"nebulosity," "nebulous light," and "nebulous matter," when he means luminosity and luminous matter. In ante-spectroscopic days the terms nebula and cluster were used almost indiscriminately, a nebula being looked upon as simply an irresolvable cluster, and this error still survives in many astronomical text-books and compilations, but Dr. Boeddicker should have avoided it. When we consider that the majority of the stars in the cluster which we call the Milky Way are of the Sirian type, we see how misleading is the use of the terms nebulous light and nebulous matter. A. T.

THE THEORY OF SUBSTITUTIONS AND ITS APPLICATIONS TO ALGEBRA.

The Theory of Substitutions and its Applications to Algebra. By Dr. Eugen Netto. Translated by F. N. Cole, Ph.D. (Mich.: Ann Arbor, 1892.)

THE theory of substitutions abstractly considered is concerned with the enumeration and classification of the permutations of a set of n different letters $x_1, x_2, \ldots, x_n$. It is scarcely apparent at first sight that a far-reaching mathematical theory could be built on a basis so simple, still less that there should be any connection between this and the complicated question of the solution of algebraical equations by means of radicals. It may be worth while, in order to excite the interest of mathematical readers in the work before us, to mention one or two points in the Theory of Substitutions which will give an inkling of the nature of its connection with the interesting problem just mentioned.

The operation of replacing—say in any function $\phi(x_1, x_2, x_3)$—any permutation of the letters, say x_1, x_2, x_3, by any other, say x_1, x_3, x_2, is called a *substitution*. This operation is denoted explicitly by $\begin{pmatrix} x_1, x_2, x_3 \\ x_1, x_3, x_2 \end{pmatrix}$, or shortly by a single letter s. Thus $s\phi(x_1, x_2, x_3) = \phi(x_1, x_3, x_2)$; and again: If t denote the substitution $\begin{pmatrix} x_1, x_2, x_3 \\ x_3, x_2, x_1 \end{pmatrix}$, $t\phi(x_1, x_3, x_2) = \phi(x_3, x_1, x_2)$. We may indicate the successive application of the two substitutions s and t by multiplying the symbols st in the order of application: thus $st\phi(x_1, x_2, x_3) = \phi(x_3, x_1, x_2)$ and $ts\phi(x_1, x_2, x_3) = \phi(x_2, x_3, x_1)$. In particular, the repetition of the same substitution may be represented by powers of the symbol; thus $s^2\phi(x_1, x_2, x_3) = \phi(x_1, x_2, x_3)$. The identical substitution $\begin{pmatrix} x_1, x_2, x_3 \\ x_1, x_2, x_3 \end{pmatrix}$ is represented by unity. The total number of different substitutions of n letters is obviously $n!$; consequently, if we form the consecutive powers of any substitution we shall ultimately arrive at a power s^m which will be the identical substitution, m being some positive integer not exceeding $n!$: m is called the *order* and n the degree of the substitution.

If among the substitutions of any given degree we can select a set which have the property that the product of any two furnishes another substitution belonging to the set, we obtain what is called a *group of substitutions*. The whole of the $n!$ substitutions of n letters obviously form a group, and the identical substitution by itself forms a group. It is easy, however, to see that in general there are other groups among the substitutions of a given degree. Consider, for example, any rational function $\phi(x_1, x_2, \ldots, x_n)$ which is not wholly asymmetric: there must exist a set of substitutions each of which leaves the value of ϕ unaltered. A substitution which is the product of any number of these must also leave ϕ unaltered: hence the set in question forms a group. We have here a fundamental point in the theory of substitutions, viz., the existence of a group of substitutions and the correlation therewith of rational functions which are unaltered by all the substitutions of the group. The group is said to belong to all the functions which it leaves unaltered; and these functions are said to form a family which is characterized by the group. Thus the group of a wholly asymmetric function is the identical group consisting of the substitution ^{1}; the group of the wholly symmetric functions consists of the whole of the $n!$ substitutions of the n^{th} degree; the group of the alternating functions consists of all those substitutions which are equivalent to an even number of transpositions, and so on. It is obvious that every rational function determines a group of substitutions, and it may be shown that, conversely, for every group of substitutions we may construct an infinity of rational functions which are unaltered by the substitutions of the group. The significance of this correlation between a group and a family of functions depends on the following important theorem, which is due in substance to Lagrange. If ψ be a rational function which is unaltered by all the substitutions of the group of ϕ (in other words, if the group of ψ contain the group of ϕ) then ψ can be expressed as a rational function of ϕ, and the n elementary symmetric functions

$$C_1 = \Sigma x_1,\ C_2 = \Sigma x_1, x_2, \ldots, C_n = x_1, x_2 \ldots x_n.$$

A particular case of this is the theorem that if the groups of ψ and ϕ be identical, then each can be expressed as a rational function of the other, and of the elementary symmetric functions. A limiting case of this theorem is the familiar result that every rational symmetric function can be expressed as a rational function of the elementary symmetric functions. As a special example consider the the two wholly asymmetric functions $\psi = ax_1 + bx_2$, $\phi = a/x_1 + b/x_2$: these both belong to the identical group, since they are changed by every substitution of the letters x_1, x_2. Hence ψ can be rationally expressed as a function of ϕ, C_1, C_2. The actual expression is in fact

$$\psi = \{2(a-b)^2C_2 - (a^2+b^2)C_1{}^2 + (a+b)C_1C_2\phi\}\{-(a+b)C_1 + 2C_2\phi\}/(a-b)^2(C_1{}^2 - 4C_2).$$

The application of the theory of substitutions is limited in the first instance to rational functions. Its use in the theory of the solution of algebraical equations by means of radicals is based on the following important result in the theory of irrational functions. Any root of a solvable equation $f(x) = 0$ can be expressed as a rational integral function of certain elements $V_1, V_2, \ldots, V_1$, the coefficients of which are rational functions of the coefficients of $f(x)$ and of primitive roots of unity. The quantities $V_1, V_2, \ldots, V_n$ are on the one hand rational integral functions of the roots of $f(x) = 0$ and of primitive roots of unity, and on the other hand are determined by a series of equations

$$V_\alpha p_\alpha = F_\alpha(V_{\alpha-1}, V_{\alpha-2}, \ldots, V_v),$$

where p_α is a prime number and F is a rational function of the $V-s$. For example, in the case of the cubic

$x^3 - C_1x^2 + C_2x - C_3 = 0$, if $\Delta = -27C_3^2 + 18C_1C_2C_3 - 4C_1^3C_3 - 4C_2^3 + C_1^2C_2^2$, $S = 2C_1^3 - 9C_1C_2 + 27C_3$, $T = 9C_3 - 3C_1C_2$, the relations in question are

$$V_3^2 = -27\Delta,\ V_2^3 = \tfrac{1}{2}(S + V_3),\ V_1^3 = \tfrac{1}{2}(S - V_3);$$
$$V_1 = x_1 + \omega^2 x_2 + \omega x_3,\ V_2 = x_1 + \omega x_2 + \omega^2 x_3,$$
$$V_3 = T + 3\omega(x_1^2x_3 + x_2^2x_1 + x_3^2x_2) + 3\omega^2(x_1^2x_2 + x_2^2x_3 + x_3^2x_1);$$
$$x_1 = \tfrac{1}{3}\{C_1 + V_1 + V_2\},\ x_2 = \tfrac{1}{3}\{C_1 + \omega V_1 + \omega^2 V_2\},$$
$$x_3 = \tfrac{1}{3}\{C_1 + \omega^2 V_1 + \omega V_2\}.$$

By means of this theorem and certain elementary principles of the theory of substitutions an elegant and simple demonstration can be given of Abel's theorem that the solution by radicals of the general equation of the n^{th} degree is impossible when $n > 4$: see § 217 of the work before us.

Although the theory of substitutions bears, as we have just shown, on some of the oldest and most interesting of the problems of algebra, it has been comparatively little studied, especially by English speaking mathematicians. Dr. Cole has therefore rendered us a service of great importance by translating one of the standard treatises on this subject. Of the three that were at his disposal we think that he has chosen the one most likely to be useful to a beginner. While Serret in his "Higher Algebra" and Jordan in his "Traité" treat the theory from an abstract and more general point of view, Dr. Netto constantly associates with the substitution the function on which it is supposed to operate. This gives a powerful concrete aid to the comprehension of the propositions of the abstract theory and also helps the student to grasp their application. The great danger in subjects of such generality is that the stream of theorems is apt to run off the mind of the learner without soaking in, like water off the proverbial duck's back.

Dr. Netto's book will be found to contain all the ordinary theorems regarding the classification of substitutions, *e.g.* the existence of groups, transitive and intransitive, primitive and non-primitive, simple and compound; the theory of the algebraic relations between the values of multiple-valued functions and between functions belonging to or included in the same family; and also a considerable number of theorems regarding special groups. The applications embrace the theory of resolvents in general and of the Galois resolvent in particular; the general theory of the solvability of equations by means of radicals; the theory of the group of an equation and a discussion of the criteria of solvability; besides special applications to the cyclotomic and Abelian equations, and to equations three roots of which are connected by a rational relation.

The translation has been admirably done, both from the linguistic and from the mathematical point of view. We found, it is true, here and there passages which were somewhat obscure; but in every case, on comparing with the original, we found the rendering to be absolutely faithful. Such obscurities therefore must be charged either to the author, or to the nature of the subject, or to the idiosyncrasy of the critic, and not to the translator. We congratulate Mr. Cole on the successful completion of his arduous task, and heartily recommend the result to every lover of the most ancient and the most beautiful of all the sciences.

G. CH.

THE BRAIN IN MUDFISHES.

Das Centralnervensystem von Protopterus annectens; *eine vergleichend Anatomische Studie.* Von Dr. Rudolf Burckhardt. (Berlin: R. Friedländer und Sohn, 1892.)

THE Mudfishes, Dipnoi, from many peculiarities in their structure, have attracted the especial attention of anatomists and zoologists. Important monographs on Lepidosiren have been written by Owen and Wiedersheim, whilst Huxley, Günther, and Beauregard have described the anatomy of Ceratodus. Serres, in 1863, made a contribution to the anatomy of the nervous system of Protopterus, Fulliquet in 1886, and Parker in 1888, have also added to our knowledge of its structure; and now Dr. Burckhardt has published a well-illustrated monograph on the central nervous system of *Protopterus annectens.* He had obtained an ample supply of this fish from Herr W. Jezler, a merchant whose business engagements had taken him to the neighbourhood of Bathurst, Senegambia. On more than one occasion Dr. Burckhardt had received living fish, so that he was able to study the microscopic anatomy by the use of the most recent technical methods, and has thus added materially to our knowledge of the brain of this animal.

The author found, in the anterior horn of grey matter of the spinal cord, remarkably large nerve-cells, which possessed both branching protoplasm processes and an axial-cylinder process. In the lateral and posterior horns nerve-cells somewhat smaller in size were seen. The medulla oblongata gave origin to nerves which he names hypoglossal, vagus, glosso-pharyngeal, acusticofacialis, and trigeminus. He also describes two slender nerves as abducens and trochlearis, so that the Dipnoi are not, as some have said, destitute of these nerves. The cerebellum formed the anterior boundary of the 4th ventricle. Large nerve-cells, corresponding to those of Purkinje in the mammalian brain, were not seen. The mid-brain was distinct, and gave origin to a root of the trigeminus, to the optic tract and to the oculo-motor nerve: grey matter containing nerve-cells was grouped around the aqueduct of Sylvius.

Whilst Protopterus corresponded closely with the lowest vetebrates in the regions of the mid and hind brains it presented striking peculiarities in the pineal region. The roof of the 3rd ventricle was complicated, and possessed a velum, which represented a middle choroid plexus; a conarium, and a structure like that which Edinger has named "Zirbelpolster." The epiphysis (*Zirbel*) was attached to the skull by the arachnoid membrane.

The fore brain was well developed, and divided into two hemispheres. He recognized in it a posterior ventral swelling, which, because it contained cells similar to those found in the dentate gyrus (*fascia dentata*) of the higher brains, he describes as a lobus hippocampi. He distinguished a fissure which separated the lobus olfactorius from the pallial part of the hemisphere, so that he harmonizes the fore brain in its fundamental divisions with the mammalian brain as described by Broca and Turner. He directs attention to an elevation ventrad of the lobus olfactorius, which he calls the lobus post-olfactorius. This lobe is also found in the brains of