

## THE THEORY OF QUASI-SASAKIAN STRUCTURES

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### Introduction

On a contact manifold of dimension  $2n + 1$  there exists, by definition, a global 1-form  $\eta$  such that  $\eta \wedge (d\eta)^n \neq 0$ . An almost contact manifold also carries a 1-form  $\eta$  but it is not necessarily of maximal rank. The purpose of this paper is to explore the meaning of the rank of  $\eta$ . To this end, we initiate the study of normal almost contact metric manifolds with closed fundamental 2-form  $\phi$ . Such manifolds will be called quasi-Sasakian manifolds.

§ 1 presents the basic definitions and some results from the theory of almost contact structures. Beginning with § 2 we develop the theory of quasi-Sasakian structures. In § 2 a large class of examples is given and in § 3 we discuss the meaning of the rank of  $\eta$ . The result is that if  $\eta$  has rank  $2p + 1$  and the determined almost product structure is integrable then the manifold is locally the product of a Sasakian (normal contact metric) manifold and a Kaehler manifold. That is to say,  $\eta$  having rank  $2p + 1$  means, loosely speaking, that the space is split locally into a Sasakian piece where  $\eta \wedge (d\eta)^p \neq 0$  and a Kaehler piece whose fundamental 2-form is  $\phi - d\eta$  properly restricted. § 4 gives some geometric results on quasi-Sasakian manifolds and § 5 characterizes the case where  $d\eta = 0$ , the latter characterization being necessary in the study of the topology of cosymplectic manifolds [1], [2].

### 1. Almost contact manifolds

All manifolds considered will be  $C^\infty$  and connected. A superscript will denote the dimension of the manifold, for example  $M^{2n+1}$ , and  $\mathcal{E}^{2n+1}$  will denote the module of vector fields over  $M^{2n+1}$ . When we speak of an almost contact manifold, quasi-Sasakian manifold, etc., we mean the manifold together with the corresponding structure.

A  $(2n + 1)$ -dimensional manifold carrying a global 1-form  $\eta$  such that  $\eta \wedge (d\eta)^n \neq 0$  is said to have a *contact structure* with  $\eta$  as its *contact form*. On the other hand, a manifold  $M^{2n+1}$  has an *almost contact structure*  $(\phi, \xi, \eta)$ .

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if it carries a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$ , and a 1-form  $\eta$  such that

$$(1.1) \quad \begin{aligned} \eta(\xi) &= 1, & \phi\xi &= 0, \\ \eta \circ \phi &= 0, & \phi^2 &= -I + \xi \otimes \eta; \end{aligned}$$

this is equivalent to a reduction of the structural group of the tangent bundle of  $M^{2n+1}$  to  $U(n) \times 1$  (see [9]). From equations (1.1) we see that the maps  $-\phi^2$  and  $\xi \otimes \eta$  form an almost product structure on  $M^{2n+1}$  with decomposition  $\mathcal{E}^{2n+1} = \mathcal{E}^{2n} \oplus \mathcal{E}^1$ .

Furthermore, an almost contact manifold  $M^{2n+1}$  admitting a Riemannian metric  $g$  such that

$$(1.2) \quad \begin{aligned} g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y) \\ g(X, \xi) &= \eta(X) \end{aligned}$$

where  $X, Y \in \mathcal{E}^{2n+1}$ , is said to have an *almost contact metric structure*  $(\phi, \xi, \eta, g)$ . It follows from (1.1) that

$$g(\phi X, Y) = -g(X, \phi Y)$$

that is in an almost contact metric manifold with structure tensors  $(\phi, \xi, \eta, g)$ ,  $\phi$  is skew-symmetric with respect to  $g$ . We define a 2-form  $\Phi$  by

$$\Phi(X, Y) = g(X, \phi Y)$$

and call it the *fundamental 2-form* of the almost contact metric structure. If  $M^{2n+1}$  has a contact structure with contact form  $\eta$  then it has an underlying almost contact metric structure  $(\phi, \xi, \eta, g)$  such that

$$\Phi = d\eta$$

called an *associated almost contact metric structure* [9].

Let  $M^{2n+1}$  be an almost contact manifold. S. Sasaki and Y. Hatakeyama [10] defined an almost complex structure  $J$  on  $M^{2n+1} \times R^1$  by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right)$$

where  $f$  is a  $C^\infty$  real-valued function on  $M^{2n+1}R^1$  and  $X \in \mathcal{E}^{2n+1}$ . Considering the Nijenhuis torsion  $[J, J]$  of  $J$ , they computed  $[J, J]((X, 0), (Y, 0))$  and  $[J, J]((X, 0), (0, d/dt))$  which gave rise to four tensors  $N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}$  given by

$$\begin{aligned}
 N^{(1)}(X, Y) &= [\phi, \phi](X, Y) + d\eta(X, Y)\xi \\
 (1.3) \quad N^{(2)}(X, Y) &= (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X) \\
 N^{(3)}(X) &= (\mathcal{L}_{\xi}\phi)X \\
 N^{(4)}(X) &= -(\mathcal{L}_{\xi}\eta)(X)
 \end{aligned}$$

where  $\mathcal{L}_X$  denotes the Lie derivative with respect to  $X$ . The result is that  $J$  is integrable if and only if  $N^{(1)} = 0$ ; in particular,  $N^{(1)} = 0$  implies  $N^{(2)} = N^{(3)} = N^{(4)} = 0$  [10]. An almost contact structure is said to be *normal* if  $N^{(1)} = 0$ , that is, if the almost complex structure on  $M^{2n+1} \times R^1$  is integrable. A normal contact metric structure is called a *Sasakian structure*.

We now state some results from the theory of almost contact structures which are required later. The first two are due to A. Morimoto [7], [8].

**Proposition 1.1.** *Suppose that  $M^{2n+1}$  is the bundle space of a principal circle bundle over a complex manifold  $M^{2n}$  and that there exist a connexion form  $\eta$  on  $M^{2n+1}$  and a 2-form  $\Omega$ , the curvature form of  $\eta$ , of bidegree  $(1, 1)$  on  $M^{2n}$  such that  $d\eta = \pi^*\Omega$ , where  $\pi: M^{2n+1} \rightarrow M^{2n}$  is the bundle projection map. Then we can find a linear transformation field  $\phi$  and a vector field  $\xi$  on  $M^{2n+1}$  such that  $(\phi, \xi, \eta)$  is a normal almost contact structure.*

For later use we give the definitions of  $\phi$  and  $\xi$  in this proposition. Let  $J$  be the almost complex structure on  $M^{2n}$ , that is  $J^2 = -I$ . Then  $\phi$  is given by  $\phi X = \tilde{\pi}J\pi_*X$  where  $\tilde{\pi}$  denotes the horizontal lift with respect to the connexion given by  $\eta$ . The vector field  $\xi$  is defined by requiring that it be vertical (i.e.,  $\pi_*\xi = 0$ ) and that  $\eta(\xi) = 1$ .

We say that the vector field  $\xi$  on an almost contact manifold  $M^{2n+1}$  is *regular* if for every point  $m \in M^{2n+1}$  there is a (coordinate) neighborhood  $U_m$  of  $m$  such that every orbit of  $\xi$  passes through  $U_m$  at most once. If the orbits of  $\xi$  are closed curves,  $\xi$  is called a (regular) *closed vector field*.

Morimoto [8] showed that if  $\xi$  is a regular closed vector field, then the only normal almost contact manifolds are those constructed above.

**Proposition 1.2.** *If  $M^{2n+1}$  is a normal almost contact manifold with  $\xi$  a regular closed vector field, then  $M^{2n+1}$  has a principal circle bundle structure over a complex manifold  $M^{2n}$  as described in Proposition 1.1.*

As a corollary we have that if  $M^{2n+1}$  is a compact regular normal almost contact manifold, then it has a circle bundle structure over a complex manifold as in Proposition 1.1.

The almost complex structure tensor  $J$  in this theorem is given by  $JX = \pi_*\phi\tilde{\pi}X$ . The operator  $J$  is well-defined; for, if  $\bar{X}$  is a vector field on  $M^{2n+1}$  then  $\phi\bar{X}$  is a horizontal vector field with respect to the connexion determined by  $\eta$ . Thus,  $\phi$  is invariant under the right translations of  $M^{2n+1}$  by the action of  $S^1$  and hence  $JX(\pi(m))$  is independent of the choice of  $m$  on the fibre over  $\pi(m)$ . In the proof of Theorem 2.4 below we will show that  $J^2 = -I$ .

If the manifold  $M^{2n}$  in Proposition 1.1 is only an almost complex manifold, then  $\phi$  and  $\xi$  as given, together with the connexion form  $\eta$  define an almost contact structure on the bundle space  $M^{2n+1}$ . Hatakeyama [4] proved the following proposition.

**Proposition 1.3.** *The almost contact structure on the circle bundle  $M^{2n+1}$  given in Proposition 1.1 is normal if and only if the almost complex structure on the base manifold  $M^{2n}$  is integrable and the curvature form  $\Omega$  of the connexion form  $\eta$  is of bidegree  $(1, 1)$ .*

## 2. Quasi-Sasakian structures

**Definition.** An almost contact metric structure  $(\phi, \xi, \eta, g)$  is called *quasi-Sasakian* if it is normal and its fundamental 2-form  $\Phi$  is closed, that is, for every  $X, Y \in \mathcal{E}^{2n+1}$

$$(2.1) \quad \begin{aligned} [\phi, \phi](X, Y) + d\eta(X, Y)\xi &= 0, \\ d\Phi &= 0, \quad \Phi(X, Y) = g(X, \phi Y). \end{aligned}$$

There are many types of quasi-Sasakian structures ranging from the cosymplectic case,  $d\eta = 0$  ( $\text{rank } \eta = 1$ ), to the Sasakian case,  $\eta \wedge (d\eta)^n \neq 0$  ( $\text{rank } \eta = 2n + 1$ ,  $\Phi = d\eta$ ). The 1-form  $\eta$  has *rank*  $r = 2p$  if  $(d\eta)^p \neq 0$  and  $\eta \wedge (d\eta)^p = 0$ , and has *rank*  $r = 2p + 1$  if  $\eta \wedge (d\eta)^p \neq 0$  and  $(d\eta)^{p+1} = 0$ . We also say that  $r$  is the *rank* of the quasi-Sasakian structure.

We shall first show that there are no quasi-Sasakian structures of even rank.

**Lemma 2.1.** *If  $(\phi, \xi, \eta, g)$  is a normal almost contact metric structure, then*

$$d\eta(X, \xi) = 0$$

for every  $X \in \mathcal{E}^{2n+1}$ .

*Proof.* The coboundary formula for  $d$  gives

$$\begin{aligned} d\eta(X, \xi) &= X(\eta(\xi)) - \xi(\eta(X)) - \eta([X, \xi]) \\ &= -\xi(\eta(X)) - \eta([X, \xi]) \\ &= -(\mathcal{L}_\xi \eta)(X) = 0 \end{aligned}$$

since  $\eta(\xi) = 1$  and by normality (see formula (1.3)),  $(\mathcal{L}_\xi \eta)(X) = 0$ .

**Theorem 2.2.** *There are no quasi-Sasakian structures of even rank.*

*Proof.* Let  $X_1, \dots, X_{2p} \in \mathcal{E}^{2n+1} = \mathcal{E}^{2n} \oplus \mathcal{E}^1$  be vector fields such that  $(d\eta)^p(X_1, \dots, X_{2p}) \neq 0$ . By Lemma 2.1 we may assume without loss of generality that  $X_1, \dots, X_{2p} \in \mathcal{E}^{2n}$ , from which

$$\begin{aligned} (\eta \wedge (d\eta)^p)(\xi, X_1, \dots, X_{2p}) &= \eta(\xi)((d\eta)^p(X_1, \dots, X_{2p})) \\ &= (d\eta)^p(X_1, \dots, X_{2p}) \neq 0 \end{aligned}$$

where we have used the facts that  $\eta(\xi) = 1$  and  $\eta(X_1) = \dots = \eta(X_{2p}) = 0$  for  $X_1, \dots, X_{2p} \in \mathcal{E}^{2n}$ .

We now give some examples of quasi-Sasakian structures of odd rank. In fact we shall exhibit a large class of quasi-Sasakian manifolds.

Let  $M^{2n}$  be a Kaehler manifold with metric  $g'$ . Let  $\Omega$  be the fundamental 2-form and  $J$  be the almost complex structure tensor. S. Kobayashi [6] has shown that the set of all principal circle bundles over  $M^{2n}$  can be given a group structure isomorphic to the cohomology group  $H^2(M^{2n}, Z)$ , where  $Z$  is the ring of integers. Using this result we can prove the following theorem.

**Theorem 2.3.** *Let  $M^{2n}$  be a Kaehler manifold. If there exists a 2-form  $\Psi^*$  of bidegree  $(1, 1)$  and rank  $p$  representing an element of  $H^2(M^{2n}, Z)$ , then there exists a quasi-Sasakian structure of rank  $2p + 1$  on the corresponding principal circle bundle.*

*Proof.* Let  $M^{2n+1}$  denote the bundle space and  $\pi : M^{2n+1} \rightarrow M^{2n}$  the projection map. Let  $\eta'$  be a connexion form on  $M^{2n+1}$ . Then there exists a 2-form  $\Psi^{**}$  on  $M^{2n}$  such that  $d\eta' = \pi^*\Psi^{**}$ . However, the characteristic class  $[\Psi^*]$  of  $M^{2n+1}$ ,  $[\Psi^*] \in H^2(M^{2n}, Z)$ , is independent of the choice of connexions (Kobayashi [6]), so that  $[\Psi^*] = [\Psi^{**}]$ . Thus, there exists a 1-form  $\omega$  on  $M^{2n}$  such that  $\Psi^* - \Psi^{**} = d\omega$ . Hence

$$\pi^*\Psi^* = \pi^*\Psi^{**} + \pi^*d\omega = d(\eta' + \pi^*\omega).$$

Now  $\pi^*\omega$  is horizontal and *ad*-equivariant (i.e.  $\pi^*\omega \circ R_s = ad(s^{-1})\pi^*\omega$ , where  $R_s$  is right translation by  $s \in S^1$ ). Since  $S^1$  is abelian,  $ad(s^{-1})$  is the identity map, so  $\pi^*\omega \circ R_s = \pi^*\omega$ . Hence, if we set  $\eta = \eta' + \pi^*\omega$ ,

$$\eta \circ R_s = \eta,$$

since  $\eta'$  is *ad*-equivariant. Moreover, if  $\xi$  is a vertical vector field such that  $\eta'(\xi) = 1$ , then  $\eta(\xi) = 1$ , since  $(\pi^*\omega)(\xi) = \omega(\pi_*\xi) = 0$ . Thus,  $\eta$  is a connexion form on  $M^{2n+1}$  with  $d\eta = \pi^*\Psi^*$ , the curvature form of  $\eta$ , and hence  $\eta \wedge (d\eta)^p \neq 0$ . For, if  $X_1, \dots, X_{2p}$  are  $2p$  linearly independent horizontal vector fields,

$$\begin{aligned} (\eta \wedge (d\eta)^p)(\xi, X_1, \dots, X_{2p}) &= \eta(\xi)((d\eta)^p(X_1, \dots, X_{2p})) \\ &= (\pi^*\Psi^*)^p(X_1, \dots, X_{2p}) \\ &= \Psi^{**p}(\pi_*X_1, \dots, \pi_*X_{2p}) \neq 0. \end{aligned}$$

Define  $\phi$  by  $\phi X = \tilde{\pi}J\pi_*X$  where  $\tilde{\pi}$  denotes the horizontal lift with respect to the connexion  $\eta$ . Then, since  $\xi$  is vertical,  $\phi\xi = 0$ ; moreover  $\eta \circ \phi = 0$ . An easy computation gives  $\phi^2X = -X + \eta(X)\xi$ . Hence, we have an almost contact structure on  $M^{2n+1}$ . Now define a metric  $g$  on  $M^{2n+1}$  by  $g(X, Y) = g'(\pi_*X, \pi_*Y) + \eta(X)\eta(Y)$ . Then, since  $g'$  is hermitian, one can verify that  $g$  satisfies equations (1.2), so we have an almost contact metric structure on

$M^{2n+1}$ . Defining the fundamental 2-form  $\Phi$  by  $\Phi(X, Y) = g(X, \phi Y)$  we see that

$$\begin{aligned}\Phi(X, Y) &= g'(\pi_* X, \pi_* \phi Y) + \eta(X)\eta(\phi Y) \\ &= g'(\pi_* X, J\pi_* Y) = \Omega(\pi_* X, \pi_* Y)\end{aligned}$$

so that  $\Phi = \pi^*\Omega$ , and  $d\Phi = 0$  since  $M^{2n}$  is Kaehlerian. Finally, since  $\Psi^*$  is of bidegree  $(1, 1)$  and  $M^{2n}$  is Kaehlerian, it follows from Proposition 1.3 that the almost contact metric structure is normal.

That quasi-Sasakian manifolds actually exist may be seen by taking  $M^{2n}$  to be the Kaehlerian product of Kaehler manifolds  $M^{2p}$  and  $M^{2q}(p + q = n)$  and letting  $\Psi^*$  denote the fundamental 2-form of  $M^{2p}$  extended to be a form on  $M^{2n}$  vanishing over  $M^{2q}$ .

We shall now show that if  $\xi$  is a regular closed vector field on a quasi-Sasakian manifold  $M^{2n+1}$ , then  $M^{2n+1}$  has a circle bundle structure as in Theorem 2.3.

**Theorem 2.4.** *If  $M^{2n+1}$  has a quasi-Sasakian structure  $(\phi, \xi, \eta, g)$  of rank  $2p + 1$  with  $\xi$  a regular closed vector field, then  $M^{2n+1}$  has a principal circle bundle structure over a Kaehler manifold, the characteristic class of  $M^{2n+1}$  being  $[\Psi^*]$  where  $d\eta = \pi^*\Psi^*$ ;  $\Psi^*$  is of bidegree  $(1, 1)$  and rank  $p$ .*

*Proof.* By Proposition 1.2,  $M^{2n+1}$  has a circle bundle structure over a complex manifold  $M^{2n}$ . Let  $\pi : M^{2n+1} \rightarrow M^{2n}$  be the bundle projection map and  $\tilde{\pi}X$  the horizontal lift of a vector field  $X$  on  $M^{2n}$  with respect to the connexion given by  $\eta$ . The almost complex structure tensor  $J$  on  $M^{2n}$  is given by  $JX = \pi_*\phi\tilde{\pi}X$  and  $J$  is well-defined as we saw in §1. A direct computation gives  $J^2 = -I$ . Indeed,

$$\begin{aligned}J^2X &= \pi_*\phi\tilde{\pi}\pi_*\phi\tilde{\pi}X \\ &= \pi_*\phi(\phi\tilde{\pi}X - \eta(\phi\tilde{\pi}X)\xi) \\ &= \pi_*(-\tilde{\pi}X + \eta(\tilde{\pi}X)\xi) = -X.\end{aligned}$$

Now define a metric  $g'$  on  $M^{2n}$  by  $g'(X, Y) = g(\tilde{\pi}X, \tilde{\pi}Y) - \eta(\tilde{\pi}X)\eta(\tilde{\pi}Y)$ , and a 2-form  $\Omega$  on  $M^{2n}$  by  $\Omega(X, Y) = g'(X, JY)$ . Then

$$g'(JX, JY) = g'(X, Y), \quad \Omega(X, Y) = \Phi(\tilde{\pi}X, \tilde{\pi}Y)$$

where  $\Phi$  is the fundamental 2-form of the structure  $(\phi, \xi, \eta, g)$ . Thus,  $g'$  is hermitian and  $\pi^*\Omega = \Phi$ . Now  $\Phi$  has rank  $n$ , and hence  $\Omega$  does also. Furthermore  $0 = d\Phi = d\pi^*\Omega = \pi^*(d\Omega)$  implies  $d\Omega = 0$  since  $\pi^*$  is injective. Thus,  $M^{2n}$  is Kaehlerian. Finally there exists a 2-form  $\Psi^*$  on  $M^{2n}$  (the curvature form of  $\eta$ ) such that  $d\eta = \pi^*\Psi^*$  which by Proposition 1.3 is of bidegree  $(1, 1)$ . Moreover, the characteristic class of  $M^{2n+1}$  is  $[\Psi^*] \in H^2(M^{2n}, \mathbb{Z})$ .

**Remark.** Combining the above results with the well-known Boothby-Wang fibration [3] it is seen that if  $M^{2n+1}$  has a quasi-Sasakian structure of rank  $2p + 1$  with  $\xi$  a regular closed vector field, and  $[\mathcal{Q}] \in H^2(M^{2n}, Z)$ , then it also has a Sasakian structure.

### 3. Locally product quasi-Sasakian manifolds

Before proceeding with the results of this section, we require some new notation and to possibly alter the quasi-Sasakian structure to a more canonical form. Let  $(\phi, \xi, \eta, g')$  be a quasi-Sasakian structure of rank  $2p + 1$  on a manifold  $M^{2n+1}$ . Let  $\mathcal{E}^{2p}$  denote the submodule of  $\mathcal{E}^{2n+1}$  on which  $(d\eta)^p \neq 0$  and  $\phi\mathcal{E}^{2p} = \mathcal{E}^{2p}$ ; since  $d\eta(X, \xi) = 0$  by Lemma 2.1,  $\mathcal{E}^{2p}$  is a submodule of  $\mathcal{E}^{2n}$ . Let  $\mathcal{E}^{2q}$  denote the orthogonal complement of  $\mathcal{E}^{2p}$  in  $\mathcal{E}^{2n}$  and define maps  $\phi$  and  $\theta$  by

$$\phi = \begin{cases} \phi|_{\mathcal{E}^{2p}} & \text{on } \mathcal{E}^{2p} \\ 0 & \text{on } \mathcal{E}^{2q} \\ 0 & \text{on } \mathcal{E}^1 \end{cases}, \quad \theta = \begin{cases} 0 & \text{on } \mathcal{E}^{2p} \\ \phi|_{\mathcal{E}^{2q}} & \text{on } \mathcal{E}^{2q} \\ 0 & \text{on } \mathcal{E}^1 \end{cases}$$

then  $\phi = \phi + \theta$ . Observe that  $\phi$  and hence  $\theta$  are not unique since the choice of  $\mathcal{E}^{2p}$  is not so. Now, if necessary, define a new metric  $g$  on  $M^{2n+1}$  by requiring that  $g$  agree with  $g'$  on  $\mathcal{E}^{2q}$ , and  $\mathcal{E}^1$  and satisfy  $g(X, \phi Y) = d\eta(X, Y)$  for  $X, Y \in \mathcal{E}^{2p}$ . It is easy to verify that  $(\phi, \xi, \eta, g)$  is a quasi-Sasakian structure of rank  $2p + 1$  on  $M^{2n+1}$ . We shall work with this structure in this paper.

The maps  $-\phi^2 + \xi \otimes \eta$  and  $-\theta^2$  define an almost product structure with decomposition  $\mathcal{E}^{2n+1} = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$ , where  $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \mathcal{E}^1$ . Similarly the maps  $-\phi^2$  and  $-\theta^2 + \xi \otimes \eta$  give an almost product structure with decomposition  $\mathcal{E}^{2n+1} = \mathcal{E}^{2p} \oplus \mathcal{E}^{2q+1}$ , where  $\mathcal{E}^{2q+1} = \mathcal{E}^{2q} \oplus \mathcal{E}^1$ . The integrability of these almost product structures is discussed below in detail.

**Theorem 3.1.** *If  $M^{2n+1}$  has a quasi-Sasakian structure of rank  $2p + 1$  with  $[\theta, \theta] = 0$  for some  $\theta$ , then  $M^{2n+1}$  is locally the product of a Sasakian manifold  $M^{2p+1}$  and a Kaehler manifold  $M^{2q}$ ,  $q = n - p$ .*

*Proof.* It is well-known that  $[\theta, \theta] = 0$  if and only if  $[-\theta^2, -\theta^2] = 0$ ; but this is just the integrability condition for a locally product structure (the decomposition here is  $\mathcal{E}^{2n+1} = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$ ). Let  $x^\alpha$  ( $\alpha = 1, \dots, 2p + 1$ ),  $x^\alpha$  ( $\alpha = 2p + 2, \dots, 2n + 1$ ) be coordinates such that  $\{\partial/\partial x^\alpha\}$  is a basis of  $\mathcal{E}^{2p+1}$ , and  $\{\partial/\partial x^\alpha\}$  is a basis of  $\mathcal{E}^{2q}$ . Thus if  $\{x^\alpha, x^\alpha\}$  and  $\{y^\alpha, y^\alpha\}$  are coordinates for two overlapping coordinate neighborhoods, then

$$\left| \begin{array}{cc} \frac{\partial y^\alpha}{\partial x^\alpha} & \frac{\partial y^\alpha}{\partial x^\alpha} \\ \frac{\partial y^\alpha}{\partial x^\alpha} & \frac{\partial y^\alpha}{\partial x^\alpha} \end{array} \right| \neq 0.$$

However, since we have a locally product structure, the  $y^\alpha$ 's depend only on the  $x^\alpha$ 's, and the  $y^\alpha$ 's only on the  $x^\alpha$ 's; hence

$$\frac{\partial y^\alpha}{\partial x^\alpha} = 0, \quad \frac{\partial y^\alpha}{\partial x^\alpha} = 0.$$

Therefore,

$$\left| \frac{\partial y^\alpha}{\partial x^\alpha} \right| \neq 0, \quad \left| \frac{\partial y^\alpha}{\partial x^\alpha} \right| \neq 0.$$

Hence the system of subspaces defined by  $x^\alpha = \text{constant}$ , for each  $\alpha$ , is an atlas determining a manifold  $M^{2p+1}$ ; similarly, the system of subspaces defined by  $x^\alpha = \text{constant}$ , for each  $\alpha$ , is an atlas determining a manifold  $M^{2q}$ . Locally,  $M^{2p+1}$  is the product of  $M^{2p+1}$  and  $M^{2q}$ , and the localized modules of vector fields over  $M^{2p+1}$  and  $M^{2q}$  are (isomorphic to)  $\mathcal{E}^{2p+1}$  and  $\mathcal{E}^{2q}$ , respectively.

Now  $\eta \wedge (d\eta)^p \neq 0$  on  $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \mathcal{E}^1$ , so  $\eta|_{\mathcal{E}^{2p+1}} \wedge (d(\eta|_{\mathcal{E}^{2p+1}}))^p \neq 0$  over  $M^{2p+1}$  giving a contact structure. Since  $\phi$  and  $\phi$  agree on  $\mathcal{E}^{2p}$  and vanish on  $\mathcal{E}^1$ ,  $(\phi, \xi, \eta)|_{\mathcal{E}^{2p+1}}$  satisfy equations (1.1) on  $M^{2p+1}$ . Furthermore,  $g|_{\mathcal{E}^{2p+1}}$  satisfies equations (1.2). Hence,  $(\phi, \xi, \eta, g)|_{\mathcal{E}^{2p+1}}$  is an associated almost contact metric structure. To show that  $M^{2p+1}$  is Sasakian, it remains only to show that the structure is normal. For  $X, Y \in \mathcal{E}^{2p+1}$

$$\begin{aligned} [\phi, \phi](X, Y) + d(\eta|_{\mathcal{E}^{2p+1}})(X, Y)\xi \\ = [\phi, \phi](X, Y) - 2[\phi, \theta](X, Y) + [\theta, \theta](X, Y) + d\eta(X, Y)\xi \\ = -2[\phi, \theta](X, Y) \end{aligned}$$

since  $\phi = \phi - \theta$ ,  $[\theta, \theta] = 0$ , and by normality  $[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0$ . Continuing the computation we have

$$\begin{aligned} [\phi, \phi](X, Y) + d(\eta|_{\mathcal{E}^{2p+1}})(X, Y)\xi &= -2[\phi, \theta](X, Y) \\ &= -(\phi\theta[X, Y] + \theta\phi[X, Y] + [\phi X, \theta Y] + [\theta X, \phi Y] \\ &\quad - \phi[X, \theta Y] - \theta[X, \phi Y] - \phi[\theta X, Y] - \theta[\phi X, Y]) \\ &= 0 \end{aligned}$$

where each term in the last expression vanishes because  $X, Y \in \mathcal{E}^{2p+1}$  and  $\theta$  is zero on  $\mathcal{E}^{2p+1}$ ,  $X \in \mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \mathcal{E}^1$  implies  $\phi X = \phi X + \theta X \in \mathcal{E}^{2p}$ , and the distributions  $-\phi^2 + \xi \otimes \eta$  and  $-\theta^2$  are integrable so that  $[X, Y] \in \mathcal{E}^{2p+1}$ .

Finally, define a 2-form  $\Theta$  by  $\Theta(X, Y) = g(X, \theta Y)$ . Since  $\theta = \phi - \psi$  we have  $\Theta = \phi - \psi$ , and hence  $d\Theta = 0$ . Furthermore,  $\theta$  has rank  $2q$ , so  $\Theta^q \neq 0$ ,  $(\theta|_{\mathcal{E}^{2q}})^2 = -I$ ,  $[\theta, \theta] = 0$  and  $g|_{\mathcal{E}^{2q}}$  is hermitian. Thus,  $\theta|_{\mathcal{E}^{2q}}$  and  $g|_{\mathcal{E}^{2q}}$  give  $M^{2q}$  a Kaehler structure.

We can also obtain the converse of this theorem, so we again have a large class of examples of quasi-Sasakian manifolds.

**Theorem 3.2.** *If a manifold  $M^{2n+1}$  is (locally) the product of a Sasakian manifold  $M^{2p+1}$  and a Kaehler manifold  $M^{2q}$ , then  $M^{2n+1}$  has a quasi-Sasakian structure of rank  $2p + 1$ .*

*Proof.* Let  $\mathcal{E}^{2n+1}$ ,  $\mathcal{E}^{2p+1}$  and  $\mathcal{E}^{2q}$  denote the localized modules of vector fields on  $M^{2n+1}$ ,  $M^{2p+1}$  and  $M^{2q}$  respectively. Then, since  $M^{2n+1}$  is locally the product of  $M^{2p+1}$  and  $M^{2q}$ , we have  $\mathcal{E}^{2n+1} \cong \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$ . Let  $(\phi', \xi', \eta', g_p)$  be an associated almost contact metric structure to the Sasakian structure on  $M^{2p+1}$ , and  $(\theta', g_q)$  an associated almost hermitian structure on  $M^{2q}$  (i.e.  $\theta'^2 = -I$ ,  $g_q(\theta'X, \theta'Y) = g_q(X, Y)$ ,  $X, Y \in \mathcal{E}^{2q}$ , and  $\Theta'(X, Y) = g_q(X, \theta'Y)$  where  $\Theta'$  is the fundamental 2-form of the Kaehler structure on  $M^{2q}$ ).

We shall write  $X \in \mathcal{E}^{2n+1}$  as  $X_1 + X_2$  where  $X_1 \in \mathcal{E}^{2p+1}$  and  $X_2 \in \mathcal{E}^{2q}$  (under the isomorphism  $\mathcal{E}^{2n+1} \cong \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$ ). Define a 1-form  $\eta$  on  $M^{2n+1}$  by  $\eta(X) = \eta'(X_1)$  and take  $\xi \in \mathcal{E}^{2n+1}$  to be equal to  $\xi' \in \mathcal{E}^{2p+1}$ ; then  $\eta(\xi) = \eta'(\xi') = 1$ . Now define new maps  $\phi, \theta, \phi : \mathcal{E}^{2n+1} \rightarrow \mathcal{E}^{2n+1}$  by  $\phi X = \phi'X_1$ ,  $\theta X = \theta'X_2$ ,  $\phi = \phi + \theta$ ; then a direct verification gives  $\phi\xi = 0$ ,  $\eta \circ \phi = 0$  and  $\phi^2 = -I + \xi \otimes \eta$ . Defining a metric  $g$  on  $M^{2n+1}$  by  $g = g_p + g_q$ , we obtain equations (1.2) by direct computation using the facts that  $g_p$  satisfies equations (1.2) and  $g_q$  is hermitian. Thus,  $M^{2n+1}$  has an almost contact metric structure.

Now since  $M^{2n+1}$  has a locally product structure, we have coordinates  $\{x^a, x^a\}$  and basis vector fields  $\{\partial/\partial x^a, \partial/\partial x^a\}$  as in the proof of Theorem 3.1. With respect to this basis the components  $\phi_a^b$  of  $\phi$  are functions of the  $x^a$ 's alone ( $\phi_a^b \partial/\partial x^b = \phi \partial/\partial x^a = \phi' \partial/\partial x^a$ ) and similarly for the components  $\theta_a^b$  of  $\theta$ . Using these facts a direct computation yields  $[\phi, \phi] + \xi \otimes d\eta = 0$  giving the normality of the structure on  $M^{2n+1}$ .

Finally let  $\Phi$ , given by  $\Phi(X, Y) = g(X, \phi Y)$ , denote the fundamental 2-form of the structure on  $M^{2n+1}$ . Since  $M^{2q}$  is Kaehlerian,  $d\Theta' = 0$ , and since  $\phi = \phi + \theta (= \phi' + \theta')$ ,  $\Phi = d\eta' + \Theta'$ . Thus  $d\Phi = d(d\eta' + \Theta') = 0$ . Hence, the almost contact metric structure defined above is quasi-Sasakian; since  $\phi'$  has rank  $2p$ , so has  $\phi$  and therefore the structure has rank  $2p + 1$ .

It should be remarked that quasi-Sasakian structures with  $[\phi, \phi] = 0$  (decomposition  $\mathcal{E}^{2n+1} = \mathcal{E}^{2p} \oplus \mathcal{E}^{2q+1}$ , where  $\mathcal{E}^{2q+1} = \mathcal{E}^{2q} \oplus \mathcal{E}^1$ ) are of special interest. In fact, if  $[\phi, \phi] = 0$  then  $\phi$  is the zero map and we therefore have only the rank 1 case (cosymplectic). For, since  $[\phi, \phi] = 0$  gives an integrable distribution,  $X, Y \in \mathcal{E}^{2p}$  implies  $[X, Y] \in \mathcal{E}^{2p}$ , and hence  $[\theta, \theta](X, Y) = 0$ . Now, by normality

$$\begin{aligned} -d\eta(X, Y)\xi &= [\phi, \phi](X, Y) \\ &= [\phi, \phi](X, Y) + 2[\phi, \theta](X, Y) + [\theta, \theta](X, Y) \\ &= 2[\phi, \theta](X, Y) = 0. \end{aligned}$$

But if  $X$  or  $Y$  is in  $\mathcal{E}^{2q+1}$  then  $d\eta(X, Y) = g(X, \phi Y) = 0$ . Thus,  $d\eta(X, Y) = 0$  for every  $X, Y \in \mathcal{E}^{2n+1}$  giving the cosymplectic case.

The integrability of the almost product structure determined by  $-\phi^2$ , and  $\xi \otimes \eta$  (decomposition  $\mathcal{E}^{2n+1} = \mathcal{E}^{2n} \oplus \mathcal{E}^1$ ) also occurs only in the cosymplectic case. For, we know that  $[-\phi^2, -\phi^2] = 0$  ( $[\xi \otimes \eta, \xi \otimes \eta] = 0$ ) if and only if  $[\phi, \phi] = 0$ , and hence it follows from the normality condition,

$$[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0 ,$$

that  $[\phi, \phi] = 0$  if and only if  $d\eta = 0$ .

Returning to the case  $[\theta, \theta] = 0$ , let us suppose that  $\xi$  is a regular closed vector field and see what effect the integrability condition has on the base space of Theorems 2.3 and 2.4.

**Theorem 3.3.** *Let  $M^{2n+1}$  be a quasi-Sasakian manifold of rank  $2p+1$  with  $\xi$  a regular closed vector field. If  $M^{2n+1}$  has the locally product structure of Theorem 3.1 (i.e.  $[\theta, \theta] = 0$ ), then the base manifold  $M^{2n}$  of the circle bundle  $M^{2n+1}$  is locally the Kaehlerian product of two Kaehler manifolds  $M^{2p}$  and  $M^{2q}$ ,  $p+q=n$ .*

*Proof.* Using the usual notation, define maps  $P$  and  $Q$  over  $M^{2n}$  by  $PX = \pi_*\phi\tilde{\pi}X$  and  $QX = \pi_*\theta\tilde{\pi}X$  where  $\tilde{\pi}X$  is the horizontal lift of  $X$  with respect to the connexion  $\eta$  on  $M^{2n+1}$ . Then, since  $\phi = \psi + \theta$ , the almost complex structure tensor  $J$  on  $M^{2n}$  satisfies  $J = P + Q$ . Since  $\psi\theta = \theta\psi = 0$ , it can be verified that  $PQ = QP = 0$  and  $-P^2 + (-Q^2) = I$ , and hence  $-P^2$  and  $-Q^2$  are projection maps. Thus, to show that  $M^{2n}$  has a locally product structure one needs only to show that  $[Q, Q] = 0$ , for, then  $[-Q^2, -Q^2] = [-P^2, -P^2] = 0$ . Thus, if  $X$  and  $Y$  are vector fields on  $M^{2n}$

$$\begin{aligned} [Q, Q](X, Y) &= Q^2[X, Y] + [QX, QY] - Q[X, QY] - Q[QX, Y] \\ &= \pi_*\theta^2\tilde{\pi}[X, Y] + [\pi_*\theta\tilde{\pi}X, \pi_*\theta\tilde{\pi}Y] - \pi_*\theta\tilde{\pi}[X, \pi_*\theta\tilde{\pi}Y] \\ &\quad - \pi_*\theta\tilde{\pi}[\pi_*\theta\tilde{\pi}X, Y] \\ &= \pi_*\theta^2[\tilde{\pi}X, \tilde{\pi}Y] + \pi_*[\theta\tilde{\pi}X, \theta\tilde{\pi}Y] - \pi_*\theta[\tilde{\pi}X, \theta\tilde{\pi}Y] \\ &\quad - \pi_*\theta[\theta\tilde{\pi}X, \tilde{\pi}Y] \\ &= 0 . \end{aligned}$$

The spaces of which  $M^{2n}$  is locally the product will be denoted by  $M^{2p}$  and  $M^{2q}$ ; we now show that these spaces are Kaehlerian. Since  $-P^2$  and  $-Q^2$  are projection maps we have  $P^2|_{\mathcal{E}^{2p}} = -I|_{\mathcal{E}^{2p}}$  and  $Q^2|_{\mathcal{E}^{2q}} = -I|_{\mathcal{E}^{2q}}$  giving almost complex structures on  $M^{2p}$  and  $M^{2q}$ ; furthermore, we have  $[P, P] = 0$  and  $[Q, Q] = 0$  so these are complex structures. If  $g'$  is the Kaehler metric on  $M^{2n}$ , then for  $X, Y \in \mathcal{E}^{2p}$

$$g'|_{\mathcal{E}^{2p}}(PX, PY) = g'(JX, JY) = g'(X, Y) = g'|_{\mathcal{E}^{2p}}(X, Y) .$$

Similarly  $g'|_{\mathcal{E}^{2q}}(QX, QY) = g'|_{\mathcal{E}^{2q}}(X, Y)$  for  $X, Y \in \mathcal{E}^{2q}$ . Thus, the restrictions of  $g'$  to  $\mathcal{E}^{2p}$  and  $\mathcal{E}^{2q}$  give hermitian metrics on  $M^{2p}$  and  $M^{2q}$ , respectively. Define 2-forms  $\Omega_1$  and  $\Omega_2$  by  $\Omega_1(X, Y) = g'(X, PY)$  and  $\Omega_2(X, Y) = g'(X, QY)$ . Then, since  $J = P + Q$ , the fundamental 2-form  $\Omega$  on  $M^{2n}$  is equal to  $\Omega_1 + \Omega_2$ . Since  $P$  has rank  $2p$  and  $Q$  rank  $2q$ ,  $\Omega_1^p \neq 0$  on  $M^{2p}$  and  $\Omega_2^q \neq 0$  on  $M^{2q}$ . Finally, since  $\Omega_1|_{\mathcal{E}^{2p}} = \Omega|_{\mathcal{E}^{2p}}$  and  $\Omega_2|_{\mathcal{E}^{2q}} = \Omega|_{\mathcal{E}^{2q}}$ ,  $d(\Omega_1|_{\mathcal{E}^{2p}}) = 0$  and  $d(\Omega_2|_{\mathcal{E}^{2q}}) = 0$ .

Now  $\nabla_X \theta = 0$  for every  $X$  implies  $[\theta, \theta] = 0$  where  $\nabla$  is covariant differentiation with respect to the Riemannian connexion determined by the metric  $g$  of the quasi-Sasakian structure. Thus, if the stronger hypothesis is imposed we have a locally product structure as above. We show that it is actually a *locally decomposable* Riemannian structure, i.e., if  $\{x^\alpha, x^\alpha\}$  are the coordinates introduced above, then  $g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$ ,  $\alpha, \beta = 1, \dots, 2p+1$ , depends only on the  $x^\alpha$ 's, and  $g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$ ,  $\alpha, \beta = 2p+2, \dots, 2n+1$ , only on the  $x^\alpha$ 's.

**Lemma 3.4.**  $\nabla_X \theta = 0$  implies  $\nabla_X \theta^2 = 0$ .

*Proof.*  $(\nabla_X \theta)Y = \nabla_X \theta Y - \theta \nabla_X Y$ .

Hence

$$\begin{aligned} (\nabla_X \theta^2)Y &= \nabla_X \theta^2 Y - \theta^2 \nabla_X Y \\ &= \theta \nabla_X \theta Y + (\nabla_X \theta) \theta Y + \theta (\nabla_X \theta) Y - \theta \nabla_X \theta Y \\ &= (\nabla_X \theta) \theta Y + \theta (\nabla_X \theta) Y \end{aligned}$$

from which the lemma follows.

Let  $F = -\psi^2 + \xi \otimes \eta + \theta^2$ , that is,  $F$  is the difference of the projection maps  $-\psi^2 + \xi \otimes \eta$  and  $-\theta^2$ . It is known (Yano [12], p. 221) that a necessary and sufficient condition for a locally product Riemannian space to be locally decomposable is

$$\nabla_X F = 0$$

for every  $X$ . But in our case  $F = -\psi^2 + \xi \otimes \eta + \theta^2 = I + 2\theta^2$  and our result follows from the lemma. We state the result formally.

**Theorem 3.5.** *If  $M^{2n+1}$  has a quasi-Sasakian structure of rank  $2p+1$  with  $\nabla_X \theta = 0$  for every  $X \in \mathcal{E}^{2n+1}$ , then  $M^{2n+1}$  is a locally decomposable Riemannian manifold with the locally product structure of Theorem 3.1.*

**Corollary 3.6.** *The Riemannian structure of a cosymplectic manifold is locally decomposable.*

#### 4. Some geometric results

In the last section we considered the distributions  $-\theta^2$  and  $-\psi^2 + \xi \otimes \eta$  with  $[\theta, \theta] = 0$ ; here we shall work with a general quasi-Sasakian structure,

that is, we have the three distributions  $-\phi^2$ ,  $-\theta^2$ ,  $\xi \otimes \eta$ . We begin with some important and interesting lemmas.

**Lemma 4.1.** *The fundamental vector field  $\xi$  of a quasi-Sasakian structure is a Killing vector field.*

*Proof.* By normality  $N^{(3)}$  vanishes, that is,  $\mathcal{L}_\xi \phi = 0$ ; hence for  $X, Y \in \mathcal{E}^{2n+1}$

$$(\mathcal{L}_\xi \Phi)(X, Y) = (\mathcal{L}_\xi g)(X, \phi Y).$$

On the other hand

$$\mathcal{L}_\xi \Phi = d\iota_\xi \Phi + \iota_\xi d\Phi = 0$$

since  $d\Phi = 0$  and  $(\iota_\xi \Phi)X = \Phi(\xi, X) = g(\xi, \phi X) = 0$ . Furthermore, by normality,  $N^{(4)}$  vanishes, that is,  $\mathcal{L}_\xi \eta = 0$ . Hence, a computation gives

$$(\mathcal{L}_\xi g)(X, (\xi \otimes \eta)Y) = 0.$$

Thus  $(\mathcal{L}_\xi g)(X, \phi Y) = 0$  and  $(\mathcal{L}_\xi g)(X, (\xi \otimes \eta)Y) = 0$ . Now since the map  $\phi + \xi \otimes \eta$  is non-singular,  $\mathcal{L}_\xi g = 0$  and hence  $\xi$  is a Killing vector field.

**Lemma 4.2.**  $\mathcal{L}_\xi \psi = 0$  and  $\mathcal{L}_\xi \theta = 0$ .

*Proof.*  $\mathcal{L}_\xi \Psi = 0$ , since  $d\Psi = 0$  and  $(\iota_\xi \Psi)X = 0$ . Hence

$$0 = (\mathcal{L}_\xi \Psi)(X, Y) = \xi(g(X, \psi Y)) - g([\xi, X], \psi Y) - g(X, \psi[\xi, Y]).$$

However, by Lemma 4.1

$$0 = (\mathcal{L}_\xi g)(X, \psi Y) = \xi(g(X, \psi Y)) - g([\xi, X], \psi Y) - g(X, [\xi, \psi Y]).$$

Therefore,  $0 = [\xi, \psi Y] - \psi[\xi, Y] = (\mathcal{L}_\xi \psi)Y$ . On the other hand, since  $\mathcal{L}_\xi \phi = 0$ , we have  $\mathcal{L}_\xi \theta = \mathcal{L}_\xi \phi - \mathcal{L}_\xi \psi = 0$ .

**Lemma 4.3.**  $\nabla_Y \xi = -\frac{1}{2}\phi Y$  for any  $Y \in \mathcal{E}^{2n+1}$  ( $\nabla$  is covariant differentiation with respect to the Riemannian connexion).

*Proof.* Since  $\nabla$  is covariant differentiation with respect to the Riemannian connexion

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = d\eta(X, Y).$$

Using the fact that  $\xi$  is a Killing vector field, that is,

$$0 = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]),$$

and the identity

$$Xg(Y, Z) = g(Y, \nabla_X Z) + g(\nabla_Y X, Z)$$

we obtain

$$(\nabla_X \eta)(Y) = - (\nabla_Y \eta)(X).$$

Hence,

$$d\eta(X, Y) = - 2(\nabla_Y \eta)(X),$$

so  $g(X, \phi Y) = - 2g(X, \nabla_Y \xi)$  from which since  $X$  is arbitrary

$$\nabla_Y \xi = - \frac{1}{2} \phi Y$$

as desired.

The questions of the distributions  $-\phi^2$ ,  $-\theta^2$ ,  $\xi \otimes \eta$  being parallel along one another, being flat and being geodesic were discussed in detail in [1]. Here we will prove an interesting curvature theorem.

Let  $X_m, Y_m$  be tangent vectors at  $m \in M^{2n+1}$  and  $K(X_m, Y_m)$  denote the sectional curvature at  $m$  determined by the plane section spanned by  $X_m$  and  $Y_m$ . Let  $R_{XY}$  denote the curvature transformation, that is,

$$R_{XY} = \nabla_{[X, Y]} + \nabla_Y \nabla_X - \nabla_X \nabla_Y.$$

**Theorem 4.4.** *If  $M^{2n+1}$  has a quasi-Sasakian structure of rank  $2p + 1$  then at every  $m \in M^{2n+1}$*

$$K(\xi_m, X_m) = \begin{cases} \frac{1}{4}, & X_m \in \mathcal{E}^{2p+1}(m), X_m \notin \mathcal{E}^1(m) \\ 0, & X_m \in \mathcal{E}^{2q}(m). \end{cases}$$

*Proof.* Without loss of generality we may take  $X_m$  to be a unit vector orthogonal to  $\xi_m$ . We now have

$$\begin{aligned} g(R_{\xi X} \xi, X) &= g(\nabla_{[\xi, X]} \xi, X) + g(\nabla_X \nabla_\xi \xi, X) - g(\nabla_\xi \nabla_X \xi, X) \\ &= g(-\frac{1}{2} \phi[\xi, X], X) - g(-\frac{1}{2} \nabla_\xi \phi X, X) \\ &= g(-\frac{1}{2} \phi[\xi, X], X) - g(-\frac{1}{2} \nabla_\phi \xi, X) - g(-\frac{1}{2} [\xi, \phi X], X) \\ &= -g(\frac{1}{4} \phi^2 X, X) \\ &= \begin{cases} \frac{1}{4}, & X \in \mathcal{E}^{2p} \\ 0, & X \in \mathcal{E}^{2q} \end{cases} \end{aligned}$$

from which the result follows. We have used Lemmas 4.2 and 4.3 in the computation.

In the Sasakian case (rank  $2n + 1$ ) the theorem reduces to that of Hatakeyama, Ogawa, Tanno [4].

**Corollary 4.5.** *A quasi-Sasakian manifold of constant curvature is either Sasakian or cosymplectic (locally flat).*

**Corollary 4.6.** *A quasi-Sasakian manifold of strictly positive curvature is Sasakian.*

### 5. Characterization of the cosymplectic case

The following formula in the theory of Sasakian manifolds is proved in [11]:

$$(5.1) \quad (\nabla_X \Phi)(Y, Z) = \frac{1}{2}(\eta(Y)g(X, Z) - \eta(Z)g(X, Y)).$$

In this section its quasi-Sasakian analogue is given in order to determine the meaning of the vanishing of  $\nabla_X \Phi$  (equivalently  $\nabla_X \phi$ ).

**Proposition 5.1.** *On a quasi-Sasakian manifold*

$$(5.2) \quad \begin{aligned} (\nabla_X \Phi)(Y, Z) &= \frac{1}{2}(\eta(Y)g(X, Z) - \eta(Z)g(X, Y)) \\ &\quad + \frac{1}{2}(\eta(Y)g(\theta^2 X, Z) - \eta(Z)g(\theta^2 X, Y)). \end{aligned}$$

The proof is a very lengthy computation but is similar to that of (5.1).

**Theorem 5.2.** *A quasi-Sasakian manifold  $M^{2n+1}$  is cosymplectic (rank 1) if and only if  $\nabla_X \Phi = 0$  for every  $X \in \mathcal{E}^{2n+1}$ .*

*Proof.* The condition is clearly sufficient; for,  $\nabla_X \Phi = 0$  for every  $X$ , implies  $[\phi, \phi] = 0$  and hence by normality we have

$$d\eta(X, Y)\xi = -[\phi, \phi](X, Y) = 0$$

for every  $X, Y \in \mathcal{E}^{2n+1}$ . Necessity follows from Proposition 5.1; for, if  $d\eta = 0$  on  $M^{2n+1}$ , then  $\phi$  is the zero map on  $\mathcal{E}^{2n+1}$  and  $\theta = \phi$ . Thus (5.2) becomes

$$\begin{aligned} (\nabla_X \Phi)(Y, Z) &= \frac{1}{2}(\eta(Y)g(X, Z) - \eta(Z)g(X, Y)) \\ &\quad + \frac{1}{2}(\eta(Y)g(-X + \eta(X)\xi, Z) \\ &\quad - \eta(Z)g(-X + \eta(X)\xi, Y)) \\ &= 0. \end{aligned}$$

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