The Theory of Group Characters and Matrix Representations of Groups

Second Edition

Dudley E. Littlewood

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CONTENTS

I. MATRICES

1.1. Linear transformations 1
1.2. Matrices 2
1.3. The transform of a matrix 4
1.4. Rectangular matrices and vectors 5
1.5. The characteristic equation of a matrix 6
1.6. The classical canonical form of a matrix 7
1.7. The classical canonical form; multiple characteristic roots . 8
1.8. Various properties of matrices 14
1.9. Unitary and orthogonal matrices 15

II. ALGEBRAS

2.1. Definition of an algebra over the complex numbers . . 22
2.2. Change of basis and the regular matrix representation . . 23
2.3. Simple matrix algebras 25
2.4. Examples of associative algebras 25
2.5. Linear sets and sub-algebras 26
2.6. Modulus, idempotent and nilpotent elements . . . 26
2.7. The reduced characteristic equation 27
2.8. Reduction of an algebra relative to an idempotent . . 29
2.9. The trace of an element 31

III. GROUPS

3.1. Definition of a group 32
3.2. Subgroups 33
3.3. Examples of groups 34
3.4. Permutation groups 36
3.5. The alternating group 37
3.6. Classes of conjugate elements 38
3.7. Conjugate and self-conjugate subgroups 40
3.8. The representations of an abstract group as a permutation group . 41

IV. THE FROBENIUS ALGEBRA

4.1. Groups and algebras 43
4.2. The group characters 45
4.3. Matrix representations and group matrices . . . 48
4.4. Characteristic units 56
4.5. The relations between the characters of a group and those of a subgroup 57

V. THE SYMMETRIC GROUP

5.1. Partitions 59
5.2. Frobenius's formula for the characters of the symmetric group . 61
5.3. Characters and lattices 67
5.4. Primitive characteristic units and Young tableaux . . 71

VI. IMMANANTS AND S-FUNCTIONS

6.1. Immanants of a matrix 81
6.2. Schur functions 82
6.3. Properties of S-functions 87
6.4. Generating functions and further properties of S-functions . 98
6.5. Relations between immanants and S-functions . . . 118

VII. S-FUNCTIONS OF SPECIAL SERIES

7.1. The function $\Phi(q, x)$ 122
7.2. The functions $(1-x)^{-N}$ and $(1-x^r)^{-m}$ 126
7.3. S-functions associated with $f(x^r)$ 131

VIII. THE CALCULATION OF THE CHARACTERS OF THE SYMMETRIC GROUP

8.1. Frobenius's formula 137
S-functions of special series 138
Recurrence relations 140
Congruences 142
Classes for which the orders of the cycles have a common factor . 143
Graphs and lattices 146
Orthogonal properties 146

IX. GROUP CHARACTERS AND THE STRUCTURE OF GROUPS

9.1. The compound character associated with a subgroup . . 147
9.2. Deduction of the characters of a subgroup from those of the group 150
9.3. Determination of subgroups: necessary criteria that a compound character should correspond to a permutation representation of the group 155
9.4. The properties of groups and character tables . . . 159
9.5. Transitivity 164
9.6. Invariant subgroups 171

X. CONTINUOUS MATRIX GROUPS AND INVARIANT MATRICES

10.1. Invariant matrices 178
10.2. The classical canonical form of an invariant matrix . . 193
10.3. Application to invariant theory 203

XI. GROUPS OF UNITARY MATRICES

11.1. Introductory 210
11.2. Fundamental formula for integration over the group manifold . 211
11.3. Simplification of integration formulae for class functions . 217
11.4. Verification of the orthogonal properties of the characters of the unitary group 222
11.5. Orthogonal matrices and the rotation groups . . . 223
11.6. Relations between the characters of D and D' . . . 225

11.7. Integration formulae connected with D and D' . . . 227
11.8. The characters of the orthogonal group 233
11.9. Alternative forms for the characters of the orthogonal group . 238
11.10. The difference characters of the rotation group . . 245
11.11. The spin representations of the orthogonal group . . 248
11.12. Complex orthogonal matrices and groups of matrices with a quadratic invariant 260

APPENDIX

Tables of Characters of the Symmetric Groups . . . 265
Tables of Characters of Transitive Subgroups. Alternating Groups . 272
General Cyclic Group of Order n 273
Other Transitive Subgroups 273
Some Recent Developments 285

BIBLIOGRAPHY 301

SUPPLEMENTARY BIBLIOGRAPHY 306

INDEX 309

CORRIGENDUM

p. 23. **The regular matrix representation**

This representation will not be *simply* isomorphic if there exists an element x of the algebra for which $ax = 0$ for all a of the algebra. The corresponding matrix X would be identically zero. A simply isomorphic representation, however, may be obtained in any case by adjoining a modulus to the algebra before obtaining the regular representation.