THE THEORY OF SCALE RELATIVITY*

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Abstract

Basing our discussion on the relative character of all scales in nature and on the explicit dependence of physical laws on scale in quantum physics, we apply the principle of relativity to scale transformations. This principle, in combination with its breaking above the Einstein-de Broglie wavelength and time, leads to the demonstration of the *existence of a universal, absolute and impassable scale in nature, which is invariant under dilatation*. This lower limit to all lengths is identified with the Planck scale, which now plays for scale the same role as is played by light velocity for motion. We get new scale transformations of a Lorentzian form and generalize the de Broglie and Heisenberg relations. As a consequence the high energy length and mass scales now decouple, energy and momentum tending to infinity when resolution tends to the Planck scale, which thus plays the role of the previous zero point. This theory solves the problem of divergence of charge and mass (self-energy) in electrodynamics, implies that the *four* fundamental couplings (including gravitation) converge at the Planck energy, improves the agreement of GUT predictions with experimental results, and allows one to get precise estimates of the values of the fundamental coupling constants.

1 Introduction

Since the Galilean analysis about the nature of inertial motion, the theory of relativity has been developed by extending its application domain to coordinates systems involved in more and more general states of motion: this was partly achieved in Einstein's special and general theories of relativity. Hence the principle of general relativity states that "the laws of physics must be of such a nature that they apply to systems of reference in any kind of motion" [1].

However, as pointed out by Levy-Leblond, [2] the abstract *principle of relativity* should be distinguished from any of its possible realizations as concrete *theories of relativity*. This point of view may still be generalized into a framework in which

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relativity is considered as a general method of thinking in sciences [3]: it consists in analysing how the results of measurements and (cor)relations between them are dependent on the particular conditions under which the measurements have been performed. Assuming that these results are measured in some "reference system" (e.g. coordinate systems in case of position and time measurements), these conditions may be characterized as "states" of the reference systems.

Such states of reference systems play a special role in physics: they being defined as characteristics of the reference systems themselves, no absolute value can be attributed to them, but only *relative* ones, since they can be defined and described only with respect to *another* reference system. Two systems at least are needed to define them. As a consequence the transformation laws between reference frames will be of the greatest importance in a relativity theory.

In such a frame of thinking, the most fundamental relativity is the relativity of positions and instants. It is usually expressed in terms of homogeneity and isotropy of space and uniformity of time and actually makes up the basis of the whole of physics. It states that there is no preferential origin for a coordinate system and is finally included in special relativity through the Poincaré group.

Then Einstein's relativity is, strictly, a theory of "motion relativity", since the particular relative state of coordinate systems which the special and general theories of relativity have extensively analysed (in the classical domain) is their state of motion.

We suggest in this paper that *scale* (i.e. *resolution* with which measurements have been performed) *may also be defined as a relative state of reference systems*, and that Einstein's principle of relativity can be generalized by requiring that *the laws of physics apply to any systems of coordinates, whatever their state of scale.* In other words, we shall require *scale covariance* of the equations of physics. The *quantum behavior of microphysics* may to some extent be reinterpreted as *a manifestation of scale relativity.* But in its present form quantum field theory corresponds, rather, to a Galilean version of such a scale relativity theory, especially in the renormalization group approach.

Indeed we first recall how the renormalization group may be applied to space-time itself, yielding an anomalous dimension[1] for space and time variables. Then we demonstrate in a general way that the principle of relativity alone, in its Galilean form (i.e. without adding any extra postulate of invariance), is sufficient to derive the Lorentz transformation as a general solution to the (special) relativity problem. Once applied to scale, and owing to the fact that physical laws become explicitely scale-dependent only for resolutions below the de Broglie length and time (namely, that scale relativity is broken at the de Broglie transition), this reasoning leads to the existence of an absolute, universal scale which is invariant under dilatations and so cannot be exceeded. Then, after having identified this scale as the Planck scale, we attempt to develop a theory based on this new structure: the Einstein-de Broglie and Heisenberg relations are generalized, and first implications concerning the domain of high energy physics are considered.

[1] This dimension is 'anomalous' in the sense that it cannot be obtained from simple dimensional analysis. It is actually a 'scale dimension', defined as the difference between the fractal dimension and the topological dimension. It becomes a variable in scale-relativity, identified with a fifth dimension that we have called 'djinn' in subsequent papers.

2 The Dependence on Scale of Microphysics

Starting from the Planck/Einstein work at the beginning of the century, the development of the quantum theory has forced physicists to admit that the microscopic world behaves in a radically new way compared to the classical way. One of the main property of microphysics irreducible to being classical is described by the Heisenberg relations, which imply a profound *scale dependence* of physical laws in the quantum domain. When $\Delta p \gg p_0$ (i.e. $\Delta x \ll \lambda_{dB}$, the de Broglie length), p becomes of the order of Δp and the position-momentum Heisenberg relation $\Delta p \,.\, \Delta x \approx \hbar$ becomes $p \approx \hbar/\Delta x$. Similarly when $\Delta E \gg E_0$ (i.e. $\Delta t \ll \tau_{dB}$, the de Broglie time), E becomes of the order of ΔE and the time-energy Heisenberg relation $\Delta E \,.\, \Delta t \approx \hbar$ becomes $E \approx \hbar/\Delta t$. In the presently best accepted interpretation of quantum mechanics, this behaviour is understood as a consequence of the uncontrollable interaction between the measurement apparatus and the system to be measured. However it is remarkable that the Heisenberg relations are universal and independent from any particular measurement process. They may be derived from the general law that the momentum probability amplitude and position probability amplitude are reciprocal Fourier transforms.

So we have proposed a different interpretation [3]: that the quantum behaviour is a consequence of a fundamental and universal dependence of space-time itself on resolution, which is revealed in any measurement: namely, that the quantum space-time has fractal properties. This leads to the view that scale should be explicitely introduced into the fundamental laws of physics, and that this goal may be achieved by identifying it as a state of the reference system (i.e. of measurement apparatus in the language of quantum mechanics). In this frame of thought, the Heisenberg relations tell us that the results of measurements of momentum and energy are relative to the state of scale of the reference system.

The scale dependence of microphysics has already taken on reinforced importance in the study of the asymptotic behaviour of quantum field theories, presently best described by the renormalization group methods [4]-[8], which led to some important results, like asymptotic freedom of QCD, the variation of coupling constants with scale and their convergence at the "Grand Unified Theory" scale $\approx 10^{15}$ GeV. These methods have for the first time explicitly introduced scale into physical equations. Indeed it is remarkable that below the Compton length of the electron, there is a strong "degeneration" of space and time variables: the velocity becomes disqualified as a pertinent mechanical variable (all velocities are close to the velocity of light) and the classical laws of mechanics are actually replaced by dilatations in terms of Lorentz γ factors. Then, at high energy, the laws of scale actually take the place of the laws of motion.

There are, however, several additional elements in our proposal with respect to the standard renormalization group approach. The first is that it is argued that scale, like motion, may be considered as a state of coordinate systems which can never be defined in an absolute way, and thus comes under a relativity theory. In this respect we may identify the theory of the renormalization group as a Galilean version of scale relativity. The second is that the renormalization group is, strictly, only a semi-group (one integrates the small scales to get the larger ones) [9], while one may hope it to be completed in the future by an inverse transformation, at least for some elementary physical systems. This would mean being able to deduce the small scale structure from the larger scale: this is exactly what a fractal generator makes [3, 10, 11]. The third new element is that the relativistic analysis of scale, once applied to space and time variables themselves, finally leads to a completely new structure of physical laws, as will be demonstrated in the following.

3 Galilean Scale Relativity

One of the main characteristics of scale which point toward the need for a scale relativity theory is the nonexistence of an *a priori* absolute scale. Just as one may write for velocities in Galilean motion relativity :

$$v = v_2 - v_1 = (v_2 - v_0) - (v_1 - v_0), \tag{1}$$

one may, in present physics, write for a scale ratio

$$\varrho = \frac{\Delta x_2}{\Delta x_1} = \frac{\Delta x_2/\Delta x_0}{\Delta x_1/\Delta x_0}. \tag{2}$$

It is indeed clear that one can never define the length of an object without comparing it to another object: only scale ratios, i.e. dilatations, have a physical meaning. The expression (2) may be written under the same additive group form as Eq. (1), in a logarithmic representation:

$$\ln \varrho = \ln\left(\frac{\Delta x_2}{\Delta x_1}\right) = \ln\left(\frac{\Delta x_2}{\Delta x_0}\right) - \ln\left(\frac{\Delta x_1}{\Delta x_0}\right). \tag{3}$$

So the "scale state" $V = \ln(\Delta x_2/\Delta x_1)$ appears like a "scale-velocity" or "zoom", in agreement with our principle that it should describe the state of resolution of the coordinate system in the same way as the velocity describes its state of motion. Just as one can speak only of the velocity of a system *relative* to another one, the scale of a system can be defined only by its ratio to the scale of another system. Eq. (3) may now be written in exactly the same form as Eq. (1):

$$V = V_2 - V_1 = (V_2 - V_0) - (V_1 - V_0). \tag{4}$$

Concerning the problem of units, notice the difference of status between motion and scale laws. While velocity is expressed in terms of a physical unit (e.g. m.s^{-1}), the scale state is expressed in terms of a mathematical unit, i.e. the adopted logarithm base. Indeed the same behaviour is obtained (whatever base b is) using the more general definition:

$$V = \frac{\ln(\Delta x_2/\Delta x_1)}{\ln b} = \log_b\left(\frac{\Delta x_2}{\Delta x_1}\right). \tag{5}$$

It will be seen hereafter that this leads to a new kind of dimensional analysis.

Consider now a field φ which transforms under a dilatation $q = \Delta x/\Delta x'$ following a power law:

$$\varphi' = \varphi\, q^{\delta}. \tag{6}$$

In a renormalization group description, the power δ is identified with the anomalous dimension of the field φ [9]. In a fractal interpretation of the same phenomenon, we get $\delta = D - D_T$, where D is the fractal dimension and D_T the topological dimension [3, 11].

We are particularly interested here in the case where the "field" φ is space-time itself. Let us briefly remind the present state of things concerning this approach [3]. Assume that we consider a system having first a de Broglie length $\lambda_0 = \hbar/p_0$, and that we perform successive measurements at given time intervals with a resolution Δx in order to determine its velocity and then its average momentum and the length of its trajectory. If $\Delta x \gg \lambda_0$, the momentum perturbation implied by Heisenberg's relation is $\Delta p \ll p_0$, so that the result will remain $\approx p_0$, independent of scale. One gets the usual classical trajectory whose length does not depend on resolution. On the contrary when $\Delta x \ll \lambda_0$, $\Delta x \,.\, \Delta p \approx \hbar$ implies that the measured momentum will keep practically no trace of the initial one, i.e. $\Delta p \gg p_0$ so that $p = p_0 + \Delta p \approx \Delta p$; finally the momentum will be a direct function of resolution, $p \approx \hbar/\Delta x$, and the new de Broglie length of the system after the measurement becomes of the order of Δx. The length of the particle path now diverges as Δx^{-1} [12, 13, 3]. This means that the length $\mathcal{L}$, integrated along the (fractal) path of a particle, diverges for resolutions Δx smaller than the de Broglie length (or time) λ as

$$\mathcal{L} = \mathcal{L}_0 \, \frac{\lambda}{\Delta x}, \tag{7}$$

corresponding to the particular case $\delta = D - D_T = 1$ for $D = 2$ and $D_T = 1$.

The same behaviour is found for the temporal coordinate around the de Broglie time of the system [3]. This result is obtained when one takes into account not only the transition to relativistic velocities, but also particle-antiparticle pair creations. Owing to the fact that the whole set of virtual pairs contribute in the self-energy of a particle (say, of the electron) and then in the nature of the particle itself, and extending the Feynman-Wheeler-Stückelberg interpretation of antiparticles as particles which run backward in time, we have suggested that, if one wants to compute the full proper time $\mathcal{T}$ elapsed on the particle, one must *add* the proper times elapsed on all the members of the virtual pairs to that of the "bare" particle. Thus one finds a temporal coordinate diverging with energy, i.e. in an equivalent way with the inverse of time resolution when $\Delta t < \tau = \hbar/E$ as

$$\mathcal{T} = \mathcal{T}_0 \, \frac{\tau}{\Delta t}. \tag{8}$$

A similar result is obtained from localized solutions of the Dirac equation: from the requirement that the solution should be localized into an interval $\Delta x \approx c\Delta t \approx \hbar c/E$, one may compute the rate of negative and positive energy solutions. One finds $P_-/P_+ \approx (E - mc^2)/(E + mc^2)$. Then considering that this set of positive and negative solutions is nothing but the manifestation of a fractal trajectory which runs backward in time for $\Delta t < \tau$ leads to (8). This makes Lorentz covariant the reinterpretation of the de Broglie scale as a universal space-time transition from $\delta = 0$ to an anomalous dimension $\delta = 1$, since it applies to all four space-time coordinates. In terms of the renormalization group, the de Broglie scale may be identified as the correlation length of space-time.

Keeping all these results in mind, let us write Eq. (6) in a linear form by passing once again to a logarithmic representation:

$$\ln\left(\frac{\varphi'}{\varphi_0}\right) = \ln\left(\frac{\varphi}{\varphi_0}\right) + \delta \times \ln\left(\frac{\Delta x}{\Delta x'}\right), \tag{9}$$

this being assumed to hold when $\Delta x \ll \lambda$. Our comparison with motion relativity may then be pursued. The Galilean transform between two coordinate systems reads

$$x' = x + vt, \tag{10}$$

$$t' = t. \tag{11}$$

We may now get a consistent description in which, as conjectured, resolution acts as a "scale-velocity", while the anomalous dimension (i.e. here the fractal dimension minus 1) plays the role of a "scale-time". Indeed, setting

$$X = \ln\left(\frac{\varphi}{\varphi_0}\right), \tag{12}$$

from the linear relation

$$X = X_0 + \delta \times \ln\left(\frac{\lambda}{\Delta x}\right), \tag{13}$$

we may define the state of scale V as

$$V = \ln\left(\frac{\lambda}{\Delta x}\right) = \frac{d(\ln\varphi)}{d\delta} = \frac{dX}{d\delta}, \tag{14}$$

in the same way as the velocity of an object is defined as $u = dx/dt$ in motion relativity. Note the different approach with respect to the usual definition for the fractal dimension $\delta = \partial \ln\varphi / \partial \ln(\lambda/\Delta x)$.[2] The scale law (13) is the equivalent for scale of free motion at constant velocity, which is at the basis of the definition of inertial motion. Likewise we suggest that a coordinate supersystem [3] (i.e. defined by its states of motion *and of scale*) into which Eq. (13) holds, may be called "scale-inertial", and that we may set a principle of (special) scale relativity, which states that *the laws of nature are identical into all scale-inertial supersystems of coordinates.*

The anomalous dimension δ is assumed to be invariant (e.g. for space-time coordinates we find the universal value $\delta = 1$, itself coming from the universality of the Heisenberg relations: in that case, δ is a constant), as time is invariant in Galilean relativity. This is translated by the equations of the Galilean "scale-inertial" transformation

$$X' = X + V\,\delta, \tag{15}$$

$$\delta' = \delta. \tag{16}$$

In such a Galilean frame, the law of composition of scale states is the direct sum

$$W = U + V, \tag{17}$$

which corresponds to the direct product $\Delta x''/\Delta x = (\Delta x''/\Delta x')\,.\,(\Delta x'/\Delta x)$ for resolutions. Finally, with the three equations (15)-(17), we have put the scale relativity problem in exactly the same mathematical form (Galileo group) as that of motion relativity in classical mechanics.

But one should also keep in mind that the hereabove "inertial scaling" holds only under some upper cut-off λ, contrarily to the motion case where it is universal.

[2] More precisely the fractal dimension is $D_F = D_T + \delta$, where D_T is the topological dimension.

This may be expressed by writing, instead of (13), a formula including a transition from scale dependence to scale independence, such as

$$X = X_0 + \delta \times \ln \left[1 + \left(\frac{\lambda}{\Delta x} \right)^2 \right]^{1/2}. \tag{18}$$

More generally one may introduce a parameter k which characterizes the speed of transition and replace V in (15) by Z, defined as

$$Z = \frac{1}{k} \ln \left[1 + \left(\frac{\lambda}{\Delta x} \right)^k \right] = \ln(1 + e^{kV})^{1/k}. \tag{19}$$

Then for $\Delta x \gg \lambda$, $Z = 0$ and for $\Delta x \ll \lambda$, $Z = V$. For k small, the transition between these two regions is slow, while it is sudden (singular point) for $k \to \infty$. Strictly this description of the transition is only a model, since its details depend on the physical system considered: for example, in many situations the transition may imply exponentially decreasing "Yukawa-like" terms.

4 A New Derivation of the Lorentz Transformation

As remarked by Levy-Leblond [2], very little freedom is allowed for the choice of a relativity group, so that the Poincaré group is an almost unique solution to the problem [14]. In his original paper, Einstein derived the Lorentz transformation from the (sometimes implicit) successive assumptions of (i) linearity; (ii) the invariance of c, the light velocity in vacuum; (iii) the existence of a composition law; (iv) the existence of a neutral element; and (v) reflection invariance.

But one may demonstrate that the postulate of the invariance of some absolute velocity is not necessary for the construction of the special theory of relativity. Indeed it was shown by Levy-Leblond [2] that the Lorentz transformation may be obtained through six *successive* constraints: {1} homogeneity of space-time (translated by linearity of the transformation of coordinates), {2} isotropy of space-time (translated by reflection invariance), {3} group structure (i.e. {3.1} existence of a neutral element, {3.2} of an inverse transformation and {3.3} of a composition law yielding a new transformation which is a member of the group, viz. which is internal) and {4} the causality condition. The last group axiom, associativity, is in fact straightforward in this case and leads to no constraint.

Actually this set of hypotheses is still overdetermined to derive the Lorentz transformation. We shall indeed demonstrate hereafter that the Lorentz transformation may be obtained *from the only assumptions of* {a} *linearity*; {b} *internal composition law and* {c} *reflection invariance.* All the other assumptions, in particular the postulate of the existence of an inverse transformation which is a member of the group, may be *derived as consequences* of these purely mathematical constraints. The importance of this result, especially concerning scale relativity, is that we do not have to postulate a full group law in order to get the Lorentz behavior: the hypothesis of a semi-group structure is sufficient.

Let us start from a linear transformation of coordinates:

$$x' = a(v)x - b(v)t, \tag{20}$$

$$t' = \alpha(v)t - \beta(v)x. \tag{21}$$

In these equations and in the whole section, the coordinates x and t do not denote *a priori* lengths and times, but may refer to any kind of variables having the mathematical properties considered. Equation (20) may be written as $x' = a(v)[x-(b/a)t]$. But we may *define* the "velocity" v as $v = b/a$, so that, without any loss of generality, linearity alone leads to the general form

$$x' = \gamma(v)\,[x - vt], \tag{22}$$

$$t' = \gamma(v)\,[A(v)t - B(v)x], \tag{23}$$

where $\gamma(v) = a(v)$, and A and B are new functions of v. Let us now perform two successive transformations of the form (22,23):

$$x' = \gamma(u)\,[x - ut], \tag{24}$$

$$t' = \gamma(u)\,[A(u)t - B(u)x], \tag{25}$$

$$x'' = \gamma(v)\,[x' - vt'], \tag{26}$$

$$t'' = \gamma(v)\,[A(v)t' - B(v)x']. \tag{27}$$

This results in the transformation

$$x'' = \gamma(u)\gamma(v)\,[1 + B(u)v]\,\left[x - \frac{u + A(u)v}{1 + B(u)v}\,t\right], \tag{28}$$

$$t'' = \gamma(u)\gamma(v)\,[A(u)A(v) + B(v)u]\,\left[t - \frac{A(v)B(u) + B(v)}{A(u)A(v) + B(v)u}\,x\right]. \tag{29}$$

Then the principle of relativity tells us that the composed transformation (28,29) keeps the same form as the initial one (22, 23), in terms of a composed velocity w given by the factor of t in (28). We get four conditions:

$$w = \frac{u + A(u)v}{1 + B(u)v}, \tag{30}$$

$$\gamma(w) = \gamma(u)\gamma(v)[1 + B(u)v], \tag{31}$$

$$\gamma(w)A(w) = \gamma(u)\gamma(v)[A(u)A(v) + B(v)u], \tag{32}$$

$$\frac{B(w)}{A(w)} = \frac{A(v)B(u) + B(v)}{A(u)A(v) + B(v)u}. \tag{33}$$

Our third postulate is reflection invariance. It reflects the fact that the choice of the orientation of the x (and x') axis is completely arbitrary, and should be indistinguishable from the alternative choice $(-x,\ -x')$. With this new choice, the transformation (24,25) becomes $\{-x' = \gamma(u')(-x-u't),\ t' = \gamma(u')[A(u')t+B(u')x]\}$ in terms of the value u' taken by the relative velocity in the new orientation. The requirement that the two orientations be indistinguishable yields $u' = -u$. This leads to parity relations for the three unknown functions γ, A and B [2]:

$$\gamma(-v) = \gamma(v), \quad A(-v) = A(v), \quad B(-v) = -B(v). \tag{34}$$

Combining Eqs. (30), (31) and (32) yields the relation

$$A\left[\frac{u + A(u)v}{1 + B(u)v}\right] = \frac{A(u)A(v) + B(v)u}{1 + B(u)v}. \tag{35}$$

Making $v = 0$ in this equation gives

$$A(u)[1 - A(0)] = uB(0). \tag{36}$$

Making $u = 0$ yields only two solutions, $A(0) = 0$ or 1. The first case gives $A(u) = uB(0)$. $B(0) \neq 0$ is excluded by reflection invariance (34); then $A(u) = 0$. But (33) becomes $A(w) = B(w)u$ so that $B(w) = 0$: this is a case of complete degeneration to only one efficient variable since $t' = 0 \; \forall u$, which can thus be excluded. We are left with $A(0) = 1$, which implies $B(0) = 0$, and the existence of a neutral element is demonstrated. Let us make now[3] $v = -u$ in (35) after accounting for (34), and introduce a new even function $F(u) = A(u) - 1$, which verifies $F(0) = 0$. We obtain

$$2F(u)\frac{1 + F(u)/2}{1 - uB(u)} = F\left[\frac{uF(u)}{1 - uB(u)}\right]. \tag{37}$$

We shall now use the fact that B and F are continuous functions and that $B(0) = 0$. This implies that $\exists \eta_0 > 0$ such that in the interval $-\eta_0 < u < \eta_0$, $1 - uB$ and $1 + F/2$ become bounded to $k_1 < 1 - uB(u) < k_2$ and $k_3 < 1 + F(u)/2 < k_4$ with k_1, k_2, k_3 and $k_4 > 0$. The bounds on $1 + F/2$ and $1 - uB$ allow us to bring the problem back to the equivalent equation, $2F(u) = F[uF(u)]$. The continuity of F at $u = 0$ reads, owing to the fact that $F(0) = 0$:[4] $\forall \varepsilon$, $\exists \eta$ such that $|u| < \eta \Rightarrow |(F(u)| < \varepsilon$.

Start with some $u_0 < \eta$ yielding $F(u_0) = F_0 = 2^{-n} < \varepsilon$. Then $F(u_0 F_0) = 2F_0$. Set $u_1 = u_0 F_0$ and iterate. After p iterations, one gets $F(u_p) = F[2^{p[(p-1)/2-n]}u_0] = 2^{p-n}$. In particular one gets after n iterations: $F[2^{-n(n+1)/2}u_0] = 1$ if n is an integer. (In the general case where n is not integer, one gets after Int$[n]$ iterations a value of F larger than $1/2$). This is in contradiction with the continuity of F, since $u_n < u_0 < \eta$ while $F(u_n) > \varepsilon$. Then the only solution is $F = 0$ in a finite non null interval around the origin, and from step to step whatever the value of u, so that

$$A(u) = 1. \tag{38}$$

As a consequence (35) becomes $B(u)v = B(v)u$, a relation which finally constraints the B function to be

$$B(v) = \kappa v, \tag{39}$$

where κ is a constant. At this stage of our demonstration, the law of transformation of velocities is already fixed to the Einstein-Lorentz form:

$$w = \frac{u + v}{1 + \kappa uv}, \tag{40}$$

[3] A misprint in the published version ($v = u$) has been corrected here.

[4] A misprint in the published version ($F(u) = 0$) has been corrected here.

and it is easy to verify that a full group law is verified, i.e. the existence of an identity transformation and of an inverse transformation are ensured, without having been presupposed. Consider now the γ factor. It verifies the condition

$$\gamma\left(\frac{u+v}{1+\kappa uv}\right) = \gamma(u)\gamma(v)(1+\kappa uv). \tag{41}$$

Let us consider the case $u = -v$. Equation (41) reads $\gamma(0) = \gamma(v)\gamma(-v)(1-\kappa v^2)$. For $v = 0$ it becomes $\gamma(0) = [\gamma(0)]^2$, implying $\gamma(0) = 1$, and we get

$$\gamma(v)\gamma(-v) = \frac{1}{1-\kappa v^2}. \tag{42}$$

The final step to the Lorentz transformation is straighforward from reflection invariance, which implies $\gamma(v) = \gamma(-v)$ (Eq. 34) and fixes the γ factor in its Lorentz-Einstein form:

$$\gamma(v) = \frac{1}{\sqrt{1-\kappa v^2}}. \tag{43}$$

The case $\kappa < 0$ yields a non-ordered group (applying two successive positive velocities may yield a negative one), and we are left with the only two physical solutions, the Galileo ($\kappa = 0$) and Lorentz ($\kappa = c^{-2} > 0$) groups. Three of their properties, $-$ existence of a neutral element, of an inverse element and commutativity (for one space dimension) $-$ have not been postulated, but deduced from our initial axioms.

Let us end this section by a brief but important comment. We have shown that, once we have set the hypothesis of linearity, the Lorentz transformation may be obtained through the only postulates of internal composition and reflection invariance. Linearity is not a constraint by itself: indeed it corresponds to the simplest possible choice (i.e. when searching for a transformation which would satisfy a given law, one may first look for a linear one, and then look for non linearity only in case of failure, or later as a generalization). With regard to the other two postulates, they may be seen as a *direct translation of the Galilean principle of relativity.* Indeed the hypothesis that the composed coordinate transformation ($K \to K''$) and the transformation in the reversed frame ($-K \to -K'$) must keep the same form as the initial one ($K \to K'$) is nothing but an application of the Galilean principle of relativity ("the laws of nature must keep the same form in different inertial reference systems") *to the laws of coordinates transformation themselves,* which are clearly part of these laws to which the principle should apply. So the general solution to the problem of inertial motion, without adding any postulate to the way it might have been stated in the Galileo and Descartes epoch, is actually Einstein's special relativity, whose Galilean relativity is a special case ($c = \infty$).

5 Lorentzian Scale Transformation

In the preceding section, we have recalled that the general solution to the (special) relativity problem is the Lorentz group. In the case of motion relativity the Lorentz transformation for systems in inertial motion is now one of the most solid base of physics. What about scale ?

We have argued in Secs. $1-3$ that scale (resolution) also came under a relativity theory. Set in a general way, the problem of scale transformation now consists in looking for a two-variable transformation $\ln \varphi' = f_1(\ln \varphi, \delta)$, $\delta' = f_2(\ln \varphi, \delta)$, depending on one parameter, the scale state $V = \ln(\lambda/\Delta x)$.

Let us analyse how the mathematical axioms on which was founded the above derivation of the Lorentz transformation are physically translated in the case of scale. In the theory of motion relativity *linearity* may be derived from the homogeneity of space and time (which is itself an application of the principle of relativity to positions and instants). In scale relativity, the things that play the role of lengths and times are now respectively the logarithm of some field, $\ln \varphi$, and the anomalous dimension or fractal codimension δ (see Sec. 3). The uniformity of these variables is not *a priori* straighforward, even though it is already assured in the scale laws of present physics. But *linearity,* as already specified, may be inferred from a hypothesis of simplicity. More precisely linearity is the simplest choice to make, and so comes as a provisional specialization of the present theory. It is clear that a generalization to nonlinear transformations must be considered in the future (we have suggested that such an achievement would imply to use the tool of fractal space-times) [3], but this departs from the frame of the present work.

The second axiom, the existence of an *internal composition law,* is a direct application of the principle of relativity: there is no difference here between motion and scale. *Reflection invariance* means that one may equally work with either $\ln(\varphi/\varphi_0)$ or $\ln(\varphi_0/\varphi)$, to which would respectively correspond scale states $\ln(\Delta x/\Delta x_0)$ and $\ln(\Delta x_0/\Delta x)$; this is indeed straightforward. Finally the case $\kappa < 0$ is clearly also excluded for scale, since when applying two successive dilatations we indeed expect the final product not to be a contraction.

So from our result that the general solution to the linear relativity problem is Lorentz, *we conclude that the laws of scale transformation must also take a Lorentzian form*, instead of the Galilean form, which was up to now assumed to be self-evident.

Let us now explicitly compare Lorentzian scale transformation to motion transformations. While the composition of velocities follows an additive group law, the composition of scales follows a multiplicative group law. It is easy to come back to a multiplicative group by taking the logarithm of scale ratios, as shown in Sec. 3 (Galilean case, which is also the case of the standard renormalization group).

Start with the Einstein-Lorentz law of composition of "velocities":

$$w = \frac{u+v}{1+(uv/c^2)}, \tag{44}$$

where u, v, w are *dimensioned* quantities and c is an universal *constant.* This may be written in a dimensionless way by setting $U = u/c$, $V = v/c$, $W = w/c$:

$$W = \frac{U+V}{1+UV}. \tag{45}$$

Let us now write U, V and W, which are pure numbers, as logarithms of other dimensionless quantities. This may be done *into any base* for the logarithms, say K, by setting $U = \log_K \nu$, $V = \log_K \varrho$, $W = \log_K \mu$, i.e.

$$u = \frac{c}{\ln K} \ln \nu, \tag{46}$$

and similar formulas for v and w. So (45) now becomes

$$log_K \mu = \frac{\log_K \varrho + \log_K \nu}{1 + \log_K \varrho \times log_K \nu}. \tag{47}$$

We may now divide both members of this equation by $\ln K$ and we get

$$\ln \mu = \frac{\ln \varrho + \ln \nu}{1 + (\ln \varrho \ln \nu / \ln^2 K)}. \tag{48}$$

This formula is formally identical to the initial one, Eq. (44) [and to the general structure (40)], with the difference that $\ln K$ is itself a pure number, while c was a dimensioned quantity. Now identifying μ, ν, ϱ and K to scale ratios, we see that (48) becomes the scale-relativistic generalization of the usual law of dilatation: this means that the successive application of two dilatations ϱ and ν now yields the dilatation μ instead of the usual product $\varrho\,\nu$.

We get a new law for the transformation of the field φ, which generalizes (9):

$$\log_K \left(\frac{\varphi'}{\varphi_0} \right) = \frac{\log_K(\varphi/\varphi_0) + \delta \times \log_K \varrho}{(1 - \log_K^2 \varrho)^{1/2}}. \tag{49}$$

The anomalous dimension, which was previously invariant, becomes now a function of the resolution and of the field:

$$\delta' = \frac{\delta + \log_K \varrho \log_K(\varphi/\varphi_0)}{(1 - \log_K^2 \varrho)^{1/2}}. \tag{50}$$

However these laws still cannot be considered as the definitive laws of scale relativity, since they do not incorporate the classical / quantum transition. This is done in the following.

6 Scale Relativity Broken

As already specified, scale relativity[5], contrarily to motion relativity, is not a universal principle of nature. The fact that scales (or resolutions) can be defined only by their ratios is indeed universal, but this is of no consequence in the classical domain ($\Delta x \gg \lambda_{dB}$). There, resolution reduces to precision, and improving the precision of measurements improves the precision of results, but does not change the physics. The situation changes in the quantum and quantum-relativistic domains, the transition to which corresponds to the de Broglie length and time (see Ref. 3 and Sec. 3).

Hence scale relativity must be a broken principle above the de Broglie scales $\lambda_{dB} = (\hbar/mv)(1 - v^2/c^2)^{1/2}$ and $\tau_{dB} = (\hbar/mc^2)(1 - v^2/c^2)^{1/2}$. In order to simplify the argument, let us look at the high energy degenerated case, where only one space-time variable may be considered, say $\Delta x \approx c\Delta t$, so that $c\tau_{dB}(= \hbar c/E)$ becomes equal to the Compton length $\lambda_0 = \hbar/mc$ in the rest frame of a system of mass m. Let us explicitely introduce this particular scale in the composition law (48).

[5]In this section, "scale relativity" stands for "special scale relativity". It is the special scale relativity symmetry (involving the log-Lorentz scale transformations of previous section) whose breaking toward Galilean-like scale relativity is analysed here.

We start from the scale λ_0 and first apply a contraction which leads to a new scale λ, then apply another contraction by a factor ϱ leading to a final scale λ'. The "Galilean" character of λ_0 allows us to take it as reference for all scale ratios (with the exception of ϱ which relates λ to λ'). We thus set in (48) $\mu = \lambda'/\lambda_0$, $\nu = \lambda/\lambda_0$ and $K = \Lambda/\lambda_0$ (this explicit writing of dilatations as scale ratios introduces a Galilean structure), and the composition law now takes the form

$$\ln\left(\frac{\lambda'}{\lambda_0}\right) = \frac{\ln\left(\frac{\lambda}{\lambda_0}\right) + \ln\varrho}{1 + \ln\varrho \ln\left(\frac{\lambda}{\lambda_0}\right) / \ln^2\left(\frac{\Lambda}{\lambda_0}\right)}. \tag{51}$$

We verify that the dilatation which relates λ_0 to any scale λ remains equal to their ratio, as in the classical case, while it is no more true of two scales both different from λ_0. In fact (51) may be inverted and understood as the function $\varrho(\lambda, \lambda'; \lambda_0)$ which yields the dilatation allowing to go from one scale λ to another scale λ' [see Eq. (53)]. This dilatation factor now depends on the initial de Broglie scale, λ_0.

But consider now the behaviour of the particular length Λ. Assume that we start with this length, i.e. $\lambda = \Lambda$ and that we apply to it the dilatation or contraction ϱ. From (51), we find that this results into a length given by $\ln(\lambda'/\lambda_0) = \ln(\Lambda/\lambda_0)$, i.e. $\lambda' = \Lambda$ *whatever the value of the de Broglie scale* λ_0. Starting from any scale larger than Λ, and applying any finite contraction, we get a scale larger than Λ. The scale Λ can be the result only of infinite contraction or of an infinite product of contractions, i.e. it plays the same role as the zero point of the previous theory. In terms of the renormalization group theory, it is a fixed point for space-time itself.

Hence the principle of relativity, once applied to scales, combined with the existence of the de Einstein-Broglie transition, leads to the existence of an absolute length in nature, which is invariant under dilatations and contractions. Motion relativity immediately ensures that this will be also true for time, and that an invariant time interval $T = \Lambda/c$ exists in nature. A particular case of scale transformations is the Lorentz length contraction and time dilatation: as a consequence it is straighforward that Λ and Λ/c will be also invariant under a Lorentz transformation, i.e. independent of the relative velocity of the reference system in which they are observed.

One might be disturbed by the fact that K is not an absolute constant, contrarily to the structure expected from a pure special relativity theory. However, once λ_0 is fixed (and it is fixed by the state of motion of the system considered, since the de Broglie length and time are Lorentz-covariant by construction), λ_0/Λ is a constant: *scale relativity relies on motion relativity.* Conversely it is rather satisfactory that, in the same way as motion relativity led to the existence of an absolute and unexceedable velocity, scale relativity leads to the existence of an absolute, invariant limit for all lengths and times. The final point to be elucidated is the nature of Λ. We suggest in the following that it is nothing but the Planck scale.

7 On the Absolute[6] Character of the Planck Scale

The Planck length already plays a very special role in physics: it is the characteristic scale for which all forces of nature are expected to become equivalent, while the concept of a space-time continuum seems to lose its physical meaning for smaller resolutions. It has been proposed [15, 16] that the topology of space-time may become extremely complicated (foamlike) at that scale, the continuum itself being broken.

Even though a bundle of physical arguments makes clear that the Planck scale must play a central role in microphysics, all the previous approaches to the problem were worked out in a frame where the scaling laws themselves were not questioned, i.e. in which it was considered evident that applying a dilatation q to a scale Δx yields a new scale $\Delta x' = q \times \Delta x$. This is reminiscent of classical Galilean physics in which it seemed also self-evident that throwing an object with a velocity v with respect to a body moving with velocity u relative to the ground finally yields a velocity $w = u + v$.

Let us now consider the question from the point of view, adopted here, of scale relativity. The Planck length scale $(\hbar G/c^3)^{1/2}$ is particular in that *its expression depends on no particular physical object*, but only on the three fundamental constants of physics, G, $\hbar$ and c. While we have insisted at the beginning of this paper on the relativity of all scales, the Planck scale is the only one which is in fact absolute in its definition, i.e. independent from particular physical bodies or systems.

If we admit that the three constants G, $\hbar$ and c are indeed universal and unvarying, even at the time and length scales of the Universe ($\hbar$ is known to vary by less than 4×10^{-13} per year and G by less than 10^{-11} per year, i.e. respectively less than $\approx 0.4\,\%$ and $10\,\%$ over the age of the Universe) [17], one may be upset by the fact that, in present physics, a "Planck rod" $(\hbar G/c^3)^{1/2}$, when submitted to a dilatation q becomes $q(\hbar G/c^3)^{1/2}$ in spite of its universality, and when observed from a reference system in which it moves with velocity v, is submitted to Lorentz contraction and becomes $\sqrt{(\hbar G/c^3)(1 - v^2/c^2)}$. Even if it is admitted that physics may drastically change when the Planck scale is crossed, it is still admitted that scales smaller than the Planck scale do exist in nature.

We take here a radically different position: based on the absolute character of the definition of the Planck scale, we suggest identifying it as the invariant scale Λ, which was derived above from the application to scale of the principle of relativity. *The Planck length becomes a universal scale which remains invariant upon dilatations.* It now plays for scale the same part as the velocity of light already plays for motion. The concept of a resolution smaller than the Planck length also loses any physical meaning, since the Planck length is now a limit which can not be exceeded (toward lower resolutions).

Strictly Λ could be identified with the Planck length, times any pure and constant number, but this would destroy the formal simplicity of the construction (only the exchanges $G \to 2G$ and $\hbar \to h$ remain uncertain, but $\hbar$ is preferable to h since

[6]The adjective "absolute", used here and throughout the text to characterize the properties of the Planck length-scale in the new scale-relativistic interpretation, is incorrect. It should be replaced by "universal" and/or "invariant". Indeed the Planck scale, as any other physical object or concept, is not defined in an absolute way, but relatively to other concepts (here, fundamental constants). This is also clear from its value, which depends on units.

the actual transition lengths around which physics changes are indeed the reduced Compton length $\hbar/mc$ rather than h/mc); so we set, while waiting for possible future experimental verification:

$$\Lambda = \sqrt{\frac{\hbar G}{c^3}}. \tag{52}$$

It should be noted that, taken together, the three fundamental constants G, $\hbar$ and c do nothing but fix the arbitrary part of our units of length, time and mass. Motion relativity has supplied us with a conceptual frame in which lengths and times are logically related: one now deduces length units from time units and c fixed. (It would be most consistent in fact to fix $c = 1$ and definitively measure lengths e.g. in nanoseconds). Scale relativity, if confirmed by experiments, will achieve the same for times themselves (at least in principle, since the bad present precision on G prevents one from doing this explicitly for the moment; a precise determination of the constant of gravitation now becomes an urgent task). Setting the Planck time $\Lambda/c = 1$, *all length and time intervals in nature become dimensionless real numbers larger than one*. In such a system, one would get $G = 1/\hbar$. The final step to setting also the Planck mass to 1 demands the determination of the ratios of the (low energy) masses of all elementary particles over the Planck mass, and, if one wants to be completely consistent, the understanding of the values of the three remaining coupling constants at a given scale. It will be shown hereafter that scale relativity allows to take some steps towards the achievement of this grand program.

Before going on, let us write the complete new transformation, in the case where all scales considered are smaller than the de Broglie scale λ_0 (assumed fixed) of the system. Let ϱ be the dilatation which allows to go from Δx to $\Delta x'$ ($\Delta x \leq \lambda_0$[7] and $\Delta x' \leq \lambda_0$); let $\varphi = \varphi(\Delta x)$ be some scale-dependent field and $\delta = \delta(\Delta x)$ its anomalous dimension; we set $\varphi' = \varphi(\Delta x')$, $\delta' = \delta(\Delta x')$ and define an arbitrary reference value for the field, φ_0; the new transformations for the dilatation, the field and the anomalous dimension read, in terms of the Planck scale Λ,

$$\ln \varrho = \frac{\ln\left(\frac{\Delta x'}{\Delta x}\right)}{1 - \ln\left(\frac{\lambda_0}{\Delta x'}\right) \ln\left(\frac{\lambda_0}{\Delta x}\right) / \ln^2\left(\frac{\lambda_0}{\Lambda}\right)}, \tag{53}$$

$$\ln\left(\frac{\varphi'}{\varphi_0}\right) = \frac{\ln\left(\frac{\varphi}{\varphi_0}\right) + \delta \;\; \ln \varrho}{\sqrt{1 - \ln^2 \varrho / \ln^2\left(\frac{\lambda_0}{\Lambda}\right)}}, \qquad \delta' = \frac{\delta + \ln \varrho \ln\left(\frac{\varphi}{\varphi_0}\right) / \ln^2\left(\frac{\lambda_0}{\Lambda}\right)}{\sqrt{1 - \ln^2 \varrho / \ln^2\left(\frac{\lambda_0}{\Lambda}\right)}}. \tag{54}$$

Letting $\Lambda \to 0$ yields back the standard ("Galilean") scaling transformation $\varphi' = \varphi\, \varrho^\delta$ and $\delta' = \delta$. The standard transformation is also obtained as an approximation in the limit $\ln \varrho \ll \ln(\lambda_0/\Lambda)$.

Equations (53, 54) hold only in the quantum case ($\Delta x \leq \lambda_0$ and $\Delta x' \leq \lambda_0$). Going to a scale larger than the de Broglie scale leads to scale independence: $\delta = 0$, φ independent of scale and "Galilean" dilatation law $\varrho = \Delta x'/\Delta x$. As already noted in Sec. 3, a precise description of this transition from scale dependence to scale independence depends on the particular physical system considered. A useful model (see Sec. 3, Eq. 19) consists in replacing, in Eqs. (53, 54), $\ln(\lambda_0/\Delta x)$ by

[7] A misprint in the published version ($\Delta x \leq \Delta x_0$) has been corrected here and hereafter.

$(1/k)\ln[1+(\lambda_0/\Delta x)^k]$ (and similar changes for all scales refered to λ_0), where k is a parameter which allows one to describe the steepness of the transition. Indeed for *fixed* λ_0 and $\Delta x \ll \lambda_0$, one gets $(1/k)\ln[1+(\lambda_0/\Delta x)^k] \approx \ln(\lambda_0/\Delta x)$, while for $\Delta x \gg \lambda_0$, $(1/k)\ln[1+(\lambda_0/\Delta x)^k] \approx 0$. To be complete, one should also replace δ by some function $\Delta(\Delta x)$, with $\Delta = \delta$ for $\Delta x \ll \lambda_0$ and $\Delta = 0$ for $\Delta x \gg \lambda_0$.

We also recall that Eqs. (53, 54) apply in the first place to the case where φ represents either the length measured along a quantum particle trajectory (non relativistic case), or its integrated proper time (relativistic case + reinterpretation of particle-antiparticle pairs as part of a fractal trajectory running backward in time), and more generally any of the four space-time coordinates (once integrated along the fractal path). Then Δx *may represent any of the four coordinates' resolution,* Δx i, $i = 0$ to 3, and more generally of some combination of them, in particular the resolution of the classical invariant, Δs.

To conclude this section, let us examine some of the consequences of Eqs. (53,54). We have recalled in Sec. 3 that, in standard quantum mechanics, the (integrated) coordinates were divergent as $\varphi = \varphi_0(\lambda_0/\Delta x)$ when $\Delta x < \lambda_0$, as a consequence of Heisenberg's relations. Equation (54) tells us that they will now tend to infinity, not for $\Delta x \to 0$, but for $\Delta x \to \Lambda$. The standard relation corresponds to a constant value $\delta = 1$, i.e. to a fractal dimension $D = 1 + \delta = 2$. In the new theory, the anomalous dimension δ is now subject to scale relativistic effects: its expression now implies the scale-relativistic equivalent of Lorentz so-called "γ-factors". The value $\delta = 1$ holds only at $\Delta x = \lambda_0$, then increases toward lower scales. Consider the particular simplifying choice $\varphi = \varphi_0 = \varphi(\lambda_0)$: with $\delta(\lambda_0) = 1$, we get

$$\delta(\Delta x) = \frac{1}{\sqrt{1 - \ln^2\left(\frac{\lambda_0}{\Delta x}\right) / \ln^2\left(\frac{\lambda_0}{\Lambda}\right)}}, \tag{55}$$

i.e. *the scale γ-factor is directly given by the anomalous dimension at the new scale*, $\delta(\Delta x)$, so that one gets the new law

$$\varphi' = \varphi_0 \left(\frac{\lambda_0}{\Delta x}\right)^{\delta(\Delta x)}, \tag{56}$$

to be compared to the standard one, $\varphi' = \varphi_0(\lambda_0/\Delta x)^{\delta(\lambda_0)}$.

A second scale-relativistic effect may occur even for δ factors close to 1. As may be seen from Eq. (54), δ may increase with length φ even in the "non scale-relativistic" case $\ln(\lambda_0/\Delta x) \ll \ln(\lambda_0/\Lambda)$ (i.e. the scale-relativistic equivalent of the motion-relativistic relation $v \ll c$). Can this behaviour give rise to inconsistencies? One expect to get a large effect (i.e. $\delta' - \delta \approx 1$) for very large distances $\varphi = L$, such that $\ln(L/\lambda_0) \approx \ln^2(\lambda_0/\Lambda)/\ln(\lambda_0/\Delta x)$.[8] This actually corresponds to large macroscopic distances for which a quantum description has become inadequate for long (e.g., even with the extreme choice $\Delta x = \Lambda, \lambda_0 \approx 10^{-13}$ m yields $L \approx 10^9$ m since $\Lambda \approx 10^{-35}$ m),[9] so that we expect this behaviour not to contradict well established physical results.

Another unexpected effect is the *decrease* of the anomalous dimension when $\varphi \to 0$. However this decrease is not inconsistent with the theory, since the singularity

[8] A misprint in the published version (the second log term was to the square) is corrected here.
[9] Instead of $L \approx 10^{11}$ m as given in the published version.

$\delta = 0$ would be reached for $\varphi \ll \Lambda$, which is now excluded by the theory itself. Note that these problems are avoided by the hereabove particular choice $\varphi = \varphi_0$ for the reference point of the field φ.

The new structures found in scale relativity imply profound changes of many other fundamental basic relations. Indeed the requirement that any space and time resolution in nature be larger than the Planck length and time imply that this should be required also of any length and time interval. This is a radical change of the nature of space-time itself, which is expected to have consequences for the whole of physics. In particular, the scale transformations (53,54) rely on the concept of de Broglie length and time. But it is immediately clear that the theory cannot be self-consistent if their usual definition is kept. Indeed for masses larger than the Planck mass, the Compton length (i.e. c times the rest frame de Broglie time) would become smaller than Λ, which is a now forbidden behaviour. The next section is devoted to this crucial problem of the mass (more generally energy-momentum) scale in the new theory of scale-relativity.

8 Scale-Relativistic Mechanics

8.1 A new invariant

Let us attempt to clear up the problem which is now set before us. In the classical, then relativistic theory of motion, the laws of mechanics set the relations between energy-momentum and the essential variable in the inertial case, i.e., velocity. Our claim here is that, in the quantum domain, the classical concept of motion becomes inoperative, to the advantage of the concept of scale, and velocity is disqualified as an essential variable, its place being taken by resolution. So it becomes logical to expect the energy-momentum / velocity relations to be replaced in the quantum theory by energy-momentum / scale relations: and this is precisely what the de Broglie ($< p > . \lambda = \hbar$) and Heisenberg ($\sigma_p \, . \, \sigma_x \geq \hbar/2$) relations are. The way by which one may obtain these relations *as consequences* of the principle of scale relativity and then generalize them in the Lorentzian case is clearly to construct a scale relativistic mechanics.

In the frame of standard quantum mechanics, we have recalled that the Heisenberg relations $\Delta p \, . \, \Delta x \approx \hbar$ and $\Delta E \, . \, \Delta t \approx \hbar$ can be reinterpreted in terms of some internal length $\mathcal{L}$ which becomes scale dependent (fractal) as $\mathcal{L} \approx \mathcal{L}_0(\lambda/\Delta x)^\delta$ for $\Delta x < \lambda$ (λ being the de Broglie length $\hbar/ < p >$), and of some internal time $\mathcal{T}$ such that $\mathcal{T} \approx \mathcal{T}_0(\tau/\Delta t)^\delta$ for $\Delta t < \tau$ (τ being the de Broglie time $\hbar/ < E >$), with $\delta = 1$ in both cases. Let us consider the one-dimensional case, with φ denoting either $\mathcal{L}$ or $\mathcal{T}$ in the following. If we assume the classical coordinate system to be fixed (origin, axes orientation and state of motion), $d \ln \varphi$ and $d\delta$ are independent of each other and invariant in this "Galilean" frame.

Consider now the frame of scale relativity. The variables $\ln \varphi$ and δ respectively play in scale relativity the roles played by position x and time t in motion relativity. It is well known that a formulation of special (motion) relativity equivalent to the requirement of Lorentz covariance is the requirement of invariance of the Minkowski metric element $ds^2 = c^2 dt^2 - dx^2$ (we remain in the one-space-dimension case in order to simplify the argument). In the same way, neither $d\delta$ nor $d \ln \varphi$ remains invariant in scale relativity. The new scale invariant is (for λ_0 fixed and resolution

$\Delta x < \lambda_0$):

$$d\sigma^2 = \ln^2\left(\frac{\lambda_0}{\Lambda}\right)\, d\delta^2 - \frac{d\varphi^2}{\varphi^2}. \tag{57}$$

Under this form a physical interpretation of the new invariant is difficult, since δ is not a directly measurable quantity. However the Minkowski invariant may also be expressed in terms of velocity as $ds^2 = c^2dt^2(1 - v^2/c^2) = (c^2/v^2 - 1)dx^2$. In scale relativity, the state of motion v is replaced by the state of scale $\ln(\lambda_0/\Delta x)$, so that the new invariant may be expressed in terms of quantities which are measurable (at least in principle): the length (or time) φ and the measurement resolution Δx:

$$d\sigma^2 = \left(\frac{\ln^2(\lambda_0/\Lambda)}{\ln^2(\lambda_0/\Delta x)} - 1\right)\frac{d\varphi^2}{\varphi^2}. \tag{58}$$

This result confirms our initial conjecture [3] that the space-time of microphysics is of a radically new nature compared to the classical space-time: a proper description of it implies an explicit intervention of resolution.

Let us proceed further in our construction of a relativistic "scale mechanics". The experience of special (motion) relativity may still be followed advantageously [18]. We first assume that scale physical laws emerge from a least action principle. Once the state of motion fixed, we expect the action to be the integral over $d\delta$ of some Lagrange function $L = L(\ln\varphi, \ln(\lambda_0/\lambda), \delta)$, (to be compared with $L = L(x, v, t)$ in motion relativity) and its differential $L\, d\delta$ to be given, up to some multiplicative constant, by the invariant $d\sigma$ [18]. If we denote as $V = \ln(\lambda_0/\lambda)$ the scale state, "conservative" quantities (prime integrals) $\partial L/\partial V$ and $V\partial L/\partial V - L$ will emerge from the uniformity of $\ln\varphi$ and δ respectively. But note that these quantities are not "conservative" here in terms of time independence: the uniformity of δ implies that L does not depend explicitely on δ, so that here "conservative" means that these quantities do not depend explicitely on the anomalous dimension δ, which plays for scale the structural role played by time for motion.

8.2 Generalized de Broglie and Compton relations

Let us consider first the uniformity of $\ln\varphi$. It implies the existence of a conservative quantity, a "scale momentum" $\mathcal{P}$, which is a function of the scale state $\ln(\lambda_0/\lambda)$:

$$\mathcal{P}(\lambda) = \mu\,\frac{\ln\left(\frac{\lambda_0}{\lambda}\right)}{\sqrt{1 - \frac{\ln^2(\lambda_0/\lambda)}{\ln^2(\lambda_0/\Lambda)}}}, \tag{59}$$

where μ is the constant, to be later determined, which comes from the fact that the action and the metrics invariant are equal only to some proportionality factor (this factor is equal to $-mc$ in motion relativity [18]). A similar relation is obtained for the time variable, in terms of de Broglie (τ_0) and Planck (Λ/c) times:

$$\mathcal{E}(\tau) = \mu\,\frac{\ln\left(\frac{\tau_0}{\tau}\right)}{\sqrt{1 - \frac{\ln^2(\tau_0/\tau)}{\ln^2(c\tau_0/\Lambda)}}}. \tag{60}$$

These two relations are the scale-relativistic equivalent of the motion relativistic equation for momentum, $p = mv/\sqrt{1 - v^2/c^2}$.

In order to know the meaning of this result, one first notes that physics must be invariant under the choice of the logarithm base. Then the form of (59, 60) implies that $\mathcal{P}$ is itself a logarithm of some dimensionless quantity. Now (59) has been obtained from the uniformity of a space variable, from which the usual (motion) momentum also derives as conservative quantity in classical mechanics, and (60) from the uniformity of time, from which the concept of conservative energy derives in classical mechanics. We then suggest that $\mathcal{P}$ is related to the classical momentum (case of a space variable), leading to write $\mathcal{P} = \ln(p/p_0)$, and $\mathcal{E}$ to the classical energy (case of time variable), so that $\mathcal{E} = \ln(E/E_0)$.

Consider now the limit $\Lambda \to \infty$: this limit should give us back standard quantum mechanics, i.e. (59, 60) must be identifiable with already-known equations of quantum mechanics. Indeed (59) becomes $p/p_0 = (\lambda_0/\lambda)^\mu$ for the space variable and (60) becomes $E/E_0 = (\tau_0/\tau)^\mu$ for the time variable. We recognize in these equations the two Einstein-de Broglie relations, $p\lambda = p_0\lambda_0 =$ constant and $E\tau = E_0\tau_0 =$ constant, *provided the constant μ is definitively set to the value $\mu = 1$.* Since λ and τ are themselves defined up to some multiplicative factor, we may choose them in such a way that the universal constants $p_0\lambda_0$ and $E_0\tau_0$ are the same. This defines the reduced Planck constant $\hbar$ (or h with a different choice for the remaining arbitrary scale factor) and we get:

$$p\lambda = p_0\lambda_0 = \hbar, \tag{61}$$

$$E\tau = E_0\tau_0 = \hbar. \tag{62}$$

Consider now the Lorentzian case where $\Lambda \neq 0$: this leads us to infer that the full equations (59, 60), in which must be set $\mathcal{P} = \ln(p/p_0)$, $\mathcal{E} = \ln(E/E_0)$ and $\mu = 1$, are the scale-relativistic generalization of the de Broglie relations which we were seeking. They indeed own the expected property that momentum and energy now tend to infinity when the generalized de Broglie length and time tend to the Planck length and time. We may sum up these results by a comparison between the four structures of Galilean / Lorentzian, motion / scale relativity: Galilean motion relativity yields the momentum/velocity Descartes relation $p = mv$, whose scale equivalent is the momentum/wavelength de Broglie relation $\mathcal{P} = \mu V$, [with $\mathcal{P} = \ln(p/p_0)$, $V = \ln(\lambda_0/\lambda)$ and $\mu = 1$ it reads $p/p_0 = \lambda_0/\lambda$]; Einstein motion relativity yields $p = mv/\sqrt{1 - v^2/c^2}$ while scale relativity generalizes the de Broglie relation as:

$$p = p_0 \left(\frac{\lambda_0}{\lambda}\right)^{1/\sqrt{1-V^2/C_0^2}}, \tag{63}$$

where we have set $V = \ln(\lambda_0/\lambda)$ and $\mathbb{C}_0 = \ln(\lambda_0/\Lambda)$, and where we have given up the logarithmic form adopted in Eq. (59).

Actually, this result applies only to high energy physics ($\Delta t < \hbar/m_e c^2$, $\Delta x \approx c\Delta t < \hbar/m_e c$, where m_e is the electron mass). In this case λ may be identified with the Compton length of a given system, (or more generally, to remain Lorentz covariant, with $c\tau_{dB}$, i.e., the Lorentz contracted Compton length). Equation (63) provides us with a new relation between the momentum-energy scale and the length scale. In standard high energy quantum mechanics, the length and mass scales are directly inverse: there is an inverse correspondence between any mass scale m (equivalently an energy mc^2) and a length r through the relation $mr \approx \hbar/c$. So the asymptotic behavior of the various quantum theories, which is so crucial for their ultimate understanding, corresponds to both $r \to 0$ and $p \to \infty$. In scale relativity it now corresponds to $r \to \Lambda$ and $p \to \infty$: the experimental consequences of this new length-momentum relation will be considered in Sec. 9.

The fact that we admit that (63) applies to the de Broglie lengths themselves, while the de Broglie scale is used as "input" in the scale transformations, implies some difficulty of interpretation. We are now comparing de Broglie lengths one with another, rather than assuming λ_0 fixed and then measuring the system at some resolution Δx. So we need one new universal scale which will serve as reference for all other scales. The Compton length of the electron clearly plays this role in microphysics. It corresponds to the less massive of all elementary particles and thus to the transition from non-(motion-)relativistic to relativistic quantum behaviour. At this scale, all velocities become relativistic, and the concept of well-defined position loses its physical meaning, since the first occurrence of particle-antiparticle pair creation-annihilation starts the domain of elementary particle physics: *it is from the electron Compton length λ_e onward that the fundamental coupling constants and the particle rest masses begin to vary.* Thus the electron Compton length clearly plays the role of a zero point for the whole domain of relativistic quantum fields, so that we can write (63) in terms of a new relation between mass $m > m_e$ and Compton length $\lambda_c < \lambda_e$:

$$\ln\left(\frac{m}{m_e}\right) = \frac{\ln\left(\frac{\lambda_e}{\lambda_c}\right)}{\sqrt{1 - \frac{\ln^2(\lambda_e/\lambda_c)}{\ln^2(\lambda_e/\Lambda)}}}. \tag{64}$$

This introduces the fundamental number $\mathbb{C}_e = \ln(\lambda_e/\Lambda) = \ln(m_{\mathbb{P}}/m_e)$ [with $\mathbb{C}_e = 51.52797(7)$ from the presently known values of $\hbar$, c and G [19], which serve to define the Planck mass $m_{\mathbb{P}} = (\hbar c/G)^{1/2}$; the number into parentheses following numerical results is by convention the error on the last digits]. It is straightforward to verify on (64) that now the Compton length is limited by Λ when the mass scale tends to infinity. Applying a Lorentz transformation to (64), one finds, as expected, that (63) is only an asymptotic formula ($p \gg m_e c$), so that the strict relation $p\lambda_{dB} = \hbar$ remains true in the non-relativistic domain, $\lambda_{dB} > \lambda_e$.

8.3 Generalized Heisenberg relations

The problem of a generalization of the Heisenberg relations is set in a somewhat different way from the de Broglie and Compton problems since we now deal with inequalities rather than with strict equalities. However a similar behaviour is expected for them, i.e. we expect σ_p to tend to infinity when σ_x now tends to the Planck scale. Actually a full treatment of the problem would imply a proper generalization of the whole structure of quantum mechanics: this huge technical problem goes outside the scope of this paper. Our hope is that the setting of the self-consistency of our generalization of the two basic quantum relations, de Broglie's and Heisenberg's, will ensure the possibility of a self-consistent generalization of the whole formalism of quantum mechanics. Let us briefly consider a possible way in that direction.

We have already shown in Ref. [3] that, starting from the hypothesis that the quantum space-time is such that the lines which define the possible particle trajectories have a fractal dimension $D = 1+\delta$, one gets generalized Heisenberg inequalities: $(\sigma_p/p_0)(\sigma_x/\lambda_0)^{\delta} \geq 1$, this holding for all four space-time coordinates. The usual

Heisenberg relation is recovered, as expected, for the particular value $\delta = 1$. This generalization, which was purely formal in Ref. [3], is endowed with physical meaning now that we have introduced a fractal dimension (equivalently, an anomalous dimension in a renormalization group approach) which is allowed to vary. Note that such a relation is not incompatible with Heisenberg's: the usual Heisenberg inequality remains true even for $\delta > 1$, though the inequality becomes stronger as δ increases, i.e. $\sigma_p \,.\, \sigma_x \gg \hbar$. Since we have demonstrated hereabove that, after a dilatation (or contraction), the scale-γ-factor is precisely equal to the anomalous dimension at the new scale, the generalized Heisenberg relations finally keep, as expected, a form similar to de Broglie's in the new theory (for $\sigma_x \leq \lambda_0$):

$$\ln\left(\frac{\sigma_p}{p_0}\right) \geq \frac{\ln\left(\frac{\lambda_0}{\sigma_x}\right)}{\sqrt{1-\frac{\ln^2\left(\frac{\lambda_0}{\sigma_x}\right)}{\ln^2\left(\frac{\lambda_0}{\Lambda}\right)}}}, \tag{65}$$

with an equivalent expression holding for time and energy.

8.4 On the nature of charge

Let us come back to our construction of a scale-relativistic mechanics and build the "conservative" quantity independent of δ which comes from the homogeneity of the anomalous dimension itself, i.e., $V\partial L/\partial V - L$. *This is a completely new quantity*[10] which had no theoretical existence in Galilean scale relativity, since there δ either was itself undefined (classical case) or was an absolute constant (standard quantum case). This is the scale equivalent (from the viewpoint of the mathematical structure) of the relativistic expression for energy, $E = mc^2/\sqrt{1 - v^2/c^2}$. It reads

$$\mathbb{E} = \frac{\ln^2\left(\frac{\lambda_0}{\Lambda}\right)}{\sqrt{1-\frac{\ln^2\left(\frac{\lambda_0}{\lambda}\right)}{\ln^2\left(\frac{\lambda_0}{\Lambda}\right)}}}. \tag{66}$$

This expression should involve *a priori* an arbitrary multiplicative factor μ, but this factor was already set to 1 by the identification of the "scale-momentum" to the logarithm of a ratio of motion momenta. Once again the requirement of invariance under the logarithm form of this equation leads us to set $\mathbb{E} = \ln^2(\mathcal{E}/\mathcal{E}_0)$. Then we get:

$$\frac{\mathcal{E}}{\mathcal{E}_0} = \left(\frac{\lambda_0}{\Lambda}\right)^{\delta^{1/2}}, \tag{67}$$

where $\delta = 1/\sqrt{1 - \ln^2(\lambda_0/\lambda)/\ln^2(\lambda_0/\Lambda)}$. The remarkable result, which is reminiscent of what happened in motion special relativity, is the *emergence of a non zero value for this new physical quantity at large scale* ($\delta = 1$), i.e. a quantity which must still exist in the classical scale-independent domain:

$$\mathcal{E}_{00} = \frac{\lambda_0 \mathcal{E}_0}{\Lambda}. \tag{68}$$

[10] It has been named "complexergy" in subsequent works.

We do not know, *a priori*, what the dimensional equation of $\mathcal{E}$ is. Let us tentatively express it in units of energy. It then seems logical to identify $\mathcal{E}_0$ as the rest energy mc^2 and λ_0 as the Compton length $\hbar/mc$, so that we arrive at the conclusion that any quantum system, in particular any particle, owns a physical property which is in some way equivalent to an internal energy given by the Planck energy:

$$\mathcal{E}_{oo} = m_{\mathbb{P}} c^2 = \left(\frac{\hbar c^5}{G}\right)^{1/2}. \tag{69}$$

How can this be interpreted ? Let us consider the Newtonian gravitational force between two bodies of masses m_1 and m_2. It reads: $F_g = G m_1 m_2 / r^2$. The Coulomb force between two electric charges e_1 and e_2 reads: $F_{em} = \alpha \hbar c Z_1 Z_2 / r^2$ in units where Z_1 and Z_2 are dimensionless integers. But let us now write the Coulomb force in a form such that the charges are expressed in units of *masses.* We get $F_{em} = G(Z_1\sqrt{\alpha}\, m_{\mathbb{P}})(Z_2\sqrt{\alpha}\, m_{\mathbb{P}})/r^2$. A similar transformation may be (at least formally) made for the weak and strong forces, with the fine structure constant α replaced by the SU(2) and SU(3) coupling constants α_2 and α_3. Under this form we see that indeed the various charges in nature other than the gravitational one correspond to an internal energy of order $m_{\mathbb{P}} c^2$: they would even be strictly equal to the Planck energy provided the high energy common bare coupling constant was equal to 1 [see Sec. 9 about the convergence of U(1), SU(2) and SU(3) couplings at high energy].

This fact, often expressed in terms of the high value of the ratio of electric over gravitational forces ($\approx 4 \times 10^{42}$), is a well known structure of present physics. The new point here is the following: the mere existence of the electromagnetic field presently relies, in its main lines, on experimental grounds [18] and does not seem to be made necessary from fundamental principles. This is to be compared to the present status of the gravitational field: the principle of relativity, once applied to accelerated motion, leads to Einstein's general relativity whose equations are the simplest and most general equations which are invariant under continuous and differentiable transformations of coordinates; they introduce space-time curvature as a universal property of nature whose manifestations are what we call gravitation. In this sense, one may say that the principle of (motion) relativity leads to the demonstration of the unavoidable existence of gravitation in nature. We suggest that the hereabove result is a first step towards a similar demonstration concerning electromagnetism. Indeed one may interpret it by saying that, once applied to scale dilatations, the principle of relativity implies the existence of some force of nature additional to gravitation, of strength $F_{em} \approx G m_{\mathbb{P}}^2/r^2 = \hbar c/r^2$. However the full understanding of these structures must clearly await a proper generalization of scale relativity to fields, i.e. to nonlinear scale transformations.

9 Implications for High Energy Physics

9.1 Introduction

It is well known that Galilean motion relativity is recovered in the limit $c \to \infty$ of Einstein special relativity. Is it strictly true? Starting from special relativity, one gets an expanded formula for energy given by $E = mc^2 + (1/2)mv^2 + (1/c^2)\cdots$.

Taking the limit $c \to \infty$ indeed makes all the last terms vanish and yields the classical kinetic energy, but *it also yields a term of infinite internal energy.* Thus, if one admits our argument of Sec. 4, according to which special relativity could have been derived from the Galilean principle of relativity alone, classical mechanics was already faced with a problem of energy divergence (even if this was not explicitely realized) which the Einstein-Poincaré-Lorentz theory of relativity has solved. Does scale relativity, which aims at understanding from first principles the quantum behavior of microphysics, and which clearly has something to do with electromagnetism (see previous section), solve the old problem of the divergences of electric charge and self- electromagnetic energy ?

The present section is aimed at analysing this important issue and at first considering a not less important question: that of possible experimental verifications of the theory. Having arrived at this point of our argument, the reader may indeed legitimately ask himself whether scale relativity is a pure theoretical construction whose consequences are only to be looked for only at the presently unobservable Planck scale, or whether experimental consequences are to be expected in the energy range reached by existing particle accelerators. After a simplified reminder of the current status of the charge and divergence problem in the standard model, we shall show that scale relativity yields new predictions at observable energy yet ($E < 100$ GeV), which may be used to test the theory.

9.2 The divergence of mass and charge : a reminder

In classical electrodynamics, the electrostatic energy of a system of point charges is given in terms of the scalar potential ϕ by:

$$U = \frac{1}{2} \sum_i e_i \phi_i. \tag{70}$$

Once applied to the self-interaction of one electron, this gives one an electrodynamical self-energy:

$$E_{em} = \frac{1}{2} \frac{e^2}{r} = \frac{1}{2} \frac{\alpha \hbar c}{r}. \tag{71}$$

So in classical electrodynamics, when $r \to 0$, the electromagnetic contribution to mass becomes infinite while the charge e (or in a similar way the coupling constant α) remains constant.

Let us now recall the state of the question in the frame of quantum electrodynamics (QED). For distances smaller than the Compton scale of the electron, the nature of the problem of the interaction between two nearby charges radically changes. Indeed, while the electromagnetic interaction was mediated only by photons at scales larger than the Compton length λ_c, this is no more the case when the distance between charges becomes smaller than λ_c. The new behavior is due to the phenomenon of creation and annihilation of virtual electron-positron pairs, which mainly occurs, as expected from the Heisenberg relations, for time intervals smaller than $\Delta t \approx \hbar / 2 m_e c^2$.

The most efficient way to get this high energy behaviour is the renormalization group method [6, 7, 8, 9]. As discussed at the beginning of this paper, it is already very close from the point of view adopted in scale relativity, namely that of a physics explicitly dependent on scale; we additionally require in the present paper that it be made scale-covariant. Consider the renormalization group Callan-Symanzik equation [6, 7] for the electromagnetic coupling constant variation:

$$\frac{d\alpha}{d\left(\ln \frac{r}{\lambda}\right)} = \beta(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \dots \tag{72}$$

It may be obtained through very simple reasoning. One assumes that the coupling α is explicitely scale-dependent and that it remains the only relevant parameter in determining the physics at any given scale, so that even its infinitesimal variation during an infinitesimal scale variation is a mere function of α itself: this yields the first equality of Eq. (72). Then one assumes $\alpha \ll 1$, which allows one to expand the β function in terms of powers of α. Finally the identification of the lowest order terms with the result from the perturbative approach implies the vanishing of the constant and linear terms, so that we deal with a *marginal* field: this yields the second equality of Eq. (72). Now introducing the notation $\bar{\alpha} = \alpha^{-1}$ for the inverse coupling, we get the differential equation it satisfies:[11]

$$\frac{d\bar{\alpha}}{d\left(\ln \frac{\lambda}{r}\right)} = \beta_0 + \frac{\beta_1}{\bar{\alpha}} + \dots, \tag{73}$$

whose second order solution may be written as

$$\bar{\alpha} = \bar{\alpha}_0 + \beta_0 \ln\left(\frac{\lambda_0}{r}\right) + \frac{\beta_1}{\beta_0}\, \ln\left[1 + \beta_0 \alpha_0 \ln\left(\frac{\lambda_0}{r}\right)\right] + \dots \tag{74}$$

The success of the renormalization group approach is demonstrated by the fact that the lowest order solution automatically includes infinite sums of terms of the form $\alpha^n \ln^n(\lambda/r)$, which correspond to arbitrarily high orders in the "radiative correction" perturbative method. These remarkable results, obtained from so simple a method, apply to the coupling of QED (from the electron to the W/Z scale); to the two couplings of the electroweak theory, α_1 [U(1) group] and α_2 [SU(2) group]; and to the coupling α_3 of Quantum ChromoDynamics (QCD) at high energies, for which the condition $\alpha_3 \ll 1$ remains fulfilled. [Note that to the second order, the actual renormalization group equations for α_1, α_2 and α_3 are coupled: $d\bar{\alpha}_i/d\ln(\lambda/r) = \beta_i + \sum_j \beta_{ij}\alpha_j$] [20, 21].

Let us come back to the divergence problem. The renormalization group results, connected to the success of the electroweak and QCD theories allowed one to make important progress in its understanding. The QED lowest order result:

$$\alpha(r) = \frac{\alpha(\lambda)}{1 - \frac{2\alpha(\lambda)}{3\pi} \ln\left(\frac{\lambda}{r}\right)}, \tag{75}$$

still leads to a small scale divergence of the electric charge, but now at the "Landau ghost", the scale which makes the denominator to vanish. But one important result has set the problem in a renewed way and led to the hope that it may be solved in a frame requiring unification of electroweak and strong force at high energy [22, 23, 24]: the three running couplings have been found to converge at some high energy

[11] A misprint in the published version on the second term (β_0 instead of β_1) is corrected here.

scale, the so-called Grand Unified Theory (GUT) scale. However, before turning to this point, we remark that one may also remain dissatisfied if electrodynamics alone can not be set as a self-consistent theory, which is the case in the present quantum theory.

Let us first remind the lowest order QCD result [25]:

$$\alpha_3 = \frac{\pi}{\left(\frac{11}{2} - \frac{n_f}{3}\right) \ln\left(\frac{\lambda_s}{r}\right)} \tag{76}$$

where n_f is the number of quark families ($n_f = 6$ in the present standard model) and where λ_s is an integration constant, whose experimental estimation presently lies around 150 MeV [25]. This form for α_3, a consequence of the SU(3) group structure (which has eight generators, identified as the eight intermediate gluons), led to the important discovery of QCD asymptotic freedom [26, 27, 28], i.e. $\alpha_3 \to 0$ when $r \to 0$. The integration constant may also be expressed in terms of the value of α_3 at some given scale, often taken as the W boson scale (≈ 80 GeV). With this choice and adopting $n_f = 6$, one gets an inverse coupling given to lowest order by

$$\bar{\alpha}_3 = \bar{\alpha}_3(\lambda_W) + \left(\frac{7}{2\pi}\right) \ln \frac{\lambda_W}{r}, \tag{77}$$

where $\bar{\alpha}_3(\lambda_W) = 9.35 \pm 0.80$ [29].

Concerning the electromagnetic and weak interactions, they are now unified in the electroweak theory [30, 31, 32]. Its U(1) × SU(2) group structure implies $1+3$ generators which, once mixed, yield the W^+ and W^- bosons from one side and the Z boson and the photon from the other side, and two couplings, α_1 and α_2, which are related to the electromagnetic coupling via the weak mixing angle θ_w (which can be defined by $\cos\theta_w = M_W/M_Z$):

$$\bar{\alpha}_1 = \frac{3}{5}\,\bar{\alpha}\,\cos^2\theta_w, \quad \bar{\alpha}_2 = \bar{\alpha}\,\sin^2\theta_w. \tag{78}$$

So the inverse fine structure constant is (formally) given by $\bar{\alpha} = \bar{\alpha}_2 + 5\,\bar{\alpha}_1/3$, while the Fermi weak constant equals $G_F = \pi\alpha/(\sqrt{2}M_W^2 \sin^2\theta_w)$.[12] Current experimental determinations of the basic parameters of the model from LEP are [33]: $M_Z = 91.177\ (31)$ GeV, $M_W = 79.9\ (4)$ GeV and $\sin^2\theta_w = 0.2302\ (21)$. Still unknown, as well in the model as experimentally, are the number N_H and the mass(es) of Higgs boson doublets [25].

The two running couplings are given to the lowest order for $r < \lambda_W$ by [29]

$$\bar{\alpha}_1 = \bar{\alpha}_1(\lambda_W) - \left(\frac{2}{\pi} + \frac{N_H}{20\pi}\right) \ln\left(\frac{\lambda_W}{r}\right), \tag{79}$$

$$\bar{\alpha}_2 = \bar{\alpha}_2(\lambda_W) + \left(\frac{5}{3\pi} - \frac{N_H}{12\pi}\right) \ln\left(\frac{\lambda_W}{r}\right), \tag{80}$$

while the variation of $\bar{\alpha}$ from the electron to the W scale is given by [34]

$$\bar{\alpha}(M_W) = \bar{\alpha} - \frac{2}{3\pi}\sum_f Q_f^2 \ln\left(\frac{M_W}{M_f}\right) + \frac{1}{6\pi}, \tag{81}$$

[12] In the published version, the numerical factor $\sqrt{2}/8$ was erroneous. We give here the correct factor $\pi/\sqrt{2}$.

where the sum is done over all elementary particles, of charges Q_f and mass M_f. This yields $\bar{\alpha}(M_W) - \bar{\alpha} = -9.2 \pm 0.3$ [35, 36]. Combining this result with the measured value of Fermi's constant, the values of the couplings at the W scale are thus estimated to be $\bar{\alpha}_1(\lambda_W) = 59.17 \pm 0.35$ and $\bar{\alpha}_2(\lambda_W) = 29.07 \pm 0.58$ [29].

Let us finally come to grand unified theories. From Eqs. (77), (79) and (80) we may plot the variation of the three couplings from the W scale to higher energies, i.e. smaller resolutions. This yields the remarkable result of the convergence of the three couplings at some high energy of the order of $10^{14} - 10^{15}$ GeV, which is a very strong argument in favor of a complete unification of electromagnetic, weak and strong (color) forces at this scale [22, 23, 24]. This convergence is ensured under the "great desert hypothesis", which assumes that there is no new particle (no new physics) between the electroweak scale $\approx m_W$ and the unification scale m_X. Second order terms in the solutions to the renormalization group coupled equations [20, 21] do not change these conclusions, their contribution being presently smaller than the errors on the couplings at the W scale.

GUTs achieved at first a lot of successes:

1. In their frame, the quantization of charge finds a natural explanation [23].

2. The value of the b quark mass may be predicted from its expected equality with the τ lepton at energy m_X, and from its evolution with scale deduced from the renormalization group equations [37]; one finds M_b/M_τ (pred) $= 2.75 \pm 0.37$, to be compared with the observed ratio M_b/M_τ (obs)$= 2.38 \pm 0.06$.

3. The number of generations is constrained to be $n = 3$ (a larger number would have made the hereabove prediction unacceptably high) [37]: this has been later definitively confirmed by primordial nucleosynthesis and by LEP [33].

4. The possible values of the low energy fine structure constant are constrained to be $< 1/25$ [38], and even better $1/120 < \alpha < 1/170$ [39].

5. The value of Weinberg's mixing angle may be predicted: at unification one has $\alpha_1 = \alpha_2$, and one may introduce a running effective angle, such that $\sin^2 \hat{\theta}_W(m_X) = \bar{\alpha}_2(m_X)/\bar{\alpha}(m_X) = 3/8$, while the renormalization group yields a scale variation given by [34]

$$\sin^2 \hat{\theta}_w(m_W) = \frac{3}{8} \left[1 - \frac{109}{9} \frac{\alpha(m_W)}{2\pi} \ln\left(\frac{m_X}{m_W}\right)\right] \tag{82}$$

   The effective and measured values are related by $\sin^2 \hat{\theta}_w(m_W) = .9907 \sin^2 \theta_w$ [36]. This allows one to predict that $\sin^2 \theta_w(m_W) = 0.210$ [36], which was in good agreement with the measured value at the time of the prediction, 0.23 ± 0.02.

6. The decay of the proton was predicted [22], with a lifetime $\approx m_X^4/M_P^5 \geq 10^{30}$ years for $m_X \approx 10^{15}$ GeV, also in good agreement with known experiments at that time.

We recall that most of these results hold in a large class of theories, not only in the simplest GUT, based on the SU(5) group [23]. Unfortunately these great hopes were soon dashed: the increase of precision of experimental results led to unacceptable disagreements with the predictions. The present value of $\sin^2\theta_w(m_W)$ as measured by LEP is $\sin^2\theta_w(m_W) = 0.2302 \pm 0.0021$ [33], more than 10σ off the theoretical prediction; the experimental proton lifetime is now known to be $> 10^{32}$ years, this requiring that $m_X \gg 10^{15}$ GeV, while agreement for the mixing angle would require $m_X \approx 10^{13}$ GeV.

This rather long reminder is intended to set the frame in which we shall now consider some fundamental implications of scale relativity. We shall demonstrate that the new structure of space-time we are proposing improves strongly the situation: it solves the divergence problem, yields a theoretical understanding on the nature of the GUT scale which allows one to predict its value, and reconciles GUT predictions with experimental results without introducing new particles.

9.3 Solution to the divergence problem

Because of the previous identity in standard quantum theory between length scale and mass(-energy) scale [$\ln(m/m_0) \approx \ln(r_0/r)$ for high energy in the rest frame], the renormalization group equations are currently written indifferently in terms of $\ln m$ or $\ln r$. Actually the momentum representation, being easier to work out and holding closer to experimental data, is systematically used in quantum field theory rather than the position one (which may be obtained from the momentum representation through a Fourier transform). In present quantum mechanics, the momentum and position solutions to (72) differ only by some constants. This is no more true in scale relativity, and one should now specify which changes are to be brought to the renormalisation group equations.

One must in this respect distinguish between the cases of *relevant* fields and of *marginal* fields. Fields which vary with scale as power laws (Eq. 6), for which we have been able to establish a parallel with motion relativity laws, are precisely cases of relevant fields. The lowest order term in their renormalization group equation is linear:

$$\frac{d\varphi}{d\ln\left(\frac{\lambda}{r}\right)} = \delta \times \varphi, \tag{83}$$

this yielding the "Galilean" solution $\varphi = \varphi_0\,(\lambda/r)^\delta$. Note that the scale-relativistic solution would correspond to the equation

$$\frac{d\varphi}{d\ln\left(\frac{\lambda}{r}\right)} = \delta \times \varphi \left[1 + \frac{\ln^2(\varphi/\varphi_0)}{\delta^2\ln^2(\lambda/\Lambda)}\right]^{3/2}, \tag{84}$$

whose first non linear term is in $(\varphi\ln^2\varphi)$, so that such a form of the β function could not have been guessed from the usual pure power expansion.

Conversely the lowest order term in the β function for marginal fields (which is the case of coupling constants and masses) is to the square:

$$\frac{d\alpha}{d\ln\left(\frac{\lambda}{r}\right)} = \beta_0\,\alpha^2. \tag{85}$$

This means that this lowest order term is beyond any order of the expansion of Eq. (84), so that we conclude that the renormalization group equation for marginal fields is unchanged in scale relativity (special, i.e. linear, case). However, an important point should be noted: this conclusion holds only for the renormalization group equation expressed in terms of *length* scale: indeed the renormalization group method, in its general definition [8, 9], as well as the scale relativistic approach, essentially aims at describing the way physical laws change when going from one *spatio-temporal* scale to a larger one. But the relation from length-time scale to mass-energy-momentum scale is generalized in scale relativity (Eq. 64). So, while we obtain the usual solution (Eq. 74) in terms of length scale, we get in terms of mass scale the new relation (lowest order)

$$\bar{\alpha}(m) = \bar{\alpha}(m_W) + \beta_0 \frac{\ln(m/m_W)}{\sqrt{1 + \ln^2(m/m_W)/\mathbb{C}_W^2}} \tag{86}$$

where $\mathbb{C}_W = \ln(\lambda_W/\Lambda) = 39.876(6)$. *At the absolute limit* $r = \Lambda$, i.e. $m \to \infty$, *the charge is now finite.* So QED becomes a self-consistent theory in the frame of scale relativity. In the same way, masses were previously divergent even to first order as

$$m = m_0 \left[1 + \kappa\alpha_0 \ln\left(\frac{\lambda_0}{r}\right) \right], \tag{87}$$

while they now remain finite in scale relativity. We shall in the following let the question of mass determination be open for future works, and shall focus mainly on the coupling constant problem.

9.4 New predictions

In order to fix the ideas about the way scale relativity is expected to yield new testable predictions, let us consider some numerical values. The amplitude of scale-relativistic corrections will be given by Lorentz-like "scale γ factors" depending on "$V/\mathbb{C}$ " ratios, i.e. $\ln(\lambda_0/\lambda)/\ln(\lambda_0/\Lambda)$. For example, from the electron scale (0.511 MeV) to the W scale (79.9 GeV) , one already gets $V/\mathbb{C}_e = 0.232$, i.e. a $\gamma = 1.028$, which is not negligible. From the W scale to the GUT scale ($\approx 10^{14}$ GeV), one gets $V/\mathbb{C}_W = 0.7$, i.e. the large value $\gamma = 1.4$.

This last result allows us to introduce our first new prediction. In the standard model, there is no understanding of why a new scale is needed in addition to the W/Z one, and so no purely theoretical prediction of its value. Conversely scale relativity naturally introduces a new fundamental scale in nature. Indeed the new relation between the mass scale and the length scale (case $r \approx ct$) is such that the Planck mass $m_{\mathbb{P}}$ does not correspond any more to the Planck length Λ. We know that to the Planck length now corresponds an infinite mass, and we thus expect a new fundamental length $\lambda_{\mathbb{P}}$ to emerge (see Fig. 1). Let us compute it starting from the W scale (we recall that in the new theory lengths are no more absolute; now only scale dilatations from one scale to another have physical meaning):

$$\ln\left(\frac{\lambda_W}{\lambda_{\mathbb{P}}}\right) = \frac{\ln(m_{\mathbb{P}}/m_W)}{\sqrt{1 + \ln^2(m_{\mathbb{P}}/m_W)/\,\mathbb{C}_W^2}}. \tag{88}$$

A first estimate of this scale may be obtained by neglecting the fact the W length scale should be itself subjected to a scale-relativistic correction. To this approximation we have $\mathbb{C}_W = \ln(\lambda_W/\Lambda) \approx \ln(m_{\mathbb{P}}/m_W)$, so that the denominator of (88),

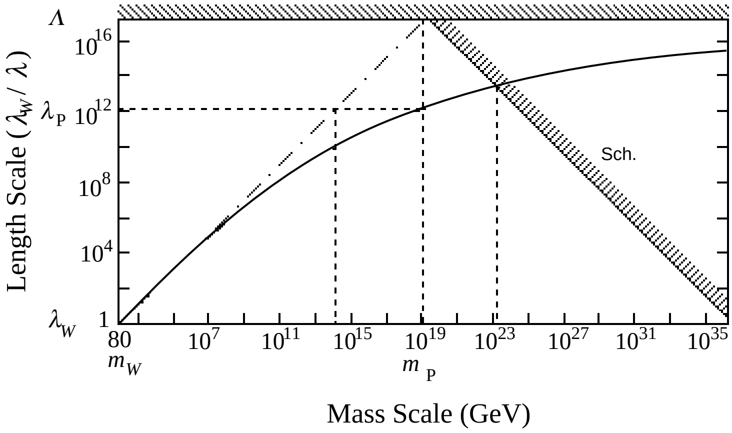

Figure 1: The mass-energy / length relation in scale relativity. In the new theory the Planck length, $\Lambda = (\hbar G/c^3)^{1/2}$ becomes an absolute and impassable limit to all lengths in nature. The new relation is such that now the Planck space-time scale corresponds to infinite energy-momentum. The Planck mass ($m_{\mathbb{P}} = 1.22105(8) \times 10^{19}$ GeV) then allows one to define a new universal length scale, $\lambda_{\mathbb{P}}$, which is found to be 10^{-12} times the W scale. This length-scale corresponds, in the previous standard theory (dot-dashed line), to an energy of 10^{14} GeV, i.e. precisely the value of the GUT scale deduced from the convergence of coupling constants. To the right of the diagram the relation which corresponds to the Schwarzschild horizon $r_s = 2Gm/c^2$ is shown. It is now reached at an energy of $\approx 10^{23}$ GeV.

i.e. the scale γ factor of the new length $\lambda_{\mathbb{P}}$ is found to be $\sqrt{2}$. So $\lambda_{\mathbb{P}}$ is expected to be $1.4 10^{12}$ times smaller than the W scale, which corresponds to 10^{14} GeV in the standard theory: in other words the new fundamental scale introduced by scale relativity is precisely the GUT scale where the U(1), SU(2) and SU(3) couplings are known to converge.

A more precise computation, accounting for the fact that the ratio λ_e/λ_W is itself no more equal to m_W/m_e, yields essentially the same result. Indeed at such scales the correction is still small: one gets $\log(\lambda_e/\lambda_W) = 5.059(3)$, $\log(m_W/m_e) = 5.194(3)$, so that $\gamma = 1.409$ to be compared with $\sqrt{2} = 1.414....$

This result has an important consequence. Indeed one may express it in another way, by computing the energies corresponding to the unification scale. As expected from the fact that the mass corresponding to $\lambda_{\mathbb{P}}$ is the Planck mass, we find that the energy at which the α_1 and α_3 couplings cross themselves (Eqs. 77, 79 and 86) is $m_{13} = 1.1 \times 10^{19}$ GeV/c^2, in excellent agreement with the value of the Planck energy $m_{\mathbb{P}} = 1.22105(8) \times 10^{19}$ GeV/c^2. This means that now not only the three electromagnetic, weak and color couplings converge at about the same energy, but also the gravitational one (see Fig. 2). Indeed the gravitational coupling α_g varies with mass scale as $\alpha_g = (m/m_{\mathbb{P}})^2$ when $m = (m_0^2 + p^2/c^2)^{1/2} \gg m_0$,[13] so that $\bar{\alpha}_g$ reaches the common value $\bar{\alpha}_1 \approx \bar{\alpha}_2 \approx \bar{\alpha}_3 \approx 40$ at energy $\approx m_{\mathbb{P}}/\sqrt{40} \approx 1.9 \times 10^{18}$ GeV. One may also directly study the crossing of the gravitational coupling with the three others. We find that they cross at a scale $\lambda = 1.8 \times 10^{-12} \lambda_W$, (which corresponds in the standard theory to 4.4×10^{13} GeV), respectively at $\bar{\alpha} = 42.4$,

[13]A misprint in the published version ($\ll m_0$) has been corrected here.

42.8 and 39.7.

These results resolve the discrepancy concerning the GUT prediction of the mixing angle and proton lifetime: the solution comes from the fact that it is length-scale which occurs in the weak angle theoretical prediction, while the proton lifetime prediction depends on mass-scale. The length-scale where $\alpha_1 = \alpha_2$ (Eqs. 79 and 80) is found to be 10^{11} times smaller than that of the W (previously 10^{13} GeV), so that one gets a prediction $\sin^2 \theta_W(W) = 0.232 \pm 0.004$ (see Eq. 82), which compares well with the experimental value 0.230 ± 0.002 [33, 36]. (Note that this may imply that there is no strictly common unification point, as recently remarked by some authors [40]). But this unification range, $10^{-11} - 10^{-12}$ times the W scale, now corresponds in mass to $10^{17} - 10^{19}$ GeV/c^2. Hence the proton life-time theoretical expectation, which varies as m_X^4, becomes larger than 10^{37} years, far greater than the present experimental lower limit.

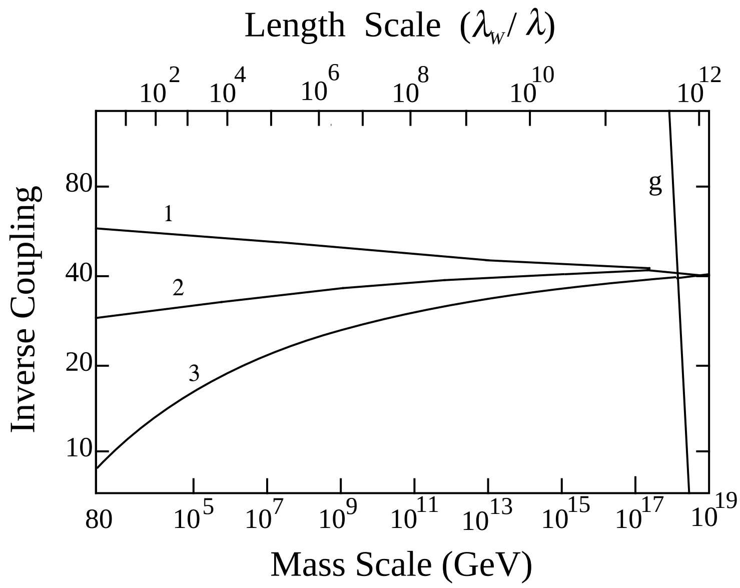

Figure 2: Variation with length- and mass-scales of the inverse of the four coupling constants, U(1), SU(2), SU(3) and gravitational (g). The mass-scale goes from the W boson mass (79.9 ± 0.4 GeV) to the Planck mass, $m_{\mathbb{P}} = 1.22105(8) \times 10^{19}$ GeV. The convergence point of couplings (i.e. the GUT scale in the standard model), at a length-scale about 10^{-12} times the W length-scale, now corresponds in scale relativity to the Planck energy (see Fig. 1 and text). In the new theory the *four* fundamental couplings converge towards the same scale.

Another comment about the new structure is that it points toward a common origin of all forces of nature. If we take Fig. 2 at face value, one starts at very high energy from a purely gravitational regime, then going to lower energies one finds a first decoupling with the color field, and finally with the electroweak field which separates in U(1) and SU(2).

Let us point out another remarkable fact. In the standard theory, the Planck length is equal to the Compton length of a Planck mass, but also to the Schwarzschild radius of a Planck mass (to a factor of 2): $\Lambda = \hbar/m_{\mathbb{P}}c = Gm_{\mathbb{P}}/c^2$. This implies that a point mass larger than the Planck mass is a black hole of radius larger than the Planck length. Hence the Planck scale is not only a domain where the quantum and gravitational phenomena are expected to become of the same order, but also a domain where the gravitational field itself is very strong: from this partly comes the difficulty of elaborating a theory of quantum gravity. This is radically changed in the new theory, so that the quantum gravity problem is now set in completely different terms. Indeed the Planck mass $m_{\mathbb{P}}$ now corresponds to the new scale $\lambda_{\mathbb{P}}$, so that the gravitational potential at the Planck energy scale is in scale relativity $Gm_{\mathbb{P}}/c^2\lambda_{\mathbb{P}} \approx 10^{-5}$. This is a weak field situation, i.e. typically of the order of the potential at the solar limb. The mass-length relation (Eq. 64) eventually crosses the black hole horizon-mass relation ($r_s = 2Gm/c^2$), but at a far larger energy of 1.3×10^{23} GeV (see Fig. 1).

Let us now consider the question of the theoretical determination of the values of the fundamental charges. The standard quantum theory, thanks to the renormalization group approach, already arrived at a magnificent clarification of the problem: the low energy charges (equivalently the low energy couplings) result from some high energy unifying value at the GUT scale, and from a variation from high to low energy which is completely determined by the renormalization group equations, in terms of elementary particle masses. Hence one of the couplings may be estimated from the other two: for example, starting from $\alpha_1(m_W)$ and $\alpha_3(m_W)$, one may find, thanks to Eqs. (77) and (79), λ_X and α_X at the GUT scale, and then predict $\alpha_2(m_W)$ from Eq. (80).

Scale relativity brings several improvements to the solution of this problem. The first one comes from the fact that we may now use the intersections of the electroweak and color couplings with the gravitational coupling, which is completely known, additionally to their own intersections. Starting, for example, from the intersection point of α_g and α_3, and admitting as a first step that it also yields the zero point of α_1 and α_2, allows us to deduce the low energy fine structure constant to 6% of its measured value and the Fermi constant to 10%. We may also use the fact that one of the free parameters of the standard theory, the GUT scale, is theoretically known in scale relativity.

But the best improvement would come from an expectation for the value of the "bare" common coupling at high energy. In this paper, we shall make a conjecture about this value and we shall present some justifications of it based upon some new structures which can be found in the ($\bar{\alpha}$, $\log r$) plane. Attempts at a theoretical understanding of the origin of this value will be presented in a forthcoming work [42]. Our conjecture is that there exist a universal high energy charge whose value is equal to

$$\sqrt{\alpha_{\mathbb{P}}} = \frac{1}{2\pi}. \tag{89}$$

This would yield a common inverse coupling $\bar{\alpha}_{\mathbb{P}} = 4\pi^2 = 39.478418....$ Let us consider the arguments in favor of this suggestion.[14]

One argument is the (α_g, α_3) intersection at which we have obtained $\bar{\alpha}_{3g} = 39.7 \pm 0.8$. Another argument is given by the electroweak couplings. At first sight they do not support our conjecture, since they meet at an inverse coupling value

[14]We have given in subsequent works theoretical arguments supporting this conjecture: see e.g. L. Nottale, Chaos, Solitons and Fractals 7, 877 (1996); 16, 539 (2003).

of about ≈ 42. Consider, however, a formal electromagnetic inverse coupling $\bar{\alpha} = \bar{\alpha}_2 + \frac{5}{3}\bar{\alpha}_1$. Below the unification scale, it becomes equal to 8/3 of the common inverse coupling $\bar{\alpha}_1 = \bar{\alpha}_2$, and the solution to its renormalization group equation is (lowest order)

$$\bar{\alpha}(r) = \bar{\alpha}(\lambda_W) - \left(\frac{5}{3\pi} + \frac{N_H}{6\pi}\right) \ln\left(\frac{\lambda_W}{r}\right), \tag{90}$$

where we recall that N_H is the (unknown) number of Higgs doublets. The value of the fine structure constant at the W scale is known to 0.2%, $[\bar{\alpha}(\lambda_W) = 127.8 \pm 0.3$, see Eq. (81)], so that we may compute the value of $\bar{\alpha}(\Lambda)$, the inverse formal fine structure constant at the absolute limiting Planck scale, and then of $\bar{\alpha}_1(\Lambda) = \bar{\alpha}_2(\Lambda) = 3\bar{\alpha}(\Lambda)/8$. We find, in this frame of a pure electroweak theory (1 Higgs doublet assumed), that

$$\bar{\alpha}_1(\Lambda) = \bar{\alpha}_2(\Lambda) \approx \frac{3}{8}\bar{\alpha}(\lambda_W) - \frac{5}{8\pi}\,\mathbb{C}_W = 40.0 \pm 0.5, \tag{91}$$

which also supports the $4\pi^2$ conjecture. Conversely the conjecture becomes testable, since it allows us to predict the low energy value of the SU(3) coupling with a precision which is a huge improvement on the presently known value. Starting from the hypothesis that $\bar{\alpha}_3(m_{\mathbb{P}}) = 4\pi^2$ and going back to the W scale from the renormalization group equation, we find from a first order calculation $\bar{\alpha}_3(\lambda_W) = 0.113$. However second order terms [20, 21] are not negligible to this precision. Including them yields $\bar{\alpha}_3(\lambda_W) = 0.1165 \pm 0.0005$, where the error comes from a rough estimate of the contribution from third order terms. This prediction is compatible with present experimental results: $(0.120 \pm 0.012\ )$ [33], $(0.107 \pm 0.011\ )$ [36], but being far more precise, will allow one to test the theory when the experimental error decreases.[15]

Equations (90, 91) do not lead to a similar prediction, owing to the fact that the low energy fine structure constant is currently known to a high precision ($\bar{\alpha} = 137.0359914(11)$ [41]). However it allows us to get its value to 0.5%, namely $\bar{\alpha} = 137.7 \pm 0.7$ (first order, 1 Higgs doublet assumed), 138.1 ± 0.7 (second order, 1 Higgs doublet), 136.0 ± 0.7 (second order, 0 Higgs doublet), and to predict that there can be no more than 1 Higgs doublet, since each additional doublet would contribute to $+2.1$ in the final result.[16]

Let us finally consider an additional argument in favor of the $1/2\pi$ conjecture. It is remarkable that well definite structures seem to emerge in the new $(\bar{\alpha}, \log r)$ plane. Hence the fundamental ratios of the Planck to W/Z mass scales, which are now the two fundamental symmetry-breaking scales yield, from the current values of weak boson masses (see Subsec. 9.2),

$$\ln\frac{m_{\mathbb{P}}}{m_W} = 39.567, \quad \ln\frac{m_{\mathbb{P}}}{m_Z} = 39.436, \tag{92}$$

[15] Ten years later, the measurement of the SU(3) coupling has been improved by a factor of 6, and its value continue to support the theoretical prediction: $\alpha_3 = 0.1172(20)$ [PDG2002] and $\alpha_3 = 0.1155(20)$ [NNLO].

[16] Thanks to the improvement of the estimate of the variation of α between the electron scale and the W/Z scale, we obtain now (in 2003) $\alpha^{-1} = 137.04 \pm 0.03$ for one Higgs doublet, which continue to support the conjecture that the bare unified coupling (at Planck length-scale, that is at infinite energy) is such that $(3/8)\bar{\alpha}(\infty) = 4\pi^2$. See references in note 14 and: L. Nottale, Electromagnetic Phenomena T. 3, No.1 (9), 24 (2003).

which tightly enclose the $\bar{\alpha}_{\mathbb{P}} = 4\pi^2$ value.[17] Similar structures also seem to relate the W/Z scale, but now in length, to the charge at the *electron* scale, which is the third fundamental scale in the theory. Indeed we find:

$$\ln \frac{\lambda_e}{\lambda_W} = 11.650, \quad \ln \frac{\lambda_e}{\lambda_Z} = 11.772, \tag{93}$$

which enclose the inverse electric charge value, $\bar{e} = \sqrt{\bar{\alpha}} = 11.706$.[18] More tentatively, note that the λ_P to Λ scale ratio may also be related to the low energy electric charge, since it is given to first approximation by $\mathbb{C}_W(1 - 1/\sqrt{2}) = 11.679$. We shall show elsewhere [42] that these remarks may be turned into precise formulas which allow one to predict with high precision the W and Z boson masses, and which brings new insights on the nature of the electric charge.[19]

10 Conclusion

What are the uncertainties in our construction ? Our theory is based on the following postulates:

1. Scale, as motion already does, may be defined as a relative state of reference systems, so that scale transformations, i.e. dilatation and contractions, come under the principle of relativity; the logarithm of the resolution with which a measurement is performed is the measure of such a state, and plays in scale relativity the part played by velocity in motion relativity.

2. The renormalization group method may be applied to space-time itself (in an enlarged sense: it is applied to the length or time virtually "measured" along a space or space-time particle path, i.e. to the internal quantum structure of a particle).

3. The couple of variables $(\ln \mathcal{L}, \delta)$, i.e. the logarithm of length (or time) as defined above, and the renormalization group anomalous dimension, play the same role in scale laws as do length and time in motion laws.

Once these postulates are accepted, we believe we have demonstrated in a general way that the general solution to the scale relativity problem implies the existence of an impassable, absolute and limiting scale which is invariant under dilatations. We could also have started from the *postulate* that the Planck length and time are invariant under dilatation. This would have given us the same theory, but with

[17] This relation has been borne out by the subsequent developments of the scale-relativity theory. Indeed, the identification of gauge transformations with transformations of the internal resolution variables has led to a proof of the quantization of charge based on the new meaning of the Planck scale established in this paper, implying new relations between couplings and mass scales, of the form $\alpha \ln(m_{\mathbb{P}}/m) = 1$. When one takes in this relation the bare value $\alpha = 1/4\pi^2$, one obtains a mass-scale $\exp(-4\pi^2)m_{\mathbb{P}} = 87.389$ GeV, of the order of the W/Z mass scale. See references quoted in notes 14 and 16.

[18] This numerical value corresponds to a choice $4\pi\epsilon_0 = 1$.

[19] Only the first of these remarks has subsequently acquired a profound physical meaning through a theoretical development based on first principles (see note 17). The two other relations given here (which involve the square-root of the coupling) have remained numerical coincidences devoid of physical meaning.

the *result* that scale relativity should be broken above some particular transitional scale, to be identified as the de Broglie length and time (otherwise one would get an invariant dilatation instead of an invariant limiting scale).

Anyway the essence of our proposal may be traced back to the basic question: Does such a limiting scale, invariant under dilatation, exist in nature ? If it does, this is an universal law of nature, and the consequences of its existence must concern the whole of physics. Even if the theory which is presented here was to be proved insufficient in some of its aspects, one could not escape the need to build such a theory and to make the whole of physics scale-relativistic.

Even in the restricted framework which has been considered in the present paper, there is a lot of work still to be done. We have examined only the case of one independent space or time variable, while proper account of the full space-time should be taken: this must include in particular the questions of angles and rotations. Solving the problem of the transformation of probability amplitudes is also an urgent task;[20] indeed our scale relativistic approach needs to be generalized to quantum systems less simple than those considered here (free particles), in particular those to which no well defined de Broglie scale can be attributed.

We may, however, by now remark that there are some domains of physics which would clearly be profoundly affected by this new structure, among them primordial cosmology and unified theories (including gravitation). We shall particularly consider the cosmological implications of scale relativity in a forthcoming work [42].

To conclude, we recall the encouraging successes which have been obtained:

(a) The theory solves the old problem of the divergence of charge and self-energy of particles.

(b) It implies that the *four* fundamental coupling constants of physics converge on about the same energy, which is now the Planck energy.

(c) It brings agreement, without introducing new interactions or particles, to the predictions of GUT's concerning the Weinberg mixing angle and the decay of the proton, which were previously both mutually contradictory and inconsistent with experimental results.

(d) It allows to fix to better than 10% the value of at least two out of the three electromagnetic, weak and strong coupling constants and opens a new avenue for the investigation of the nature of charge and the structure of elementary particles: from the conjecture that the (now finite) "bare" charge is $1/2\pi$, we are able to predict a precise value for the strong coupling at the W scale.

Let us finally note that the precision of scale relativistic predictions is highly dependent on the current error on the Planck length value, which is itself dominated by the error on the gravitational constant. So it becomes urgent to have a precision determination of G.

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[20] See Ref. [42], p. 247, for an attempt of description of scale-relativistic transformations of probability densities.

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