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The Third Law of Black Hole Mechanics: A Counterexample

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Abstract. The collapse of a spherically symmetric charged thin shell in a Reissner-Nordstrøm field can lead to an extreme black hole. No contradiction to the assumption of Cosmic Censorship results.

1. Introduction

The third law of black hole mechanics is the conjecture, first formulated by Bardeen et al. [1], that reads: "It is impossible by any procedure, no matter how idealized, to reduce $\varkappa$ to zero by a finite sequence of operations."

Here

$$\varkappa = \frac{\sqrt{m^2-a^2-e^2}}{2m^2-e^2+2m\sqrt{m^2-a^2-e^2}}$$

is the so-called surface gravity of the black hole of total mass m, specific angular momentum a and total charge e. Zero $\varkappa$ means that the hole is extreme, i.e.,

$$m^2 = a^2 + e^2 .$$

We devise a counterexample to this conjecture: a process that is capable of producing an extreme black hole in a finite interval of advanced time. The process can be described as follows. Consider asymptotic observers throwing a thin shell of charged incoherent matter towards a Reissner-Nordstrøm black hole in a spherically symmetric way. In the past of the shell, the values of the mass and charge parameters of the spacetime are m_1 and e_1, in its future, they are m_2 and e_2. The shell starts with total *injection energy* E (energy with respect to the asymptotic observers) and total charge e, which satisfy the conservation laws:

$$e_2 = e_1 + e, \qquad m_2 = m_1 + E$$

(see Sect. 2). The motion of such shells has been thoroughly studied [2–5]. One particular result is that the shell implodes unless a minimal, or *bounce radius*, R_b, is

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reached, and then it explodes back to infinity, if the parameters $e_{1,2}$, $m_{1,2}$ are chosen properly (see [3]). The shells of this sort can be divided into two classes:

(1) $R_b < m_1 + \sqrt{m_1^2 - e_1^2}$.

That is, the shell crosses the horizon of the original black hole. In this case, the equations of motion lead inevitably to $m_2 > |e_2|$ (unless E is negative) and to

$$m_1 + \sqrt{m_1^2 - e_1^2} < m_2 + \sqrt{m_2^2 - e_2^2}.$$

Thus, there is always a new, underextreme, horizon lying over the old one and the third law as well as the law of cosmic censorship are satisfied [5].

(2) $m_1 + \sqrt{m_1^2 - e_1^2} < R_b$.

The shell stops imploding over the old horizon. In this case, m_2 can be larger, equal or smaller than $|e_2|$ and a new horizon can form over the shell not only for $m_2 > |e_2|$ but also for $m_2 = |e_2|$. This has been recognized long ago by Kuchař [2] in the subcase $m_1 = e_1 = 0$ (Minkowski-space inside the shell), where, for all values of e and E satisfying $e = E$, an extreme horizon forms over the shell. (At the time Kuchař studied thin shells nobody ever dreamt about Black Hole Thermodynamics, and Kuchař was not aware of inventing an Interesting Counterexample.)

The plan of the paper is as follows: In Sect. 2, we calculate those aspects of the case (2) that are relevant to our goal, generalizing Kuchař's result to any $m_1 > |e_1|$, whereas Sect. 3 contains a discussion.

An important discussion point can be made already here. Our counterexample shows that black hole thermodynamics has peculiar features not encountered in that of ordinary bodies. The generality and unity of the "generalized thermodynamics" is thereby not destroyed, however: one only has to consider the third law as a less general statement than the other laws.

2. The Model

The dynamics of a charged thin shell of incoherent matter in the field of a non-extreme Reissner-Nordstrøm black hole has been studied in [2–4]. We use the results of these papers.

One word concerning the notation: the inner and outer spacetimes with respect to the shell and the corresponding quantities are labelled by the indices 1 and 2, respectively. Thus, the metrics are

$$ds_{1,2} = f_{1,2}(r_{1,2}) dv_{1,2}^2 - 2dr_{1,2} dv_{1,2} - r_{1,2}^2 d\theta_{1,2}^2 - r_{1,2}^2 \sin^2\theta_{1,2} d\varphi_{1,2}^2, \tag{1}$$

where

$$f_{1,2}(r) = 1 - \frac{2m_{1,2}}{r} + \frac{e_{1,2}^2}{r^2}. \tag{2}$$

We assume that

$|e_1| < m_1$,

in order to have a generic black hole with the horizon H_1 at $r_1 = m_1 + \sqrt{m_1^2 - e_1^2}$ inside the shell.

The trajectory of the shell is given by the functions

$$r_{1,2} = R(s), \qquad v_{1,2} = V_{1,2}(s),$$

s denoting the proper time along the curves $\theta_{1,2} = \text{const}$, $\varphi_{1,2} = \text{const}$. $R(s)$, $V_1(s)$, and $V_2(s)$ are determined by the dynamical equations:

$$\dot{R}^2 = \left[A - B(e_2)\frac{m_2}{R}\right]^2 - f_2(R), \tag{3}$$

$$\dot{V}_{1,2}^2 = \frac{1}{\sqrt{\dot{R}^2 + f_{1,2}(R)} - \dot{R}}, \tag{4}$$

where

$$A = \frac{m_2 - m_1}{m}, \tag{5}$$

$$B(e_2) = \frac{m^2 + e_2^2 - e_1^2}{2mm_2}, \tag{6}$$

and m is the total rest mass of the shell [2–4]. The functions $V_{1,2}(s)$ as given by (4) remain regular at those Killing horizons through which the shell implodes ($\dot{R} < 0$) and diverge at those Killing horizons through which it explodes ($\dot{R} > 0$). This is in consonance with the meaning of v_1 and v_2 as advanced time coordinates.

In the absence of pressure, the total rest mass m of the shell remains constant throughout the motion. The total charge e of the shell is also a constant, given by

$$e = e_2 - e_1 \tag{7}$$

[2–4].

We assume that the shell starts from infinity ($R(-\infty) = \infty$) with a non-zero radial velocity u:

$$u = \dot{R}(-\infty) < 0. \tag{8}$$

Then, (3) and (2) imply

$$A = \sqrt{1 + u^2}, \tag{9}$$

while from (5) we have

$$m_2 = m_1 + E, \tag{10}$$

$$E = m\sqrt{1 + u^2} \tag{11}$$

is the total *injection energy* of the shell.

Finally, let the outer metric deviate only slightly from an extremal one:

$$\frac{|e_2|}{m_2} = 1 + \varepsilon, \qquad \varepsilon \ll 1. \tag{12}$$

In all equations, we ignore powers of ε higher than the first. We can then write

$$B(e_2)=B+\frac{m_2}{m}\varepsilon, \tag{13}$$

$$B \quad =\frac{m_2^2+m^2-e_1^2}{2mm_2}. \tag{14}$$

Equation (3) becomes

$$R^2\dot{R}^2=(A^2-1)R^2-2m_2\left(AB-1+\frac{m_2}{m}A\varepsilon\right)R$$
$$+m_2^2\left[B^2-1+2\left(\frac{m_2}{m}B-1\right)\varepsilon\right]. \tag{15}$$

The discriminant Δ of the right-hand side of (15) can be brought to the form

$$\Delta=\tfrac{1}{4}m^2(u^2-u_c^2)^2+m_2^2(u^2+u_c^2)\varepsilon. \tag{16}$$

Here, we have introduced the abbreviation

$$u_c=-\frac{1}{m}\sqrt{m_1^2-e_1^2}. \tag{17}$$

Distinguish the following cases: 1. $u_c<u<0$, 2. $u<u_c$, 3. $u=u_c$. In cases 1 and 2, there is a finite range of ε containing zero so that the second term in Δ is small compared to the first:

$$\varepsilon \ll \frac{m^2(u^2-u_c^2)^2}{4m_2^2(u^2+u_c^2)}.$$

Then, also, $\Delta>0$. Thus, (15) becomes

$$R^2\dot{R}^2=u^2(R-R_a)(R-R_b),$$

where R_a and R_b are the two roots of the right-hand side of (15), R_b being the bigger root. The solution $R(s)$ to this equation can be found readily (see, e.g. [6]), but it is rather lengthy, which is why we do not reproduce it. We concentrate on those properties of it that we shall need:

1) The function $R(s)$ is analytic in the range $R_b<R(s)<\infty$, where its derivative satisfies $\dot{R}(s)<0$.

2) $\lim_{s\to-\infty} R(s)=+\infty$, $\lim_{s\to-\infty} \dot{R}(s)=u$.

3) $R(0)=R_b$, $\dot{R}(0)=0$.

The extension of the shell motion through $s=0$ is given by time inversion, $R(-s)$, so that R_b is a turning (or bounce) point (see [3]).

For case 3, we have with $\varepsilon=0$:

$$R^2\dot{R}^2=u^2(R-m_2)^2.$$

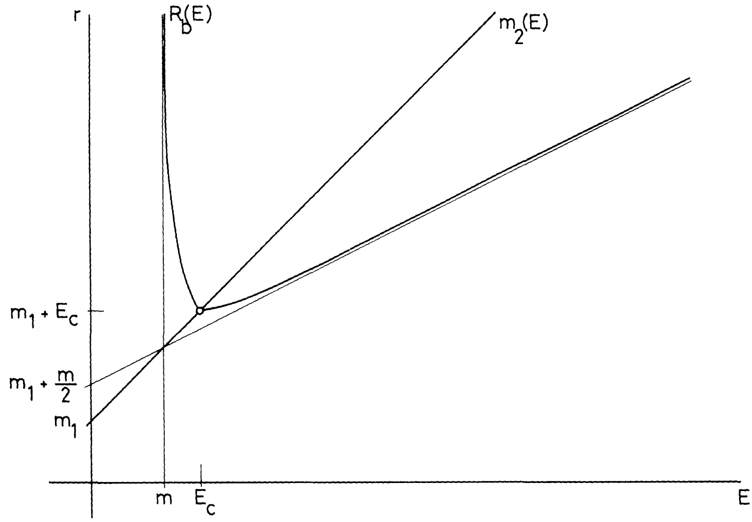

Fig. 1. The plot of R_b and m_2 against E for $\varepsilon=0$. The R_b-curve reaches its minimum at $E=E_c$. Its asymptotes are $E=m$ and $r=m_1+\frac{1}{2}m+\frac{1}{2}E$, and it has a cusp at the minimum

This time, the solution $R(s)$ has the properties:

1. The function $R(s)$ is analytic in the range $m_2<R(s)<\infty$, where $\dot{R}(s)<0$.
2. $\lim_{s\to-\infty} R(s)=+\infty$, $\lim_{s\to-\infty} \dot{R}(s)=u$.
3. $\lim_{s\to+\infty} R(s)=m_2$, $\lim_{s\to+\infty} \dot{R}(s)=0$.

Thus, the shell implodes forever, reaching the radius m_2 only asymptotically. In the first two cases, we calculate R_b and obtain:

1. $m<E<E_c$

$$R_b=m_1+\frac{1}{2}\left(E-m+\frac{m^2u_c^2}{E-m}\right)+K_1\varepsilon, \tag{18}$$

2. $E_c<E<\infty$

$$R_b=m_1+\frac{1}{2}\left(E+m+\frac{m^2u_c^2}{E+m}\right)+K_2\varepsilon, \tag{19}$$

where K_1 and K_2 are positive. E_c is defined by

$$E_c=m\sqrt{1+u_c^2}\,.$$

This will henceforth be referred to as the *critical injection energy.*

For $\varepsilon=0$, we plot R_b and m_2 (which is the radius of the horizon H_2 for the outer spacetime in this case) against E in Fig. 1. Notice that, for small injection energies, $m<E<E_c$, the turning point R_b lies over the horizon at $r_2=m_2$ so that no horizon forms over the shell, and the state of the original black hole is not altered. However, if we exceed the threshold at E_c, the shell goes through $R=m_2$ and the

part of the null hypersurface $r_2 = m_2$ which lies over the shell does appear in our spacetime. It forms a part of a new event horizon (see, e.g., [7]), H_2,

$$H_2 = \partial(I^-(\mathscr{I}_1^+)).$$

The situation is illustrated by Fig. 2. H_2 is the event horizon with respect to the observers who threw in the shell, because they move along a trajectory near $\hat{\mathscr{I}}_1^-$ and $\mathscr{I}_1^+$. The local appearence of the old horizon, H_1, is independent of whether the shell is thrown in or not, at least in the past of the Cauchy horizon H_3 (there are arguments suggesting that the region in the future of H_3 should not be taken very seriously [8]). However, from the global point of view, its status changes to that of an apparent horizon [9].

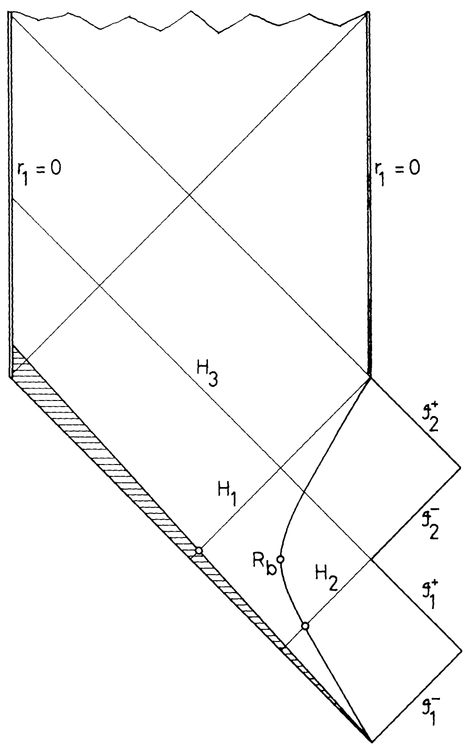

Fig. 2. Penrose diagram of our model. The shaded region is a hypothetical star from whose collapse the original horizon H_1 results

Notice that the Eqs. (3) and (4) are invariant under the transformation

$$R(s) \to R(s), \qquad V_1(s) \to V_1(s) + T,$$
$$V_2(s) \to V_2(s),$$

where T is a constant. This transformation changes the time delay between the collapse of a hypothetical star which forms the initial black hole under the shell and the collapse of the shell (see Fig. 2). Using this freedom we can make the advanced time interval between the intersection of the star surface with H_1 and that of the shell with H_2 arbitrarily large. Then, in a sense, H_2 will come very close to H_1, and the biography of our black hole with horizon H_2 can be sketched as follows: It is born as a light cone of an event at the center of the star, growing towards its surface. After crossing the surface, it coincides with H_1 to a good approximation for as long time as we wish (choosing our T), and imitates a static state of a black hole with mass m_1 and charge e_1. Then, it again begins to grow till it reaches the point of intersection with the shell. From now on, it is an exactly static black hole with mass m_2 and charge $e_2 = m_2$.

It boils down to this: we have produced an extreme black hole by throwing in a shell with an arbitrary total rest mass m, an injection energy E satisfying $E > E_c$, and a total charge $e = m_1 - e_1 + E$. This conclusion is at odds with the third law in the formulation quoted above.

3. Discussion

The critical energy E_c can be written as follows

$$E_c = \sqrt{m^2 + P_c^2}\,, \tag{20}$$

where $P_c = mu_c$ is the *critical momentum*

$$P_c = \sqrt{m_1^2 - e_1^2}\,. \tag{21}$$

Hence, the absolute lower bound for the energy of the shell that can bring the hole to the extreme is

$$E_{\text{a.l.b.}} = P_c = \sqrt{m_1^2 - e_1^2}\,, \tag{22}$$

as seen from (20) when $m \to 0$, $u \to -\infty$. This energy can be made arbitrarily small, if the original hole is taken sufficiently close to the extreme.

In particular, if $m_1 = e_1 = 0$ (flat spacetime inside the shell), an arbitrarily small injection energy E and charge $e = E$ suffice (in our idealized model) to produce an extremal Reissner-Nordstrøm black hole. The assumption of incoherent matter, however, will not be justified for small E, because the corresponding Schwarzschild radius $r_2 = m_2 = E$ will be extremely small and the densities in the shell high. Real matter cannot be considered as incoherent all the way down to it. Nevertheless, for large E this poses no problem.

We should mention that we are not the first to conjure up an extreme black hole from the flat spacetime; this has been essentially done already by Kuchař [2].

Our example shows the relation between the third law and the cosmic censorship in a new light. Sometimes it has been felt that the former follows from the latter [1]. This is not the case here. For consider $E > E_c$ and let ε increase from small negative to small positive values. All quantities change continuously and there is nothing exceptional in our equations at the point $\varepsilon = 0$. Hence, by a small change in the charge e of the shell, we can produce an underextreme, extreme, or overextreme Reissner-Nordstrøm spacetime outside the shell so that, for $\varepsilon \leq 0$, there is a new horizon, H_2, and, for $\varepsilon > 0$, there is none. On the one hand, the minimal radius R_b of the shell is at any rate larger than $m_1 + \sqrt{m_1^2 - e_1^2}$ and the singularity of the outer metric (which is located at $r_2 = 0$) will not appear in our spacetime. On the other hand, the singularity of the inner metric (at $r_1 = 0$) remains always safely hidden under the old Horizon H_1 (see Fig. 2).

Many things militate against our model having direct astrophysical implications. To mention but one: it is difficult to find a highly charged black hole [10] and for neutral black holes the necessary injection energies are enormous. Let us discuss the rôle of some other idealizations we indulged in.

a) Infinitely Thin Shell

This is just a simplifying assumption, not crucial to the feasibility of the construct. To see this, at least qualitatively, consider a model consisting of two shells, both being undercritical with respect to the original black hole and only the second capable of bringing the hole to the extreme. The only conceivable difficulty is that the first shell would remain too far from the hole so that the mass of the second shell would have to be enormous in order to produce a horizon outside the first one. However, Fig. 1 shows that the subcritical shells probe as deep into the spacetime as the overcritical ones do. The upshot of the detailed, but rather tedious calculation (which we spare the reader) is that it confirms the hunch that the two shell model also works.

b) Incoherent Matter

This assumption can be satisfied by real matter to a good approximation, provided the required densities are not too high. The largest energy density which is necessary for our model to work is reached in the shell as it crosses the horizon H_2. Suppose that the shell has a finite thickness ℓ. (The thin shell approximation is good, if $\ell \ll m_1$.) For the density ϱ of the shell energy at the horizon H_2, we have

$$\varrho \sim \frac{E}{l m_2^2} = \frac{E}{l(m_1 + E)^2}.$$

Thus, ϱ can be made arbitrarily small by chosing E and/or m_1 sufficiently large.

c) Neglect of Quantum Effects

As is shown in [10], the charge losses of a Reissner-Nordstrøm black hole of mass m_2 by the electron-positron pair creation are low, if $m_2 > 10^5 M_{\odot}$. We assume that this also holds for shells which collapse to such holes in spite of their motion. This seems fair, because the pairs are created by the field and not by the matter of the shell itself. Thus, we need not worry about pair creation in the event that m_1 and/or E are big enough.

d) Spherical Symmetry

The import of the exact spherical symmetry for our counterexample can be adequately estimated, e.g., by investigating small nonspherical perturbations of the model. Slight rotation of the hole, small deformations of the shell and weak electromagnetic and gravitational radiation come then into play. Stability of our model, at least in its observable regions, remains a problem which deserves further study.

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