of heterogeneous mental constructs.* Certainly, this is the ideal but possible case.

As we have seen, very often under the impact of figural laws the image may detach itself and escape from the formal-conceptual control.

(d) Piaget and Inhelder

Piaget and Inhelder have devoted a comprehensive study to the relationships between images and operations (that is, logical structures) (Piaget and Inhelder, 1966). In their view, though images and concepts represent two distinct categories, there is a profound interaction between them. In this interaction, the operations fulfil a leading role which grows with age.

A special situation takes place in the case of geometry in their view. "... in the special case of geometrical operations, the role of which is just to describe the spatial figures and their transformations, there is a homogeneity between the symbolized (le symbolisé) consisting in spatial operations and the representing symbol (le symbolisant imagé) which is itself of a spatial nature: it follows the privileged situation of the geometrical intuition, the double nature of which, both operational and imaginatory, reaches an intimate synthesis more than in any other domain ..." "The geometrical intuition reaches this adequate synthesis only by subordinating the imagined elements to its operational nucleus and this subordination implies a development." (Piaget and Inhelder, 1966, pp. 394–395) (my translation).

It seems that Piaget and Inhelder have also had the intuition of the total fusion between the conceptual and the figural aspects in the special case of geometrical thinking. The fact that they have reached this conclusion after considerable evidence constitutes a strong support to our theory. On the other hand, they did not draw either the general, theoretical or the didactical implications of this finding. In their work, it remains a marginal remark.

Didactical Implications

Let us mention a number of didactical aspects implied by the theory of figural concepts. Some of them are already known to the teachers from their teaching experience but not related to a general theory.

Image and definition

As we have already emphasized, the relationship between object and definition is basically different in empirical sciences and mathematics. While in empirical sciences the definition is ultimately dictated by the properties of the respective category of objects, in mathematics it is the definition which imposes directly, or via deduction, the properties of the corresponding category of objects. Accordingly, the interpretation of the figural component of a geometrical figure should remain entirely subjected to the formal constraints. This idea is not always understood, and it is very often forgotten by the student. The figural component tends to liberate itself from the formal control and to behave autonomously in conformity with Gestalt patterns (for example, the finding that many students, after accepting the proof of a theorem as an absolute guarantee for the validity of a theorem, require additional checks for every particular sub-class of the respective class of

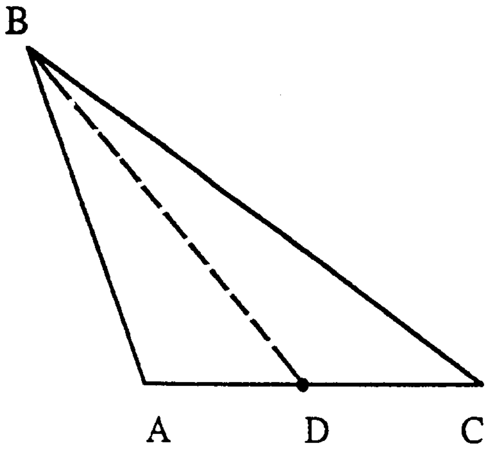

Fig. 6.

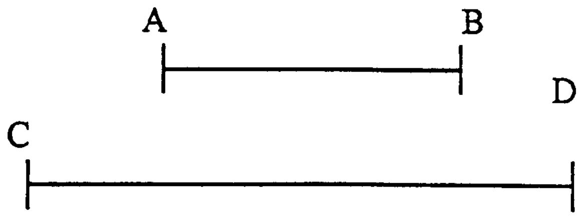

Fig. 7.

figures). This difficulty in manipulating figural concepts, that is, the tendency to neglect the definition under the pressure of figural constraints, represents a major obstacle in geometrical reasoning.

It follows from the didactical point of view, that the student should be especially trained in coping with this type of conflictual situations.

Students may not be able to draw correctly the altitude from vertex B and draw, instead, BD, despite the fact that they know the definition of the altitude in a triangle (Figure 6).

They should be made aware of the definition and asked to carry out the task correctly, *according to the definition* and not according to what seems to them to be imposed by the image.

This is certainly a trivial example but many such conflictual examples should be used, systematically, in the classroom in order to emphasize the predominance of the definition over the figure in using and interpreting the figural concept.

Yet another example: comparing the set of points in the segments AB and CD, one has to cope with the conflict between the claim that in CD there are more points, and the claim that the two sets are equivalent (Figure 7).

The correct interpretation of the notion of point is that it is a *figural concept.*

Conceptually speaking, a point is a zero-dimensional entity. *Figurally* (spatially), a point indicates a position. But because a position cannot be represented otherwise than through an image, the points get dimensions (bi-dimensional representation). The figural concept loses its ideal purity and this generates the conflict. When we affirm that a segment contains an infinity of points, we refer to an infinity of zero-dimensional entities. The expression "an infinity of zero-dimensional entities" has an ideal meaning: it deals with pure figural concepts. At the same time, the figural component (the position) tends to get, automatically, a certain pictorial substantialization which leads to the tacit belief in the non-equivalence of the two sets of points.

A high-school student should be made aware of the conflict and its source, in order to emphasize, in his mind, the necessity to rely in mathematical reasoning ultimately on the formal constraints.

All this leads to the conclusion that the *processes of building figural concepts in the student's mind should not be considered a spontaneous effect of usual geometry courses.*

The integration of conceptual and figural properties in unitary mental structures, with the predominance of the conceptual constraints over the figural ones, is not a natural process. It should constitute a continuous, systematic and main preoccupation of the teacher.

The concept of locus

In the previous lines, we have claimed that, in order to produce an adequate integration of figure and concept in geometrical reasoning, with the predominance of the formal constraints, conflictual situations should be used: the student should be trained to follow carefully the requirements of the definition, sometimes in apparent contradiction to the suggestions of the figure. A second aspect to be mentioned, with regard to the crystallization of figural concepts, is the explicit use of various loci. It is in this case of loci that the profound, intimate relationship between logical and figural aspects is explicitly applied. A locus is a figure (a line or surface) all the points of which satisfy a certain property and all the points corresponding to the respective property belong to the respective figure.

For instance, the quality of a circle to be a figural concept is determined by the complete correspondence between its points and a certain relationship metrically or algebraically defined. *All* the points of the circle are equidistant (the radius r) from a point C (the center) and all the points equidistant from C are situated on the respective circle. Algebraically, one has $(x-a)^2+(y-b)^2=r^2$. It is not possible to invent (or discover) properties of the circle which could not be derived from the definition. Though the circle is *an image*, a spatial representation, its existence and its properties are entirely imposed by an abstract, formal definition. Nothing is true figurally which is not true *and* provable conceptually and vice versa.

Briefly speaking, the systematic use of loci with their explicitly stated double nature in our opinion represents an important didactical tool to deepen the

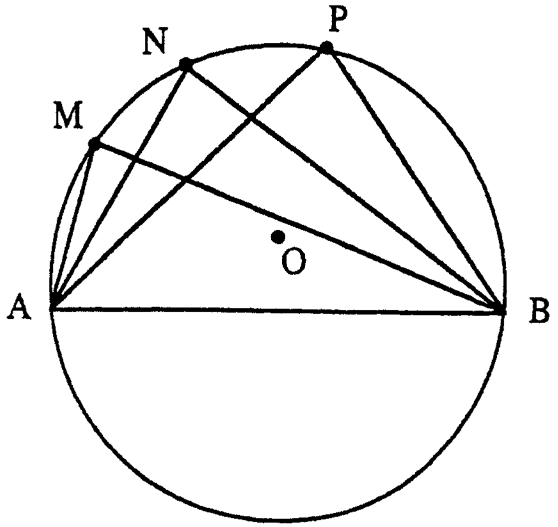

Fig. 8.

understanding of the nature of figural concepts.

Let me add an example: let us consider a circle with center O. Let us choose two points, A and B, on the circle and draw several angles, the sides of which pass through A and B, and having their vertices on the circle (Figure 8). Let us compare the angles M, N, P.

It is difficult to compare the angles figurally, directly. They seem to be of different magnitudes. But we know that the measure of an angle with its vertex on a circle is equal to the half of the arc determined by its sides. The three angles M, N, and P consequently are equal.

We deal here with figural concepts because every part of the image (angles, sides, points, the circle, the arc) are simultaneously images and concepts, the images being controlled by the respective definitions. But, in the dynamics of the reasoning process, the image by itself seems to be unable to answer the question. It is *through the theorem* that the equality of the angles is determined.

Reciprocally: all the vertices of the angles, the sides of which pass through the same points of the circle (and are of the same magnitude with an angle the vertex of which is on the circle) are situated on the same circle. Our belief is that by confronting figural impressions with formal constraints one helps to improve the conceptual control and, at the same time, one stimulates the symbiosis between the figural and the conceptual constraints.

Logic and image should be inseparable in geometrical reasoning and this can be beautifully and explicitly seen in a locus problem. The figural elements become an integral part of the logical reasoning process as if they themselves would be genuine concepts. If a certain discrepancy emerges it is usually due to some figural "disobedience", as an effect of figural, extra-logical forces.

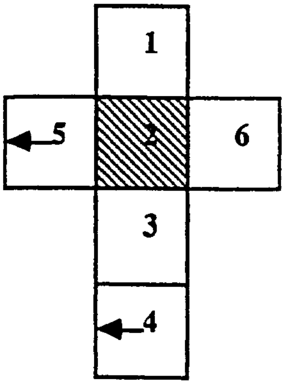

Fig. 9.

Remarks

A last remark refers to the possibility to practice, with the students, mental activities in which the cooperation between the figural and the conceptual requires a special endeavor. In such activities, the student has to learn to mentally manipulate geometrical objects by resorting simultaneously to operations with figures and to logical conditions and operations.

Such a type of activities, already referred to in the present paper, consists of (a) asking the students to draw the image obtained by unfolding a geometrical body (actually perceived or mentally represented), (b) asking the students to identify the geometrical body which could be obtained by imagining the folding back of a bi-dimensional drawing and (c) asking the students to indicate the edges which will match when the tri-dimensional object will be reconstructed.

Some of such tasks are relatively easy but others are very complex. For instance, it is relatively easy to determine that the drawing in Figure 9 represents the unfolding of a cube. The symmetry of the image is certainly helpful and the folding back of the faces 1, 3, 4, 5, 6 is mentally performed as a unique task (with face 2 representing the base). In this case, the figural and the conceptual components are naturally well integrated and consequently what one manipulates is a figural concept with its elements. Matching the corresponding edges is also not a difficult task in the case of adjacent edges (in the drawing). It is more difficult to see that the marked edges (by arrows) meet also in the folded cube.

A still more complex task would be to identify the drawing in Figure 10 as the unfolding of a cube. It is also rather difficult to see that the marked edges match in the reconstructed cube. In such mental activities, one does not simply internally imitate external manipulatory acts. It is a mental construction which requires not only to "see" figures but also to modify their positions; to imagine their transformed positions; to imagine the effect of the transformation on adjacent figures. For instance, when raising square 4 so as to become perpendicular to

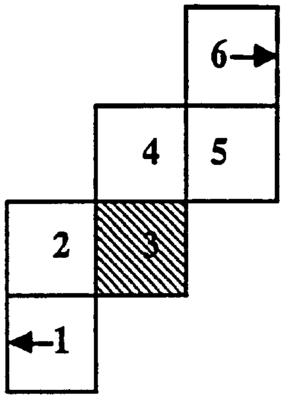

Fig. 10.

square 3 (the chosen base), one transports also 5 and 6. Square 4 being kept in the standing position one folds square 5, etc. The effects of the successive transformations have to be kept in mind and coordinated until the original solid is reconstructed.

What is the contribution of figural manipulations and the contribution of the logical operations? The actual literature does not answer to this fundamental question, simply because images and concepts are considered basically distinct categories of mental activities. When investigating various types of mental transformations of tri-dimensional objects (like rotations or unfolding and folding back), one deals with such operations as if they would be of a mere pictorial nature.

As a matter of fact, things are not and cannot be so. It is because we deal with faces of a cube (in the above example) that the edges are equal, that the faces are squares, that we deal with right angles, etc. All this is *tacit* knowledge, implied in the mental operations. Without such a tacit conceptual control the entire operation would be meaningless.

What we claim is that this type of complex mental activities, which sometimes put a high strain on the intellectual process, represents an excellent opportunity for training the capacity of handling figural concepts in geometrical reasoning.

Such a training is aimed to improve the following abilities: (a) the constructive cooperation of the figural and conceptual aspects in a geometrical problem solving activity; (b) the ability to keep in mind and coordinate as many as possible figural-conceptual items; (c) the ability to organize the mental process in meaningful subunits so as to reduce the memory load; and (d) the ability to predict and integrate the effect of each transformation on the road to the solution.

SUMMARY AND CONCLUSIONS

An attempt has been made to interpret geometrical figures as mental entities which possess *simultaneously* conceptual and figural properties.

Figural concepts are abstract, general, ideal, pure, logically determinable entities, though they still reflect and manipulate mentally representations of spatial properties (like shape, position, metrically expressed magnitudes). Very often, figures tend to retain and impose on the reasoning process their apparent striking features according to Gestalt or graphical representation constraints. Accordingly, the conceptual (axiomatic-deductive) control is weakened and the solving or interpretation process is vitiated.

Since it is in principle completely controlled conceptually, the geometrical figure may participate actively in a formal, rigorous mathematical reasoning.

The term "figural concept", introduced by us, is intended to emphasize the fact that we deal with a particular type of mental entities which are not reducible, neither to usual images – perceptive or entencephalic – nor to genuine concepts. We deal with *figures*, the properties of which are completely fixed –directly or indirectly – by definitions in the frame of a certain axiomatic system. In our interpretation, the conceptual control, ideally, should be intrinsic and thus the image and the concept should merge in a unique mental object. In mathematical reasoning we also resort explicitly to definitions and theorems in order to direct our reasoning or to check our assumptions and conclusions. But, usually *in the process of mathematical invention* we try, we experiment, we resort to analogies and inductive processes by manipulating not crude images or pure, formal axiomatic constraints, but *figural concepts, images intrinsically controlled by concepts.* Without the notion of figural concepts, the processes of problem solving and invention in geometry could not be satisfactorily described and explained.

During the process of invention it is mainly by intuition that we are basically inspired and not by explicit logical chains of arguments. By looking constantly for analytical, formal justifications like theorems and definitions, the flow of productive ideas would be disturbed or even inhibited.

It is by resorting mainly to figures *intrinsically* controlled by conceptual constraints, that the process of invention in geometry can progress creatively. Certainly, the formal framework represented by axioms, definitions, theorems and proofs, has to be invoked from time to time in order to check our steps. And this mainly because psychologically the symbiosis between the conceptual and the figural components is very often not absolute.

Although a figural concept consists of a unitary entity (a concept expressed figurally) it potentially remains under the double and sometimes contradictory influence of the two systems to which it may be related – the conceptual and the figural one. Ideally, it is the conceptual system which should absolutely control the meanings, the relationships and the properties of the figure. As a matter of fact, very often the figure disobeys the dictates of the concept and the interpretation of its properties is shaped by figural Gestalt patterns.

Many mistakes students make in their geometrical reasoning may be explained by this kind of split (or lack of congruence) between the conceptual and the figural aspect of the figural concepts. The figural structure may dominate the dynamics of reasoning instead of being controlled by the corresponding formal constraints. As a consequence, many students do not understand the genuine nature of a geometrical proof and tend to experience the need for supplementing it with empirical verifications.

Images and concepts interact in the cognitive activity of a person (a child or an adult) cooperating sometimes or conflicting in other situations. But the development of figural concepts generally is not a natural process. One of the main reasons that geometry is such a difficult topic in school programs is that figural concepts do not develop naturally towards their ideal form.

Consequently, one of the main tasks of mathematics education (in the domain of geometry) is to create types of didactical situations which would systematically ask for a strict cooperation between the two aspects, up to their fusion in unitary mental objects. We have already mentioned above some types of activities: more emphasis on loci and problems using them and, on the contrary, problems in which the figural patterns naturally tend to disobey the conceptual constraints (leading to conflicts), or problems with unfoldings and reconstructions in which the cooperation between the logical demands and the figural representations is so difficult. Many other situations may be considered but we do not yet possess enough experimental evidence referring to the whole matter.

The existence of figural concepts, *in addition* to images and concepts, is also relevant for the information processing interpretation of cognition. The possibility of complete congruence between logical and figural constraints in a certain category of mental entities represents a strong argument in favor of the propositional theory: a common, interpretative structure has to be postulated which makes this congruence possible.

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