THE THEORY OF INTEGRATION IN A SPACE OF AN INFINITE NUMBER OF DIMENSIONS.

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in Copenhagen.

Contents.

§ 1. Introduction.

The object of the present paper is to study in greater detail than has been done before a certain space of an infinite number of dimensions, in which a theory of integration can be developed. The space in question was first considered by Daniell in connection with his studies on integration in abstract spaces[1]; since then it has been investigated by Wiener, Steinhaus, Paley and Zygmund, Carlson and myself.[2] The theory has applications to analytical problems and to problems in the calculus of probabilities.

Let us consider a real or complex function $f(x_1, x_2, x_3, \ldots)$ depending on a sequence of real variables; such a function is called *periodic* with the periods 1, 1, 1, ... if for arbitrary integers $n_1, n_2, n_3, \ldots$ we have always

$$f(x_1, x_2, x_3, \ldots) = f(x_1 + n_1,\ x_2 + n_2,\ x_3 + n_3, \ldots).$$

We restrict ourselves to the consideration of such functions. As always when dealing with periodic functions, it is convenient to consider the functions as defined not in the usual »open» space but in the closed space which we obtain by replacing the coordinate axes by circles. Thus the space with which we have to deal in the present paper is that closed space or *torus-space* which we obtain from the space of all real sequences $x_1, x_2, x_3, \ldots$ by reduction of the coordinates mod. 1. This reduced space is denoted throughout by Q_ω.

It is not usual to speak about a space before relations between its points have been defined. We give these relations in the next few sections by introducing the notions of intervals, limit points and closed and open sets and by proving the classical covering theorems. These sections contain what may be called the *topology* of the space Q_ω upon which the whole theory is based. It is also possible to introduce a distance between the points of Q_ω but not without violating the symmetry of the variables, and we do not need the notion. The fact that we consider the variables as arranged in a fixed order is only a matter of convenience.

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[1] P. J. Daniell [1], [2].

[2] N. Wiener [1], H. Steinhaus [1], [2], R. E. A. C. Paley-A. Zygmund [1], [2], [3], F. Carlson [1], B. Jessen [1], [2], [3].

Other spaces than the space Q_ω can be treated in a similar way; such spaces have been considered by Daniell, Feller and Tornier, and Kolmogoroff[1]; a quite general result has been announced by Ulam.[2] The theory is also related to a type of integration in functional space introduced by Wiener.[3]

The idea of the present exposition is to develop the theory of functions in the space Q_ω in close analogy to the theory of the n-dimensional torus-space Q_n, obtained from the ordinary n-dimensional space by reduction of the coordinates mod. 1. This is possible; in fact, as soon as intervals and the measure of intervals have been defined, the ordinary definitions of exterior and interior measure, and so also the ordinary definition of measurable and integrable functions, may be applied. The proof that the measure and so the integral have the ordinary properties is obtained, without the trouble of repeating all arguments, by using a simple *transferring principle;* the corresponding principle for n dimensions has been used by Lebesgue, F. Riesz and de la Vallée-Poussin.[4] It would be easy to prove that my definition leads to the same notion as Daniell's which was based on Young's definition of the Lebesgue integral. — The most interesting part of the theory is that which deals with such problems as have no analogue for functions of a finite number of variables; for instance, the problem of what meaning can be attached to an infinitely multiple integral. — A main point in the theory is the establishment of a theory of Fourier series for functions in Q_ω. A solution of this problem in the case of continuous functions was given by Bohr in his second paper on almost periodic functions; in the present paper we attach to any integrable function in Q_ω a Fourier series of the form

$$\sum c_{p_1, p_2, \ldots, p_n} e^{2\pi i(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)},$$

(where the summation is both over the p's and over n) and we prove that this series determines the function uniquely; we also prove the Parseval and Riesz-Fischer theorems for these series.

Daniell and Wiener used the theory of the space Q_ω as an example and special properties of the space were not given. Steinhaus gave two applications of the theory; he pointed out that a theory of measure in Q_ω would make it

[1] P. J. Daniell [2], W. Feller-E. Tornier [1], A. Kolmogoroff [2] 24—30.

[2] S. Ulam [1]. It is of interest to remark that the main results of the present paper, in particular the theorems of §§ 13 and 14, hold also in the case considered by Ulam. The proofs must be rearranged.

[3] See R. E. A. C. Paley †-N. Wiener-A. Zygmund [1].

[4] H. Lebesgue [1] 365; F. Riesz [1] 497; C. de la Vallée-Poussin [1].

possible to generalise the theory of probabilities by a sequence of choices of real signs ± 1 (where the two signs are supposed to be equally probable) to the case of sequences of complex signs. This leads in our notation to considerations concerning the orthogonal system $e^{2\pi i x_k}$. Translating a beautiful result of Kolmogoroff from the language of the calculus of probabilities into the language of real functions, Steinhaus proved an interesting convergence theorem for series of the form

$$\sum_{k=1}^{\infty} a_k e^{2\pi i x_k};$$

he also proved a theorem on the analytical continuation of power series. Paley and Zygmund developed the theory for both real and complex signs in a systematic way; they obtained results not only for power series, but also for Fourier series and Dirichlet series; Carlson added certain results concerning Dirichlet series.

The present author was led to the theory in connection with some investigations by Bohr concerning the distribution of the values of the Riemann zeta-function, which were carried out in collaboration with the author.[1] These investigations have a certain connection with those just mentioned, but here it is the case of complex signs that occurs. It proved advantageous to consider the zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1}$$

in relation to the general class of functions

$$\zeta(s, x_1, x_2, x_3, \ldots) = \prod_{k=1}^{\infty} (1 - e^{2\pi i x_k} p_k^{-s})^{-1}.$$

In the present paper I apply the theory of the space Q_ω not to the zeta-function itself but to other almost periodic cases for which the details are simpler. The results of Steinhaus and some of the results of Paley and Zygmund are of importance for these applications; I give a slightly simplified exposition of these results before I give my own applications.

§ 2. The Torus-space of an Infinite Number of Dimensions.

We start from the space of all sequences $x_1, x_2, x_3, \ldots$ of real numbers. Reducing the coordinates of this space mod. 1 we obtain a certain closed space;

[1] H. Bohr-B. Jessen [1], [2], [3].

we call it the torus-space Q_ω. We shall have in the sequel to consider functions defined in this space. Through the reduction of the coordinates mod. 1 the coordinate axes in the original space become circles; we call these circles the *coordinate circles* of Q_ω and denote them by $c_1, c_2, c_3, \ldots$; they all have the perimeter 1. The letter x_k is in what follows used in two different senses: both to denote a point of the coordinate circle c_k and to denote the abscissa of this point on c_k; in the latter case x_k is only determined mod. 1. The point of Q_ω which is determined by the coordinate points, or simply coordinates, $x_1, x_2, x_3, \ldots$ is denoted by

$$x = (x_1, x_2, x_3, \ldots).$$

It will be convenient for us to have a fixed notation for what is usually called the *product* of a finite or infinite number of (arbitrary) sets $A_1, A_2, A_3, \ldots$ The product $A = (A_1, A_2, A_3, \ldots)$ is defined as the set of all symbols $x = (x_1, x_2, x_3, \ldots)$, where x_k belongs to A_k. In this sense the torus-space Q_ω is the product of the coordinate circles $c_1, c_2, c_3, \ldots$ or

$$Q_\omega = (c_1, c_2, c_3, \ldots).^1$$

Of the greatest importance for the present paper is now the definition of an *interval* in the space Q_ω. Denote as an arc b on a circle c either an ordinary open arc (perhaps the circle with exception of one point) or the circle itself. The obvious thing to do would be to denote as an interval in Q_ω any set of points obtained by choosing on any coordinate circle c_k an arc b_k and then forming the product

$$I = (b_1, b_2, b_3, \ldots) \tag{2.1}$$

of these arcs. This definition, however, turns out to be very unsatisfactory. Our theory depends entirely on the fact, that we admit as intervals only those sets of the form (2.1) for which only a *finite* number of the arcs $b_1, b_2, b_3, \ldots$ are ordinary arcs, the rest of them being the coordinate circles themselves. The lengths of the arcs are called the *edge-lengths* of the interval; so for an interval in our sense the edge-lengths are all $= 1$ with the exception of a finite number of them which are $\leqq 1$. The space itself is an interval and its edge-lengths are all $= 1$.

[1] In the applications we shall also consider spaces of the form (R_n, Q_ω) where R_n is, as usual, a Euclidean space,. It was found convenient not to complicate the general theory by the consideration of this case.

It is often convenient to consider Q_ω as the product of the n-dimensional torus-space

$$Q_n = (c_1, c_2, \ldots, c_n)$$

and the infinite-dimensional torus-space

$$Q_{n,\omega} = (c_{n+1}, c_{n+2}, \ldots).$$

We write then in accordance with our general notation $Q_\omega = (Q_n, Q_{n,\omega})$. If $x' = (x_1, x_2, \ldots, x_n)$ and $x'' = (x_{n+1}, x_{n+2}, \ldots)$ are two points of Q_n and $Q_{n,\omega}$, we denote the corresponding point of Q_ω by $x = (x', x'')$. The points x' and x'' are said to be the *projections* of x on the spaces Q_n and $Q_{n,\omega}$. If we project all the points of a set in Q_ω on Q_n or $Q_{n,\omega}$ we get the projection of the set itself. The projections of the interval (2.1) on Q_n and $Q_{n,\omega}$ are the intervals

$$I' = (b_1, b_2, \ldots, b_n)$$

and

$$I'' = (b_{n+1}, b_{n+2}, \ldots);$$

so we have $I = (I', I'')$. If n is large enough we have $I'' = Q_{n,\omega}$ and so $I = (I', Q_{n,\omega})$. For an arbitrary set A in Q_ω the relation $A = (A', A'')$ will generally not be true. If we have $A = (A', Q_{n,\omega})$ or $A = (Q_n, A'')$ the set A is called a *cylinder;* in the first case its base is the set A' in Q_n, in the second case the set A'' in $Q_{n,\omega}$.

Our theory of functions in the space Q_ω will be developed in close analogy to the theory of the space Q_n, but it should be observed that the theory of the space Q_ω contains that of the space Q_n. In fact any function $f(x_1, x_2, \ldots, x_n)$ in Q_n may as well be considered as a function in Q_ω, which does not depend on the variables $x_{n+1}, x_{n+2}, \ldots$ and to any set A' in Q_n corresponds the cylinder $A = (A', Q_{n,\omega})$ in Q_ω. Functions which only depend either on the variables $x_1, x_2, \ldots, x_n$ or on the variables $x_{n+1}, x_{n+2}, \ldots$ play an important rôle in the theory.

§ 3. Limit Points. Covering Theorems.

A sequence of points $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)$ in Q_ω is said to be *convergent* and to have the limit point $x = (x_1, x_2, x_3, \ldots)$ if $x_k^{(n)} \to x_k$ as $n \to \infty$ for any fixed k (but not necessarily uniformly in k); we then write $x^{(n)} \to x$ as $n \to \infty$. Let the sequence $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ be convergent to the limit point x and let I_x be an interval surrounding x; then it follows immediately that $x^{(n)}$ must lie in I_x

for all sufficiently large n. The converse of this is also true: Suppose that the sequence $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ and the point x have the property that for any interval I_x containing x the points $x^{(n)}$ of the sequence lie ultimately in I_x, then the sequence must be convergent and must have the limit point x. The Weierstrass-Bolzano theorem is true for the space Q_ω, that is: *Any sequence* $x^{(1)}, x^{(2)}, x^{(3)}, \ldots$ *of points in* Q_ω *contains a convergent subsequence.* This is proved in the ordinary way by means of the diagonal method.

If the limit point of every convergent sequence of points of a set A belongs to A then we say that A is a closed set; a set A is said to be open if its complementary set $Q_\omega - A$ is closed. An interval is evidently an open set, but the same is not true for a set of the form (2. 1) where an infinite number of the arcs b_k may be ordinary arcs. We shall now prove the following more general theorem: *A set* A *of points in* Q_ω *is open if and only if corresponding to any of its points* x *it contains an interval* I_x *surrounding* x.

The one part of this theorem is obvious: If the set A contains corresponding to any of its points x an interval I_x surrounding x, then no point x of A can be a limit point for the complementary set $Q_\omega - A$; so this set must be closed and hence A open. In order to prove the converse suppose A to be open and $x = (x_1, x_2, x_3, \ldots)$ to be a point of A, and consider for any n the interval $I^{(n)} = (b_1, b_2, \ldots, b_n, c_{n+1}, c_{n+2}, \ldots)$ where b_k for $1 \leqq k \leqq n$ is the arc on c_k which has its midpoint in x_k and the length $\frac{1}{n}$; then it is clear that $I^{(n)}$ must belong to A for all sufficiently large n; for if not we should have for any n a point $x^{(n)}$ of $Q_\omega - A$ lying in $I^{(n)}$ and $Q_\omega - A$ would not be closed since evidently $x^{(n)} \to x$ as $n \to \infty$.

If we add to an arbitrary set A all points outside A which are limit points for sequences of points of A we obtain the closure $\bar{A}$ of A, which is the smallest closed set containing A. If $A = (e_1, e_2, e_3, \ldots)$ where e_k is an arbitrary set on the coordinate circle c_k we have $\bar{A} = (\bar{e}_1, \bar{e}_2, \bar{e}_3, \ldots)$. If I is an interval we call $\bar{I}$ the corresponding closed interval.

Finally we call to mind the two classical *covering theorems:*

1. The covering theorem of Lindelöf. *If to any point* x *of a set* A *in* Q_ω *there corresponds an interval* I_x *surrounding* x *then we can find a finite or enumerable number of these intervals which will cover* A.

In order to prove this we denote as a rational point on each coordinate circle a point with rational abscissa and as a rational arc an arc whose endpoints are rational (the circle itself shall also be considered as a rational arc). An interval I in Q_ω is now called a rational interval, when the defining arcs are all rational; evidently there is only an enumerable number of rational intervals in Q_ω. Now any interval I_x surrounding a point x in Q_ω certainly contains a rational interval surrounding x. This proves the theorem.

2. The covering theorem of Borel. *If the set A considered is a closed set, then a finite number of the intervals will always be enough to cover A.*

A simple way of proving this theorem is to deduce it from Lindelöf's theorem. Suppose that A is covered by a sequence of intervals $I^{(1)}, I^{(2)}, I^{(3)}, \ldots$ and denote by $A^{(n)}$ the part of A which lies outside the first n intervals $I^{(1)}, I^{(2)}, \ldots, I^{(n)}$; then $A^{(n)}$ is evidently closed. We have to prove that $A^{(n)}$ vanishes for all sufficiently large n; this, however, is clear; for from the Weierstrass-Bolzano theorem it easily follows that a sequence of closed sets $A^{(1)} \geqq \geqq A^{(2)} \geqq A^{(3)} \geqq \cdots$, none of which vanishes, must have a point in common; and such a point does not exist in our case.

We could of course carry this study of the space Q_ω further; we shall only add the following theorem, which is in itself of no great interest but which we shall require later on: Let A' denote an arbitrary set in Q_n and let the set $A = (A', Q_{n,\omega})$ in Q_ω be enclosed in an open set O in Q_ω; then there exists an open set U' in Q_n which contains A' and is such that $U = (U', Q_{n,\omega})$ is contained in O. The proof is as follows: For a fixed point x' of A' we consider the set $(x', Q_{n,\omega})$ in Q_ω; this set is contained in O, so to any of its points x there corresponds an interval I_x surrounding x and contained in O; now the set $(x', Q_{n,\omega})$ is closed, so by Borel's theorem a finite number of the intervals I_x will cover the set and we may conclude that there exists an interval I' in Q_n containing x' such that $(I', Q_{n,\omega})$ is contained in O. This proves the theorem. We have also the corresponding theorem, that if a set of the form (Q_n, A''), where A'' is a set in $Q_{n,\omega}$, is contained in an open set O in Q_ω, then A'' is contained in an open set U'' such that $U = (Q_n, U'')$ lies in O.

§ 4. Continuous and Semi-continuous Functions.

A function $f(x) = f(x_1, x_2, x_3, \ldots)$ defined in Q_ω is said to be *continuous* (vollstetig) in Q_ω, if $f(x^{(n)}) \to f(x)$ whenever $x^{(n)} \to x$. It follows easily: A function $f(x)$ is continuous if and only if for any x in Q_ω and for any $\varepsilon > 0$ there exists an interval I_x surrounding x such that the inequality

$$|f(x) - f(x')| < \varepsilon$$

holds for all points x' in I_x. If we keep ε fixed and vary x, it follows at once from the covering theorem of Borel (since Q_ω is closed) that any continuous function $f(x)$ must be bounded in Q_ω. A real function $f(x)$ is continuous if, and only if, for any real a the points where $f(x) \geqq a$ or $f(x) \leqq a$ both form a closed set. The function attains both its upper and its lower bound.

We shall give one more property of continuous functions which also follows immediately from Borel's covering theorem: If $f(x) = f(x_1, x_2, x_3, \ldots)$ is continuous in Q_ω then there exists for any $\varepsilon > 0$ a number n and a number $\delta > 0$, such that the inequality

$$|f(x_1, x_2, x_3, \ldots) - f(x_1', x_2', x_3', \ldots)| < \varepsilon$$

holds whenever the n inequalities $|x_k - x_k'| < \delta$ for $1 \leqq k \leqq n$ are fulfilled. This theorem shows that any continuous function in Q_ω may be uniformly approximated by continuous functions, each of which depends only on a finite number of the variables.

A real function $f(x)$ defined in Q_ω is called *semi-continuous* from above in Q_ω if $\overline{\lim_{n\to\infty}} f(x^{(n)}) \leqq f(x)$ whenever $x^{(n)} \to x$; a function is semi-continuous from above if, and only if, for any real a the points where $f(x) \geqq a$ form a closed set. A function $f(x)$ is semi-continuous from below if $-f(x)$ is semi-continuous from above.

§ 5. The Lebesgue Measure.

The way to a *theory of measure* in Q_ω is now rather obvious; we need only attach to any (open) interval I (in the sense defined above) as measure the product of its edge-lengths and then apply the ordinary Lebesgue definitions. These definitions are the following:

Let A be any set in Q_ω and consider all coverings of A with a (finite or) enumerable number of intervals I; determine for each such covering the sum of the measures of the covering intervals; the set of numbers obtained certainly contains the number 1 since Q_ω itself is a covering interval. We call its lower bound $m_e A$ the *exterior Lebesgue measure* of the set A; the *interior Lebesgue measure* $m_i A$ of A is defined by the relation $m_i A = 1 - m_e(Q_\omega - A)$.

We have $0 \leqq m_e A \leqq 1$ and hence also $0 \leqq m_i A \leqq 1$. Further $m_i A \leqq m_e A$ or $m_e A + m_e(Q_\omega - A) \geqq 1$; for two coverings of A and $Q_\omega - A$ form together a covering of Q_ω. Then, however, it follows from Borel's covering theorem that a finite number of the intervals considered will cover Q_ω; now these intervals may for some fixed n all be written in the form $I = (I', Q_{n,\omega})$ where I' denotes an interval in Q_n, whose measure is equal to that of I. Now as these intervals I' must cover Q_n the sum of their measures must be $\geqq 1$ and so the theorem is proved.

If the interior measure is equal to the exterior measure, the set A is said to be measurable in the Lebesgue sense with the infinite-dimensional measure

$$mA = m_i A = m_e A.$$

Now it must of course be proved, that *intervals are measurable sets* and that their measure is equal to that already defined. This could easily be proved directly but follows also from the more general remark that if A' denotes any set in Q_n then the exterior and interior measure of the cylinder $A = (A', Q_{n,\omega})$ will be equal to the exterior and interior n-dimensional measure of A'. In particular the two sets A and A' will be measurable together and with the same measure. In order to prove this we suppose first that A' is measurable; then the relation $m_e A + m_e(Q_\omega - A) \geqq 1$ in connection with the relations

$$mA' + m(Q_n - A') = 1, \quad mA' \geqq m_e A, \quad m(Q_n - A') \geqq m_e(Q_\omega - A)$$

give

$$m_e A = mA', \quad m_e(Q_\omega - A) = m(Q_n - A')$$

and so $m_i A = m_e A = mA'$. Suppose now that A' is not measurable; it is evidently enough to prove that $m_e A = m_e A'$ and since $m_e A \leqq m_e A'$ it is enough to prove that $m_e A + \varepsilon > m_e A'$ for any $\varepsilon > 0$. Consider now a (finite or) enumerable number of intervals I covering A such that the sum of their measures is $< m_e A + \varepsilon$; denote the open set composed of these intervals by O; then we have clearly $m_e A + \varepsilon > m_e O$; now we have proved before that there exists an open set U' in Q_n containing A' so that $U = (U', Q_{n,\omega})$ is contained in O; but this proves the theorem since then by the case already considered

$$m_e A + \varepsilon > m_e O \geqq mU = mU' \geqq m_e A'.$$

The theorem is to be considered as showing that *n-dimensional measure* (in the space Q_n) *is a particular case of the infinite-dimensional measure.* It follows especially that any closed interval $\bar{I}$ in Q_ω is also measurable and that its measure is equal to the product of its edge-lengths.

In exactly the same way it can be proved that if A'' denotes an arbitrary set in $Q_{n,\omega}$ and $A = (Q_n, A'')$ is the corresponding cylinder in Q_ω, then the two sets A'' and A have the same exterior and interior measure (now both measures are infinite-dimensional). The proof depends on the theorem (not yet established) that any open infinite-dimensional set is measurable. Observe also the following theorem which is very easy to prove: If A' and A'' are arbitrary *measurable* sets in Q_n and $Q_{n,\omega}$, then the set $A = (A', A'')$ in Q_ω is also measurable and we have $mA = mA' \cdot mA''$.

A special rôle is played by the null-sets in Q_ω, that is the measurable sets of measure o. The sum of an enumerable number of null-sets is again a null-set. If Q_ω^* is a subset of Q_ω which differs from Q_ω only by a null-set and if A^* denotes the common part of Q_ω^* and an arbitrary set A in Q_ω, then it is easily seen that $m_e A = m_e A^*$ and $m_i A = m_i A^*$. As an example of a null-set we mention the set of all points in Q_ω whose coordinates are not all irrational.

§ 6. The Construction of Nets.

After the above discussion it will not be surprising that the measure introduced has all the properties of the n-dimensional measure in the space Q_n. This could be proved by repeating all the ordinary proofs. It is, however, much easier to use a simple transferring principle, which gives everything without the trouble of repeating all arguments. This transferring principle depends entirely on the concept of a *net*, introduced with such great success in the theory of real functions by de la Vallée-Poussin. In the case of functions of a finite number of variables the importance of the nets is rather that they give a simple technique for proving theorems which have themselves nothing to do with the nets and could have been proved without them. In the theory of the space Q_ω it seems (as we shall see later on) that the use of nets is the only way of obtaining the deeper results. We shall have to use nets not only for the proof of the transferring principle but also later on; we therefore postpone the proof of the transferring principle and deal in this section only with the construction of nets.

We shall speak of a *dissection* of a circle c into arcs b, when we leave out of c a finite number of points; the extreme cases where no point, or only one point, is left out are also to be considered as dissections. When we denote the arcs by b they are supposed to be open; the corresponding closed arcs are denoted by $\bar{b}$. Now consider the first n coordinate circles c_k which form the torus-space Q_n and take a dissection of each of them into arcs b_k; then we obtain a dissection D' of Q_n into intervals, which we shall denote by I', by taking all intervals of the form

$$I' = (b_1, b_2, \ldots, b_n)$$

where each b_k is an arc from the dissection of c_k considered. We obtain a *subdissection* of a circle c by leaving out more points of c; accordingly a subdissection of Q_n is obtained by taking subdissections of the coordinate circles c_k or at least of some of them. We now define a *net in* Q_n as a sequence of dissections $D'_1, D'_2, D'_3, \ldots$ where D'_{m+1} is always a subdissection of D'_m and where further the fundamental condition is fulfilled that the maximum of the edge-lengths of all intervals in D'_m tends to zero as $m \to \infty$.

This definition is easily generalised to the case of the space Q_ω. In order to obtain a dissection D of Q_ω we take a dissection of each of the coordinate circles c_k into arcs b_k but so that only a *finite* number of these dissections are real dissections no points being left out from the rest of the circles. Then again we may form the set of all intervals

$$I = (b_1, b_2, b_3, \ldots)$$

and these sets I are indeed intervals in Q_ω according to our definition. It follows at once that any dissection D of Q_ω may also be considered as generated by a dissection of the space Q_n where n is sufficiently large, in the sense that the intervals I of the dissection D are all of the form $I = (I', Q_{n,\omega})$, the intervals I' forming a dissection of Q_n. Now consider a sequence of dissections $D_1, D_2, D_3, \ldots$ of Q_ω where D_{n+1} is always a subdissection of D_n; we then say that this sequence forms a *net in* Q_ω if for any fixed k the maximal length of the arcs of that dissection of c_k which corresponds to D_n tends to zero as $n \to \infty$ (but evidently not uniformly in k). With this definition of a net in Q_ω the fundamental property holds that if $\overline{I}_1 \geqq \overline{I}_2 \geqq \overline{I}_3 \geqq \cdots$ denotes any sequence of closed intervals such that I_n for any n belongs to D_n, then these intervals will have exactly one point in common. Obviously there *exist* nets in Q_ω. The set of all points which lie on the boundary of some interval in a net form a null-set.

§ 7. The Transferring Principle.[1]

By means of the construction of nets it is now easy to prove the transferring principle referred to above. Let Q and q be two torus-spaces of the kind

[1] Cf. H. Lebesgue [1] 367; F. Riesz [1] 497; C. de la Vallée-Poussin [1].

considered; they need not be of an infinite number of dimensions; suppose for instance that Q is Q_ω and q a circle of length 1. We shall prove that there exists an application of Q on q which *conserves the measure* in the following precise sense:

There exists a one-one application of the points of Q with the exception of a null-set on the points of q with the exception of a null-set, with the property that corresponding sets in Q and q have always the same exterior and interior measure.

For our present object (the proof that the measure in Q_ω has all the properties of the measure on a circle) the existence of one such application in the special case mentioned above would be sufficient; the way in which it is constructed is, however, of importance for the applications, and it is also of importance that we do not restrict ourselves to this case alone.

We consider a sequence of dissections $D_1, D_2, D_3, \ldots$ which form a net in Q and a sequence of dissections $d_1, d_2, d_3, \ldots$ which form a net in q. The intervals of D_n will be denoted by I_n, the intervals of d_n by i_n. Now we suppose that the two nets in Q and q *correspond* in the sense that for any n we have a one-one correspondence between the intervals I_n of D_n and the intervals i_n of d_n with the following two properties: Corresponding intervals I_n and i_n have always the same measure; to an interval I_{n+1} contained in an interval I_n always corresponds an interval i_{n+1} contained in the corresponding interval i_n. Evidently there *exist* corresponding nets in Q and q.

We now consider any sequence

$$\bar{I}_1 \geqq \bar{I}_2 \geqq \bar{I}_3 \geqq \cdots \tag{7.1}$$

of closed intervals in Q, one from each of the dissections D_n; this sequence determines uniquely a point x of Q but there are points of Q which are determined by more than one sequence (7.1); those are the points x which lie on the boundary of some interval I_n. Now to any sequence (7.1) by the correspondence of the two nets there corresponds a sequence

$$\bar{i}_1 \geqq \bar{i}_2 \geqq \bar{i}_3 \geqq \cdots \tag{7.2}$$

of closed intervals in q, which determines uniquely a certain point t in q. So if we let two points x and t of Q and q correspond if they are determined by corresponding sequences (7.1) and (7.2), we obtain an application of Q on q. This application is not a one-one application except in trivial cases — the possibilities are illustrated in fig. 1 — but we shall show that if we only consider the application for pairs of corresponding points x and t, each of which has only one corresponding point, the exceptional sets are null-sets and the application has the desired properties. The proof of this is simple, but it requires a certain care and we may of course during the proof only apply the few properties of the measure which we have already established in § 5.

It is convenient to delay for a moment the omission of the exceptional sets and first consider the application as it is defined by the correspondence between the sequences (7. 1) and (7. 2). For any set S in Q we may consider the set s of all points in q which correspond to some point of S (this does not imply that S contains all points in Q which correspond to some point of s);

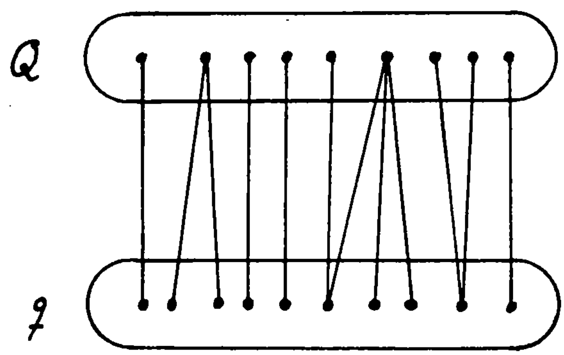

Fig. 1.

we can now prove very easily that $m_e s \leqq m_e S$. Consider for a given $\varepsilon > 0$ a (finite or) enumerable number of intervals I covering S, so that the sum of the measures of these intervals is $< m_e S + \varepsilon$. Take now first all the closed intervals $\bar{I}_1$ of D_1 which belong to one of these covering intervals I, then all the closed intervals $\bar{I}_2$ of D_2 which belong to one of the intervals I without being contained in one of the intervals $\bar{I}_1$ already selected, and continue this process; the sum of the measures of all the closed intervals $\bar{I}_n$ obtained in this way is clearly $< m_e S + \varepsilon$; furthermore for each sequence (7. 1) which defines a point x of S, the first interval $\bar{I}_n$ of the sequence which belongs to one of the intervals I is among the selected intervals $\bar{I}_n$. Now consider the set in q composed of all the corresponding intervals $\bar{i}_n$; this set must contain s; now any interval $\bar{i}_n$ can be enclosed in an open interval whose measure is only slightly greater; so we get $m_e s < m_e S + \varepsilon$ and since ε was arbitrary, $m_e s \leqq m_e S$. We have of course the corresponding result that if r is a set in q and R the set of all points in Q which correspond to the points of r, then $m_e R \leqq m_e r$.

Now consider first the set S_1 of all points in Q which have more than one corresponding point in q, and denote by s_1 the set of all corresponding points in q; the set S_1 is known to be a null-set; since $m_e s_1 \leqq m_e S_1$ we see that s_1 must also be a null-set. Similarly if r_1 denotes the set of points in q with more than one corresponding point in Q, and if R_1 is the corresponding set in Q, we have that r_1, and hence also R_1, is a null-set. Now denote by Q^* the set obtained from Q by leaving out the two null-sets S_1 and R_1 and by q^* the set obtained from q by leaving out the sets s_1 and r_1; then Q^* and q^* are formed by all the pairs of points x and t each of which has the other point as its only corresponding point. Q^* and q^* differ from Q and q by null-sets and the one-one application of Q^* on q^* preserves the exterior and interior measure; in fact, if S and s are corresponding sets in Q^* and q^*, we have both $m_e s \leqq m_e S$ and $m_e S \leqq m_e s$ and hence $m_e S = m_e s$, and the corresponding result for the interior measure follows by considering the complementary sets with respect to Q^* and q^*, using the remark that the omission of a null-set does not alter either the exterior or the interior measure.

Thus the theorem is proved; together with the above remark on null-sets the transferring principle proves that *the measure in Q_ω has all the properties of the measure on a circle;* we emphasize especially the main theorem: If $A_1, A_2, A_3, \ldots$ denotes a finite or enumerable sequence of measurable sets in Q_ω, then the common part and the sum of these sets are again measurable. If no two of the sets have common points we have further

$$m(A_1 + A_2 + A_3 + \cdots) = mA_1 + mA_2 + mA_3 + \cdots.$$

From this theorem it follows that any *open* set A in Q_ω must be measurable; in fact, if we choose for each point x of A an interval I_x surrounding x and contained in A, it follows from Lindelöf's theorem that A is the sum of a finite or enumerable number of these intervals. From this it follows that also any *closed* set, as complementary set to an open set, must be measurable.

§ 8. The Jordan Measure.

In the general theory we use exclusively the infinite-dimensional Lebesgue measure; in the applications, however, the corresponding Jordan measure also plays an essential rôle. We define this measure imitating the ordinary definitions.

Let A denote an arbitrary set in Q_ω. We consider all coverings of A by a *finite* number of intervals I and determine for each such covering the sum of the measures of the covering intervals. We define the *exterior* Jordan measure of the set A to be the lower bound $\mu_e A$ of the numbers obtained; the *interior* Jordan measure is defined by the relation $\mu_i A = 1 - \mu_e(Q_\omega - A)$. We have evidently $0 \leqq \mu_e A \leqq 1$ and hence also $0 \leqq \mu_i A \leqq 1$; the relation $\mu_i A \leqq \mu_e A$ is for the Jordan measure an elementary relation.

If the interior Jordan measure is equal to the exterior, we say that the set A is measurable in the Jordan sense with the infinite-dimensional Jordan measure

$$\mu A = \mu_i A = \mu_e A.$$

The properties of the Jordan measure are, exactly as in the n-dimensional case, most easily treated by means of the Lebesgue measure. Let A be an arbitrary set in Q_ω and let $\bar{A}$ denote the closure of A in the sense defined above; since $\bar{A}$ is closed, it is measurable in the Lebesgue sense and we shall now prove that

$$\mu_e A = m\bar{A}.$$

That $\mu_e A \leqq m\bar{A}$ follows at once from Borel's covering theorem; in fact, if we have a covering of $\bar{A}$ by a (finite or) enumerable set of intervals I, then a finite number of these intervals will cover $\bar{A}$ and so A. On the other hand, if A is covered by a finite number of intervals I, then the corresponding closed intervals $\bar{I}$ will cover $\bar{A}$; so $\mu_e A \geqq m\bar{A}$ and the two inequalities together give the result. By consideration of the complementary sets we get immediately the corresponding result for the interior Jordan measure that

$$\mu_i A = m\underline{A}$$

where $\underline{A} = Q_\omega - \overline{(Q_\omega - A)}$ denotes the open kernel of A (that is, the open set composed of all interior points of A). The two relations together show that a set A is measurable in the Jordan sense if and only if the closed set $\bar{A} - \underline{A}$, which we may denote as the *boundary* of A, has the Lebesgue measure, and so also the Jordan measure, zero. From this result it follows at once that the sum and common part of a finite number of sets measurable in the Jordan sense are again measurable in the Jordan sense.

The infinite-dimensional Jordan measure is more nearly related to the n-dimensional measure than we should perhaps think at first sight. Let A denote an arbitrary set in Q_ω and let A_n for each n denote the projection of A on the n-dimensional torus-space Q_n; then the sequence of numbers

$$\mu_e A_1,\ \mu_e A_2,\ \mu_e A_3,\ \ldots$$

is evidently decreasing (in the wide sense); we shall prove that

$$\lim_{n\to\infty} \mu_e A_n = \mu_e A.$$

As the interior Jordan measure is defined by means of the exterior, this gives us a definition of the infinite-dimensional Jordan measure by means of the n-dimensional.[1] The proof is immediate; for in the first place we have $\mu_e A_n \geqq \mu_e A$ for all n; and on the other hand if $\varepsilon > 0$ is given, then there exists a covering of A by a finite number of intervals such that the sum of their measures is $< \mu_e A + \varepsilon$; for all sufficiently large n these intervals have the same measure as their projections on Q_n; so for large n we have $\mu_e A_n < \mu_e A + \varepsilon$. A corresponding determination of the exterior Lebesgue measure in Q_ω is not possible.

§ 9. The Definite and Indefinite Integrals.

On the basis of the Lebesgue measure the notions of *measurable* and *integrable* functions $f(x) = f(x_1, x_2, x_3, \ldots)$ can now immediately be introduced.

Suppose first that $f(x)$ is a *real* function in Q_ω. If now for any a the set of points in which $f(x) \geqq a$ is a measurable set, then we call $f(x)$ a measurable function. Suppose this to be true and consider an arbitrary scale

$$\cdots < y_{-2} < y_{-1} < y_0 < y_1 < y_2 < \cdots$$

of increasing numbers for which $y_{-n} \to -\infty$ and $y_n \to \infty$ as $n \to \infty$ and further the number

$$\overline{\text{bound}}\ (y_{n+1} - y_n) \tag{9.1}$$

is finite; we denote by m_n the measure of that part of Q_ω in which we have $y_n \leqq f(x) < y_{n+1}$ and form the series

$$\sum_{n=-\infty}^{\infty} y_n m_n. \tag{9.2}$$

[1] This definition was used in H. Bohr—B. Jessen [1] 65—69.

If now for some scale this series is absolutely convergent, then it is absolutely convergent for any scale of the kind considered; the function $f(x)$ is then called integrable and its integral is defined as the limit of the sum of the series (9. 2) when the number (9. 1) tends to zero in an arbitrary way. A *complex* function $f(x)=u(x)+iv(x)$ is called measurable or integrable if the two functions $u(x)$ and $v(x)$ are both measurable or integrable respectively. In the latter case we define the *infinite-dimensional integral*

$$\int_{Q_\omega} f(x)\, dw_\omega \tag{9.3}$$

of $f(x)$ over Q_ω by integrating the real and imaginary parts separately.

Any *continuous* function $f(x)$ is integrable; for a continuous and real function is always bounded and the set of points in which $f(x) \geqq a$ is for any a closed and so measurable. It should also be observed that if a function $f(x)$ in Q_ω depends only on the variables $x_1, x_2, \ldots, x_n$, then it is measurable considered as a function in the n-dimensional torus-space Q_n if and only if it is measurable considered as a function in Q_ω; the functions are also integrable together and the integral over Q_ω is equal to the integral over Q_n. This remark may be considered as an *expression of the n-dimensional integral* (for functions in Q_n) *as a particular case of the infinite-dimensional.* Similarly if a function depends only on the variables $x_{n+1}, x_{n+2}, \ldots$, then it is integrable considered as a function in $Q_{n,\omega}$ if and only if it is integrable over Q_ω and the integrals are equal. We shall sometimes allow ourselves to write

$$\int_{Q_n} f(x)\, dw_n \quad \text{or} \quad \int_{Q_{n,\omega}} f(x)\, dw_{n,\omega}$$

instead of (9. 3) in these cases.

Exactly as in the n-dimensional case, the circumstances in a set of measure zero are of no importance in the integration. Two functions which differ only in a null-set will therefore in the following not be considered as different functions and a function will also be considered as defined in Q_ω if it is only defined outside a set of measure zero.

The integral has evidently *the same fundamental properties as the ordinary Lebesgue integral.* This follows immediately from the transferring principle; in fact if we denote by $x=x(t)$ the application of a circle q of length 1 on Q_ω

constructed by means of corresponding nets in Q_ω and q, then for any real function $f(x)$ in Q_ω the corresponding function $\varphi(t)=f(x(t))$ on q will have the same distribution of its values as $f(x)$ in the sense that for any real a the sets of points where we have $f(x) \geqq a$ and $\varphi(t) \geqq a$ will have the same exterior and the same interior measures. This, however, implies that the two functions $f(x)$ and $\varphi(t)$ will always be measurable or integrable together and in the latter case always with the same integral.

In addition to the definite integral (9. 3), for the considerations of the present paper the *indefinite* integral of an integrable function $f(x)$ in Q_ω is also very important. Let E denote an arbitrary measurable set in Q_ω; we consider that function defined in Q_ω which in E is equal to $f(x)$ and is 0 elsewhere; this function is again integrable; we denote its integral over Q_ω as the integral

$$F(E) = \int_E f(x)\, d\, w_\omega$$

of $f(x)$ over the set E. The function $F(E)$ of the variable set E is called the indefinite integral of $f(x)$. From the transferring principle follows at once the main theorem of Lebesgue:

A function $F(E)$ defined for all measurable sets E in Q_ω is the indefinite integral of an integrable function $f(x)$ in Q_ω if and only if it is additive and absolutely continuous, that is if, firstly, for any two sets E_1 and E_2 without common points we have

$$F(E_1 + E_2) = F(E_1) + F(E_2)$$

and, secondly, to any $\varepsilon > 0$ there corresponds an $\eta > 0$, so that

$$|F(E)| < \varepsilon \quad \text{when} \quad m\,E < \eta.$$

The indefinite integral determines uniquely the integrated function $f(x)$; but how shall we obtain $f(x)$ when $F(E)$ is given? The Lebesgue theory of symmetric derivatives certainly cannot be generalised to our case (at least there is no obvious generalisation); this follows from the fact that there are no symmetric neighbourhoods for the points of Q_ω, all intervals in Q_ω (except Q_ω itself) being highly unsymmetric. We may, however, always differentiate $F(E)$ on any *net* in Q_ω; so the construction of a net turns out to be of greater importance for the theory than merely to supply a simple technique of proofs. The differentiation theorem which we obtain is as follows:

Suppose that the sequence of dissections $D_1, D_2, D_3, \ldots$ of Q_ω form a net and denote by $\Delta_n(x)$, for any value of n, the »stepfunction» which in any interval I_n of the n-th dissection D_n is equal to the corresponding quotient

$$\frac{F(I_n)}{mI_n};$$

then the sequence of functions $\Delta_n(x)$ will tend to $f(x)$ as $n \to \infty$ almost everywhere in Q_ω.

The proof of this theorem follows immediately. On a circle q of length 1 we construct a net $d_1, d_2, d_3, \ldots$ which corresponds to the net $D_1, D_2, D_3, \ldots$ in Q_ω. We denote by $x = x(t)$ the corresponding application of q on Q_ω and by $\Phi(e)$ the indefinite integral of the function $\varphi(t) = f(x(t))$ on q; then we have evidently for any interval I_n in Q_ω and the corresponding interval i_n on q the relation

$$\frac{F(I_n)}{mI_n} = \frac{\Phi(i_n)}{mi_n};$$

hence to the sequence of functions $\Delta_n(x)$ in Q_ω there corresponds by the application of Q_ω on q a sequence of functions which tends to $\varphi(t)$ almost everywhere in q; but then the sequence $\Delta_n(x)$ must tend to $f(x)$ almost everywhere in Q_ω.

In the last theorem we have the fundamental starting point for a deeper study of integrable functions of an infinite number of variables. It shows that the relationship between these functions and integrable functions of a finite number of variables is not so distant as might have been expected from the beginning, each of the functions $\Delta_n(x)$ being in fact a function depending only on a finite number of the variables $x_1, x_2, x_3, \ldots$. This state of affairs is finally only a characteristic reflection of the definition of an interval upon which the theory is based.

§ 10. The Riemann Integral.

In the applications we shall also make use of an infinite-dimensional Riemann integral. We introduce this integral, imitating the ordinary definitions.

Let $f(x)$ be an arbitrary function in Q_ω which is real and bounded, and let us form, for any dissection D of Q_ω, the two sums

$$s(D) = \Sigma g(I)\, mI \quad \text{and} \quad S(D) = \Sigma G(I)\, mI$$

where the summations are over all intervals I of D, and $g(I)$ and $G(I)$ denote the lower and upper bounds of $f(x)$ in I. It is easily seen that

$$\overline{\text{bound}}\ \{s(D)\} \leqq \underline{\text{bound}}\ \{S(D)\}$$

where the upper and lower bounds are with respect to all dissections D of Q_ω. These two numbers define the lower and upper Riemann integrals of $f(x)$ over Q_ω. If they are equal, we call $f(x)$ integrable *in the Riemann sense*; in this case it is easily seen that $f(x)$ is measurable and that the integral is equal to the Lebesgue integral. When we say that a function is integrable in the Riemann sense, it is always understood that $f(x)$ is real and bounded.

The simplest way of dealing with the Riemann integral is to reduce it to the Lebesgue integral. If we introduce to a given function $f(x)$, which we suppose to be real and bounded, the two functions

$$\varphi(x) = \overline{\text{bound}}\ \{g(I_x)\} \quad \text{and} \quad \Phi(x) = \underline{\text{bound}}\ \{G(I_x)\}$$

where the upper and lower bounds are with respect to all intervals I_x surrounding x, it is seen that these functions $\varphi(x)$ and $\Phi(x)$ are semi-continuous from below and above respectively. The integrals of $\varphi(x)$ and $\Phi(x)$ are simply the lower and upper Riemann integrals of $f(x)$. We therefore obtain the usual criterion for the integrability in the Riemann sense that $\Phi(x) - \varphi(x)$ must be a null-function.

We shall also use the following remark: A function $f(x)$ is integrable in the Riemann sense if, and only if, there exists corresponding to any $\varepsilon > 0$ two *continuous* functions $a(x)$ and $A(x)$ such that

$$a(x) \leqq f(x) \leqq A(x) \text{ for all } x$$

and

$$\int_{Q_\omega} (A(x) - a(x))\, dw_\omega < \varepsilon.$$

It is always possible to choose for $a(x)$ and $A(x)$ functions depending only on a finite number of the variables $x_1, x_2, x_3, \ldots$; if we wish we may take $a(x)$ and $A(x)$ as finite trigonometrical polynomials.

A function $f(x)$ which takes only a finite number of values is integrable in the Riemann sense if, and only if, the sets of points where it takes its values are all measurable in the Jordan sense.

§ 11. An Important Lemma.

We shall sometimes meet, in the following considerations, functions $f(x)$ in Q_ω which have the property that for any two points x in Q_ω which differ only in a *finite* number of coordinates the function is either not defined in the two points or is defined and has the same values in both points. We shall call this property the *property* S. A set of points in Q_ω will be said to have the property S if the characteristic function of the set has the property, that is if any two points in Q_ω which differ only in a finite number of coordinates either both belong to the set or both belong to the complementary set. Another way of expressing that a function $f(x)$ has the property S is by saying that for *any* n the function does not depend on the variables $x_1, x_2, \ldots, x_n$ but may be considered as a function in $Q_{n,\omega}$; from this formulation it is natural to conclude that the function cannot depend on anything but must be a constant. If the function $f(x)$ is *measurable,* this is actually true in the sense that the function must be a constant almost everywhere. This theorem, and the corresponding theorem for measurable sets, is a very useful lemma for many considerations:

A measurable function $f(x)$ in Q_ω which has the property S must be a constant almost everywhere. A measurable set with the property S has either the measure 0 *or the measure* 1.

It is clearly sufficient to prove the theorem for sets. The proof follows at once from the differentiation theorem given above. Let A denote a measurable set in Q_ω with the property S and let $f(x)$ be the characteristic function of A; then for any value of n the corresponding function $\Delta_n(x)$ must be constant and $= mA$. This follows at once from the fact that for any value of n we have $A = (Q_n, A'')$, denoting by A'' the projection of A on $Q_{n,\omega}$; now A is measurable; it follows that also A'' must be measurable and that $mA = mA''$. On the other hand, if $I = (I', Q_{n,\omega})$ denotes an arbitrary interval in Q_ω with its base I' in Q_n, then we have $AI = (I', A'')$ and consequently $mAI = mI'\,mA'' = mI\,mA$. But this proves that we have for any interval I_n of the dissection to which $\Delta_n(x)$ belongs

$$\int_{I_n} f(x)\, dw_\omega = m\, A\, I_n = m\, I_n\, m\, A$$

and consequently $\Delta_n(x) = m\, A$ in all points of Q_ω. Hence since $\Delta_n(x) \to f(x)$ almost everywhere and since $f(x)$ only takes the two values 0 and 1, we must have either $m\, A = 0$ or $m\, A = 1$.

There is a corresponding theorem that if a measurable function of a single real variable has arbitrarily small periods then it must be a constant almost everywhere. This theorem is familiar. The application of the same idea to the space Q_ω is due to Steinhaus[1] who used the above lemma to prove an interesting theorem on analytic continuation of power series. I shall quote this theorem which is a standard example for the application of the lemma.

Let us consider all power series of the form

$$(11.1) \qquad f(z, x) = \sum_{k=1}^{\infty} a_k\, e^{2\pi i x_k} z^{k-1}$$

where the a_k are given numbers; all these series have the same radius of convergence; suppose this radius r to be > 0 and finite. Then to any point $x = (x_1, x_2, x_3, \ldots)$ in Q_ω there corresponds by (11.1) an analytic function $f(z, x)$ in $|z| < r$. Now consider those of these functions which are *not continuable* (for which the circle $|z| = r$ is a natural boundary). The corresponding set of points x in Q_ω has clearly the property S and it is also measurable; the latter assertion follows from the remark that it is the set of points x in Q_ω for which

$$\overline{\lim_{p \to \infty}} \left[\frac{|f^{(p)}(z, x)|}{p!}\right]^{\frac{1}{p}} = \frac{1}{r - |z|}$$

for all points $z = a + ib$ in $|z| < r$ where a and b are rational. Consequently we may conclude that the measure of the set is always either 0 or 1. Steinhaus proved that it is always the latter case that occurs and so gave a very natural interpretation of the theorem that *almost all power series are not continuable.* A more general (and more difficult) theorem for Dirichlet series was proved by Paley and Zygmund; I shall deal with a generalisation of this theorem in § 22.

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[1] H. Steinhaus [2].

§ 12. Application of Fubini's Theorem.

Our first problem will be to give an extension of Fubini's theorem, concerning the reduction of an integral over a space to simple integrations, to the case of the infinite-dimensional integral. For this purpose we need the following application of Fubini's theorem for functions of a finite number of variables.

Suppose that the coordinate sequence $x_1, x_2, x_3, \ldots$ has been divided in some way into a finite number of sets

$$\begin{array}{l} x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},\ \ldots \\ x_1^{(2)},\ x_2^{(2)},\ x_3^{(2)},\ \ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_1^{(n)},\ x_2^{(n)},\ x_3^{(n)},\ \ldots; \end{array}$$

some of these sets (but evidently not all) may be finite. If the point $x=(x_1, x_2, x_3, \ldots)$ describes Q_ω then each of the points $x^{(\nu)}=(x_1^{(\nu)}, x_2^{(\nu)}, x_3^{(\nu)}, \ldots)$ where $1 \leqq \nu \leqq n$ describes a certain torus-space which we may denote by $Q^{(\nu)}$ and which may be of a finite or infinite number of dimensions; we write as usual $x=(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and $Q_\omega=(Q^{(1)}, Q^{(2)}, \ldots, Q^{(n)})$. Then we have the following theorem:

If $f(x)$ denotes an arbitrary integrable function in Q_ω, then we have

$$\int_{Q_\omega} f(x)\, d\,w_\omega = \int_{Q^{(n)}} d\,w^{(n)} \ldots \int_{Q^{(2)}} d\,w^{(2)} \int_{Q^{(1)}} f(x^{(1)}, x^{(2)}, \ldots, x^{(n)})\, d\,w^{(1)}$$

where the integrations, carried out from the right to the left, are always possible almost everywhere and always lead to integrable functions of the remaining variables.

The proof follows at once from the transferring principle. In each of the torus-spaces $Q^{(\nu)}$ we construct a sequence of dissections $D_1^{(\nu)}, D_2^{(\nu)}, D_3^{(\nu)}, \ldots$ which form a *net* in $Q^{(\nu)}$ and we construct also a corresponding net of dissections $d_1^{(\nu)}, d_2^{(\nu)}, d_3^{(\nu)}, \ldots$ on a circle $q^{(\nu)}$ of length 1; then by these corresponding nets is determined an application $x^{(\nu)}=x^{(\nu)}(t^{(\nu)})$ of $q^{(\nu)}$ on $Q^{(\nu)}$, where $t^{(\nu)}$ denotes the parameter on $q^{(\nu)}$. Now for any m the dissections $D_m^{(1)}, D_m^{(2)}, \ldots, D_m^{(n)}$ generate a certain dissection D_m of Q_ω, each interval of D_m being the product of intervals, one from each of the dissections $D_m^{(\nu)}$; in the same way the dissections $d_m^{(1)}, d_m^{(2)}, \ldots, d_m^{(n)}$ generate a dissection d_m of the n-dimensional space q described by the point $t=(t^{(1)}, t^{(2)}, \ldots, t^{(n)})$. The two sequences $D_1, D_2, D_3, \ldots$ and $d_1, d_2, d_3, \ldots$ evidently determine corresponding nets in Q_ω and q and the application $x=x(t)$ of q on Q_ω to which these nets give rise is exactly the application $(x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (x^{(1)}(t^{(1)}), x^{(2)}(t^{(2)}), \ldots, x^{(n)}(t^{(n)}))$. So if we apply the ordinary theorem of Fubini for the space Q_n to the function $\varphi(t)=f(x(t))$ we obtain the theorem.

§ 13. Infinitely Multiple Integrals.

We are now able to give the promised *extension of Fubini's theorem* to functions of an infinite number of variables. Let $f(x)$ denote an integrable function in Q_ω; we write $Q_\omega=(Q_n, Q_{n,\omega})$; then it follows from the theorem of the last section that the integral

$$(13.1)\qquad \int\limits_{Q_n} f(x)\, d w_n = \int\limits_{c_n} d x_n \ldots \int\limits_{c_2} d x_2 \int\limits_{c_1} f(x_1, x_2, x_3, \ldots)\, d x_1$$

exists as an integrable function of the variables $x_{n+1}, x_{n+2}, \ldots$ defined almost everywhere in $Q_{n,\omega}$; it also follows that the integral of this function over $Q_{n,\omega}$ is equal to the integral of $f(x)$ over Q_ω which we denote by A. It is more convenient for the following considerations to consider the integral (13.1) not as a function in $Q_{n,\omega}$ but as a function in Q_ω which does not depend on the variables $x_1, x_2, \ldots, x_n$; denoting this function by $f_{n,\omega}(x)$ we have for all n

$$\int\limits_{Q_\omega} f_{n,\omega}(x)\, d w_\omega = A.$$

The theorem which we are going to prove states that the sequence of functions $f_{1,\omega}(x), f_{2,\omega}(x), f_{3,\omega}(x), \ldots$ is *convergent* almost everywhere in Q_ω and that its limit function is exactly the constant A; in other words, we have the theorem:

If $f(x)$ is an arbitrary integrable function in Q_ω, then

$$\int\limits_{Q_\omega} f(x)\, d w_\omega = \lim_{n\to\infty} \int\limits_{c_n} d x_n \ldots \int\limits_{c_2} d x_2 \int\limits_{c_1} f(x_1, x_2, x_3, \ldots)\, d x_1$$

for almost all points of Q_ω.

The proof is not quite obvious. It is evidently sufficient to consider the case where $f(x)$ is real. It is also sufficient to prove that we have

$$\overline{\lim_{n \to \infty}} f_{n,\omega}(x) \leqq A$$

almost everywhere in Q_ω, for then applying this to $-f(x)$ instead of $f(x)$ it follows that we have also

$$\underline{\lim_{n \to \infty}} f_{n,\omega}(x) \geqq A$$

almost everywhere in Q_ω. Now the function

$$\Phi(x) = \overline{\lim_{n \to \infty}} f_{n,\omega}(x)$$

has evidently the property S of § 11; in fact, since the function $f_{n,\omega}(x)$ does not depend on the variables $x_1, x_2, \ldots, x_n$, a change in a finite number of the variables will not alter $\Phi(x)$; hence it follows by the fundamental lemma that $\Phi(x)$ must be a constant, say $\Phi(x) = B$ almost everywhere. In order to prove that we have $A \geqq B$ we prove the following lemma:

Let E denote the set of points where

$$\overline{\text{bound}} \{f_{n,\omega}(x)\} = \overline{\text{bound}} \{f_{1,\omega}(x), f_{2,\omega}(x), f_{3,\omega}(x), \ldots\} > K; \tag{13. 2}$$

then

$$\int_E f(x)\, dw_\omega \geqq K\, m E. \tag{13. 3}$$

Let us first prove that this lemma implies $A \geqq B$; in fact, if we choose $K < B$, it follows from the relation $\Phi(x) > K$ almost everywhere in Q_ω that the relation (13. 2) must also be true almost everywhere in Q_ω; consequently the set E only differs from Q_ω by a null-set and so it follows from (13. 3) that we have

$$A = \int_{Q_\omega} f(x)\, dw_\omega \geqq K$$

and since this is true for any $K < B$ we get $A \geqq B$.

We prove the lemma by making $N \to \infty$ in the following more elementary lemma:

Let E_N denote the set of points where

(13.4) $$\overline{\text{bound}}\ \{f_{1,\omega}(x),\ f_{2,\omega}(x),\ \ldots,\ f_{N,\omega}(x)\} > K;$$

then

(13.5) $$\int\limits_{E_N} f(x)\, dw_\omega \geqq K\, m E_N.$$

Let B_n denote the set of points in Q_ω in which $f_{n,\omega}(x) > K$; then we have evidently

$$E_N = B_N + B_{N-1} B_N^* + B_{N-2} B_N^* B_{N-1}^* + \cdots + B_1 B_N^* B_{N-1}^* \cdots B_2^*$$

where for a moment the star is used to indicate the complementary set with respect to Q_ω. We write for abbreviation

$$A_n = B_n B_N^* B_{N-1}^* \cdots B_{n+1}^*.$$

Now the set B_n is for any n a cylinder with its base in $Q_{n,\omega}$; it follows that also the set A_n must be such a cylinder; let us write $A_n = (Q_n, A_n'')$, where A_n'' is the projection of A_n on $Q_{n,\omega}$; then we get immediately

$$\int\limits_{A_n} f(x)\, dw_\omega = \int\limits_{A_n''} dw_{n,\omega} \int\limits_{Q_n} f(x)\, dw_n = \int\limits_{A_n''} f_{n,\omega}(x)\, dw_{n,\omega} = \int\limits_{A_n} f_{n,\omega}(x)\, dw_\omega.$$

Now A_n is contained in B_n; so $f_{n,\omega}(x) > K$ in A_n and thus we get

$$\int\limits_{A_n} f(x)\, dw_\omega \geqq K m A_n.$$

Adding these relations for $n = 1, 2, \ldots, N$ and using the fact that no two of the sets A_n have common points, we get the relation (13.5)[1].

[1] The idea of this proof was suggested by a similar proof of Kolmogoroff; see A. Kolmogoroff [1]. Exactly the same proof gives the following theorem: Let $f(t)$ be an integrable function on a circle q of length 1 and let us write

$$f_m(t) = \frac{1}{m} \sum_{\mu=1}^{m} f\left(t + \frac{\mu}{m}\right).$$

Then

$$f_{2^n}(t) \to \int\limits_q f(t)\, dt$$

as $n \to \infty$ almost everywhere on q. The proof of the ergodic theorem of Birkhoff is based on the same idea but is more complicated; see A. Khintchine [1].

§ 14. Representation of a Function as the Limit of an Integral.

There is a theorem which is in a certain sense *dual* to that of the last section. We consider again an arbitrary integrable function $f(x)$ in Q_ω and now form for any n the integral

$$\int\limits_{Q_{n,\,\omega}} f(x)\, d w_{n,\,\omega};$$

this integral is according to § 12 an integrable function of the variables $x_1, x_2, \ldots, x_n$ defined almost everywhere in Q_n. It is, however, more convenient for our purposes to consider it not as a function in Q_n but as a function in Q_ω; this function, which does not depend on the variables $x_{n+1}, x_{n+2}, \ldots$, will be denoted by

$$f_n(x).$$

Now we have the following theorem:

The sequence of functions $f_1(x), f_2(x), f_3(x), \ldots$ is convergent almost everywhere in Q_ω and its limit function is exactly the function $f(x)$ itself.

Using the result of the last section we may also say: If $f(x)$ is an arbitrary integrable function in Q_ω, then

$$f(x) = \lim_{n \to \infty} \left\{ \lim_{p \to \infty} \int\limits_{c_{n+p}} d x_{n+p} \ldots \int\limits_{c_{n+2}} d x_{n+2} \int\limits_{c_{n+1}} f(x_1, x_2, x_3, \ldots)\, d x_{n+1} \right\}$$

almost everywhere in Q_ω.

In order to prove this[1] we shall have to use the theorem on differentiation on nets. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ denote a sequence of positive numbers such that the series $\sum\limits_{n=1}^{\infty} \varepsilon_n$ is convergent, and let $\delta_1, \delta_2, \delta_3, \ldots$ denote a sequence of positive numbers tending to zero. Our aim is to construct a sequence of dissections $D_1, D_2, D_3, \ldots$ of Q_ω which form a net and which have the following properties: For any n the dissection D_n shall be generated by a certain dissection of

[1] The following proof was suggested to me by the proof of a special case of the theorem given by Prof. F. Riesz and communicated to me by Dr. Kalmár. We shall apply the theorem to this special case in § 18.

Q_n (that is each interval of D_n is a cylinder with an interval in Q_n as base); further, denoting by $\Delta_n(x)$ the function defined in § 9 which belongs to D_n and by E_n the set of points in Q_ω where

$$(14.1) \qquad |\Delta_n(x) - f_n(x)| < \delta_n,$$

we must have $mE_n > 1 - \varepsilon_n$.

Suppose that we have such a net in Q_ω; then our theorem is immediately proved. In fact, we know from § 9 that $\Delta_n(x) \to f(x)$ almost everywhere in Q_ω; so what we shall prove is that the properties of the net imply $\Delta_n(x) - f_n(x) \to 0$ almost everywhere in Q_ω. This, however, is clear, for on account of the assumed convergence of the series $\sum_{n=1}^{\infty} \varepsilon_n$ almost all points x of Q_ω must *ultimately* belong to the sets of the sequence $E_1, E_2, E_3, \ldots$ and for any such point the property follows, since $\delta_n \to 0$ as $n \to \infty$.

It remains to construct a net in Q_ω with the required properties; we do this by induction. Suppose that the dissections $D_1, D_2, \ldots, D_{n-1}$ have already been constructed. D_{n-1} is generated by a certain dissection of Q_{n-1} hence also by a dissection of Q_n. We now take a sequence of dissections $D^{(1)}, D^{(2)}, D^{(3)}, \ldots$ of Q_ω beginning with D_{n-1} and generated by dissections of Q_n, so that these dissections form a net in Q_n. For an interval $I^{(\nu)}$ of $D^{(\nu)}$ we have, since $f_n(x)$ depends only on $x_1, x_2, \ldots, x_n$,

$$\int_{I^{(\nu)}} f(x)\, dw_\omega = \int_{I^{(\nu)}} f_n(x)\, dw_\omega;$$

so the function $\Delta^{(\nu)}(x)$ defined by

$$\Delta^{(\nu)}(x) = \frac{1}{m\, I^{(\nu)}} \int_{I^{(\nu)}} f(x)\, dw_\omega$$

in any interval $I^{(\nu)}$ of $D^{(\nu)}$ is equal to the corresponding function where $f(x)$ has been replaced by $f_n(x)$. Hence, since the sequence $D^{(1)}, D^{(2)}, D^{(3)}, \ldots$ was generated by a net in Q_n and since $f_n(x)$ does not depend on $x_{n+1}, x_{n+2}, \ldots$, it follows from the theorem on differentiation on a net in Q_n that we must have $\Delta^{(\nu)}(x) \to f_n(x)$ almost everywhere in Q_ω; thus for any sufficiently large value of ν we have by Egoroff's theorem

$$|\Delta^{(\nu)}(x)-f_n(x)| < \delta_n$$

in a set of measure $> 1-\varepsilon_n$, and consequently (14. 1) will be fulfilled if we take $D_n = D^{(\nu)}$. In this way we get a sequence of dissections $D_1, D_2, D_3, \ldots$ where D_{n+1} is always a subdissection of D_n but we cannot be sure that these dissections form a *net* in Q_ω; this, however, can easily be obtained; we only need in the choice of D_n to take care that all edge-lengths of the intervals of the generating dissection of Q_n are (say) $< \frac{1}{n}$, which will be true if we choose ν large enough.

§ 15. Strong Convergence.

So far the only notion of convergence with which we have worked has been that of convergence almost everywhere. In this sense we obtained in §§ 9, 13 and 14 (with the notations there used) the results

$$(15.1) \qquad \Delta_n(x) \to f(x), \quad f_{n,\omega}(x) \to A \quad \text{and} \quad f_n(x) \to f(x).$$

There are, however, other notions of convergence which are of greater importance for most applications, above all the notion of *strong convergence*. It is therefore not without interest that the limit relations (15.1) are also true when the arrow is used to denote convergence in this sense.[1]

We say that the (measurable) function $f(x)$ belongs to the class L^p where $p \geqq 1$ if $|f(x)|^p$ is integrable. If $f(x)$ and $g(x)$ both belong to L^p we define their *distance* $D_p(f, g)$ by

$$D_p(f, g) = \left[\int_{Q_\omega} |f(x)-g(x)|^p \, dw_\omega\right]^{\frac{1}{p}}.$$

If $h(x)$ is any function of L^p it follows by Minkowski's inequality that we have $D_p(f, g) \leqq D_p(f, h) + D_p(g, h)$ which is the usual triangle inequality.

[1] I had originally proved the theorems $f_{n,\omega}(x) \to A$ and $f_n(x) \to f(x)$ using the notion of convergence on the average (dem Masse nach) which is weaker than both the notion of strong convergence and the notion of convergence almost everywhere. It was pointed out to me by Prof. F. Riesz that the (well-known) argument used above would give the same theorems for the more convenient notion of strong convergence. Finally it was Prof. Daniell who suggested to me that the theorems should be true for convergence almost everywhere.

A sequence of functions $h_1(x)$, $h_2(x)$, $h_3(x)$, ... is said to converge strongly in the class L^p towards the function $f(x)$ if $D_p(f, h_n) \to 0$ as $n \to \infty$. Such a sequence cannot at the same time converge strongly to another function $g(x)$; in fact from $D_p(f, g) \leqq D_p(f, h_n) + D_p(g, h_n)$ would follow $D_p(f, g) = 0$ which is possible only when $f(x) - g(x) = 0$ almost everywhere.

We have now the following theorem:

If $f(x)$ belongs to the class L^p where $p \geqq 1$, then the relations (15.1) are all true not only in the sense of convergence almost everywhere but also in the sense of strong convergence in the class L^p.

For the first relation $\Delta_n(x) \to f(x)$ this follows at once from the corresponding theorem for functions of a finite number of variables by the transferring principle. Let the sequence of functions $\Delta_n(x)$ correspond to the net $D_1, D_2, D_3, \ldots$; we construct, just as at the end of § 9, a corresponding net $d_1, d_2, d_3, \ldots$ on a circle q of length 1; if then the application of q on Q_ω obtained in this way is denoted by $x = x(t)$, the function $\varphi(t) = f(x(t))$ on q will also belong to L^p; and to the sequence $\Delta_n(x)$ there corresponds a sequence of functions in q which tends strongly to $\varphi(t)$ in the class L^p.

Now from this theorem we may immediately deduce the corresponding result for the sequences $f_{n,\omega}(x)$ and $f_n(x)$. Suppose (as we may) that the net $D_1, D_2, D_3, \ldots$ to which the sequence $\Delta_n(x)$ belongs is such that D_n is always generated by a dissection of Q_n; then the result follows at once from the two inequalities

$$D_p(f_{n,\omega}, A) \leqq D_p(f, \Delta_n) \quad \text{and} \quad D_p(f_n, \Delta_n) \leqq D_p(f, \Delta_n)$$

which we obtain from a familiar inequality by the relations

$$\int_{Q_\omega} |f_{n,\omega}(x) - A|^p \, dw_\omega = \int_{Q_{n,\omega}} |f_{n,\omega}(x) - A|^p \, dw_{n,\omega} =$$

$$= \int_{Q_{n,\omega}} \left| \int_{Q_n} (f(x) - \Delta_n(x)) \, dw_n \right|^p dw_{n,\omega} \leqq$$

$$\leqq \int_{Q_{n,\omega}} dw_{n,\omega} \int_{Q_n} |f(x) - \Delta_n(x)|^p \, dw_n = \int_{Q_\omega} |f(x) - \Delta_n(x)|^p \, dw_\omega$$

and

$$\int\limits_{Q_\omega} |f_n(x) - \varDelta_n(x)|^p \, dw_\omega = \int\limits_{Q_n} |f_n(x) - \varDelta_n(x)|^p \, dw_n =$$

$$= \int\limits_{Q_n} \left| \int\limits_{Q_{n,\omega}} (f(x) - \varDelta_n(x)) \, dw_{n,\omega} \right|^p dw_n \leqq$$

$$\leqq \int\limits_{Q_n} dw_n \int\limits_{Q_{n,\omega}} |f(x) - \varDelta_n(x)|^p \, dw_{n,\omega} = \int\limits_{Q_\omega} |f(x) - \varDelta_n(x)|^p \, dw_\omega.$$

§ 16. Majorised Convergence.[1]

Let $h_1(x)$, $h_2(x)$, $h_3(x)$, ... denote a sequence of functions in Q_ω; then the function

$$H(x) = \overline{\text{bound}} \; |h_n(x)|$$

is called the (smallest) *majorant* of the sequence. In all cases where a sequence of functions is known to be convergent it is of particular interest to study the majorant of the sequence. We shall consider this problem in the case where $h_n(x)$ is either of the three sequences $\varDelta_n(x)$, $f_{n,\omega}(x)$ or $f_n(x)$ attached to an integrable function $f(x)$ in Q_ω. Our result is given by the following theorem:

If the function $f(x)$ belongs to the class L^p where $p > 1$, then the same is true for the majorant $H(x)$ of any of the three sequences $\varDelta_n(x)$, $f_{n,\omega}(x)$ and $f_n(x)$ and we have in any of the three cases:

$$(16.1) \qquad \int\limits_{Q_\omega} (H(x))^p \, dw_\omega \leqq \left(\frac{p}{p-1}\right)^p \int\limits_{Q_\omega} |f(x)|^p \, dw_\omega.$$

This theorem is closely connected with a well-known maximal theorem of Hardy and Littlewood.[2] It is convenient, however, to attach our proof not to the latter (which is possible only when $h_n(x)$ is either $\varDelta_n(x)$ or $f_n(x)$) but to a lemma which is implicitly contained in the proof given by F. Riesz for the maximal theorem.[3] Let us first observe that it is sufficient to consider the case

[1] The theorems of this section generalise some results of Paley and Zygmund; see R. E. A. C. Paley-A. Zygmund [1] 351, [2] 462, [3] 190.

[2] G. H. Hardy-J. E. Littlewood [1].

[3] F. Riesz [3]. The lemma is obtained by combining the last part of the proof of F. Riesz with an earlier inequality of Hardy. For the simplest possible proof of this inequality, see F. Riesz [2] 167—168.

where $f(x)$ is real and $\geqq 0$; this follows from the fact that in any of the three hypotheses the majorant $H(x)$ is not decreased when we replace $f(x)$ by $|f(x)|$. The lemma is now as follows:

Suppose that the two functions $f(x) \geqq 0$ and $H(x) \geqq 0$ satisfy for any K the condition

$$\int_E f(x)\, dw_\omega \geqq K mE \tag{16.2}$$

where E denotes the set of points in which $H(x) > K$, then if $f(x)$ belongs to L^p, where $p > 1$, the same is true for $H(x)$ and

$$\int_{Q_\omega} (H(x))^p\, dw_\omega \leqq \left(\frac{p}{p-1}\right)^p \int_{Q_\omega} (f(x))^p\, d\, w_\omega. \tag{16.3}$$

From this lemma the proof of the above theorem is immediate; we only need to prove that if $f(x) \geqq 0$ the functions $f(x)$ and $H(x)$ will satisfy the conditions of the lemma. In the case $h_n(x) = f_{n,\,\omega}(x)$ this is an immediate consequence of the lemma of § 13; in the cases where $h_n(x)$ is either $\Delta_n(x)$ or $f_n(x)$ we deduce this result from the following lemma which is the analogue of that of § 13:

Suppose that $f(x)$ is real and that $h_n(x)$ is either $\Delta_n(x)$ or $f_n(x)$; then if we denote by E the set of points where

$$\overline{\text{bound}}\ \{h_n(x)\} > K, \tag{16.4}$$

we have

$$\int_E f(x)\, dw_\omega \geqq K mE. \tag{16.5}$$

The proof is simpler than that of § 13. Let us denote by B_n the set of points in Q_ω where $h_n(x) > K$. Then

$$E = B_1 + B_2 B_1^* + B_3 B_1^* B_2^* + \cdots,$$

where the star is used to indicate the complementary set with respect to Q_ω. Let us write for shortness

$$A_n = B_n B_1^* B_2^* \ldots B_{n-1}^*.$$

When $h_n(x) = \varDelta_n(x)$ the set B_n is composed of a certain number of intervals taken from the dissection of Q_ω to which $\varDelta_n(x)$ belongs; it follows that the same is true of the set A_n. Consequently we have

$$\int_{A_n} f(x)\, d w_\omega = \int_{A_n} \varDelta_n(x)\, d w_\omega .$$

When $h_n(x) = f_n(x)$ the set B_n is a cylinder with its base in Q_n; it follows that A_n also is a similar cylinder; let us write $A_n = (A'_n, Q_{n,\omega})$ where A'_n is the projection of A_n on Q_n. Then we have

$$\int_{A_n} f(x)\, d w_\omega = \int_{A'_n} d w_n \int_{Q_{n,\omega}} f(x)\, d w_{n,\omega} = \int_{A'_n} f_n(x)\, d w_n = \int_{A_n} f_n(x)\, d w_\omega .$$

In both cases the set A_n is contained in B_n; thus $h_n(x) > K$ in A_n and we get

$$\int_{A_n} f(x)\, d w_\omega \geqq K m A_n .$$

Adding these inequalities for $n = 1, 2, 3, \ldots$, we get the desired inequality (16.5).

In the case where $p = 1$ the theorem fails; it is not difficult to construct examples showing that in this case the majorant $H(x)$ need not even be integrable.

For later application we call the attention to the following result, which is an easy deduction from the above theorem:

If $e^{\lambda |f(x)|^2}$ *is integrable for some* $\lambda > 0$, *then the same is the case for the function* $e^{\lambda (H(x))^2}$ *in any of the three hypotheses and*

$$(16.6) \qquad \int_{Q_\omega} e^{\lambda (H(x))^2} d w_\omega - 1 \leqq 4 \left(\int_{Q_\omega} e^{\lambda |f(x)|^2} d w_\omega - 1 \right).$$

The proof follows at once when we apply the expansion for $e^{\lambda y^2}$ and use the fact that $\left(\frac{p}{p-1}\right)^p$ is a decreasing function of p.

§ 17. Fourier Series.

The general theory of normalised orthogonal systems, in particular the Riesz-Fischer theorem and the Parseval equation, follows for functions in Q_ω either by repeating the usual arguments word by word or more simply by using the transferring principle. In contrast to this, the generalisation of the theory of ordinary Fourier series requires new considerations, peculiar to the space Q_ω.

We consider for any of the coordinate circles c_k the system of all pure oscillations

$$(17.1) \qquad e^{2\pi i p_k x_k}; \qquad p_k = 0, \pm 1, \pm 2, \ldots$$

these functions are known to form a complete orthogonal system within the class of all functions of L^2 on the circle c_k. Any integrable function of x_k has a Fourier series, its development in terms of the system (17. 1), which conversely determines the function completely.

Now in order to get from the single systems (17. 1) to a complete orthogonal system in Q_ω one might be tempted to denote as a *pure oscillation in* Q_ω any function

$$(17.2) \qquad \prod_{k=1}^{\infty} e^{2\pi i p_k x_k} = e^{2\pi i (p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots)}.$$

It is clear that this is without meaning, since the product generally does not exist. But if we restrict ourselves to the consideration of those products where only a *finite* number of the numbers p_k are $\neq 0$, then we have in (17.2) a certain system of functions in Q_ω. If we write

$$p = (p_1, p_2, p_3, \ldots)$$

where this notation is to be reserved for sequences of the kind considered, we may denote the linear form $p_1 x_1 + p_2 x_2 + p_3 x_3 + \cdots$ occuring in (17. 2) as px; then the pure oscillations in Q_ω are all functions of the form

$$(17.3) \qquad e^{2\pi i p x}$$

where $p = (p_1, p_2, p_3, \ldots)$ is a sequence of the kind considered. By $e^{-2\pi i p x}$ we shall, of course, denote the oscillation belonging to $-p = (-p_1, -p_2, -p_3, \ldots)$.

The system (17.3) is evidently a normalised orthogonal system in Q_ω; as, moreover, the functions of the system are all bounded, we may attach to any integrable function $f(x)$ in Q_ω a *Fourier series*

$$f(x) \sim \sum_p c_p e^{2\pi i p x}$$

with respect to the system (17.3), where

$$c_p = \int_{Q_\omega} f(x) e^{-2\pi i p x} d w_\omega$$

and where the (purely formal) summation is over all sequences p. Now we have, just as in the theory of Fourier series in the space Q_n, the main theorem:

An integrable function $f(x)$ in Q_ω is always uniquely determined by its Fourier series; that is, two functions have the same Fourier series only when they are identical.

We reduce this theorem to the corresponding theorem for the space Q_n, using the remark that we obtain the Fourier series of the function

$$f_n(x) = \int_{Q_{n,\omega}} f(x) d w_{n,\omega}$$

from the Fourier series of $f(x)$ by formal integration over the variables x_{n+1}, $x_{n+2}, \ldots$, or more precisely by replacing c_p by 0 for any $p = (p_1, p_2, p_3, \ldots)$ for which the numbers $p_{n+1}, p_{n+2}, \ldots$ are not all zero. This follows at once when we calculate the Fourier constants of $f_n(x)$; in fact, if the numbers p_{n+1}, $p_{n+2}, \ldots$ are all zero, we have

$$\int_{Q_\omega} f_n(x) e^{-2\pi i p x} d w_\omega = \int_{Q_n} e^{-2\pi i p x} d w_n \int_{Q_{n,\omega}} f(x) d w_{n,\omega} = \int_{Q_\omega} f(x) e^{-2\pi i p x} d w_\omega = c_p,$$

and if the numbers $p_{n+1}, p_{n+2}, \ldots$ are not all zero we have

$$\int_{Q_\omega} f_n(x) e^{-2\pi i p x} d w_\omega = \int_{Q_n} f_n(x) d w_n \int_{Q_{n,\omega}} e^{-2\pi i p x} d w_{n,\omega} = 0.$$

Now the function $f_n(x)$ is uniquely determined by its Fourier series; this is simply the theorem for the space Q_n; since $f_n(x) \to f(x)$ almost everywhere as

$n \to \infty$ we obtain the result that $f(x)$ also must be uniquely determined by its Fourier series.

If we restrict ourselves to the consideration of functions $f(x)$ of L^2, then it follows from the theorem just proved that the system (17.3) is a *complete* system; consequently we have the Parseval equation

$$\int_{Q_\omega} |f(x)|^2 \, dw_\omega = \sum_p |c_p|^2.$$

It follows in the usual way that this equation is also true when $f(x)$ does not belong to L^2, that is, in this case both sides are infinite.

§ 18. A Special Orthogonal System.

As an example of the application of the general theory we shall prove an interesting convergence theorem for Fourier series. The theorem is:

If the Fourier series of an integrable function $f(x)$ in Q_ω is of the special form

$$f(x) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i x_k}; \tag{18.1}$$

that is, if $c_p = 0$ whenever $e^{2\pi i p x}$ is not one of the oscillations $e^{2\pi i x_k}$, then the series is convergent almost everywhere in Q_ω to the sum $f(x)$, so that we may write

$$f(x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k}.$$

The proof[1] follows at once from the discussion of the last section; in fact, since we obtain the Fourier series of the function $f_n(x)$ from the Fourier series of $f(x)$ by formal integration over the variables $x_{n+1}, x_{n+2}, \ldots$, we have

$$f_n(x) \sim \sum_{k=1}^{n} a_k e^{2\pi i x_k}$$

[1] A proof of the theorem by means of the differentiation theorem of § 9 was given by Prof. F. Riesz and communicated to me by Dr. Kalmár. It was this proof that suggested to me the proof of the theorem in § 14. I note from a letter from Prof. Zygmund that a proof on similar lines was given by Paley.

and consequently by the uniqueness theorem (formally for functions in Q_ω, actually only for functions in Q_n), since a finite sum is its own Fourier series,

$$f_n(x) = \sum_{k=1}^{n} a_k e^{2\pi i x_k};$$

this proves the theorem since $f_n(x) \to f(x)$ as $n \to \infty$.

General trigonometrical series of the form

$$\sum_{k=1}^{\infty} a_k e^{2\pi i x_k} \tag{18.2}$$

were considered by Steinhaus.[1] It is clear from the lemma of § 11 that a series of this form must always either be convergent or be divergent almost everywhere in Q_ω; in fact, since the convergence of a series is not altered if we change a finite number of its terms, the set of points in Q_ω in which the series (18. 2) is convergent must have the property S of § 11; and it is also measurable. Now the main theorem is:

A series of the form (18. 2) *is convergent almost everywhere in* Q_ω *if the series*

$$\sum_{k=1}^{\infty} |a_k|^2 \tag{18.3}$$

is convergent; it is divergent almost everywhere in Q_ω *if the series* (18. 3) *is divergent.*

This theorem is in fact only a very special case of a general theorem of Kolmogoroff[2] concerning the convergence of series whose terms »depend on chance». One of the proofs given by Kolmogoroff was translated from the language of probabilities to the language of real functions by Steinhaus. This proof, however, is not the simplest possible. A very simple way of proving the first part of the theorem is to deduce it from the theorem given above; in fact, when the series (18. 3) is convergent, the series (18. 2) is by the Riesz-Fischer theorem the Fourier series of a function $f(x)$ in Q_ω; consequently the series converges to the sum $f(x)$ almost everywhere in Q_ω. As to the second part of the theorem, a simple proof was given by Paley and Zygmund[3]; I shall give

[1] H. Steinhaus [1].

[2] See for the most general exposition A. Kolmogoroff [1].

[2] R. E. A. C. Paley-A. Zygmund [2] 464—465.

this proof in a slightly different form; it depends on the following lemma, due essentially to Zygmund:

Let Ω_n denote the set of points in Q_ω in which we have

$$|s_n(x)| = \left| \sum_{k=1}^{n} a_k e^{2\pi i x_k} \right| \leqq R;$$

then

$$\sum_{k=1}^{n} |a_k|^2 \left(m\,\Omega_n - \frac{1}{2} \right) \leqq R^2. \tag{18.4}$$

That this lemma implies the theorem is clear; for if the series (18.3) is divergent, we may deduce from (18.4) that for *any* R

$$\overline{\lim_{n \to \infty}}\; m\,\Omega_n \leqq \frac{1}{2};$$

consequently the set of points where the series (18.2) is convergent can at most have the measure $\frac{1}{2}$ and since the measure is either 0 or 1, it must be zero.

The proof of the lemma is as follows. We have

$$I = \int_{\Omega_n} |s_n(x)|^2 \, d w_\omega = \int_{\Omega_n} \sum_{k=1}^{n} a_k e^{2\pi i x_k} \sum_{l=1}^{n} \bar{a}_l e^{-2\pi i x_l} \, d w_\omega$$

$$= \sum_{k=1}^{n} |a_k|^2 \, m\,\Omega_n + \sum_{\substack{k,\, l=1 \\ k \neq l}}^{n} a_k \bar{a}_l \int_{\Omega_n} e^{2\pi i (x_k - x_l)} \, d w_\omega = I_1 + I_2 .$$

We write for shortness

$$\int_{\Omega_n} e^{2\pi i (x_k - x_l)} \, d w_\omega = b_{k,\,l} .$$

Then the numbers $b_{k,l}$ are all Fourier constants of the characteristic function of the set Ω_n; the Fourier series of this function contains in addition the constant term $m\,\Omega_n$. So we get immediately from Parseval's theorem

$$(m\,\Omega_n)^2 + \sum_{\substack{k,\, l=1 \\ k \neq l}}^{n} |b_{k,\,l}|^2 \leqq m\,\Omega_n$$

and so

$$\sum_{\substack{k,l=1\\k\neq l}}^{n} |b_{k,l}|^2 \leqq m\,\Omega_n - (m\,\Omega_n)^2 \leqq \frac{1}{4}.$$

This, however, gives for the second term of I the evaluation

$$|I_2| = \left| \sum_{\substack{k,l=1\\k\neq l}}^{n} a_k \bar{a}_l b_{k,l} \right| \leqq \left[\sum_{\substack{k,l=1\\k\neq l}}^{n} |a_k a_l|^2 \right]^{\frac{1}{2}} \left[\sum_{\substack{k,l=1\\k\neq l}}^{n} |b_{k,l}|^2 \right]^{\frac{1}{2}} \leqq \sum_{k=1}^{n} |a_k|^2 \cdot \frac{1}{2}$$

and so we have

$$I = I_1 + I_2 \geqq \sum_{k=1}^{n} |a_k|^2 \left(m\,\Omega_n - \frac{1}{2} \right).$$

On the other hand we have from the definition of the set Ω_n

$$I \leqq m\,\Omega_n \cdot R^2 \leqq R^2;$$

this gives the desired inequality (18. 4).

Making $n \to \infty$ in the last lemma we obtain the following more general lemma:

Let the series (18. 3) *be convergent, and denote by Ω the set of points in Q_ω where*

$$|f(x)| = \left| \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} \right| \leqq R;$$

then

$$\sum_{k=1}^{\infty} |a_k|^2 \left(m\,\Omega - \frac{1}{2} \right) \leqq R^2. \tag{18. 5}$$

The class of function $f(x)$ whose Fourier series is of the form (18. 1) will be denoted by G. It follows from the theorems already proved that this class is also characterised as the class of functions represented by a series (18. 2), where the series (18. 3) is convergent; in particular, a function of the class G must always belong to L^2. It follows immediately that for a function of the class G we have

$$f_{n,\omega}(x) = \sum_{k=n+1}^{\infty} a_k e^{2\pi i x_k},$$

so that $f_{n,\omega}(x)$ is also a function of G. The result of § 13, according to which $f_{n,\omega}(x)$ tends to the integral of $f(x)$ over Q_ω (in this case 0), is therefore in this particular case equivalent to the result $f_n(x) \to f(x)$. We shall prove some more theorems for functions of the class G, which play an essential rôle in the later applications.

Let us first consider a sequence of functions

$$f^{(m)}(x) = \sum_{k=1}^{\infty} a_k^{(m)} e^{2\pi i x_k} \tag{18.6}$$

belonging to G. It is clear that if this sequence converges in mean[1], then the limit function $f(x)$ must also belong to the class G; that is, we must have

$$f(x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k};$$

but the sequence need not converge to $f(x)$ almost everywhere. The interesting thing is that the converse is true; we have, in fact, the following theorem:

If a sequence of functions (18.6) belonging to G is convergent almost everywhere in Q_ω, then it is also convergent in mean and consequently[2] *to the same limit.*

The proof follows at once from the last lemma. We have to prove that

$$\lim_{m,n\to\infty} \sum_{k=1}^{\infty} |a_k^{(m)} - a_k^{(n)}|^2 = 0.$$

Now let us denote for any $R > 0$ by $\Omega^{m,n}$ the set of points in Q_ω in which $|f^{(m)}(x) - f^{(n)}(x)| \leq R$; then we have by (18.5)

$$\sum_{k=1}^{\infty} |a_k^{(m)} - a_k^{(n)}|^2 \left(m\,\Omega^{m,n} - \frac{1}{2}\right) \leq R^2.$$

Since

$$\lim_{m,n\to\infty} m\,\Omega^{m,n} = 1$$

and since R may be taken arbitrarily small, the result follows.

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[1] We use the expression »convergence in mean» instead of strong convergence in the class L^2.

[2] By the familiar theorem that a sequence which converges strongly to a function $f(x)$ always contains a subsequence which converges to $f(x)$ almost everywhere.

We shall need one more result concerning the class G, which is due to Paley and Zygmund.[1] In order to formulate this result briefly, we denote by L^* the class of functions $f(x)$ in Q_ω for which $e^{\lambda|f(x)|^2}$ is integrable for any $\lambda > 0$; this class is contained in all the classes L^p. It follows easily from the inequality $|a+b|^2 \leqq 2|a|^2 + 2|b|^2$ that the sum of two functions of L^* will again belong to L^*. A *distance* in the class L^* is defined by the expression

$$D_*^\lambda(f, g) = \left[\frac{1}{\lambda} \log \int\limits_{Q_\omega} e^{\lambda|f(x)-g(x)|^2} d w_\omega\right]^{\frac{1}{2}}.$$

This distance is a function of λ; it is easily seen that it is increasing (in the wide sense). We have the inequality $D_*^\lambda(f, g) \leqq D_*^{4\lambda}(f, h) + D_*^{4\lambda}(g, h)$ for any function $h(x)$ in L^*; this inequality takes the place of the triangle inequality. A sequence $h_1(x), h_2(x), h_3(x), \ldots$ is said to converge strongly towards $f(x)$ in the class L^* if $D_*^\lambda(f, h_n) \to 0$ for any $\lambda > 0$; strong convergence in L^* implies strong convergence in L^p for any p.

Now we have the following theorem:

A function $f(x)$ of the class G will always belong to the class L^* and we have $f_n(x) \to f(x)$ in the sense of strong convergence in L^*.

Let us write

$$B = \sum_{k=1}^{\infty} |a_k|^2, \quad B_n = \sum_{k=1}^{n} |a_k|^2 \quad \text{and} \quad B_{n,\omega} = \sum_{k=n+1}^{\infty} |a_k|^2 .$$

It is sufficient to prove that for any fixed λ the function $e^{\lambda|f_{n,\omega}(x)|^2}$ is integrable for all sufficiently large values of n, and that its integral over Q_ω tends to 1 when $n \to \infty$. We deduce this from the inequality

$$\int\limits_{Q_\omega} e^{\lambda|f_{n,\omega}(x)|^2} d w_\omega \leqq \frac{1}{1-\lambda B_{n,\omega}} \tag{18.7}$$

which is valid for $\lambda B_{n,\omega} < 1$; this inequality will also be applied later on. It is clearly sufficient to prove that

$$\int\limits_{Q_\omega} e^{\lambda|f(x)|^2} d w_\omega \leqq \frac{1}{1-\lambda B} \tag{18.8}$$

[1] See R. E. A. C. Paley-A. Zygmund [1] 342—343.

when $\lambda B < 1$; then (18.7) will follow when we replace $f(x)$ by $f_{n,\omega}(x)$. In order to obtain (18.8) we apply Parseval's equation to the function $(f_n(x))^q$ where q is any positive integer; this gives

$$\int\limits_{Q_\omega} |f_n(x)|^{2q}\, dw_\omega = \sum_{q_1+q_2+\cdots+q_n=q} \left| \frac{q!}{q_1!\, q_2! \ldots q_n!} a_1^{q_1} a_2^{q_2} \ldots a_n^{q_n} \right|^2 \leqq$$

$$\leqq q! \sum_{q_1+q_2+\cdots+q_n=q} \frac{q!}{q_1!\, q_2! \ldots q_n!} |a_1|^{2q_1} |a_2|^{2q_2} \ldots |a_n|^{2q_n} = q!\, B_n^q.$$

Now if $\lambda B_n < 1$ this implies the inequality

$$\int\limits_{Q_\omega} e^{\lambda |f_n(x)|^2}\, dw_\omega \leqq \frac{1}{1-\lambda B_n}$$

and from this we deduce (18.8) by a well-known theorem of Fatou.

§ 19. A Case of Birkhoff's Ergodic Theorem.

In the next few sections we shall make some applications of a case of the ergodic theorem of Birkhoff to functions which are almost periodic in the generalised sense of Besicovitch. The theory of these functions depends on a certain notion of a *mean value* for functions defined on the infinite line l: $-\infty < t < \infty$. I use a slightly narrower definition than that generally adopted; this makes no difference in the theory but it makes our results more precise.

Let $f(t)$ be a function defined on the infinite line l and integrable over any finite interval j on l. We shall say that it has a mean value

$$M_t\{f(t)\}$$

over l, if and only if the two limits

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{-T}^{0} f(t)\, dt \quad \text{and} \quad \lim_{T\to\infty} \frac{1}{T} \int\limits_{0}^{T} f(t)\, dt$$

both exist and have the same value; this value is then by definition the mean value $M_t\{f(t)\}$ of $f(t)$. The special rôle played by the point $t=0$ is only apparent; that is, if $M_t\{f(t)\}$ exists, then $M_t\{f(t+a)\}$ exists for any real a, and we have

$$M_t\{f(t+a)\} = M_t\{f(t)\}.$$

Now let $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ be an arbitrary sequence of real numbers which are not all zero; by μt, where t is any real number, we denote the point $\mu t = (\mu_1 t, \mu_2 t, \mu_3 t, \ldots)$; then if x is any point in Q_ω, the point $x + \mu t$ will describe a »straight line» in Q_ω when t describes the real axis. Let $f(t)$ be any function in Q_ω; then $f(x+\mu t)$ is, for a fixed μ, a function in the space (l, Q_ω) in which the variable point is (t, x).[1] It is easily seen that if $f(x)$ is measurable in Q_ω, then $f(x+\mu t)$ is measurable in (l, Q_ω); also if $f(x)$ is integrable in Q_ω, then $f(x+\mu t)$ will be integrable over any set (j, Q_ω), in (l, Q_ω), where j is an interval on l. From this it follows at once by Fubini's theorem that for almost all x in Q_ω the function $f(x+\mu t)$, considered as a function of t alone, is integrable over any interval j on l. Now we have the following theorem, which is a special case of the ergodic theorem of Birkhoff.[2]

Let $f(x)$ be integrable over Q_ω and let μ be given; then for almost all x in Q_ω the mean value

$$M_t\{f(x+\mu t)\} = \varphi(x) \tag{19.1}$$

exists; this mean value $\varphi(x)$ is integrable over Q_ω and we have

$$\int_{Q_\omega} \varphi(x)\, dw_\omega = \int_{Q_\omega} f(x)\, dw_\omega . \tag{19.2}$$

More generally we have

$$\int_{\Omega} \varphi(x)\, dw_\omega = \int_{\Omega} f(x)\, dw_\omega \tag{19.3}$$

for any measurable set Ω in Q_ω which is invariant under the translations $x + \mu t$.[3]

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[1] In this and the following sections we use freely and without further comment the extension of the general theory to spaces of the form (R_n, Q_ω) where R_n is, as usual, a Euclidean space. It is clear from the previous exposition how the theory should be developed for such spaces.

[2] See A. Khintchine [1] for the redaction which comes nearest to our formulation. Khintchine proves (19.1) for one-sided mean values only and does not formulate (19.2) or (19.3) explicitly, but it is easy by means of his arguments to obtain the result in the general form.

[3] That means which contains all points of the line $x + \mu t$, when it contains x.

It is easy to find the *Fourier series* of $\varphi(x)$ in terms of the Fourier series of $f(x)$. Let us suppose that

$$f(x) \sim \sum_p c_p e^{2\pi i p x} \quad \text{and} \quad \varphi(x) \sim \sum_p \gamma_p e^{2\pi i p x};$$

we shall then prove that $\gamma_p = c_p$ whenever $p\mu = 0$, while $\gamma_p = 0$ when $p\mu \neq 0$; here $p\mu = p_1\mu_1 + p_2\mu_2 + p_3\mu_3 + \cdots$. This result is what we should expect from a formal calculation. That $\gamma_0 = c_0$ is simply the relation (19. 2). Now let us apply the theorem to the function $f(x)e^{-2\pi i p x}$ instead of $f(x)$; then we get in the case $p\mu = 0$ instead of $\varphi(x)$ the corresponding function $\varphi(x)e^{-2\pi i p x}$, and we see that $\gamma_p = c_p$. It remains to prove that $\gamma_p = 0$ when $p\mu \neq 0$, but this follows immediately from the fact that we have $\varphi(x) = \varphi(x + \mu t)$ and consequently $\gamma_p = \gamma_p e^{2\pi i p \mu t}$ for any t.

In the special case where the numbers $\mu_1, \mu_2, \mu_3, \ldots$ are *linearly independent*, that is when $p\mu = 0$ only when $p = 0$, the function $\varphi(x)$ is seen to reduce to a constant, and we have

$$M_t\{f(x + \mu t)\} = \int_{Q_\omega} f(x)\, d w_\omega$$

for almost all x in Q_ω. The only measurable sets Ω which are invariant by the translations $x + \mu t$ are either null-sets or sets differing from Q_ω by null-sets. If in this case the function $f(x)$ is integrable *in the Riemann sense*, this result can be essentially sharpened by an immediate extension of a classical theorem of Weyl. We return to this case in § 25.

§ 20. Application to Almost Periodic Functions.

We define the upper mean value $\overline{M}_t\{h(t)\}$ of a non-negative function $h(t)$ over the infinite line l: $-\infty < t < \infty$ as the greater of the two numbers

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \int_{-T}^{0} h(t)\, dt \quad \text{and} \quad \overline{\lim_{T\to\infty}} \frac{1}{T} \int_{0}^{T} h(t)\, dt.$$

If $p \geqq 1$, we may consider the class of all real or complex functions $f(t)$ for which $\overline{M}_t\{|f(t)|^p\}$ is finite. Similarly we may consider the class of functions for which $\overline{M}_t\{e^{\lambda|f(x)|^2}\}$ is finite for any $\lambda > 0$. Let us denote these classes by M^p and M^* respectively. In these classes distances may be introduced by the formulae

$$D_p(f, g) = [\overline{M}_t\{|f(t) - g(t)|^p\}]^{\frac{1}{p}}$$

and

$$D_*^\lambda(f, g) = \left[\frac{1}{\lambda} \log \overline{M}_t\{e^{\lambda|f(t)-g(t)|^2}\}\right]^{\frac{1}{2}}.$$

Strong convergence in M^p or M^* is defined by the relations $D_p(f, h_n) \to 0$ and $D_*^\lambda(f, h_n) \to 0$ for any λ respectively. The class of B^p *a.p.* functions is now defined as the closure in the sense of strong convergence in M^p of the class of all finite trigonometrical sums $h(t) = \sum_{k=1}^{n} b_k e^{i\mu_k t}$ (with arbitrary real exponents μ_k). Similarly the class of B^* *a.p.* functions is defined as the closure of the same class in the sense of strong convergence in M^*.

B^p *a.p.* functions were studied in detail by Besicovitch[1]; we get the widest class for $p = 1$; this class is denoted simply the class of B *a.p.* functions. Any B *a.p.* function has a Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$$

where $a_k = M_t\{f(t) e^{-i\lambda_k t}\}$ while $M_t\{f(t) e^{-i\lambda t}\} = 0$ for all other λ. The mean value $M_t\{|f(t)|^p\}$ exists always for a B^p *a.p.* function and is in the case $p = 2$ determined by Parseval's equation

$$M_t\{|f(t)|^2\} = \sum_{k=1}^{\infty} |a_k|^2.$$

For B^2 *a.p.* functions the analogue of the Riesz-Fischer theorem is also true. The class of B^* *a.p.* functions is contained in the class of B^p *a.p.* functions for any p; it is introduced here only to give our results in their most precise form.

The partial sums of the Fourier series of a B^p or B^* *a.p.* function will not generally converge strongly to the function in M^p or M^* respectively, but if a trigonometrical series (of the general type considered here) converges strongly in M^p or M^*, the series will always be the Fourier series of its sum.

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[1] See A. S. Besicovitch [2].

Let us now consider the series

$$(20.1) \qquad \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{i\lambda_k t},$$

where the coefficients a_k are given and the exponents λ_k are real and all different. If the series

$$(20.2) \qquad \sum_{k=1}^{\infty} |a_k|^2$$

is convergent, the series (20. 1) is for any x in Q_ω the Fourier series of a B^2 *a.p.* function. We shall give first a theorem concerning the case where the series (20. 2) is divergent, and afterwards a more detailed theorem for the case where the series (20. 2) is convergent.

Let the series (20. 2) *be divergent. Then for almost all points x in Q_ω the series* (20. 1) *is not the Fourier series of a B a.p. function.*[1]

Let numbers $r_k^{(m)}$ for $m = 1, 2, 3, \ldots$ be taken in such a way that for each m only a finite number of the $r_k^{(m)}$ differ from zero, and moreover the two conditions $0 \leqq r_k^{(m)} \leqq 1$ for all k and m and $r_k^{(m)} \to 1$ as $m \to \infty$ for each k are satisfied. We consider the sequence of functions

$$f^{(m)}(x) = \sum_{k=1}^{\infty} r_k^{(m)} a_k e^{2\pi i x_k}$$

in Q_ω; each series is actually a finite sum. This sequence is not convergent in mean; consequently by a theorem of § 18 it is not convergent almost everywhere

[1] The corresponding theorem for ordinary Fourier series is contained as a special case in a theorem of Paley and Zygmund. See R. E. A. C. Paley-A. Zygmund [2] 466. For the special case of linearly independent exponents λ_k a much stronger result has recently been proved, namely that a series

$$\sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$$

with linearly independent exponents is the Fourier series of a B *a.p.* function only when the series

$$\sum_{k=1}^{\infty} |a_k|^2$$

is convergent; that is, when it is already the Fourier series of a B^2 *a.p.* function. See S. Bochner-B. Jessen [1].

in Q_ω either. The same is of course true for any subsequence of the sequence $f^{(m)}(x)$.

Now by the extension of the Fejér summation theorem to B *a.p.* functions we may take the numbers $r_k^{(m)}$ in such a way that if we write

$$f^{(m)}(t, x) = \sum_{k=1}^{\infty} r_k^{(m)} a_k e^{2\pi i x_k} e^{i\lambda_k t},$$

we have

$$M_t\{|f^{(m)}(t, x) - f^{(n)}(t, x)|\} \to 0 \quad \text{as} \quad m, n \to \infty$$

for any point x in Q_ω for which the series (20. 1) is the Fourier series of a B *a.p.* function. The set of points x in Q_ω for which the last relation is true is clearly measurable; it also has the property S of § 11, and it therefore follows that its measure is either 0 or 1. Let us suppose that it is 1, and show that this leads to a contradiction.

We write $\lambda_k = 2\pi\mu_k$. Then we have clearly $f^{(m)}(t, x) = f^{(m)}(x + \mu t)$ where the notation is that of § 19. Our assumption is that

$$M_t\{|f^{(m)}(x + \mu t) - f^{(n)}(x + \mu t)|\} \to 0 \quad \text{as} \quad m, n \to \infty$$

for almost all x in Q_ω. For a given $R > 0$ we denote by $\Omega^{m,n}$ the set of points x in Q_ω for which

$$M_t\{|f^{(m)}(x + \mu t) - f^{(n)}(x + \mu t)|\} \leqq R.$$

This set $\Omega^{m,n}$ is invariant for the translations $x + \mu t$; it therefore follows from (19. 3) applied to the function $f(x) = |f^{(m)}(x) - f^{(n)}(x)|$ that we must have

$$\int_{\Omega^{m,n}} |f^{(m)}(x) - f^{(n)}(x)|\, dw_\omega \leqq R\, m\,\Omega^{m,n}.$$

Since $m\,\Omega^{m,n} \to 1$ as $m, n \to \infty$ and since R may be taken arbitrarily small, by familiar arguments this contradicts the result that no subsequence of the sequence $f^{(m)}(x)$ converges almost everywhere. This completes the proof.

In the case where the series (20. 2) is convergent we have the following theorem:

Let the series (20. 2) *be convergent. Then for almost all points x in Q_ω the series* (20. 1) *is convergent almost everywhere in t and strongly in the class M^* to a certain sum $f(t, x)$. Consequently, $f(t, x)$ is B^* a.p. and has* (20. 1) *as its Fourier series.*

The first part of the theorem concerning convergence almost everywhere is an immediate deduction from § 18. The series

$$f(x) = \sum_{k=1}^{\infty} a_k \, e^{2\pi i x_k}$$

determines a function in Q_ω of the class G. Now let us write $\lambda_k = 2\pi\mu_k$ as before; then $f_n(x+\mu t)$ is simply the partial sum of the series (20. 1). Since $f_n(x) \to f(x)$ almost everywhere, it follows that $f_n(x+\mu t) \to f(x+\mu t)$ almost everywhere in the space (l, Q_ω), where (t, x) is the variable point, and this implies that for almost all x we have $f_n(x+\mu t) \to f(x+\mu t)$ almost everywhere in t. This proves the first part of the theorem, and it shows also that $f(t, x) = f(x+\mu t)$.

In order to prove the second part of the theorem, concerning strong convergence in M^*, we first observe that since $f(x)$ belongs to the class L^* of § 18, the function $f(t, x) = f(x+\mu t)$ must by Birkhoff's theorem belong to M^* for almost all x. It will be sufficient, in order to prove that $f_n(x+\mu t)$ converges strongly towards $f(x+\mu t)$ in M^* for almost all x, to prove that for any fixed $\lambda > 0$ we have

$$M_t\left\{e^{\lambda |f_{n,\omega}(x+\mu t)|^2}\right\} \to 1 \quad \text{as} \quad n \to \infty \tag{20.3}$$

for almost all x. The existence of the mean values on the left in (20. 3) is a consequence of Birkhoff's theorem, which also shows that

$$\int_{Q_\omega} M_t\left\{e^{\lambda |f_{n,\omega}(x+\mu t)|^2}\right\} d w_\omega = \int_{Q_\omega} e^{\lambda |f_{n,\omega}(x)|^2} d w_\omega .$$

We know from § 18 that the right-hand side of this relation tends to 1 as $n \to \infty$; this, however, does not imply the limit relation (20. 3). In order to obtain this limit relation we have to apply the last theorem of § 16.

For a given value of n we denote by $H_n(x)$ the majorant of the sequence $f_{n+p,\omega}(x)$; it is sufficient if we can prove that

$$M_t\left\{e^{\lambda (H_n(x+\mu t))^2}\right\} \to 1 \quad \text{as} \quad n \to \infty \tag{20.4}$$

for almost all x. Now the sequence $f_{n+p,\omega}(x)$ has the same relation to the function $f_{n,\omega}(x)$ as the sequence $f_{n,\omega}(x)$ has to $f(x)$; it follows that we have

$$(20.5) \qquad \int\limits_{Q_\omega} e^{\lambda (H_n(x))^2} dw_\omega - 1 \leqq 4\left(\int\limits_{Q_\omega} e^{\lambda |f_{n,\omega}(x)|^2} dw_\omega - 1\right).$$

We also have

$$(20.6) \qquad \int\limits_{Q_\omega} M_t\{e^{\lambda (H_n(x+\mu t))^2}\} dw_\omega = \int\limits_{Q_\omega} e^{\lambda (H_n(x))^2} dw_\omega.$$

By (20.5) the right hand side of (20.6) tends to 1 as $n \to \infty$; since the sequence $H_n(x)$ is *decreasing* (in the wide sense), this implies (20.4).

§ 21. Orthogonal Series whose Coefficients are Analytic Functions.

Let us denote by D a fixed open domain in the $s = \sigma + it$ plane. By A we shall always denote a bounded and closed set belonging to D, but not always the same. We shall make use of the fact that if A_1 is any such set and A_2 a set of the same kind whose points are all interior points of A_1, then there exists a constant C, depending only on A_1 and A_2, such that for any function $f(s)$ which is regular in D we have

$$(21.1) \qquad \overline{\text{bound}}_{A_2} |f(s)| \leqq C \int\limits_{A_1} |f(s)|\, dw_s,$$

and consequently for perhaps a different value of C

$$(21.2) \qquad \overline{\text{bound}}_{A} |f(s)| \leqq C\left[\int\limits_{A_1} |f(s)|^2 dw_s\right]^{\frac{1}{2}}.$$

This theorem immediately implies that if a sequence of regular functions $f_1(s)$, $f_2(s), f_3(s), \ldots$ converges in mean to a certain function $f(s)$ in the sense that it converges in mean in any set A, then the sequence is also *uniformly convergent* in any set A and consequently $f(s)$ is regular in D.

Now let $g_1(s), g_2(s), g_3(s), \ldots$ be a sequence of regular functions in D, such that the series

$$(21.3) \qquad \sum_{k=1}^{\infty} \int\limits_{A} |g_k(s)|^2 dw_s$$

converges for any A. This condition is by (21.2) equivalent to the apparently stronger condition, that for any A the series

$$\sum_{k=1}^{\infty} \overline{\operatorname{bound}_A} \, |g_k(s)|^2 \tag{21.4}$$

shall be convergent. We consider now the series

$$f(s, x) = \sum_{k=1}^{\infty} g_k(s) \, e^{2\pi i x_k}, \tag{21.5}$$

where s varies in D and x in Q_ω, and shall prove the following theorem:

For almost all x in Q_ω the series (21.5) is uniformly convergent in any set A and consequently its sum $f(s, x)$ is regular in D.

From the convergence of the series (21.4) for any A it follows, in particular, that the series

$$\sum_{k=1}^{\infty} |g_k(s)|^2$$

is convergent for any point s in D. This implies by § 18 that for any s the series (21.5) is convergent for almost all x to a certain sum $f(s, x)$. The function $f(s, x)$ is clearly a measurable function in the space (D, Q_ω). It also follows from § 16, taking $p = 2$, that if $H(s, x)$ denotes the majorant of the sequence

$$f_n(s, x) = \sum_{k=1}^{n} g_k(s) \, e^{2\pi i x_k},$$

we have

$$\int_{Q_\omega} (H(s, x))^2 \, dw_\omega \leqq 4 \sum_{k=1}^{\infty} |g_k(s)|^2.$$

Now $H(s, x)$ is measurable in the space (D, Q_ω); using the convergence of the series (21.3) it follows that the integral of $(H(s, x))^2$ over any set (A, Q_ω) in (D, Q_ω) must be finite. This implies, however, by Fubini's theorem that for almost all x in Q_ω the integral

$$\int_A (H(s, x))^2 \, d w_s$$

must exist for any A. On the other hand we know that for almost all x in Q_ω we have $f_n(s, x) \to f(s, x)$ for almost all s in D. Both propositions together show that for almost all x we have

$$\int_A |f(s, x) - f_n(s, x)|^2 \, dw_s \to 0 \quad \text{as} \quad n \to \infty,$$

and for any such x we know that $f_n(s, x)$ converges uniformly in any set A. This completes the proof.

§ 22. General Exponential Series.

Let us consider all exponential series of the form

$$f(s, x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{\lambda_k s} \qquad (s = \sigma + it), \tag{22.1}$$

where the coefficients a_k are given numbers and the exponents λ_k are real and all different; the exponents shall not be restricted in any other way. Suppose that the series

$$\sum_{k=1}^{\infty} |a_k|^2 e^{2\lambda_k \sigma} \tag{22.2}$$

is convergent for some σ; these σ must clearly form an interval; we suppose that this interval does not reduce to a single point. The end-points α and β of the interval (if they are finite) may or may not belong to it; in all cases we denote by (α, β) the open interval $\alpha < \sigma < \beta$. The vertical strip $\alpha < \sigma < \beta$ shall also be denoted by (α, β); this strip will be the domain D of § 21; as before A will be used to denote any bounded and closed set in D. As an immediate consequence of the result of § 21 we have the theorem:

For almost all x in Q_ω the series (22.1) is uniformly convergent in any set A in the strip (α, β) and consequently its sum $f(s, x)$ is regular in (α, β).

We wish to study these functions $f(s, x)$ in greater detail; in this section we shall prove the following result, which is a generalisation of a theorem for Dirichlet series of the usual type due to Paley and Zygmund.[1]

[1] R. E. A. C. Paley-A. Zygmund [3] 202—203.

For almost all x in Q_ω the function $f(s, x)$ is not continuable across the lines $\sigma = \alpha$ and $\sigma = \beta$.

It is no restriction in generality to suppose that all λ_k are positive, in which case we have $\alpha = -\infty$; we may also suppose that $\beta = 0$, and it is then sufficient to prove that the series (22. 1) almost all have the point $s = 0$ as a singular point. For in this case since the class of all functions $f(s + i\tau, x)$, where τ is some fixed real number, is identical with the class $f(s, x)$, it follows that the point $s = i\tau$ must also be a singular point for almost all the series (22. 1); and considering only rational values of τ we conclude that the series (22. 1) are almost all singular in all points of the line $\sigma = 0$. We shall denote by E the set of points x in Q_ω for which the series (22. 1) converges uniformly in any set A in $(-\infty, 0)$; this set E has clearly the property S of § 11.

In the proof we shall have to use the differentiated series

$$(22.3) \qquad f^{(p)}(s, x) = \sum_{k=1}^{\infty} \lambda_k^p a_k e^{2\pi i x_k} e^{\lambda_k s}$$

for $p = 1, 2, 3, \ldots$; the value $p = 0$ corresponds to the series (22. 1). For a point x in E these series are all uniformly convergent in any set A in $(-\infty, 0)$. We shall also use the associated series

$$(22.4) \qquad \sum_{k=1}^{\infty} \lambda_k^{2p} |a_k|^2 e^{2\lambda_k \sigma};$$

they all converge in $(-\infty, 0)$.

We now observe that if the series (22. 1) were not almost all singular at the point $s = 0$, then they would be almost all regular at $s = 0$. This follows from the lemma of § 11. In fact, it is clear that the set of points x in E for which the function $f(s, x)$ is regular at $s = 0$ has the property S and it is measurable since it may be defined as the set of points in E for which

$$(22.5) \qquad \overline{\lim_{p \to \infty}} \left[\frac{|f^{(p)}(-1, x)|}{p!} \right]^{\frac{1}{p}} < 1.$$

Hence by the lemma its measure is either 0 or 1.

The only case in which the theorem presents a difficulty is when the series (22. 4) are all *convergent* at the point $s = 0$. For suppose that for some p the series (22.4) were divergent for $s = 0$; denoting by ε_m a sequence of negative numbers tending to 0, it follows that we have

$$\sum_{k=1}^{\infty} \lambda_k^{2p} |a_k|^2 e^{2\lambda_k \varepsilon_m} \to \infty \quad \text{as} \quad m \to \infty$$

and this implies by a theorem of § 18 that the sequence of functions $f^{(p)}(\varepsilon_m, x)$ cannot be convergent almost everywhere in Q_ω; consequently the functions $f^{(p)}(s, x)$ and hence also the functions $f(s, x)$ cannot almost all be regular at the point $s = 0$.

Suppose now that the series (22.4) are all convergent at the point $s = 0$ and that the series (22.1) are almost all regular at this point; then the inequality (22.5) holds for almost all points x in Q_ω. The left-hand side of the inequality (22.5) is a measurable function in Q_ω, it is defined at least for all points x in E and it has the property S of § 11. It follows that it is a constant almost everywhere in Q_ω and this constant must be < 1. Let us denote it by $\frac{1}{1+\varepsilon}$; then the functions $f(s, x)$ are almost all regular in the circle $|s+1| < 1 + \varepsilon$ and *a fortiori* in the circle $|s| < \varepsilon$. From this we shall obtain a contradiction by proving directly that it involves the convergence of the series (22.2) for $\sigma < \varepsilon$.

We write for shortness

$$\sum_{k=1}^{\infty} \lambda_k^{2p} |a_k|^2 = K_p.$$

From the (supposed) convergence of this series it follows that for almost all x the series (22.3) converges at $s = 0$ and also that its sum will be the limit in mean of $f^{(p)}(\varepsilon_m, x)$ as $m \to \infty$, if ε_m denotes a sequence of negative numbers tending to 0; now we know that $f^{(p)}(\varepsilon_m, x) \to f^{(p)}(0, x)$ as $m \to \infty$ almost everywhere; hence it follows that

$$f^{(p)}(0, x) = \sum_{k=1}^{\infty} \lambda_k^p a_k e^{2\pi i x_k}. \tag{22.6}$$

On the other hand, since $f(s, x)$ is regular for $|s| < \varepsilon$, we have

$$\overline{\lim_{p\to\infty}} \left[\frac{|f^{(p)}(0, x)|}{p!} \right]^{\frac{1}{p}} \leqq \frac{1}{\varepsilon}. \tag{22.7}$$

The idea of the proof is now to deduce from this an inequality for K_p which will show the convergence of the series (22. 2) for $\sigma<\varepsilon$. We shall use for this purpose the lemma of § 18 in its second form; that is, the inequality (18. 5). Chose $B>\frac{1}{\varepsilon}$ and denote by $\Omega^{(p)}$ the set of points x in Q_ω for which

$$|f^{(p)}(0, x)| \leqq B^p \cdot p!.$$

It follows from (22. 7) that $m\,\Omega^{(p)}\to 1$ as $p\to\infty$. On the other hand, (18. 5) gives

$$K_p\left(m\,\Omega^{(p)}-\frac{1}{2}\right)\leqq B^{2p}(p!)^2.$$

From this it follows that we have for some positive constant C

(22. 8) $$K_p \leqq C B^{2p}(p!)^2.$$

Consider now the series (22. 2) at a point $\sigma=\delta$ where $\delta<\varepsilon$ and suppose that B has been taken $>\frac{1}{\varepsilon}$ but $<\frac{1}{\delta}\cdot$ Then the convergence of the series (22. 2) at the point $\sigma=\delta$ follows from (22. 8). We have, in fact,[1]

$$\sum_{k=1}^{\infty}|a_k|^2 e^{2\lambda_k\delta}<2\sum_{k=1}^{\infty}|a_k|^2\sum_{p=0}^{\infty}\frac{(2\lambda_k\delta)^{2p}}{(2p)!}=2\sum_{p=0}^{\infty}\frac{2^{2p}\delta^{2p}}{(2p)!}\sum_{k=1}^{\infty}\lambda_k^{2p}|a_k|^2$$

$$=2\sum_{p=0}^{\infty}\frac{2^{2p}\delta^{2p}}{(2p)!}K_p\leqq 2C\sum_{p=0}^{\infty}\frac{2^{2p}\delta^{2p}}{(2p)!}B^{2p}(p!)^2\leqq 2C\sum_{p=0}^{\infty}(2p+1)(\delta B)^{2p},$$

where the last series is convergent since $\delta B<1$.

§ 23. **Analytic Almost Periodic Functions.**

We wish now to study the almost periodic character of the functions $f(s, x)$ introduced in § 22. To do this we must first define when a regular function $f(s)$ in the strip (α, β) shall be called B^p *a. p.* or B^* *a. p. in the strip.*

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[1] We use $e^x<2\frac{e^x+e^{-x}}{2}=2\sum_{p=0}^{\infty}\frac{x^{2p}}{(2p)!}$ and $\frac{(p!)^2}{(2p)!}\leqq\frac{2p+1}{2^{2p}}$.

The notions we shall introduce correspond to those of almost periodicity in $[\alpha, \beta]$ in the Bohr theory; as always, (α, β) shall denote the strip (or the interval) $\alpha < \sigma < \beta$ and (α_1, β_1) any strip (or interval) inside (α, β), that is, for which $\alpha < \alpha_1 < \beta_1 < \beta$; the words »in $[\alpha, \beta]$» shall mean »in any (α_1, β_1)». We may now consider the class of all functions $f(s)$ which are regular in (α, β) and for which

$$\overline{M}_t \left\{ \int_{\alpha_1}^{\beta_1} |f(\sigma + it)|^p \, d\sigma \right\} \quad \text{or} \quad \overline{M}_t \left\{ \int_{\alpha_1}^{\beta_1} e^{\lambda |f(\sigma+it)|^2} \, d\sigma \right\}$$

is finite for any (α_1, β_1) and a fixed $p \geqq 1$ or for any (α_1, β_1) and any $\lambda > 0$ respectively. Let us denote these classes by R^p and R^*. In these classes we introduce distances by the formulae

$$D_p(f, g) = \left[\overline{M}_t \left\{ \int_{\alpha_1}^{\beta_1} |f(\sigma + it) - g(\sigma + it)|^p \, d\sigma \right\} \right]^{\frac{1}{p}}$$

and

$$D_*^\lambda(f, g) = \left[\frac{1}{\lambda} \log \overline{M}_t \left\{ \int_{\alpha_1}^{\beta_1} e^{\lambda |f(\sigma+it) - g(\sigma+it)|^2} \, d\sigma \right\} \right]^{\frac{1}{2}}.$$

These distances are functions of (α_1, β_1). Strong convergence in R^p or R^* is defined by the relations $D_p(f, h_n) \to 0$ for any (α_1, β_1) and $D_*^\lambda(f, h_n) \to 0$ for any (α_1, β_1) and any $\lambda > 0$ respectively. The class of B^p *a.p.* functions in $[\alpha, \beta]$ is now defined as the closure in the sense of strong convergence in R^p of the class of all finite exponential sums $h(s) = \sum_{k=1}^{n} b_k e^{\mu_k s}$. Similarly the class of B^* *a.p.* functions in $[\alpha, \beta]$ is the closure of the same class in the sense of strong convergence in R^*. The mean values

$$M_t \left\{ \int_{\alpha_1}^{\beta_1} |f(\sigma + it)|^p \, d\sigma \right\} \quad \text{and} \quad M_t \left\{ \int_{\alpha_1}^{\beta_1} e^{\lambda |f(\sigma+it)|^2} \, d\sigma \right\}$$

will always exist when $f(s)$ is B^p *a.p.* or B^* *a.p.* respectively.

If $\alpha_1 < \alpha_2 < \beta_2 < \beta_1$, it follows from (21.1) that we have

$$(23.1) \qquad \overline{\text{bound}}_{\alpha_2 < \sigma < \beta_2} |f(\sigma + it)|^p \leqq C \int_{t-1}^{t+1} d\tau \int_{\alpha_1}^{\beta_1} |f(\sigma + i\tau)|^p \, d\sigma$$

for any function $f(s)$ which is regular in (α, β), the constant C depending only on $\alpha_1, \alpha_2, \beta_2, \beta_1$. This implies that a B^p *a.p.* function in $[\alpha, \beta]$ is always B^p *a.p.* on any vertical line in (α, β) and even uniformly in σ in a very strong sense. It is easily seen that the Fourier series for the functions $f(\sigma + it)$ for different values of σ »go together» to form a Dirichlet series

$$f(s) \sim \sum_{k=1}^{\infty} a_k e^{\lambda_k s}$$

for $f(s)$. In analogy to (23.1) we have the further result that for some fixed C

$$(23.2) \qquad \overline{\underset{\alpha_2<\sigma<\beta_2}{\text{bound}}} \; e^{\lambda |f(\sigma+it)|^2} \leqq C \int_{t-1}^{t+1} d\tau \int_{\alpha_1}^{\beta_1} e^{\lambda C |f(\sigma+i\tau)|^2} d\sigma;$$

this follows from (21.1) and from Jensen's inequality for the convex function $e^{\lambda y^2}$ and shows that a B^* *a.p.* function in $[\alpha, \beta]$ is B^* *a.p.* on any vertical line in (α, β), again with a strong uniformity in σ. Finally we observe that if an exponential series (of general type) happens to converge strongly in R^p or R^* to a function $f(s)$ which is regular in (α, β), then it is the Dirichlet series of $f(s)$.

It was proved by Besicovitch that if a regular function $f(s)$ in (α, β) is of bounded order in $[\alpha, \beta]$ and has the property that $\overline{M}_t\{|f(\sigma + it)|^2\}$ is finite for all σ in (α, β), then it is B^2 *a.p.* on any vertical line in (α, β) if it is B^2 *a.p.* on one single line $\sigma = \sigma_0$.[1] The proof gives enough uniformity in σ to show that $f(s)$ is B^2 *a.p.* in $[\alpha, \beta]$ in our sense also.

We shall now prove the following theorem:

For almost all points x in Q_ω, the series (22.1) is strongly convergent to its sum $f(s, x)$ in the class R^*; consequently $f(s, x)$ is B^* a.p. in $[\alpha, \beta]$ with (22.1) as its Dirichlet series; that is, we may write

$$f(s, x) \sim \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{\lambda_k s}$$

for almost all x.

The proof is very similar to that of § 20. For each σ in (α, β) we consider the function $f(\sigma, x)$. Considered as a function of x, this function belongs

[1] See A. S. Besicovitch [1].

to the class G of § 18. Let $f_n(\sigma, x)$ and $f_{n,\omega}(\sigma, x)$ have the usual meaning. Now if we write $\lambda_k = 2\pi\mu_k$, we get $f(s, x) = f(\sigma, x + \mu t)$, and $f_n(\sigma, x + \mu t)$ and $f_{n,\omega}(\sigma, x + \mu t)$ will be the partial sum and the rest of the series (22.1). It will be sufficient to prove that for any fixed strip (α_1, β_1) and any fixed $\lambda > 0$ we have

$$(23.3) \qquad \overline{M}_t\left\{\int_{\alpha_1}^{\beta_1} e^{\lambda |f_{n,\omega}(\sigma, x+\mu t)|^2}\, d\sigma\right\} \to 1 \quad \text{as} \quad n \to \infty$$

for almost all x.

Let us write

$$B_{n,\omega}(\sigma) = \sum_{k=n+1}^{\infty} |a_k|^2 e^{2\lambda_k \sigma};$$

then it follows from (18.7) that we have

$$(23.4) \qquad \int_{Q_\omega} e^{\lambda |f_{n,\omega}(\sigma, x)|^2}\, dw_\omega \leqq \frac{1}{1 - \lambda B_{n,\omega}(\sigma)}$$

when $\lambda B_{n,\omega}(\sigma) < 1$. Also if $H_n(\sigma, x)$ for a fixed n denotes the majorant of the sequence $f_{n+p,\omega}(\sigma, x)$, it follows from (16.6) that

$$(23.5) \qquad \int_{Q_\omega} e^{\lambda (H_n(\sigma, x))^2}\, dw_\omega - 1 \leqq 4\left(\int_{Q_\omega} e^{\lambda |f_{n,\omega}(\sigma, x)|^2}\, dw_\omega - 1\right).$$

Since $B_{n,\omega}(\sigma) \to 0$ *uniformly* in $[\alpha, \beta]$, it follows from (23.4) and (23.5) that

$$\int_{Q_\omega} dw_\omega \int_{\alpha_1}^{\beta_1} e^{\lambda (H_n(\sigma, x))^2}\, d\sigma \to 1 \quad \text{as} \quad n \to \infty.$$

Now we have for each n by Birkhoff's theorem

$$\int_{Q_\omega} M_t\left\{\int_{\alpha_1}^{\beta_1} e^{\lambda (H_n(\sigma, x+\mu t))^2}\, d\sigma\right\} dw_\omega = \int_{Q_\omega} dw_\omega \int_{\alpha_1}^{\beta_1} e^{\lambda (H_n(\sigma, x))^2}\, d\sigma.$$

Since $H_n(\sigma, x)$ is decreasing (in the wide sense), it follows that

$$M_t\left\{\int_{\alpha_1}^{\beta_1} e^{\lambda(H_n(\sigma,\, x+\mu t))^2}\, d\sigma\right\} \to 1 \quad \text{as} \quad n \to \infty$$

for almost all x and this implies (23. 3).

The above theorem contains, in particular, *Parseval's equation*, that

$$M_t\{|f(\sigma + it,\, x)|^2\} = \sum_{k=1}^{\infty} |a_k|^2 e^{2\lambda_k \sigma}$$

holds for almost all x and it also follows that the limits defining the mean value exist uniformly in σ in $[\alpha, \beta]$. This contains a recent result of Carlson for Dirichlet series of the usual type.[1]

Let us finally consider the order of growth of B^p *a.p.* and B^* *a.p.* functions. From the existence of a mean value with respect to t of the right-hand side of (23. 1) in the case where $f(s)$ is B^p *a.p.* we conclude that $|f(\sigma + it)|^p = o(|t|)$ or $f(\sigma + it) = o\left(|t|^{\frac{1}{p}}\right)$ uniformly in $[\alpha, \beta]$. Similarly we conclude from (23. 2) that if $f(s)$ is B^* *a.p.* we have $e^{\lambda|f(\sigma+it)|^2} = o(|t|)$ for any $\lambda > 0$ or $f(\sigma + it) = o(\sqrt{\log |t|})$ uniformly in $[\alpha, \beta]$. We thus obtain the result that *for almost all* x

$$f(\sigma + it,\, x) = o(\sqrt{\log |t|})$$

uniformly in $[\alpha, \beta]$. This result was proved by Carlson for Dirichlet series of the usual type; his proof, which is more elementary than ours, is applicable also in the present case.

§ 24. Distribution Functions.

In this section we take up again the study of functions of the form

$$(24.\ 1) \qquad f(x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k}$$

[1] See F. Carlson [1]. Carlson did not obtain the result for the most general Dirichlet series of the usual type but had to make a restriction on the exponents. His method is valid also for the case where the signs $e^{2\pi i x_k}$ are replaced by real signs ± 1, in which case our argument does not apply.

but the problem we shall consider now is of a more special character. For the sake of simplicity we do not consider the general class G of § 18 but restrict the investigation to the special class of series (24. 1) for which the series

$$\sum_{k=1}^{\infty} |a_k| \tag{24.2}$$

is convergent. In this case the series (24. 1) is absolutely convergent in Q_ω and consequently its sum $f(x)$ is continuous. We call the class of functions $f(x)$ defined in this way the class H; we shall say that $f(x)$ belongs to H^* if at least 5 of the numbers a_k are not zero and shall prove that in this case the values of the function are particularly regularly distributed. Problems of this nature were treated by Bohr and the author in an elementary way[1]; the method here used yields better results and is in the main part an adaptation from a paper of Wintner.[2]

The set S of values $z = f(x)$ attained by a function of the class H is always closed and connected and it also has the property of being transformed into itself by any rotation about the origin. Consequently it is either a closed circle $|z| \leqq R$ or a closed circular ring $r \leqq z \leqq R$. In both cases R is the sum of the series (24. 2). It is easy to see that the second case occurs only when one of the numbers $|a_k|$ is greater than the sum $R - |a_k|$ of the rest of them and that in this case $r = |a_k| - (R - |a_k|) = 2|a_k| - R$. In the special case when one of the $|a_k|$ equals the sum of all the other $|a_k|$ it is convenient still to consider S as a ring with $z = 0$ as the interior boundary.

Now let E denote an arbitrary measurable set in the complex z-plane and let $\Omega = \Omega(E)$ denote the set of points x in Q_ω for which $f(x)$ belongs to E; if Ω is measurable, we denote its measure $m\Omega(E)$ by $\varphi(E)$; the set-function $\varphi(E)$ obtained in this way is called the *distribution function of* $f(x)$. It is clear that $\varphi(E)$ is not changed by a rotation about the origin. We wish to prove the following theorem:

If $f(x)$ belongs to H^*, then $\varphi(E)$ is defined for all measurable sets E and is the indefinite integral of a continuous function $F(z)$; that is, we have

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[1] H. Bohr-B. Jessen [1].

[2] A. Wintner [1]; see also S. Bochner-B. Jessen [1]. Wintner considered directly the distribution problem for almost periodic functions to which we shall apply our results in § 26. The extension to distributions with respect to a weight function is new.

(24.3) $$\varphi(E) = \int_E F(z)\, dw_z$$

for any measurable set E. The function F(z) is called the density of the distribution of f(x).

The function $F(z)$ is clearly a function of $|z|$ alone; it is zero outside and on the boundary of the set S and its integral over S is 1. It could easily be proved that $F(z)$ is positive in all interior points of S except in the special case mentioned above where S should be considered a ring with $z = 0$ as the interior boundary in which case we have $F(0) = 0$. Finally we may observe that $\varphi(E)$ is the indefinite integral of an integrable function $F(z)$ also when $f(x)$ belongs to H provided at least 2 of the numbers a_k are not zero, but there are cases where 4 of the numbers a_k are not zero and where $F(z)$ is not continuous. We shall not require these additional statements.

The theorem has the following immediate consequence of which we shall make use: that *if the set E is measurable in the Jordan sense, then the set in Q_ω is also measurable in the Jordan sense.* This follows from the fact that, since $f(x)$ is continuous, the boundary of Ω will be contained in the set of points x in Q_ω for which $f(x)$ belongs to the boundary of E.

Let $f(x)$ belong to H^* and let

$$g(x) = \sum_{k=1}^{\infty} b_k e^{2\pi i x_k}$$

be any function of H. Then we define the distribution function $\psi(E)$ of $f(x)$ *with respect to* $|g(x)|^2$ as

$$\psi(E) = \int_{\Omega(E)} |g(x)|^2\, dw_\omega .$$

We shall prove the following theorem:

The function $\psi(E)$ is also the indefinite integral of a continuous function $G(z)$, that is we have

(24.4) $$\psi(E) = \int_E G(z)\, dw_z .$$

We call $G(z)$ the density of the distribution of $f(x)$ with respect to $|g(x)|^2$.

The function $G(z)$ is also a function of $|z|$ alone; it is zero when $F(z)$ is zero, and generally positive when $F(z)$ is positive; it can be shown that the only case in which this may not be true is when $g(x) = kf(x)$ for some constant k, when we have $G(z) = |k|^2 |z|^2 F(z)$. The integral of $G(z)$ over S is equal to the integral of $|g(x)|^2$ over Q_ω; we do not suppose that it is 1.

The proof of these theorems is based on a theorem of Fourier transforms and on well-known properties of the Bessel functions $J_0(y)$ and $J_1(y)$. Let us write $z = u + iv$; then $\varphi(E)$ and $\psi(E)$ are at least always defined when E is a rectangle $u_0 \leqq u < u_1$, $v_0 \leqq v < v_1$. It is sufficient to show the existence of continuous functions $F(z)$ and $G(z)$ such that (24.3) and (24.4) hold whenever E is a rectangle of this type. Now let $\zeta = \alpha + i\beta$ be another complex variable and let us form the Fourier transforms

$$\Phi(\zeta) = \int_Z e^{-i(\alpha u + \beta v)}\, d\varphi(E) \quad \text{and} \quad \Psi(\zeta) = \int_Z e^{-i(\alpha u + \beta v)}\, d\psi(E)$$

where the integrals are Stieltjes integrals over the z-plane Z. If we can prove that $\Phi(\zeta)$ and $\Psi(\zeta)$ are *integrable* over the whole ζ-plane Z, our theorems will follow from a familiar theorem for Fourier transforms which says that in this case $\varphi(E)$ and $\psi(E)$ are the indefinite integrals of the functions

$$F(z) = \frac{1}{4\pi^2}\int_Z e^{i(u\alpha + v\beta)}\, \Phi(\zeta)\, dw_\zeta \quad \text{and} \quad G(z) = \frac{1}{4\pi^2}\int_Z e^{i(u\alpha + v\beta)}\, \Psi(\zeta)\, dw_\zeta .$$

We prove the integrability of $\Phi(\zeta)$ and $\Psi(\zeta)$ by simple calculation; we consider first the case of the function $\Phi(\zeta)$. Let us write $f(x) = u(x) + iv(x)$; applying the definition of an integral we get immediately

$$\Phi(\zeta) = \int_{Q_\omega} e^{-i(\alpha u(x) + \beta v(x))}\, dw_\omega .$$

If $a_k = |a_k|\, e^{2\pi i \varrho_k}$ the term $a_k e^{2\pi i x_k}$ in $f(x)$ will give the contributions

$$|a_k| \cos 2\pi(x_k + \varrho_k) \quad \text{and} \quad |a_k| \sin 2\pi(x_k + \varrho_k)$$

to $u(x)$ and $v(x)$ respectively and therefore, if $\zeta = |\zeta|\, e^{2\pi i \sigma}$, the contribution

$$|a_k \zeta| \cos 2\pi(x_k + \varrho_k - \sigma) \quad \text{to} \quad \alpha u(x) + \beta v(x).$$

Consequently we have

$$\Phi(\zeta) = \int\limits_{Q_\omega} \prod_{k=1}^{\infty} e^{-i|a_k\zeta| \cos 2\pi (x_k + \varrho_k - \sigma)} \, dw_\omega .$$

Now the product is uniformly convergent in Q_ω for any fixed ζ; we therefore get

$$\Phi(\zeta) = \prod_{k=1}^{\infty} \int\limits_{C_k} e^{-i|a_k\zeta| \cos 2\pi (x_k + \varrho_k - \sigma)} \, dx_k$$

or finally, using a familiar formula for the function $J_0(y)$,

$$\Phi(\zeta) = \prod_{k=1}^{\infty} J_0(|a_k \zeta|) .$$

Now we use the well-known property of the Bessel function $J_0(y)$, that $J_0(y) = O(y^{-\frac{1}{2}})$ as $y \to \infty$; since at least 5 of the numbers a_k are supposed to be different from zero and since $|J_0(y)| \leqq 1$ for all y, we conclude that $\Phi(\zeta) = O(|\zeta|^{-\frac{5}{2}})$ as $|\zeta| \to \infty$ and this proves that $\Phi(\zeta)$ is integrable over the whole ζ-plane.

For the function $\Psi(\zeta)$ the result is obtained in a similar way. We have

$$\Psi(\zeta) = \int\limits_{Q_\omega} e^{-i(\alpha u(x) + \beta v(x))} |g(x)|^2 \, dw_\omega .$$

For the function $|g(x)|^2$ we use the expansion

$$|g(x)|^2 = \sum_{k=1}^{\infty} |b_k|^2 + \sum_{\substack{k,\, l=1 \\ k \neq l}}^{\infty} b_k \bar{b}_l \, e^{2\pi i (x_k - x_l)}$$

where the series is absolutely convergent. Hence

$$\Psi(\zeta) = \sum_{k=1}^{\infty} |b_k|^2 \, \Phi(\zeta) + \sum_{\substack{k,\, l=1 \\ k \neq l}}^{\infty} b_k \bar{b}_l \, \Psi_{k,l}(\zeta),$$

where

$$\Psi_{k,l}(\zeta) = \int\limits_{Q_\omega} e^{-i(\alpha u(x) + \beta v(x))} \, e^{2\pi i (x_k - x_l)} \, dw_\omega .$$

This again yields

$$\Psi_{k,l}(\zeta) = \int_{c_k} e^{-i(|a_k\zeta| \cos 2\pi(x_k+\varrho_k-\sigma) - 2\pi x_k)}\, dx_k \int_{c_l} e^{-i(|a_l\zeta| \cos 2\pi(x_l+\varrho_l-\sigma) + 2\pi x_l)}\, dx_l$$

$$\cdot \prod_{\substack{m=1 \\ m \neq k,\, l}}^{\infty} \int_{c_m} e^{-i|a_m\zeta| \cos 2\pi(x_m+\varrho_m-\sigma)}\, dx_m$$

or finally

$$\Psi_{k,l}(\zeta) = -\, e^{-2\pi i(\varrho_k-\varrho_l)} J_1(|a_k\zeta|)\, J_1(|a_l\zeta|) \prod_{\substack{m=1 \\ m \neq k,\, l}}^{\infty} J_0(|a_m\zeta|).$$

Now we have also $J_1(y) = O(y^{-\frac{1}{2}})$ as $y \to \infty$ and $|J_1(y)| \leqq 1$ for all y. Consequently we have $\Psi_{k,l}(\zeta) = O(|\zeta|^{-\frac{5}{2}})$ as $|\zeta| \to \infty$ uniformly in k and l. This implies that also $\Psi(\zeta) = O(|\zeta|^{-\frac{5}{2}})$ as $|\zeta| \to \infty$ and this proves the theorem.

The formulae for the functions $F(z)$ and $G(z)$ which we have obtained are useful if we want to study these functions more closely. We shall need the following easy deduction from these formulae: that if $f^{(m)}(x)$ and $g^{(m)}(x)$ are functions of H^* and H respectively converging uniformly to $f(x)$ and $g(x)$, then the corresponding functions $F^{(m)}(z)$ and $G^{(m)}(z)$ must converge uniformly towards $F(z)$ and $G(z)$. We may express this more briefly by saying that $F(z)$ and $G(z)$ *depend continuously* on $f(x)$ and $g(x)$.

§ 25. Weyl's Theorem on Equal Distribution.

We shall need a definition of a mean value for functions $f(t)$ on the line l: $-\infty < t < \infty$ which is more special than that of § 19. Let $f(t)$ be real and bounded but not necessarily measurable on l. Then we say that it has a mean value

$$M_t\{f(t)\}$$

over l in the Riemann sense if there exists corresponding to any $\varepsilon > 0$ a number T such that for *any* interval j on l with $mj > T$ the lower and upper Riemann integrals of $f(t)$ over j when divided by mj differ by less than ε from $M_t\{f(t)\}$. If the characteristic function $\alpha(t)$ of a set A on l has a mean value $M_t\{\alpha(t)\}$ in this sense, we call it the relative Jordan measure of A on l; if the mean value $M_t\{\alpha(t) f(t)\}$ exists, we call it the mean value in the Riemann sense of $f(t)$ over A. It is clear when, for a class of functions $f(t)$ or sets A, the mean values or relative measures shall be said to exist uniformly for all functions or sets in the class.

With these definitions we have the following theorem which is an immediate extension to the space Q_ω of a classical theorem of Weyl; the notations are those of § 19.

If $f(x)$ is integrable over Q_n in the Riemann sense and if the numbers $\mu_1, \mu_2, \mu_3, \ldots$ are linearly independent, then

$$M_t\{f(x+\mu t)\} = \int_{Q_\omega} f(x)\, d w_\omega$$

for all x in Q_ω and the mean value exists uniformly for all x.

The proof is the same as for the theorem in the space Q_n. It is sufficient to apply the remark made in § 10 according to which a function $f(x)$ in Q_ω, when it is integrable in the Riemann sense, can be enclosed for any $\varepsilon > 0$ between two exponential polynomials $a(x)$ and $A(x)$ such that the integral over Q_ω of $A(x) - a(x)$ is smaller than ε. Since the theorem is trivial when $f(x)$ is such a polynomial (it follows by a simple computation), this remark shows its general validity.

Let us observe that if the function $f(x)$ depends only on the variables $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ (in finite or infinite number), the theorem will hold if only the numbers $\mu_{k_1}, \mu_{k_2}, \mu_{k_3}, \ldots$ are linearly independent.

§ 26. Application to a Class of Almost Periodic Functions.

It is known that if an almost periodic function has linearly independent exponents, then the Fourier series of the function is absolutely convergent. We may therefore base the study of such functions directly on their representation by means of an absolutely convergent trigonometric series with linearly independent exponents and need not use the almost periodic character explicitly.

Let the functions

$$f(x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} b_k e^{2\pi i x_k}$$

be as in § 24; that is, let $f(x)$ belong to the class H^* and $g(x)$ to the class H. We consider the functions

$$(26.1)\qquad f(t,x)=\sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{i\lambda_k t} \quad\text{and}\quad g(t,x)=\sum_{k=1}^{\infty} b_k e^{2\pi i x_k} e^{i\lambda_k t}$$

as functions of t on the line l: $-\infty < t < \infty$. The exponents λ_k or at least those of them which belong to non-vanishing coefficients a_k, shall be *linearly independent*. We want to study the distribution of the values of the function $f(t, x)$ and of $f(t, x)$ with respect to $|g(t, x)|^2$ for any x.

Let E be any set in the complex z-plane which is measurable in the Jordan sense; we know from § 24 that the set Ω in Q_ω in which $f(x)$ belongs to E is also measurable in the Jordan sense. This implies that if $\alpha(x)$ denotes the characteristic function of the set Ω, the functions $\alpha(x)$ and $\alpha(x)|g(x)|^2$ are both integrable over Q_ω in the Riemann sense. We also have

$$\varphi(E)=\int_{Q_\omega} \alpha(x)\, dw_\omega \quad\text{and}\quad \psi(E)=\int_{Q_\omega} \alpha(x)|g(x)|^2\, dw_\omega .$$

Now let $\alpha(t, x)$ denote 1 when $f(t, x)$ belongs to E and 0 elsewhere. Let us write $\lambda_k = 2\pi\mu_k$ as usual. Then we have $f(t,x)=f(x+\mu t)$, $g(t,x)=g(x+\mu t)$ and also $\alpha(t,x)=\alpha(x+\mu t)$. It therefore follows from Weyl's theorem that uniformly in x

$$\varphi(E)=M_t\{\alpha(t,x)\} \quad\text{and}\quad \psi(E)=M_t\{\alpha(t,x)|g(t,x)|^2\}.$$

We formulate this result in the following way:

The distribution functions $\varphi(E)$ and $\psi(E)$ of $f(x)$ and of $f(x)$ with respect to $|g(x)|^2$ are also for any fixed x the distribution functions of $f(t, x)$ and of $f(t, x)$ with respect to $|g(t, x)|^2$ in the sense that, for any set E which is measurable in the Jordan sense, $\varphi(E)$ is the relative Jordan measure of the set A on l in which $f(t, x)$ belongs to E and $\psi(E)$ is the mean value in the Riemann sense of $|g(t, x)|^2$ over this set A. Furthermore these relative measures and mean values exist uniformly in x for any fixed E.

This way of combining Weyl's theorem with results concerning the distribution of the values of functions of an infinite number of variables in order to obtain results for almost periodic functions is due to Bohr.[1]

[1] See for the latest exposition H. Bohr-B. Jessen [2], [3]. The simplification obtained by the use of the theory of integration in Q_ω is more essential in the case of analytical almost periodic functions with which we shall deal in § 28 than in the case considered here.

§ 27. Distribution of the Values of Certain Classes of Analytic Functions.

We consider as in § 21 a class of functions

$$f(s, x) = \sum_{k=1}^{\infty} g_k(s)\, e^{2\pi i x_k} \tag{27.1}$$

where the functions $g_k(s)$ are regular in a fixed domain D in the z-plane, but this time we suppose that for any set A the series

$$\sum_{k=1}^{\infty} \overline{\underset{A}{\text{bound}}}\, |g_k(s)|$$

converges. In this case the function $f(s, x)$ will exist as a regular function in D for *any* point x in Q_ω, the series (27.1) being uniformly convergent in any set A. The function $f(s, x)$ is obviously continuous considered as a function in the space (D, Q_ω) where (s, x) is the variable point. If s denotes a fixed point in D, we may consider $f(s, x)$ as a function in Q_ω; it is a function of the class H. The derivatives

$$f'(s, x) = \sum_{k=1}^{\infty} g'_k(s)\, e^{2\pi i x_k}$$

form clearly a class of the same kind by the familiar fact that if A_1 and A_2 are as in § 21, there exists a constant C, depending only on A_1 and A_2, such that

$$\overline{\underset{A_2}{\text{bound}}}\, |f'(s)| \leqq C\, \overline{\underset{A_1}{\text{bound}}}\, |f(s)|$$

for any function $f(s)$ which is regular in D.

We shall assume that none of the functions $f(s, x)$ reduces to a constant in D; then it follows in a familiar way that there exists, corresponding to any set A, a number $N = N(A)$ which is greater than or equal to the number of z-points of $f(s, x)$ in A for any complex z and any x in Q_ω. Here and later on z-points are always counted in their multiplicity.

Finally we shall assume that in any point s of D at least 5 of the functions $g_k(s)$ are not zero. This means that for any s the function $f(s, x)$ considered as a function in Q_ω belongs to the class H^*. The distribution function of this function in the sense of § 24 will be denoted by $\varphi(s, E)$ and its density by $F(s, z)$. We shall also consider the distribution function $\psi(s, E)$ of $f(s, x)$ with respect to the weight function $|f'(s, x)|^2$; its density will be denoted by $G(s, z)$. These functions $F(s, z)$ and $G(s, z)$ are continuous, not only considered as functions of z for a fixed s, but also considered as functions of (s, z) in the space (D, Z) where Z is used to denote the complex z-plane; this follows from the remark at the end of § 24. For any set A we have $F(s, z) = 0$ and $G(s, z) = 0$ for all s in A when z is outside the set $S = S(A)$ of values attained by all the functions $f(s, x)$ in A; this set S is either a closed circle or a closed circular ring about the origin in the z-plane.

We will now consider the distribution of the values of the functions $f(s, x)$ from another point of view. Let z be any complex number and B an arbitrary measurable set belonging to a set A in D. We denote by $n(B, z, x)$ the number of z-points of $f(s, x)$ in B; this function $n(B, z, x)$ takes for a fixed B only a finite number of values; if it is measurable over Q_ω we call its integral

$$\chi(B, z) = \int_{Q_\omega} n(B, z, x)\, dw_\omega$$

the *average frequency* with which the functions $f(s, x)$ take the value z in the set B. We wish to prove the following theorem:

The function $\chi(B, z)$ is always defined for all measurable sets B belonging to a set A in D and we have

$$\chi(B, z) = \int_B G(s, z)\, dw_s. \tag{27.2}$$

Thus the same function $G(s, z)$ which for a fixed s determines the density of the distribution of $f(s, x)$ with respect to $|f'(s, x)|^2$ will for a fixed z determine the density of the average frequency of the z-points of the functions $f(s, x)$.

Behind this theorem lies a familiar function-theoretic lemma which we shall formulate later on. The proof of the theorem is complicated by the fact that the measurability properties of the function $n(B, z, x)$ are not trivial.

It is clearly sufficient to prove the theorem in the case where B is measurable *in the Jordan sense.* We first prove that in this case the function $n(B, z, x)$ is even integrable *in the Riemann sense* over Q_ω for any z. This will be of importance for the applications. From Rouché's theorem it follows that if for a given x the function $f(s, x)$ has no z-points on the boundary C of B, then there exists an interval I_x in Q_ω surrounding x and in which $n(B, z, x)$ is constant; it is therefore sufficient to show that the set Γ of points x in Q_ω for which $f(s, x)$ has z-points on C is a null-set.

We choose a set A which contains the set C in its interior; let $\varepsilon > 0$ be given; then since the set C is a null-set we may cover it by a (finite or) enumerable number of circles C_n belonging to A such that the sum of the measures $m C_n$ of these circles is less than ε. We denote the mid-point of C_n by s_n and the radius of C_n by r_n. Now let K denote a constant such that $|f'(s, x)| \leqq K$ for all s in A and all x; then if x belongs to Γ, the inequality

$$|f(s_n, x) - z| < K r_n$$

must be true for at least one value of n. This means that the set Γ is covered by the sets Γ_n where Γ_n is the set of points x in Q_ω for which $f(s_n, x)$ belongs to the circle E_n in the z-plane which has its mid-point in the fixed point z and the radius $K r_n$. By the definition of the functions $\varphi(s, E)$ and $F(s, z)$ we have

$$m \Gamma_n = \varphi(s_n, E_n) = \int_{E_n} F(s_n, z)\, d w_z.$$

Now let M denote a constant such that $F(s, z) \leqq M$ for all s in A and all z. Then we obtain

$$m \Gamma_n \leqq M m E_n = M K^2 m C_n.$$

Consequently the sum of the measures of the sets Γ_n is less than $MK^2 \varepsilon$ and since M and K depend only on A and not on ε, it follows that Γ is a null-set.

This result has some further applications which are of importance for the proof. We know now that $\chi(B, z)$ exists for any z; we shall prove that it is a *continuous* function of z. This follows from Rouché's theorem which shows that if $z_1, z_2, z_3, \ldots$ is a sequence of points converging to z, the sequence $n(B, z_n, x)$ will converge to $n(B, z, x)$ except perhaps in the set Γ; since the sequence is uniformly bounded in Q_ω, this implies $\chi(B, z_n) \to \chi(B, z)$.

Another consequence is that the function $n(B, z, x)$ is measurable not only as a function in Q_ω for a fixed z, but also as a function of (z, x) in the space (Z, Q_ω); we shall even prove more: namely, that the function is integrable *in the Riemann sense.* From Rouché's theorem it follows that if $f(s, x)$ has no

z-points on C, then $n(B, z, x)$ is constant in an interval in (Z, Q_ω) containing (s, x); it is therefore sufficient to prove that the set Γ^* of points (z, x) for which $f(s, x)$ has z-points on C is a null-set. Now C is closed; consequently Γ^* is closed and therefore measurable; furthermore its intersection with any set (z, Q_ω) is a set Γ of the kind considered above and this implies by Fubini's theorem that Γ^* must be a null-set.

After these preparations, we now begin the proof of the formula (27.2). Since both sides of this formula are continuous functions of z, it is sufficient to prove the formula in the integrated form

$$\int_E \chi(B, z)\, dw_z = \int_E dw_z \int_B G(s, z)\, dw_s$$

and we may of course restrict ourselves to the case where E is measurable in the Jordan sense. Upon inversion of the order of integration on the right-hand side, by virtue of the definition of $G(s, z)$ this relation takes the form

$$\int_E \chi(B, z)\, dw_z = \int_B \psi(s, E)\, dw_s. \tag{27.3}$$

We base the proof on a familiar function-theoretic lemma:

Let $f(s)$ be any regular function in D and let $n(B, z)$ denote the number of z-points of $f(s)$ in B. Then

$$\int_E n(B, z)\, dw_z = \int_B \alpha(s)\, |f'(s)|^2\, dw_s$$

where $\alpha(s)$ is 1 when $f(s)$ belongs to E and 0 elsewhere.

If we apply this lemma to the functions $f(s, x)$, we obtain for each x the relation

$$\int_E n(B, z, x)\, dw_z = \int_B \alpha(s, x)\, |f'(s, x)|^2\, dw_s$$

where $\alpha(s, x)$ is 1 when $f(s, x)$ belongs to E and 0 elsewhere. Now this function $\alpha(s, x)$ is integrable, even in the Riemann sense, over the set (B, Q_ω); this follows in the usual way when we observe that the boundary of the set of points (s, x) in (D, Q_ω) for which $f(s, x)$ belongs to E is contained in the set of points for which $f(s, x)$ belongs to the boundary of E, which is closed and has the measure zero since its intersection with any set (s, Q_ω) has the measure zero. Consequently, we may integrate the last relation over Q_ω and we may also invert the order of integration. This gives immediately the relation (27.3).

§ 28. A Class of Analytic Almost Periodic Functions.

If an analytic function is almost periodic in a strip $[\alpha, \beta]$ and has linearly independent exponents, then its Dirichlet series is absolutely convergent in (α, β). We may therefore base the study of such functions directly on their representation by means of an absolutely convergent exponential series.

We consider as in § 21 a whole class of series

$$f(s, x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{\lambda_k s} \qquad (s = \sigma + it),$$

but now the series

$$\sum_{k=1}^{\infty} |a_k| e^{\lambda_k \sigma}$$

is supposed to converge in a certain interval $\alpha < \sigma < \beta$. We do not yet suppose that the exponents λ_k are linearly independent, but only that they are real and all different. We make the assumption that at least 5 of the coefficients a_k are not zero. Then the functions $f(s, x)$ satisfy all the conditions of § 27 when D denotes the strip (α, β). The result of § 27 takes in this case a particularly simple form.

Let us write $\lambda_k = 2\pi\mu_k$. Then we have $f(\sigma + it, x) = f(\sigma, x + \mu t)$. Since

$$f'(s, x) = \sum_{k=1}^{\infty} \lambda_k a_k e^{2\pi i x_k} e^{\lambda_k s},$$

we have also $f'(\sigma + it, x) = f'(\sigma, x + \mu t)$. This, however, implies that for a given $s = \sigma + it$ the distribution of $f(s, x)$ with respect to $|f'(s, x)|^2$ depends on σ only and not on t; that is, we have $\psi(s, E) = \psi(\sigma, E)$ and $G(s, z) = G(\sigma, z)$ for any t. Now let B be a rectangle $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$ where $\alpha < \alpha_1 < \beta_1 < \beta$. Then the result of the last section takes the following form:

If $n(\alpha_1, \beta_1, z, x)$ *denotes the number of z-points of* $f(s, x)$ *in* $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$, *then the integral*

$$\chi(\alpha_1, \beta_1, z) = \int_{Q_\omega} n(\alpha_1, \beta_1, z, x)\, dw_\omega ,$$

which measures the average frequency of the z-points of the functions $f(s, x)$ *in* $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$, *exists as a Riemann integral over* Q_ω *and its value is given by*

$$\chi(\alpha_1, \beta_1, z) = \int_{\alpha_1}^{\beta_1} G(\sigma, z)\, d\sigma \tag{28.1}$$

where $G(\sigma, z)$ *is the density of the distribution of* $f(\sigma, x)$ *with respect to* $|f'(\sigma, x)|^2$.

We now make the further assumption that the exponents λ_k, or at least those of them for which the corresponding coefficients a_k are not zero, are *linearly independent.* In this case we have by the theorem of Weyl

$$\chi(\alpha_1, \beta_1, z) = M_t\{n(\alpha_1, \beta_1, z, x + \mu t)\}$$

uniformly for all x. This result takes a simpler form if we introduce the following notation:

Let $f(s)$ be regular in (α, β) and let $N(\alpha_1, \beta_1, \gamma, \delta, z)$ denote the number of z-points of $f(s)$ in the rectangle $\alpha_1 < \sigma < \beta_1$, $\gamma < t < \delta$. We say that the function $f(s)$ takes the value z with the *average frequency* $\chi(\alpha_1, \beta_1, z)$ in the strip (α_1, β_1) if there exists, corresponding to any $\varepsilon > 0$, a number T such that the number $N(\alpha_1, \beta_1, \gamma, \delta, z)$ when divided by $\delta - \gamma$ differs by less than ε from $\chi(\alpha_1, \beta_1, z)$ as soon as $\delta - \gamma > T$. It is clear when for a class of functions $f(s)$ the frequency shall be said to exist uniformly for all functions in the class.

Now the number $n(\alpha_1, \beta_1, z, x + \mu t)$ is merely the number of z-points of $f(s, x + \mu t) = f(s + it, x)$ in $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$ or, what amounts to the same thing, the number of z-points of $f(s, x)$ in the rectangle obtained from $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$ by the vertical translation it. From this interpretation of $n(\alpha_1, \beta_1, z, x + \mu t)$ follows at once the following theorem.

The average frequency $\chi(\alpha_1, \beta_1, z)$ *with which the functions* $f(s, x)$ *take the value* z *in* $\alpha_1 < \sigma < \beta_1$, $-\frac{1}{2} < t < \frac{1}{2}$ *is also for each fixed* x *the average frequency with which* $f(s, x)$ *takes the value* z *in the strip* (α_1, β_1). *Furthermore we have uniformity in* x *in the existence of this average frequency for fixed* (α_1, β_1) *and* z.

The previous theorem gives for $\chi(\alpha_1, \beta_1, z)$ the expression (28. 1). It is not uninteresting to observe that in the present case the function $G(\sigma, z)$ can also be determined in another way. This follows immediately from the result of § 26, which shows that $G(\sigma, z)$ for each fixed x is the density of the distribution of $f(\sigma + it, x)$ with respect to $|f'(\sigma + it, x)|^2$. This additional remark gives a formulation of the last theorem which makes no use of the theory of integration in Q_ω and in which the point x plays only the rôle of a parameter with respect to which we have a certain uniformity. As this theorem is perhaps the most interesting result of our discussion we formulate it explicitly.

If the series

$$f(s, x) = \sum_{k=1}^{\infty} a_k e^{2\pi i x_k} e^{\lambda_k s},$$

where at least 5 of the numbers a_k *are not zero, is absolutely convergent in* (α, β) *and if those of the numbers* λ_k *for which the corresponding* a_k *are not zero are linearly independent, then there exists for any* x *an average frequency* $\chi(\alpha_1, \beta_1, z)$ *with which the function* $f(s, x)$ *takes the value* z *in the strip* $(\alpha <) \alpha_1 < \sigma < \beta_1 (< \beta)$. *The function* $\chi(\alpha_1, \beta_1, z)$ *defined in this way is independent of* x *and may be represented in the form*

$$\chi(\alpha_1, \beta_1, z) = \int_{\alpha_1}^{\beta_1} G(\sigma, z)\, d\sigma,$$

where $G(\sigma, z)$ *is a continuous function defined for* $\alpha < \sigma < \beta$ *and all* z. *In their dependence on* z *the functions* $\chi(\alpha_1, \beta_1, z)$ *and* $G(\sigma, z)$ *are functions of* $|z|$ *only. The function* $G(\sigma, z)$ *may also be determined as the density of the distribution of* $f(\sigma + it, x)$ *with respect to* $|f'(\sigma + it, x)|^2$ *for any fixed* x.

Finally, we have uniformity with respect to x *in the existence of* $\chi(\alpha_1, \beta_1, z)$ *as well as in the existence of the distribution of* $f(\sigma + it, x)$ *with respect to* $|f'(\sigma + it, x)|^2$.

The distribution of the z-points for arbitrary analytic almost periodic functions has been studied by the author in a previous paper[1]; it would be easy by means of the theory of integration in Q_ω for the case here considered to re-establish the main result of that paper.

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