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THE THERMISTOR PROBLEM FOR CONDUCTIVITY WHICH VANISHES AT LARGE TEMPERATURE

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Abstract. The thermistor problem is modeled as a coupled system of nonlinear elliptic equations. When the conductivity coefficient $\sigma(u)$ vanishes (u = temperature) one of the equations becomes degenerate; this situation is considered in the present paper. We establish the existence of a weak solution and, under some special Dirichlet and Neumann boundary conditions, analyze the structure of the set $\{\sigma(u) = 0\}$ and also prove uniqueness.

1. Introduction. A thermistor is an electric circuit device made of ceramic material whose electrical conductivity $\sigma(u)$ decreases several orders of magnitude as the temperature u increases beyond a critical temperature u^*. Denote by Ω the domain in $\mathbb{R}^N$ occupied by the thermistor, by φ the electric potential, and by $k = k(u)$ the thermal conductivity. Then

$$J = \text{electric current density} = -\sigma(u)\nabla\varphi,$$
$$q = \text{heat flux} = -k(u)\nabla u, \qquad E = -\nabla\varphi = \text{electric field}.$$

The conservation of current $\nabla \cdot J = 0$ and of energy $\nabla \cdot q = J \cdot E$ can then be written in the form

$$\nabla(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \tag{1.1}$$

and

$$\nabla(k(u)\nabla u) = -\sigma(u)\nabla\varphi \cdot \nabla\varphi = -\nabla(\sigma(u)\varphi\nabla\varphi) \quad \text{in } \Omega \tag{1.1$_a$}$$

where equation (1.1) was used in deriving the last equation in (1.1_a). Since $k(u)$ varies only slightly with u, we shall assume in the sequel that $k(u) \equiv 1$; all our results, however, extend to general $k = k(u)$. Equation (1.1_a) then becomes

$$\nabla(\nabla u + \sigma(u)\varphi\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{1.2}$$

or

$$\nabla^2 u + \sigma(u)|\nabla\varphi|^2 = 0 \quad \text{in } \Omega. \tag{1.3}$$

For the physical background of the thermistor problem and some explicit solutions we refer to [1], [9], [10], [11], and the references therein. There has been recent

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mathematical interest in the problem in case $\sigma(u)$ is uniformly positive; see [2], [3], [4], [7], [8]. Cimatti and Prodi in [2] and Cimatti in [3] considered the Dirichlet boundary conditions for both φ and u and proved existence of a solution. In [4] Cimatti extended the existence result to the case where

$$\begin{aligned} &\varphi = \varphi^0, \qquad u = u^0 \quad \text{on } \Gamma_D, \qquad \Gamma_D \subset \partial\Omega, \\ &\frac{\partial\varphi}{\partial n} = 0, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N = \partial\Omega\backslash\overline{\Gamma}_D. \end{aligned} \tag{1.4}$$

An important observation by Diesselhorst [5] that the function

$$\psi = \frac{1}{2}\varphi^2 + \int_0^u \frac{ds}{\sigma(s)} \tag{1.5}$$

satisfies the equation

$$\nabla(\sigma(u)\nabla\psi) = 0 \quad \text{in } \Omega, \tag{1.6}$$

plays a crucial role in the papers [3], [4].

In the special case

$$\begin{aligned} &\Gamma_D = \Gamma_1 \cup \Gamma_2, \quad \varphi = \varphi_i, \quad u = u_i \quad \text{on } \Gamma_i, \\ &\varphi_i \text{ and } u_i \text{ are constants and } \overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \varnothing, \\ &\frac{\partial\varphi}{\partial n} = 0, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \equiv \partial\Omega\backslash\overline{\Gamma}_D, \end{aligned} \tag{1.7}$$

Cimatti also proved uniqueness; but, in general, uniqueness is still an open problem.

More recently Howison, Rodrigues, and Shillor [8] have extended the existence result to more general boundary conditions, such as

$$\begin{aligned} &\varphi = \varphi^0 \quad \text{on } \Gamma_D^1, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial\Omega\backslash\overline{\Gamma_D^1}, \\ &u = u^0 \quad \text{on } \Gamma_D^2, \qquad \frac{\partial u}{\partial n} + \gamma u = g_0 \quad \text{on } \partial\Omega\backslash\overline{\Gamma_D^2}. \end{aligned} \tag{1.8}$$

In this paper we are interested in the case where $\sigma(u)$ vanishes for large u, i.e.,

$$\sigma(u) > 0 \quad \text{if } u < u^*, \qquad \sigma(u) = 0 \quad \text{if } u > u^* \tag{1.9}$$

for some constant u^*. This provides a good approximation to the actual engineering model of thermistors, whereby the conductivity $\sigma(u)$ drops to nearly 0 beyond some critical temperature u^*. We shall be working with the boundary conditions (1.4).

In Sec. 2 we approximate $\sigma(u)$ by a family of uniformly positive functions $\sigma_\varepsilon(u)$ and review the existence proof of a solution $(\varphi_\varepsilon, u_\varepsilon)$. We also derive a priori estimates independent of ε. In particular, we prove that

$$\int_\Omega |\nabla\sigma_\varepsilon(u_\varepsilon)|^\beta \le C \quad \text{if } \frac{1}{2(1-\alpha)} < \beta \le 1 \tag{1.10}$$

provided

$$|\sigma_\varepsilon'(u)| \le C(|u^* - u|^{-\alpha} + 1), \qquad \alpha \in \left(0, \frac{1}{2}\right). \tag{1.11}$$

In Sec. 3 we define the concept of a weak solution (φ, u) for (1.1), (1.2), (1.4) and prove that a subsequence of $(\varphi_\varepsilon, u_\varepsilon)$ converges to a weak solution.

In Sec. 4 we specialize to the boundary conditions (1.7) and prove additional properties of (φ, u). In particular, $\sigma(u(x))$ is a continuous function, the level surface

$$S = \{x \in \Omega;\ \sigma(u(x)) = 0\}$$

is piecewise analytic (analytic if $N = 2$), and $\varphi(x)$ is continuous in $\Omega \backslash S$ with jump discontinuity across S. We also prove uniqueness.

Finally, in Sec. 5 we consider special solutions with boundary conditions of the form (1.8) for which the set $\{\sigma(u(x)) = 0\}$ has nonempty interior.

2. The approximating problem. For simplicity we take $u^* = 0$ in (1.9). We shall assume that

$$\begin{gathered} 0 < \sigma(u) < M \quad \text{if } u < 0, \\ \sigma(u) = 0 \quad \text{if } u > 0, \qquad \sigma \in C^0(-\infty, \infty), \end{gathered} \tag{2.1}$$

and

$$\begin{gathered} \sigma \in C^1(-\infty, 0), \\ |\sigma'(u)| \le M_0(1 + |u|^{-\alpha}) \quad \text{if } u < 0, \text{ for some } \alpha \in \left(0, \frac{1}{2}\right); \end{gathered} \tag{2.2}$$

this implies that for $u < 0$,

$$\sigma(u) \le c|u|^{\alpha'}, \qquad \alpha' = (1 - \alpha) \in \left(\frac{1}{2}, 1\right).$$

We introduce a family of smooth functions $\sigma_\varepsilon(u)$ $(0 < \varepsilon < 1)$ which approximate $\sigma(u)$ as $\varepsilon \to 0$, each uniformly positive:

$$\begin{gathered} \varepsilon \le \sigma_\varepsilon(u) \le 2M \quad \forall u, \\ \sigma_\varepsilon(u) = \varepsilon \quad \text{if } u > 0, \qquad \sigma_\varepsilon \in C^\infty(-\infty, \infty), \end{gathered} \tag{2.3}$$

and

$$\sigma_\varepsilon(u) \to \sigma(u) \text{ as } \varepsilon \to 0, \text{ uniformly in } u \text{ in bounded intervals.} \tag{2.4}$$

We also take the σ_ε to satisfy

$$|\sigma_\varepsilon'(u)| \le 2M_0(1 + |u|^{-\alpha}) \quad \forall u, \tag{2.5}$$

with the same α as in (2.2).

We assume that $\partial\Omega$ is piecewise $C^{1+\delta}$ for some $0 < \delta < 1$, and that $\partial\Gamma_D$ is piecewise $C^{1+\delta}$. We also assume that the boundary data φ^0, u^0 can be extended into Ω so that

$$\|\varphi^0\|_{L^\infty(\Omega)} < \infty, \qquad \int_\Omega |\nabla\varphi^0|^2 < \infty, \tag{2.6}$$

$$\|u^0\|_{L^\infty(\Omega)} < \infty, \qquad \int_\Omega |\nabla u^0|^2 < \infty. \tag{2.7}$$

Finally we assume that $u^0|_{\Gamma_D}$ is smaller than the critical temperature $u^*(= 0)$, i.e.,

$$u^0|_{\Gamma_D} \le -c_* < 0. \tag{2.8}$$

If we choose the extension of $u^0|_{\Gamma_D}$ to be such that

$$\Delta u^0 = 0 \quad \text{in } \Omega, \qquad \frac{\partial u^0}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{2.9}$$

then (2.7) is of course again satisfied and further, by the maximum principle,

$$u^0 \le -c_* < 0 \quad \text{in } \overline{\Omega}. \tag{2.10}$$

Consider the elliptic system

$$\nabla(\sigma_\varepsilon(u_\varepsilon)\nabla\varphi_\varepsilon) = 0 \quad \text{in } \Omega, \tag{2.11}$$

$$\Delta u_\varepsilon + \sigma_\varepsilon(u_\varepsilon)|\nabla\varphi_\varepsilon|^2 = 0 \quad \text{in } \Omega, \tag{2.12}$$

$$\varphi_\varepsilon = \varphi^0 \quad \text{on } \Gamma_D, \qquad \frac{\partial\varphi_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{2.13}$$

$$u_\varepsilon = u^0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{2.14}$$

where $\Gamma_N = \partial\Omega\backslash\overline{\Gamma}_D$.

LEMMA 2.1. There exists a solution $(\varphi_\varepsilon, u_\varepsilon)$ of (2.11)–(2.14) in $L^\infty(\Omega)\cap H^1(\Omega)$, having the following properties:

(i) φ_ε and u_ε belong to $C^\infty(\Omega)$, and

(ii) if $\partial\Omega \in C^{m+\delta}$, $\varphi^0 \in C^{m+\delta}(\Gamma_D)$, $u^0 \in C^{m+\delta}(\Gamma_D)$ then φ_ε and u_ε belong to $C^{m+\delta}(\overline{\Omega}\backslash(\overline{\Gamma}_D\cap\overline{\Gamma}_N))$ $(m = 1, 2, \dots)$.

The proof given below is essentially due to Cimatti [4].

Proof. Introducing the change of variables

$$\psi_\varepsilon = \frac{1}{2}\varphi_\varepsilon^2 + \int_{-1}^{u_\varepsilon}\frac{dt}{\sigma_\varepsilon(t)}, \tag{2.15}$$

we can rewrite (2.11)–(2.14) in the form

$$\nabla\left(a_\varepsilon\left(\psi_\varepsilon - \frac{1}{2}\varphi_\varepsilon^2\right)\nabla\varphi_\varepsilon\right) = 0 \quad \text{in } \Omega, \tag{2.16}$$

$$\nabla\left(a_\varepsilon\left(\psi_\varepsilon - \frac{1}{2}\varphi_\varepsilon^2\right)\nabla\psi_\varepsilon\right) = 0 \quad \text{in } \Omega, \tag{2.17}$$

$$\varphi_\varepsilon = \varphi^0 \quad \text{on } \Gamma_D, \qquad \frac{\partial\varphi_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{2.18}$$

$$\psi_\varepsilon = \psi_\varepsilon^0 \equiv \frac{1}{2}\varphi^{02} + \int_{-1}^{u^0}\frac{dt}{\sigma_\varepsilon(t)} \quad \text{on } \Gamma_D, \qquad \frac{\partial\psi_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_N, \tag{2.19}$$

where

$$a_\varepsilon(s) = \sigma_\varepsilon(F_\varepsilon^{-1}(s)) \tag{2.20}$$

and $u = F_\varepsilon^{-1}(s)$ is the inverse function of

$$s = F_\varepsilon(u) = \int_{-1}^{u}\frac{dt}{\sigma_\varepsilon(t)}. \tag{2.21}$$

Define a mapping $T\colon L^2(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times H^1(\Omega)$ by $(\varphi,\psi)=T(\tilde\varphi,\tilde\psi)$ where (φ,ψ) is the solution of (2.16)–(2.19) with $a_\varepsilon(\psi_\varepsilon-\varphi_\varepsilon^2/2)$ replaced by $a_\varepsilon(\tilde\psi_\varepsilon-\tilde\varphi_\varepsilon^2/2)$. By the standard theory for elliptic equations in divergence form we know that $T(\tilde\varphi,\tilde\psi)$ is well defined and

$$\|\varphi\|_{L^\infty(\Omega)}\le\|\varphi^0\|_{L^\infty(\Omega)},\qquad \|\psi\|_{L^\infty(\Omega)}\le\|\psi_\varepsilon^0\|_{L^\infty(\Omega)}. \tag{2.22}$$

Further, multiplying the equations for φ and ψ by $\varphi-\varphi^0$ and $\psi-\psi_\varepsilon^0$ respectively, and integrating over Ω, we find that

$$\|\varphi_\varepsilon\|_{H^1(\Omega)}\le C_\varepsilon,\qquad \|\psi_\varepsilon\|_{H^1(\Omega)}\le C_\varepsilon,$$

where C_ε is a constant independent of $\tilde\varphi$, $\tilde\psi$. It follows that T maps $L^2(\Omega)\times L^2(\Omega)$ into a compact set, and one can easily verify that T is also continuous. Hence, by Schauder's fixed point theorem, T has a fixed point $(\varphi_\varepsilon,\psi_\varepsilon)$, which yields via (2.15) a solution $(\varphi_\varepsilon,u_\varepsilon)$ to (2.11)–(2.14). By elliptic estimates (see, for instance, [6]) we have that φ_ε, ψ_ε belong to $C^\rho(\Omega)$ for some $\rho\in(0,1)$ and therefore u_ε is also in the same C^ρ class. Using this fact we can deduce from (2.16), (2.17) that φ_ε, ψ_ε belong to $C^{1+\rho}(\Omega)$, and then also $u_\varepsilon\in C^{1+\rho}(\Omega)$. By the same bootstrap argument one can proceed to show that φ_ε and u_ε belong to $C^\infty(\Omega)$. The proof of the last assertion of the lemma is obtained by a similar argument.

REMARK 2.1. The assumption (2.5) was not used in the proof of Lemma 2.1.

LEMMA 2.2. The solution $(\varphi_\varepsilon,u_\varepsilon)$ satisfies:

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)}+\|u_\varepsilon\|_{L^\infty(\Omega)}\le C, \tag{2.23}$$

$$\int_\Omega|\nabla u_\varepsilon|^2\le C, \tag{2.24}$$

$$\int_\Omega\sigma_\varepsilon(u_\varepsilon)|\nabla\varphi_\varepsilon|^2\le C, \tag{2.25}$$

$$\int_\Omega|\nabla\sigma_\varepsilon^\beta(u_\varepsilon)|^2\le C_\beta\quad\text{for any } \frac{1}{2(1-\alpha)}<\beta\le1, \tag{2.26}$$

where C, C_β are constants independent of ε.

Proof. The estimate (2.23) follows from the proof of Lemma 2.1 since $\|\psi_\varepsilon^0\|_{L^\infty}\le C_0$, where C_0 is independent of ε (by (2.10) and (2.3), (2.4)). Next,

$$\begin{aligned}\int_\Omega\sigma_\varepsilon(u_\varepsilon)|\nabla\varphi_\varepsilon|^2&=\inf_{\varphi\in H^1(\Omega),\,\varphi=\varphi^0 \text{ on } \Gamma_D}\int_\Omega\sigma_\varepsilon(u_\varepsilon)|\nabla\varphi|^2\\&\le\|\sigma_\varepsilon(u_\varepsilon)\|_{L^\infty}\int_\Omega|\nabla\varphi^0|^2\le C\end{aligned}$$

since $0<\sigma_\varepsilon(t)\le 2M$; thus (2.25) holds.

To prove (2.24) and (2.26) we multiply both sides of (2.12) by $f(u_\varepsilon)-f(u^0)$ and integrate over Ω. After integrating by parts we get

$$\int_\Omega f'(u_\varepsilon)|\nabla u_\varepsilon|^2\le\int_\Omega f'(u^0)\nabla u_\varepsilon\cdot\nabla u^0+\int_\Omega(f(u_\varepsilon)-f(u^0))\sigma_\varepsilon(u_\varepsilon)|\nabla\varphi_\varepsilon|^2.$$

Using the Schwarz inequality on the first integral on the right-hand side, we obtain

$$\int_\Omega f'(u_\varepsilon)|\nabla u_\varepsilon|^2 \le \int_\Omega \frac{f'(u^0)^2}{f'(u_\varepsilon)}|\nabla u^0|^2 + C\|f(u_\varepsilon) - f(u^0)\|_{L^\infty(\Omega)}, \tag{2.27}$$

where (2.25) was used.

Taking $f(s) = s$, (2.24) follows. To prove (2.26) we take

$$f(s) = \int_0^s \left[\left|\frac{d}{ds}\sigma_\varepsilon^\beta(s)\right|^2 + 1\right] ds;$$

the condition (2.5) implies that the integrand is integrable. We then get from (2.27)

$$\int_\Omega |\nabla\sigma_\varepsilon^\beta(u_\varepsilon)|^2 \le \int_\Omega \frac{f'(u^0)^2}{f'(u_\varepsilon)}|\nabla u^0|^2 + C,$$

and $|f'(u^0)| \le C$ since $u^0 \le -c_* < 0$. Since $f'(u_\varepsilon) \ge 1$, the assertion (2.26) follows.

3. Existence of a weak solution. Consider (1.1), (1.3). Using the formula $\sigma(u)\nabla\varphi = \nabla(\sigma(u)\varphi) - \varphi\nabla\sigma$, we can rewrite these equations formally as

$$\Delta(\sigma(u)\varphi) - \nabla(\varphi\nabla\sigma(u)) = 0 \quad \text{in } H^{-1}(\Omega), \tag{3.1}$$

$$\Delta\left(u + \frac{1}{2}\sigma(u)\varphi^2\right) - \frac{1}{2}\nabla(\varphi^2\nabla\sigma(u)) = 0 \quad \text{in } H^{-1}(\Omega) \tag{3.2}$$

provided

$$\varphi \in L^\infty(\Omega), \qquad u \in H^1(\Omega), \tag{3.3}$$

$$\sigma(u),\ \sigma(u)\varphi,\ \sigma(u)\varphi^2 \in H^1(\Omega). \tag{3.4}$$

Equations (3.1), (3.2) mean that

$$\int_\Omega (\nabla(\sigma(u)\varphi)\cdot\nabla\zeta - \varphi\nabla\sigma(u)\cdot\nabla\zeta) = 0, \tag{3.5}$$

$$\int_\Omega \left(\nabla\left(u + \frac{1}{2}\sigma(u)\varphi^2\right)\cdot\nabla\zeta - \frac{1}{2}\varphi^2\nabla\sigma(u)\cdot\nabla\zeta\right) = 0 \tag{3.6}$$

for every $\zeta \in H_0^1(\Omega)$. Denote by $H_{\Gamma_D}^1(\Omega)$ the class of all functions in $H^1(\Omega)$ such that $\zeta = 0$ on Γ_D.

DEFINITION 3.1. A pair (φ, u) is called a *weak solution* of the thermistor problem (1.1), (1.3), (1.4) if (3.3), (3.4) hold; if (3.5), (3.6) hold for any $\zeta \in H_{\Gamma_D}^1(\Omega)$; and if

$$u - u^0 = 0 \quad \text{on } \Gamma_D, \qquad \sigma(u)\varphi - \sigma(u^0)\varphi^0 = 0 \quad \text{on } \Gamma_D. \tag{3.7}$$

REMARK 3.1. By the trace theorem, all the functions in (3.7) are well defined. The trace of φ may not be defined, so we have used the trace of $\sigma(u)\varphi$ instead.

REMARK 3.2. Equations (3.5), (3.6) for all $\zeta \in H_0^1(\Omega)$ mean the same thing as the equations (3.1), (3.2) (which are a weak form of (1.1), (1.3)). The additional freedom of choosing ζ in the larger class $H^1_{\Gamma_D}(\Omega)$ accounts for a weak form of the Neumann conditions

$$\frac{\partial \varphi}{\partial n} = 0, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \backslash \overline{\Gamma}_D.$$

THEOREM 3.1. Assume that $\partial\Omega$ and $\partial\Gamma_D$ are piecewise in $C^{1+\delta}$ and that (2.1), (2.2) and (2.6)–(2.8) are satisfied. Then there exists a weak solution of the thermistor problem (1.1), (1.3), (1.4).

Proof. By Lemma 2.2 there exists a sequence $\varepsilon \to 0$ and functions

$$\varphi \in L^\infty(\Omega), \qquad u \in L^\infty(\Omega) \cap H^1(\Omega), \qquad \sigma_0,\ h,\ \text{and } g \text{ in } H^1(\Omega)$$

such that

$$\varphi_\varepsilon \to \varphi \qquad \text{weakly in } (L^\infty(\Omega))^*, \tag{3.8}$$

$$u_\varepsilon \to u \qquad \text{weakly in } H^1(\Omega) \text{ and a.e. in } \Omega, \tag{3.9}$$

$$\sigma_\varepsilon(u_\varepsilon) \to \sigma_0 \qquad \text{weakly in } H^1(\Omega) \text{ and a.e. in } \Omega, \tag{3.10}$$

$$\sigma_\varepsilon(u_\varepsilon)\varphi_\varepsilon \to h \qquad \text{weakly in } H^1(\Omega) \text{ and a.e. in } \Omega, \tag{3.11}$$

$$\sigma_\varepsilon(u_\varepsilon)\varphi_\varepsilon^2 \to g \qquad \text{weakly in } H^1(\Omega) \text{ and a.e. in } \Omega. \tag{3.12}$$

Recalling (2.4) we conclude from (3.9), (3.10) that

$$\sigma_0(x) = \sigma(u(x)) \quad \text{a.e. in } \Omega.$$

Set

$$\Omega_0 = \{x \in \overline{\Omega};\ \sigma(u(x)) = 0\} \qquad (= \{x \in \overline{\Omega};\ u(x) \geq 0\}).$$

Then (3.11) implies that

$$\varphi_\varepsilon = \frac{\sigma_\varepsilon(u_\varepsilon)\varphi_\varepsilon}{\sigma_\varepsilon(u_\varepsilon)} \to \frac{h}{\sigma(u)} \quad \text{a.e. in } \Omega\backslash\Omega_0.$$

On the other hand, from (3.11) and the uniform boundedness of the φ_ε we have that $h = 0$ a.e. in Ω_0, and so $h = \sigma\varphi$ a.e. on Ω_0. Thus

$$h = \sigma(u)\varphi \quad \text{a.e. in } \Omega \tag{3.13}$$

and similarly

$$g = \sigma(u)\varphi^2 \quad \text{a.e. in } \Omega. \tag{3.14}$$

Clearly (by the trace theorem) also

$$u - u^0 = 0 \quad \text{on } \Omega_D, \qquad h - \sigma(u^0)\varphi^0 = 0 \quad \text{on } \Gamma_D.$$

To complete the proof of the theorem it remains to show that (φ, u) satisfies (3.5), (3.6). These equations of course hold for $(\varphi_\varepsilon, u_\varepsilon)$, so that it only remains to justify the passage to the limit. Since $\sigma_\varepsilon(u_\varepsilon) \to \sigma(u)$ and $\varphi_\varepsilon^k\sigma_\varepsilon(u_\varepsilon) \to \varphi^k\sigma(u)$ $(k = 1, 2)$ weakly in $H^1(\Omega)$,

$$\int_\Omega \nabla\sigma_\varepsilon(u_\varepsilon) \cdot \nabla\zeta \to \int_\Omega \nabla\sigma(u) \cdot \nabla\zeta,$$

$$\int_\Omega \nabla(\varphi_\varepsilon^k\sigma_\varepsilon(u_\varepsilon)) \cdot \nabla\zeta \to \int_\Omega \nabla(\varphi^k\sigma(u)) \cdot \nabla\zeta$$

as $\varepsilon \to 0$. Thus it remains to show that

$$\int_\Omega \varphi_\varepsilon \nabla\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\zeta \to \int_\Omega \varphi\nabla\sigma(u)\cdot\nabla\zeta \tag{3.15}$$

and

$$\int_\Omega \varphi_\varepsilon^2 \nabla\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\zeta \to \int_\Omega \varphi^2\nabla\sigma(u)\cdot\nabla\zeta . \tag{3.16}$$

Since $\nabla\sigma_\varepsilon \to \nabla\sigma$ weakly in $L^2(\Omega)$ and $\varphi_\varepsilon \to \varphi$ strongly in $L^2(\Omega\backslash\Omega_0)$ (since $\varphi_\varepsilon \to \varphi$ a.e. in $\Omega\backslash\Omega_0$ and weakly $(L^\infty(\Omega))^*$), we easily find that

$$\int_{\Omega\backslash\Omega_0} \varphi_\varepsilon \nabla\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\zeta \to \int_{\Omega\backslash\Omega_0} \varphi\nabla\sigma(u)\cdot\nabla\zeta . \tag{3.17}$$

Next, choose $\beta = 1-\delta$ $(\delta > 0)$ such that (2.26) holds. Then

$$\begin{aligned}\int_{\Omega_0} |\varphi_\varepsilon\nabla\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\zeta| &\le C\int_{\Omega_0}|\nabla\sigma_\varepsilon(u_\varepsilon)| = \frac{C}{\beta}\int_{\Omega_0}\sigma_\varepsilon^\delta(u_\varepsilon)|\nabla\sigma_\varepsilon^\beta(u_\varepsilon)| \\ &\le \frac{C}{\beta}\|\sigma_\varepsilon^\delta(u_\varepsilon)\|_{L^2(\Omega_0)}\|\nabla\sigma_\varepsilon^\beta(u_\varepsilon)\|_{L^2(\Omega_0)} \\ &\le C_1\|\sigma_\varepsilon^\delta(u_\varepsilon)\|_{L^2(\Omega_0)}\end{aligned}$$

by (2.26). By the Lebesgue dominated convergence theorem, the right-hand side converges to zero as $\varepsilon \to 0$ since $\sigma_\varepsilon(u_\varepsilon) \to \sigma(u) = 0$ a.e. on Ω_0, whereas $|\sigma_\varepsilon(u_\varepsilon)| \le 2M$. Thus

$$\int_{\Omega_0} \varphi_\varepsilon\nabla\sigma_\varepsilon(u_\varepsilon)\cdot\nabla\zeta \to 0 = \int_{\Omega_0}\varphi\nabla\sigma(u)\cdot\nabla\zeta .$$

Combining this with (3.17), the assertion (3.15) follows. The proof of (3.16) is similar.

THEOREM 3.2. The weak solution (φ, u) established in Theorem 3.1 satisfies:

$$\Delta u \le 0 \quad \text{in } \mathscr{D}'(\Omega), \tag{3.18}$$

$$u^0 \le u \le 0 \quad \text{a.e. in } \Omega. \tag{3.19}$$

Proof. The assertion (3.18) follows from $\Delta u_\varepsilon = -\sigma_\varepsilon(u_\varepsilon)|\nabla\varphi_\varepsilon|^2 \le 0$. Since $u_\varepsilon \ge u^0$ in Ω, also $u \ge u^0$ a.e. in Ω. Finally, from the uniform boundedness of the functions ψ_ε defined in (2.15) it follows that

$$\limsup_{\varepsilon\to 0} u_\varepsilon(x) \le 0 \quad \forall x \in \Omega,$$

so that $u \le 0$ a.e.

4. Additional properties of weak solutions. In this section we specialize to the boundary conditions (1.7) (with $u_1 < 0$, $u_2 < 0$) and derive more specific properties of the weak solution; we shall also prove a uniqueness theorem. Except for the proof of uniqueness we shall not actually need the assumption (2.2).

For simplicity we choose the $\sigma_\varepsilon(s)$ such that $\sigma_\varepsilon(s) = \sigma(s)$ if $s \le \max\{u_1, u_2\}$.

One can determine uniquely constants a, b such that

$$\frac{1}{2}\varphi_i^2 + \int_{-1}^{u_i} \frac{dt}{\sigma(t)} = a\varphi_i + b \qquad (i = 1, 2). \tag{4.1}$$

It then follows that

$$\psi_\varepsilon = a\varphi_\varepsilon + b \quad \text{in } \Omega, \tag{4.2}$$

since both sides satisfy the same elliptic equation $\operatorname{div}(\sigma_\varepsilon(u_\varepsilon)w) = 0$, the same Dirichlet data on Γ_D, and both have zero normal derivatives on Γ_N.

It follows (recalling (2.16)) that

$$\nabla(\tilde{a}_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon) = 0 \quad \text{in } \Omega \tag{4.3}$$

where

$$\tilde{a}_\varepsilon(\varphi) = a_\varepsilon\left(-\frac{1}{2}\varphi^2 + a\varphi + b\right) \tag{4.4}$$

and a_ε is defined in (2.20).

Setting

$$A_\varepsilon(s) = \int_0^s \tilde{a}_\varepsilon(t)\,dt \tag{4.5}$$

we deduce that the function

$$w_\varepsilon(x) = A_\varepsilon(\varphi_\varepsilon(x)) \tag{4.6}$$

satisfies

$$\begin{gathered} \nabla^2 w_\varepsilon = 0 \quad \text{in } \Omega, \qquad w_\varepsilon = A_\varepsilon(\varphi_i) \quad \text{on } \Gamma_i \qquad (i = 1, 2), \\ \frac{\partial w_\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega \backslash \overline{\Gamma_1 \cup \Gamma_2}. \end{gathered} \tag{4.7}$$

In the sequel we shall assume that

$$b_0 \equiv \int_{-1}^{0} \frac{dt}{\sigma(t)} < \infty; \tag{4.8}$$

the case

$$\int_{-1}^{0} \frac{dt}{\sigma(t)} = \infty \tag{4.9}$$

will be discussed in Remark 4.4.

Observe that, as $\varepsilon \to 0$,

$$\begin{gathered} F_\varepsilon(u) \to \begin{cases} F(u) & \text{if } u < 0, \\ \infty & \text{if } u > 0, \end{cases} \qquad \left(F(u) = \int_{-1}^{u} \frac{dt}{\sigma(t)}\right), \\ F'(u) > 0 \quad \text{if } u < 0, \qquad F(0-) = b_0, \end{gathered} \tag{4.10}$$

where (4.8) was used. Also

$$\begin{aligned} &F_\varepsilon^{-1}(s) \to F^{-1}(s) \quad \text{if } -s_0 < s < \infty, \quad -s_0 = \int_{-1}^{-\infty} \frac{dt}{\sigma(t)}, \\ &F^{-1}(s) < 0, \qquad \frac{d}{ds}F^{-1}(s) > 0 \quad \text{if } -s_0 < s < b_0, \\ &F^{-1}(s) = 0 \quad \text{if } s > b_0. \end{aligned} \tag{4.11}$$

Write

$$-\frac{1}{2}s^2 + as + b - b_0 = -\frac{1}{2}(s - s_1)(s - s_2). \tag{4.12}$$

Clearly, when s_1, s_2 are real, if $s_1 < s < s_2$ then $-s^2/2 + as + b > b_0$; and if $s < s_1$ or $s > s_2$ then $-s^2/2 + as + b < b_0$.

It follows that if

$$s_1, s_2 \text{ are real and } s_1 < s_2, \tag{4.13}$$

then the function

$$\tilde{a}(s) = \sigma\left(F^{-1}\left(-\frac{1}{2}s^2 + as + b\right)\right) \qquad \left(\tilde{a}(s) = \lim_{\varepsilon\to 0} \tilde{a}_\varepsilon(s)\right) \tag{4.14}$$

satisfies

$$\begin{gathered} \tilde{a}'(s) < 0 \quad \text{if } s < s_1, \qquad \tilde{a}'(s) > 0 \quad \text{if } s > s_2, \\ \tilde{a}(s) = 0 \quad \text{if } s_1 \le s \le s_2. \end{gathered} \tag{4.15}$$

If $s_1 = s_2$ then (4.15) remains valid, whereas if s_1, s_2 are complex then $s^2/2 - as - b > b_0$ for all s and thus

$$\tilde{a}(s) > 0 \quad \text{if } s_1, s_2 \text{ are complex.} \tag{4.16}$$

We shall first consider the case (4.13). Then, as $\varepsilon \to 0$,

$$A_\varepsilon(s) \to A(s) \tag{4.17}$$

uniformly on bounded sets,

$$\begin{gathered} A'(s) > 0 \quad \text{if } s < s_1 \quad \text{or} \quad s > s_2, \\ A(s) = A_* \equiv \int_0^{s_1} \tilde{a}(t)\,dt \quad \text{if } s_1 < s < s_2. \end{gathered} \tag{4.18}$$

The harmonic function

$$w_\varepsilon(x) = A_\varepsilon(\varphi_\varepsilon(x)) = \int_0^{\varphi_\varepsilon(x)} \tilde{a}_\varepsilon(s)\,ds \tag{4.19}$$

then satisfies

$$w_\varepsilon \to w \tag{4.20}$$

uniformly in compact subsets of $\overline{\Omega}\backslash(\overline{\Gamma}_D \cap \overline{\Gamma}_N)$, where

$$\begin{aligned} &\Delta w = 0 \quad \text{in } \Omega, \\ &w = A(\varphi_i) \quad \text{on } \Gamma_i \qquad (i = 1, 2), \\ &\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{4.21}$$

Introduce the inverse function A^{-1} of A; clearly,

$$\begin{aligned} &\frac{d}{dt}A^{-1}(t) > 0 \quad \text{if } t < A_* \quad \text{or if } t > A_*, \\ &A^{-1}(A_*) \quad \text{is the interval } \{s_1 < s < s_2\}. \end{aligned} \tag{4.22}$$

From (4.19), (4.20), (4.22) we deduce that

$$\varphi_\varepsilon(x) = A_\varepsilon^{-1}(w_\varepsilon(x)) \to A^{-1}(w(x)) \quad \text{in } \Omega\backslash S, \tag{4.23}$$

where

$$S = \{x \in \Omega;\, w(x) = A_*\}. \tag{4.24}$$

From (2.15), (4.2) we also deduce that

$$u_\varepsilon(x) \to u(x) \quad \text{in } \Omega\backslash S, \tag{4.25}$$

$$u(x) \le 0, \tag{4.26}$$

and

$$-\frac{1}{2}\varphi^2 + a\varphi + b - b_0 = \int_0^u \frac{ds}{\sigma(s)}; \tag{4.27}$$

further

$$\nabla w = \sigma(u)\nabla\varphi \quad \text{in } \Omega\backslash S. \tag{4.28}$$

The set S is a level surface of the harmonic function w, and it is therefore piecewise analytic; in case $N = 2$, S is actually an analytic curve. We are assuming here that

$$A_* \text{ lies between the number } A(\varphi_1),\ A(\varphi_2); \tag{4.29}$$

otherwise S is empty.

Set

$$\Omega_+ = \{x \in \Omega,\, w(x) > A_*\}, \qquad \Omega_- = \{x \in \Omega,\, w(x) < A_*\}; \tag{4.30}$$

each set is a connected open set. Then

$$\varphi(x) = A^{-1}(w(x)) \quad \text{in } \Omega_+ \cup \Omega_-. \tag{4.31}$$

Since $A^{-1}(w(x))$ is continuous in $\overline{\Omega}_+$ and in $\overline{\Omega}_-$ with

$$\lim_{x\to x_0,\, x\in\Omega_+} A^{-1}(w(x)) = s_2, \qquad \lim_{x\to x_0,\, x\in\Omega_-} A^{-1}(w(x)) = s_1$$

for any $x_0 \in S$, it follows that

$$\begin{aligned} &\varphi \in C^0(\Omega_+), \qquad \varphi \in C^0(\Omega_-) \quad \text{with} \\ &\lim_{x\to x_0,\, x\in\Omega_+} \varphi(x) = s_2, \qquad \lim_{x\to x_0,\, x\in\Omega_-} \varphi(x) = s_1 \quad \forall x_0 \in S. \end{aligned} \tag{4.32}$$

Recalling (4.27) we also deduce that

$$\int_0^{u(x)} \frac{ds}{\sigma(s)} \to 0 \quad \text{if } x \in \Omega\backslash S, \quad x \to x_0 \in S,$$

so that

$$u(x) \text{ is continuous across } S. \tag{4.33}$$

From (4.27) we also deduce that $u < 0$ in $\Omega\backslash S$ and $u = 0$ on S; thus

$$S \text{ is the set } \{x \in \Omega;\, \sigma(u(x)) = 0\}. \tag{4.34}$$

Theorem 4.1. Assume that $\partial\Omega$ and $\partial\Gamma_D$ are piecewise in $C^{1+\delta}$ and that (2.1), (1.7) hold with $u_1 < 0$, $u_2 < 0$. Then the limit (φ, u) of $(\varphi_\varepsilon, u_\varepsilon)$ exists and is independent of the choice of the family σ_ε, and it has the following properties:

φ and u are related by (4.27), (4.35)

$u(x)$ and $\sigma(u(x))$ are continuous in Ω, (4.36)

$\varphi(x)$ is continuous in $\Omega\backslash S$ with limits s_2, s_1 from the respective sides Ω_+, Ω_- of S, (4.37)

where

S is the A_*-level surface of the harmonic function w defined by (4.21), (4.38)

$$\sigma(u)\nabla\varphi \in L^1(\Omega\backslash S), \tag{4.39}$$

$$\nabla u \in L^\infty_{\mathrm{loc}}(\Omega), \tag{4.40}$$

and, finally,

$$\int_{\Omega\backslash S} \sigma(u)\nabla\varphi \cdot \nabla\zeta = 0 \quad \forall\zeta \in H^1(\Omega), \qquad \zeta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2. \tag{4.41}$$

Proof. We have already proved (4.35)–(4.39) ((4.39) follows from (4.28)). From (4.27),

$$\nabla u = (-\varphi + a)\sigma(u)\nabla\varphi = \left(\frac{s_1 + s_2}{2} - \varphi\right)\nabla w, \tag{4.42}$$

and since $\nabla w \in L^\infty_{\mathrm{loc}}(\Omega)$, (4.40) follows. It remains to prove (4.41). But this follows from (4.28):

$$\int_{\Omega\backslash S} \sigma(u)\nabla\varphi \cdot \nabla\zeta = \int_{\Omega\backslash S} \nabla w \cdot \nabla\zeta = \int_\Omega \nabla w \cdot \nabla\zeta = \int_{\partial\Omega} \frac{\partial w}{\partial n}\zeta = 0.$$

REMARK 4.1. From (4.42) we deduce the jump relations

$$\left[\frac{\partial u}{\partial n}\right]_S = -[\varphi]_S \frac{\partial w}{\partial n} = -\left[a\varphi\frac{\partial\varphi}{\partial n}\right]_S. \tag{4.43}$$

This implies that equation (1.3) holds in the following sense:

$$\Delta u + \sigma(u)|\nabla\varphi|^2\chi_{\Omega\backslash S} + [\varphi]_S(\nabla w \cdot n)\delta_S = 0,$$

where δ_S is the Dirac function with uniform mass distribution 1 on S.

REMARK 4.2. In establishing Theorem 4.1 we have not used condition (2.2).

REMARK 4.3. Theorem 4.1 extends to the case where (4.13) holds with $s_1 = s_2$; in this case $\varphi(x)$ is continuous across S. If s_1, s_2 are complex, then (because of (4.16)) the assertions of Theorem 4.1 hold with S the empty set.

REMARK 4.4. So far we have assumed that (4.8) holds. Since φ_ε and ψ_ε are uniformly bounded, we also have

$$\left|\int_{-1}^{u_\varepsilon(x)} \frac{ds}{\sigma_\varepsilon(s)}\right| \le C.$$

If (4.9) holds then the last inequality implies that $u_\varepsilon(x) \le -\delta$, where δ is a positive constant independent of ε. It follows that for the limiting (φ, u), $\sigma(u(x))$ is uniformly positive in Ω.

From now on we shall assume, in addition to the assumptions of Theorem 4.1, that σ satisfies (2.2). Then, by Theorem 3.1, (φ, u) is a weak solution, as defined in Sec. 3. Therefore (3.5), (3.6) hold.

We wish to prove (under some assumptions) uniqueness of the weak solution. In general, a weak solution may not be unique. For instance, if $\sigma(u)$ vanishes on a nonempty open set (examples will be given in Sec. 5), then by modifying φ in this set we get another weak solution.

Let (φ, ψ) be a weak solution of (1.1), (1.3), (1.7). We shall make several assumptions:

$$\sigma(u) \text{ is continuous in } \Omega, \tag{4.44}$$

$$\text{meas}\{\sigma(u) = 0\} = 0, \tag{4.45}$$

and

$$\text{each component of } \{\sigma(u) > 0\} \text{ is connected to } \Gamma_D \ (= \Gamma_1 \cup \Gamma_2); \tag{4.46}$$

further, setting

$$\psi = \begin{cases} \frac{1}{2}\varphi^2 + \int_0^u \frac{ds}{\sigma(s)} & \text{in } \{\sigma(u) \neq 0\}, \\ \frac{1}{2}\varphi^2 & \text{in } \Omega_0 \equiv \{\sigma(u) = 0\}, \end{cases} \tag{4.47}$$

and

$$\eta = \psi - a\varphi - b, \tag{4.48}$$

where a, b are constants such that $\eta = 0$ on Γ_D (see (4.1)), we assume that

$$\eta \in H^1(\Omega). \tag{4.49}$$

THEOREM 4.2. Let the assumptions of Theorem 4.1 and (2.2) hold. Then there exists at most one weak solution of the thermistor problem (1.1), (1.3), (1.7) satisfying (4.44)–(4.46), (4.49).

Of course, the existence of such a solution and additional properties of it were established in Theorem 4.1.

Proof. Since $\sigma\varphi$ and η belong to $H^1(\Omega)$, the same is true of $\sigma\psi$. One can easily verify that $\sigma\nabla\psi = \sigma\varphi\nabla\varphi + \nabla u$ a.e. in both $\Omega_1 \equiv \{\sigma(u) > 0\}$ and Ω_0. Using (3.5), (3.6) we then deduce that ψ satisfies the same equation (3.5) as φ, and therefore

$$\int_\Omega (\nabla(\sigma(u)\eta) \cdot \nabla\zeta - \eta\nabla\sigma(u) \cdot \nabla\zeta) = 0 \quad \forall \zeta \in H^1_{\Gamma_D}(\Omega).$$

Since $\eta = 0$ on Γ_D (in the trace class), we can take $\zeta = \eta$:

$$\int_\Omega (\nabla(\sigma(u)\eta) \cdot \nabla\eta - \eta\nabla\sigma(u) \cdot \nabla\eta) = 0, \quad \text{or} \quad \int_\Omega \sigma(u)|\nabla\eta|^2 = 0.$$

Recalling (4.46) we conclude that $\eta = 0$ a.e. in Ω_1, and consequently, by (4.45), $\eta = 0$ a.e. in Ω.

One can now proceed as in the proof of Theorem 4.1, and derive for φ a nonlinear elliptic equation: $\nabla(\tilde{a}(\varphi)\nabla\varphi) = 0$ with $\tilde{a}(\varphi)$ defined as in (4.14). But then φ must coincide with the function φ which was obtained in Theorem 4.1. Since φ is uniquely determined, also u is uniquely determined.

REMARK 4.5. Without the assumption (4.49) one can construct infinitely many weak solutions. They are obtained by taking

$$\psi = a'\varphi + b' + c\chi_{\Omega'}, \qquad c \text{ an arbitrary constant.}$$

Here Ω' is a subdomain of Ω (to be determined such that $\overline{\Omega'} \supset \Gamma_1$, $\overline{\Omega'} \cap \Gamma_2 = \emptyset$) and a', b' are constants determined by boundary conditions similar to (4.1), namely, $a\varphi_i + b$ is replaced by $a'\varphi + b' + c$ for $i = 1$ and by $a'\varphi + b'$ for $i = 2$. The function u is defined by (4.47), and $\partial\Omega' \cap \Omega$ is the set $\{\sigma(u) = 0\}$.

5. Examples. We try a solution, in $\mathbb{R}^2$, of the form

$$\varphi = \alpha\Phi(y) + \beta x, \qquad u = U(y). \tag{5.1}$$

Then (1.1), (1.3) become

$$\begin{gathered} (\sigma(U)\Phi')' = 0, \\ U'' + \sigma(U)(\alpha^2(\Phi')^2 + \beta^2) = 0, \end{gathered} \tag{5.2}$$

so that

$$\sigma\Phi' = \text{const} = C_1, \tag{5.3}$$

and

$$U'' + \sigma(U)\beta^2 + \frac{\alpha^2 C_1^2}{\sigma(U)} = 0 \quad \text{in } \{\sigma(U) > 0\}. \tag{5.4}$$

The last equation can be reduced to

$$\frac{1}{2}U'^2 + \int_0^u \left(\beta^2\sigma(s) + \frac{\alpha^2 C_1^2}{\sigma(s)}\right) ds = \text{const} = C_2, \tag{5.5}$$

or

$$U' = F(U, C_1, C_2). \tag{5.6}$$

Take for example $\alpha = 0$, $\beta = 1$ and assume that

$$\sigma(s) = \begin{cases} c|s|^\gamma & \text{if } s < 0, \\ 0 & \text{if } s > 0, \end{cases} \tag{5.7}$$

where $0 < \gamma < 1$ and c is a positive constant. Then a solution to (5.4) is given by

$$u_0(y) = \begin{cases} -y^\delta & \text{if } y > 0, \\ 0 & \text{if } -\mu < y < 0, \\ -(-\mu - y)^\delta & \text{if } y < -\mu \end{cases} \tag{5.8}$$

for any $\mu > 0$, provided

$$\delta = \frac{2}{1-\gamma}, \qquad c = \delta(\delta - 1).$$

One may perceive $(x, u_0(y))$ as a weak solution in a rectangle Ω, with boundary conditions

$$\begin{aligned} \frac{\partial\varphi}{\partial n} &= 0 \quad \text{on the horizontal edges of } \partial\Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on the vertical edges of } \partial\Omega, \end{aligned}$$

and $\varphi = \varphi_0$, $u = u_0$ (suitable functions) on the remaining edges.

In the above example

$$\text{the set } \{\sigma(u) = 0\} \text{ is a strip } \{-\mu < y < 0\}; \tag{5.9}$$

the boundary conditions are of course not of the form (1.7) (or even (1.4)).

In case $\alpha \neq 0$, for the corresponding solution of (5.2) the set $\{\sigma(u) = 0\}$ has measure zero, in general.

Let

$$f(z) = f_1(x, y) + if_2(x, y) \qquad (z = x + iy)$$

be any holomorphic function. It was observed by Howison [8] that solutions to the thermistor problem are invariant under conformal mappings of the independent variable. Thus, in particular, the pair

$$\varphi = f_1(x, y), \qquad u = u_0(f_2(x, y)),$$

where u_0 is defined by (5.8), is a solution of the thermistor problem, and $\{\sigma(u) = 0\}$ has nonempty interior.

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