The Three-Dimensional Current and Surface Wave Equations

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(Manuscript received 26 April 2002, in final form 4 March 2003)

ABSTRACT

Surface wave equations appropriate to three-dimensional ocean models apparently have not been presented in the literature. It is the intent of this paper to correct that deficiency. Thus, expressions for vertically dependent radiation stresses and a definition of the Doppler velocity for a vertically dependent current field are obtained. Other quantities such as vertically dependent surface pressure forcing are derived for inclusion in the momentum and wave energy equations. The equations include terms that represent the production of turbulence energy by currents and waves. These results are a necessary precursor for three-dimensional ocean models that handle surface waves together with wind- and buoyancy-driven currents. Although the third dimension has been added here, the analysis is based on the assumption that the depth dependence of wave motions is provided by linear theory, an assumption that is the basis of much of the wave literature.

1. Introduction

In the last several decades there has been significant progress in the understanding and numerical modeling of ocean surface waves (e.g., Longue-Higgins 1953; Phillips 1977; WAMDI Group 1988; Komen et al. 1994) and similar progress has been made in ocean circulation modeling (e.g., Bryan and Cox 1968; Blumberg and Mellor 1987; Bleck and Boudra 1986). However, the two streams are separate. More often than not, wave models do not recognize vertical current structure and ocean circulation models do not recognize surface waves as having any influence on the ocean.

In the wave modeling literature, before derivation of the surface wave equations, the equations are generally integrated from the bottom, $z = -h$, to the wave surface, $z = \eta$, and then are phase averaged. Furthermore, the slow (wind-, tide-, and density-driven) horizontal velocities are often stipulated to be independent of z a priori. The result is a mismatch between the wave models and three-dimensional circulation ocean models, which necessarily involve z-dependent horizontal velocities and other properties. For example, a recent paper by Xie et al. (2001) suggests including the conventional vertically integrated stress radiation terms from waves as forcing terms in the vertically dependent momentum equations, a strategy that is obviously incorrect. Dolata and Rosenthal (1984) did attempt to derive three-dimensional radiation stress terms but left out effects from pressure so that their results differ from mine and, after vertical integration, differ from the corresponding terms in Phillips (1977). They did not address three-dimensional effects on the wave energy equation.

To contrast the developments in this paper with conventional logic and to simplify discussion, I will temporarily address deep water ($kh \gg 1$) propagating waves such that $\eta(x, t, \psi) = a(x, t)\cos\psi$, where $\psi = kx - \sigma t$; k is the wavenumber, σ is the frequency, $c \equiv \sigma/k$ is the phase speed, and a is wave amplitude. The associated horizontal and vertical velocities are $\tilde{u} = kace^{kz}\cos\psi$ and $\tilde{w} = kace^{kz}\sin\psi$. Reviewing conventional logic, I define the phase-averaging operator

$$\overline{(\ )} = \frac{1}{2\pi}\int_0^{2\pi} (\ )\, d\psi. \tag{1}$$

Whereas $\overline{\tilde{u}} = 0$, vertical integrals such as

$$M \equiv \overline{\int_{-h}^{\eta} \tilde{u}\, dz} = \overline{\int_{0}^{\eta} \tilde{u}\, dz} \tag{2}$$

are nontrivial; thus, approximately,

$$M = ac\overline{e^{kz}|_0^\eta \cos\psi} = \frac{ka^2c}{2} \tag{3}$$

is the Stokes transport. This is said to be an Eulerian result.

One can alternately obtain the lowest-order Lagrangian velocity according to

$$u_L = \tilde{u} + \overline{\frac{\partial\tilde{u}}{\partial x}\tilde{x} + \frac{\partial\tilde{u}}{\partial z}\tilde{z}} = \overline{\frac{\partial\tilde{u}}{\partial x}\tilde{x}} + \overline{\frac{\partial\tilde{u}}{\partial z}\tilde{z}}, \tag{4}$$

where the particle displacement $(\tilde{x}, \tilde{z}) = \int_0 (\tilde{u}, \tilde{w})\, dt = ae^{kz}(-\sin\psi, \cos\psi)$, so that the Stokes drift is

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$$u_L = k^2a^2ce^{2kz} \tag{5}$$

and, in a similar way, $w_L = 0$. When (5) is integrated from $-h$ (where $kh \gg 1$) to 0, the result is identical to (3).

a. An alternate procedure

However, in this paper, I do not wish to first integrate vertically, because I seek z-dependent equations to complement the z-dependent circulation equations of three-dimensional ocean models. Furthermore, Lagrangian formulations are awkward when they involve equations more complicated than (4). Instead, let $\tilde{u}[x, z + \tilde{z}(t), t] = \tilde{u}(x, z, t) + \tilde{z}\partial\tilde{u}/\partial z$; then multiply $\tilde{u}$ by $1 + \partial\tilde{z}/\partial z$, which is the wave-distorted, normal flow area relative to the undistorted flow area. Thus,

$$\overline{\tilde{u}(x, z + \tilde{z}, t)\left(1 + \frac{\partial\tilde{z}}{\partial z}\right)} = \overline{\left(\tilde{u} + \tilde{z}\frac{\partial\tilde{u}}{\partial z}\right)\left(1 + \frac{\partial\tilde{z}}{\partial z}\right)}$$
$$= \overline{\tilde{u}\frac{\partial\tilde{z}}{\partial z}} + \overline{\tilde{z}\frac{\partial\tilde{u}}{\partial z}}. \tag{6}$$

The first term on the right is due to the velocity–flow area correlation, and the second term is due to particle wave motion–velocity gradient correlation and is the same as the second term in (4). Thus, the Stokes drift velocity is

$$u_S(x, z, t) = \frac{\partial\overline{\tilde{z}\tilde{u}}}{\partial z} = k^2a^2ce^{2kz}, \tag{7}$$

a result identical to (5). The Stokes transport, $M \equiv \int_{-1}^{0} u_s\, dz$, can be obtained directly from

$$M = \int_{-h}^{0} \frac{\partial\overline{\tilde{z}\tilde{u}}}{\partial z}\, dz = \overline{u(0)\tilde{z}(0)} = \frac{ka^2c}{2}. \tag{8}$$

The correspondence between u_L and u_S is not accidental because, using the continuity equation, it can be shown that $\overline{\tilde{x}\partial\tilde{u}/\partial x} = \overline{\tilde{u}\partial\tilde{z}/\partial z}$.

A more formal procedure will be pursued below, but the essence is the same. I will obtain the above results along with other relations that are useful when incorporating wave effects into the three-dimensional, phase-averaged equations of motion; this is done by transforming the basic equations to a sigma-coordinate system. The motivation is not to obtain final sigma-coordinate equations. Rather it is a helpful first step in the process of deriving the three-dimensional wave interaction terms in the equations of motion. If one wishes final equations in Cartesian coordinates, the reverse transformation on the final equation set, derived later in this paper, is straightforward.

Note the fact that, in much of the literature, the mean wave momentum is conceptually thought to be trapped at the surface in consequence of (2) and the third term in (8). However, it is my opinion that it is more useful to conceive of the wave momentum as distributed continuously down into the water column according to (7). Similar reasoning will be found in the paper by Longuet-Higgins (1969).

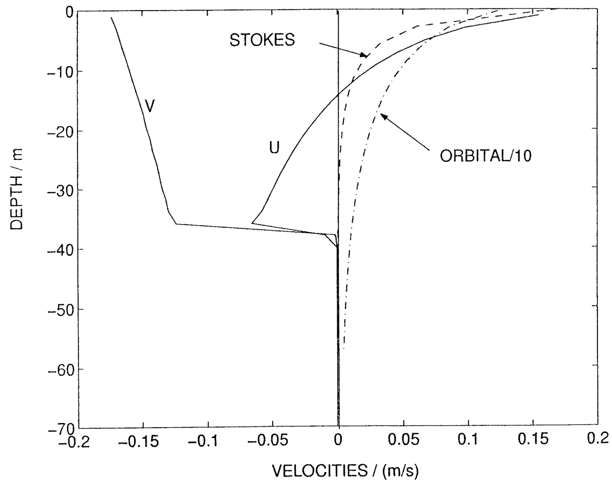

FIG. 1. Comparison of the wind-driven, mixed layer velocity components for a water side friction velocity of $u_{\tau w} = 0.02$ m s^{-1} and the Stokes drift velocity from integrals of the Pierson–Moskowitz spectrum for a wind, $U_{10} = 14.6$ m s^{-1}; also shown are the rms wave orbital velocity $(\overline{\tilde{u}^2} + \overline{\tilde{w}^2})^{1/2}$ divided by 10. The friction velocity and wind are related according to $U_{10}/u_{\tau a} = \kappa^{-1}\ln(10\text{ m}/z_0)$, where Charnock's wave roughness is taken to be $z_0 = 0.012u_{\tau a}^2/g$, $\kappa = 0.40$, and $u_{\tau a}^2 = 860\, u_{\tau w}^2$. The wind stress was initially ramped up from rest to the value $u_{\tau w}^2$ over one inertial period and then was held constant for 5 days. The initial temperature profile was linear with a vertical gradient of 0.05 K^{-1}.

In Fig. 1, I show velocity profiles calculated from a one-dimensional turbulence closure model (Mellor and Yamada 1982) for a specific wind speed of 14.6 m s^{-1} (and kinematic stress of 0.33 m^2 s^{-2}) and the Stokes drift velocity and wave orbital velocity corresponding to the same wind speed using the equilibrium wave spectrum according to Pierson and Moskowitz (1964). In the direction of the wind stress, the rms orbital velocities are an order of magnitude larger than the wind-driven velocity and the Stokes velocity. Following established convention, I refer to the wind-, baroclinic-, and Coriolis-driven velocities as ''currents'' that are distinct from the ''Stokes drift.'' Figure 1 illustrates the point that, in general, the currents are not vertically constant—a seemingly trivial statement were it not for the fact that it is a requirement of present-day wave models. Although the relative magnitudes of the different velocity components are probably correct, the calculation of currents and Stokes drift, independent of each other as in Fig. 1, is not valid, as will be demonstrated in this paper.

b. This paper

In this paper, I specialize to monochromatic waves. Later extensions will undoubtedly generalize to a spectrum of waves for use in models.

The plan of this paper is to present the governing equations for the slowly changing variables, the wave variables, and the turbulence variables in section 2. The transformation to sigma coordinates is done in section 3. The well-known linear wave solutions are given in section 4. All terms appropriate to the continuity and momentum equations are obtained in section 5, and the wave energy equation is derived in section 6. The continuity and momentum equations are vertically dependent, whereas, by virtue of assuming the linear wave solutions to be valid to lowest order, one needs only the vertically integrated wave energy equation. Section 7 establishes the correspondence between the present results and the conventional momentum and energy equations. In section 8, additional terms that result from wind pressure processes are derived. The results are summarized in section 9 together with the equations for scalars and the turbulent kinetic energy. Appendix A considers interactions between the momentum and wave energy equations and appendix B defines the nomenclature of the velocities and scalars used in this paper.

2. The basic equations

The equations of motion are

$$\frac{\partial \mathcal{U}_j}{\partial x_j} = 0 \quad \text{and} \tag{9}$$

$$\frac{\partial \mathcal{U}_i}{\partial t} + \frac{\partial \mathcal{U}_j \mathcal{U}_i}{\partial x_i} + \epsilon_{ijk} f_j \mathcal{U}_k + \frac{\partial \mathcal{P}}{\partial x_i} = -\frac{\rho g}{\rho_o}\delta_{iz}, \tag{10}$$

where $\mathcal{U}_i = (\mathcal{U}, \mathcal{V}, \mathcal{W})$, $x_i = (x, y, z)$, and $\mathcal{U}_i = \mathcal{U}_i(x_i, t)$. Here, $\mathcal{P} = \mathcal{P}(x_i, t)$ is the kinematic pressure—that is, the dynamic pressure divided by a reference density ρ_0—and f_j is the Coriolis parameter. The coordinate z and velocity w are vertically upward. We have omitted viscous terms, but they can be reclaimed where they may be important locally such as next to a smooth bottom surface. The tensor ϵ_{ijk} is nil if any of the indices are repeated, is equal to 1 when i, j, and k are any triplet in the sequence $xyzxy$, and is -1 in any other sequence.

We now decompose $\mathcal{U}_i$ into three velocity components: a ‘‘slow’’ component $\hat{u}_i$, whose time- and space scales are L and T, respectively; a wave component $\tilde{u}_i$, whose smaller time- and space scales are k^{-1} and ω^{-1}, respectively; and a random turbulence component, $\acute{u}_i$, such that

$$\mathcal{U}_i = \hat{u}_i + \tilde{u}_i + \acute{u}_i \quad \text{and} \tag{11a}$$

$$\mathcal{P} = \hat{p}_i + \tilde{p}_i + \acute{p}_i. \tag{11b}$$

The density is split into a slow and fluctuating turbulence component

$$\rho = \hat{\rho} + \acute{\rho}, \tag{11c}$$

therefore filtering out baroclinic interaction with surface waves. Using the above definitions, (10) may be written

$$\begin{aligned}\frac{\partial}{\partial t}(\hat{u}_i + \tilde{u}_i + \acute{u}_i) &+ \frac{\partial}{\partial x_j}(\hat{u}_j\hat{u}_i + \hat{u}_j\tilde{u}_i + \hat{u}_j\acute{u}_i + \hat{u}_i\tilde{u}_j + \hat{u}_i\acute{u}_j + \tilde{u}_j\tilde{u}_i \\ &+ \tilde{u}_j\acute{u}_i + \acute{u}_j\tilde{u}_i + \acute{u}_i\acute{u}_i) \\ &+ \epsilon_{ijk} f_i(\hat{u}_k + \tilde{u}_k + \acute{u}_k) + \frac{\partial}{\partial x_i}(\hat{p} + \tilde{p} + \acute{p}) \\ &= -\left(\frac{\hat{\rho}}{\rho_o} + \frac{\acute{\rho}}{\rho_o}\right) g\delta_{iz}.\end{aligned} \tag{12}$$

It is possible to write equations for the three components, and I have done so. However, as will be seen in appendix A, the slow and phase-averaged wave components interact in a somewhat complicated way. I therefore write the equations for $u_i \equiv \hat{u}_i + \tilde{u}_i$ and $p \equiv \hat{p} + \tilde{p}$ such that

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \text{and} \tag{13}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} + \epsilon_{ijk} f_j u_k + \frac{\partial p}{\partial x_i} = -\frac{\hat{\rho}}{\rho_o} g\delta_{iz} - \frac{\partial}{\partial x_j}\langle \acute{u}_i\acute{u}_j\rangle. \tag{14}$$

The Reynolds stresses are $\langle \acute{u}_i\acute{u}_j\rangle$, and so it is assumed that $\acute{u}_i$ can be extracted from u_i, a process simpler in laboratory experiments than in field experiments for which the waves are cyclic (Jensen et al. 1989; Cheung and Street 1988) and $u_i = \langle \mathcal{U}_i(t)\rangle \equiv N^{-1}\sum_{j=1}^{N} \mathcal{U}_i(t + j\tau_p)$ is a phase-conditioned average, where τ_p is the wave period and N is a large number; other dependent variables are similarly processed. Field observations present a greater challenge in separating wave and turbulence properties.

The equations governing turbulence are

$$\frac{\partial \acute{u}_k}{\partial x_k} = 0 \quad \text{and} \tag{15}$$

$$\frac{\partial}{\partial t}\acute{u}_i + \frac{\partial}{\partial x_j}(u_j\acute{u}_i + u_i\acute{u}_j + \acute{u}_j\acute{u}_i - \langle \acute{u}_j\acute{u}_i\rangle) + \frac{\partial \acute{p}}{\partial x_i} + \frac{\acute{\rho}}{\rho_o} g\delta_{iz} = \nu\nabla^2\acute{u}_i. \tag{16}$$

Notice that $\langle \acute{u}_i\rangle = 0$ and that the averages of all terms in (16) are nil. I have added the viscous term (ν is kinematic viscosity) because, when (16) is converted to an energy equation, the term becomes the important turbulence kinetic energy dissipation.

3. A transformation

There is need for separate nomenclature for the horizontal and vertical coordinates; thus, $x_\alpha \equiv (x, y)$, $x_i \equiv (x_\alpha, z)$, and $u_i \equiv (u_\alpha, w)$. Next transform the dependent variables in (13) and (14) according to

$$\phi(x_\alpha, z, t) = \phi^*(x_\alpha^*, \zeta, t^*) \quad (17)$$

and the independent variables such that

$$x_\alpha = x_\alpha^*, \quad (18a)$$

$$t = t^*, \quad \text{and} \quad (18b)$$

$$z = s(x_\alpha^*, \zeta, t^*), \quad (18c)$$

where ζ and s are general but will be constrained shortly; see (23a,b). From (17), one has

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^*}{\partial x_\alpha^*} + \frac{\partial \phi^*}{\partial \zeta}\frac{\partial \zeta}{\partial x_\alpha}, \qquad \frac{\partial \phi}{\partial z} = \frac{\partial \phi^*}{\partial \zeta}\frac{\partial \zeta}{\partial z},$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^*}{\partial t^*} + \frac{\partial \phi^*}{\partial \zeta}\frac{\partial \zeta}{\partial t},$$

and from $\partial z/\partial x_\alpha = (\partial s/\partial x_\alpha^*)(\partial x_\alpha^*/\partial x_\alpha) + (\partial s/\partial \zeta)(\partial \zeta/\partial x_\alpha) = 0$, I obtain $\partial \zeta/\partial x_\alpha = -s_\alpha/s_\zeta$; in a similar way, $\partial \zeta/\partial z = 1/s_\zeta$ and $\partial \zeta/\partial t = -s_t/s_\zeta$, where I define $s_\alpha \equiv \partial s/\partial x_\alpha^*$, $s_t \equiv \partial s/\partial t^*$, and $s_\zeta \equiv \partial s/\partial \zeta$. Putting these relations together yields

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^*}{\partial x_\alpha^*} - \frac{\partial \phi^*}{\partial \zeta}\frac{s_\alpha}{s_\zeta}, \quad (19a)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi^*}{\partial \zeta}\frac{1}{s_\zeta}, \quad \text{and} \quad (19b)$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^*}{\partial t^*} - \frac{\partial \phi^*}{\partial \zeta}\frac{s_t}{s_\zeta}. \quad (19c)$$

Using (19a,b), I now transform (13) (written as $\partial u_\alpha/\partial x_\alpha + \partial w/\partial z = 0$) and, at the same time, drop the asterisks so that

$$\frac{\partial u_\alpha}{\partial x_\alpha} - \frac{\partial u_\alpha}{\partial \zeta}\frac{s_\alpha}{s_\zeta} + \frac{\partial w}{\partial \zeta}\frac{1}{s_\zeta} = 0.$$

Next define

$$w = \mathring{\omega} + u_\alpha s_\alpha + s_t \quad (20)$$

so that, after some rearrangement,

$$\frac{\partial s_\zeta u_\alpha}{\partial x_\alpha} + \frac{\partial \mathring{\omega}}{\partial \zeta} + \frac{\partial s_\zeta}{\partial t} = 0. \quad (21)$$

Equation (21) could have been derived from a control volume formulation (visualized with help from the top panel of Fig. 2); then $\mathring{\omega}$ is the component of (the nearly vertical) velocity normal to surfaces of constant ζ.

Equation (14), for the horizontal velocity components, can be similarly manipulated and transforms to

$$\frac{\partial}{\partial t}(s_\zeta u_\alpha) + \frac{\partial}{\partial x_\beta}(s_\zeta u_\alpha u_\beta) + \frac{\partial}{\partial \zeta}(\mathring{\omega} u_\alpha) - \epsilon_{\alpha\beta z} f_z s_\zeta u_\beta$$

$$+ \frac{\partial}{\partial x_\alpha}(s_\zeta p) - \frac{\partial}{\partial \zeta}(s_\alpha p)$$

$$= -\frac{\partial}{\partial x_\beta}(s_\zeta \langle \acute{u}_\alpha \acute{u}_\beta \rangle) + \frac{\partial}{\partial \zeta}(s_\beta \langle \acute{u}_\alpha \acute{u}_\beta \rangle) - \frac{\partial}{\partial \zeta}\langle \acute{w} \acute{u}_\alpha \rangle. \quad (22)$$

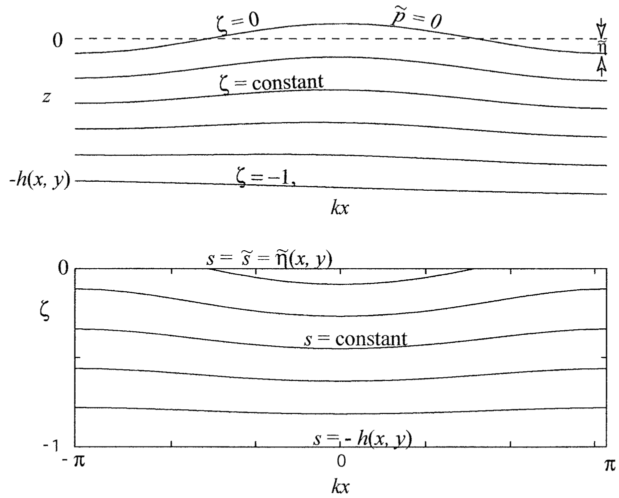

FIG. 2. An instantaneous sketch of wave particle position in (top) x, z space and (bottom) as transformed to x, ζ space according to (23) for $\hat{\eta} = 0$.

The transformation thus far is general. Now, however, (anticipating the linear wave solutions in the next section) define

$$s(x, y, \zeta, t) \equiv \hat{\eta} + \zeta D + \tilde{s} \quad \text{and} \quad (23a)$$

$$\tilde{s} \equiv a\frac{\sinh kD(1 + \zeta)}{\sinh kD}\cos\psi. \quad (23b)$$

The mean elevation is $\hat{\eta}(x, y)$; $h(x, y)$ is the bottom depth, and $D \equiv \hat{\eta} + h$ is the mean water column depth; $\hat{\eta}$, a, k, and D are slow functions of x, y, and t that vary on length scales and timescales of L and T, respectively, and $\psi \equiv k_\alpha x_\alpha - \omega t$, where k_α and ω are large in comparison with L^{-1} and T^{-1}. The "sigma" variable ζ ranges from -1 where $s = -h$ to 0 where $s = \eta = \hat{\eta} + \tilde{\eta}$. Figure 2 illustrates the transformation process for cases in which $\hat{\eta} = 0$ and there is zero atmospheric pressure.

The vertical derivatives of (23a,b) are

$$s_\zeta = \frac{\partial s}{\partial \zeta} = D + \tilde{s}_\zeta \quad \text{and} \quad (24a)$$

$$\tilde{s}_\zeta = kDa\frac{\cosh kD(1 + \zeta)}{\sinh kD}\cos\psi, \quad (24b)$$

and, in consequence of (20) at $\zeta = 0$ or $s = \eta$, $\mathring{\omega}(0) = 0$ so that $w(0) = u_\alpha(0)\partial\eta/\partial x_\alpha + \partial\eta/\partial t$ and, at $\zeta = 0$ or $s = -h$, $\mathring{\omega}(-1) = 0$ so that $w(-h) = -u_\alpha(-h)\partial h/\partial x_\alpha$.

4. The linear wave relations

The well-known linear relations are obtained from solutions of $\partial \tilde{u}_j/\partial x_j = 0$ and $\partial \tilde{u}_i/\partial t + \hat{u}_\alpha \partial \tilde{u}_i/\partial x_\alpha + \partial/\partial x_i(p + gz) = 0$ wherein only the fast terms of order kL or ωT are retained in (13) and (14). The turbulence terms are assumed to be small. Solutions whose z-

dependent arguments are $k(h + z)$ transform to $kD(1 + \zeta + \tilde{s}/D)$ in accordance with (17), (18c) and (23a). The mean sea surface is at $z = \hat{\eta}$. Thus, the wave velocities are

$$\tilde{u}_\alpha(x_\alpha, z, t, \psi) = k_\alpha ac \frac{\cosh kD(1 + \zeta + \tilde{s}/D)}{\sinh kD} \cos\psi \quad (25a)$$

and

$$\tilde{w}(x_\alpha, z, t, \psi) = kac \frac{\sinh kD(1 + \zeta + \tilde{s}/D)}{\sinh kD} \sin\psi. \quad (25b)$$

Notice that $\tilde{w} = 0$ where $\zeta = -1$.

The phase is defined as

$$\psi \equiv k_\alpha x_\alpha - \omega t. \quad (26)$$

The dispersion relation and intrinsic frequency are

$$\omega = \sigma + k_\alpha \hat{u}_{A\alpha} \quad \text{and} \quad (27a)$$

$$\sigma^2 = gk \tanh kD. \quad (27b)$$

The ''Doppler velocity'' $\hat{u}_{A\alpha}$ is, according to the analysis of Kirby and Chen (1989; and see other references in their paper), a weighted average of the vertically nonuniform current:

$$\hat{u}_{A\alpha} = 2 \int_{-1}^{0} \hat{u}_\alpha \frac{kD \cosh 2kD(1 + \zeta)}{\sinh 2kD}\, d\zeta. \quad (27c)$$

The intrinsic phase speed and group speed are

$$c \equiv \frac{\sigma}{k} = \sqrt{\frac{g}{k} \tanh kD} \quad \text{and} \quad (28a)$$

$$c_g \equiv \frac{d\sigma}{dk} = \frac{c}{2}\left(1 + \frac{2kD}{\sinh 2kD}\right). \quad (28b)$$

The surface elevation and pressure from waves are

$$\tilde{\eta}(x_\alpha, t, \psi) = a(x_\alpha, t) \cos\psi \quad \text{and} \quad (29a)$$

$$\tilde{p} + g\tilde{s} = kac^2 \frac{\cosh kD(1 + \zeta)}{\sinh kD} \cos\psi$$
$$= ga \frac{\cosh kD(1 + \zeta)}{\cosh kD} \cos\psi. \quad (29b)$$

The last term on the right is obtained using (27b).

Note that, whereas the wave velocities in (25) have been ''corrected'' to include the small term $\tilde{s}/D$, the wave pressure in (29b) does not include this term because its omission cancels an equal and opposite small error in the linear solution. Notice that (29b) satisfies the condition that $\tilde{p} = 0$ at $\zeta = 0$. I will use the condition $\tilde{p}(0) = \hat{p}(0) = 0$ to simplify the following analyses as did Phillips (1977) but will, in section 8, separately consider the consequences of nonzero atmospheric pressure.

Although equations for k_α will not be needed in this paper, they are included here for completeness. Thus, from (26), one has $k_\alpha \equiv \partial\psi/\partial x_\alpha$ and $\omega \equiv -\partial\psi/\partial t$, from which we obtain $\partial k_\alpha/\partial t + \partial\omega/\partial x_\alpha = 0$ and $\partial k_\alpha/\partial x_\beta - \partial k_\beta/\partial x_\alpha = 0$ so that the wave number vector is irrotational. These equations and $\partial\sigma/\partial x_\alpha = (\partial\sigma/\partial k)(\partial k/\partial x_\alpha) + (\partial\sigma/\partial x_\alpha)_k$ yield the well-known relation

$$\frac{\partial k_\alpha}{\partial t} + (c_{g\beta} + \hat{u}_{A\beta})\frac{\partial k_\alpha}{\partial x_\beta} = -\left(\frac{\partial\sigma}{\partial x_\alpha}\right)_k - k_\beta \frac{\partial \hat{u}_{A\beta}}{\partial x_\alpha},$$

where $c_{g\beta} = (\partial\sigma/\partial k)(k_\beta/k)$. If the right side were nil (depth and Doppler velocity are constant), then the wave number vector is invariant along a trajectory prescribed by the combined group velocity and Doppler velocity.

5. The transformed continuity and momentum equations

The next step is to phase average the terms in (21) and (22) and to define a mean velocity U_α such that

$$DU_\alpha \equiv \overline{s_\zeta u_\alpha} = \overline{(D + \tilde{s}_\zeta)(\hat{u}_\alpha + \tilde{u}_\alpha)}$$
$$= D\hat{u}_\alpha + Du_{S\alpha}, \quad (30a)$$

where $u_{s\alpha} \equiv \overline{(D + \tilde{s}_\zeta)\tilde{u}_\alpha}/D$ is the Stokes drift velocity. Referring to (25a) and $\tilde{u}_\alpha = \tilde{u}_\alpha(x, y, 1 + \zeta + \tilde{s}/D) = \tilde{u}_\alpha(x, y, 1 + \zeta) + \tilde{s}(\partial\tilde{u}_\alpha/\partial\zeta)/D$, we obtain (see section 1)

$$u_{S\alpha} = \overline{(1 + \tilde{s}_\zeta/D)\tilde{u}_\alpha} = \frac{1}{D}\frac{\partial\overline{\tilde{s}\tilde{u}_\alpha}}{\partial\zeta}$$
$$= \frac{k_\alpha}{k}\frac{(ka)^2}{2} c \frac{\cosh 2kD(1 + \zeta)}{\sinh^2 kD}$$
$$= \frac{2k_\alpha E}{c} \frac{\cosh 2kD(1 + \zeta)}{\sinh 2kD}. \quad (30b)$$

Note that $u_{s\alpha}/c$ is of order $(ka)^2$. Here and throughout the paper, I use the fact that phase averages of odd powers of $\cos\psi$ and products of $\cos\psi$ and $\sin\psi$ are nil. The expression on the right of (30b) uses (27b) and the definition, $E \equiv ga^2/2$; E will later be shown to be the sum of the kinetic and potential wave energies.

The product, $s_\zeta u_\alpha u_\beta$ in (22), when averaged, is algebraically complicated. Thus,

$$\overline{s_\zeta u_\alpha u_\beta} = \overline{s_\zeta(\hat{u}_\alpha + \tilde{u}_\alpha)(\hat{u}_\beta + \tilde{u}_\beta)}$$
$$= D\hat{u}_\alpha\hat{u}_\beta + \hat{u}_\alpha\overline{(D + \tilde{s}_\zeta)\tilde{u}_\beta}$$
$$+ \hat{u}_\beta\overline{(D + \tilde{s}_\zeta)\tilde{u}_\alpha} + D\overline{\tilde{u}_\alpha\tilde{u}_\beta}$$
$$= DU_\alpha U_\beta + D\overline{\tilde{u}_\alpha\tilde{u}_\beta}, \quad (31)$$

where the terms $-D^{-1}\overline{(D + \tilde{s}_\zeta)\tilde{u}_\alpha}\ \overline{(D + \tilde{s}_\zeta)\tilde{u}_\beta} = Du_{s\alpha}u_{s\beta}$ are of order $(ka)^4$ and are neglected.

Next define $\mathring{\omega} = \Omega(1 + \tilde{s}_\zeta/D)$ so that

$$\overline{\mathring{\omega}} = \Omega \quad \text{and} \quad (32a)$$

$$\overline{\mathring{\omega} u_\alpha} = \Omega U_\alpha, \quad (32b)$$

where $\Omega(-1) = \Omega(0) = 0$. Using (30), (31), and (32), (21) and (22), after phase averaging, may be written

$$\frac{\partial DU_\alpha}{\partial x_\alpha} + \frac{\partial \Omega}{\partial \zeta} + \frac{\partial \hat{\eta}}{\partial t} = 0 \quad \text{and} \tag{33}$$

$$\frac{\partial}{\partial t}(DU_\alpha) + \frac{\partial}{\partial x_\beta}(DU_\alpha U_\beta) + \frac{\partial}{\partial \zeta}(\Omega U_\alpha) + \epsilon_{\alpha\beta z} f_z DU_\beta + gD\frac{\partial \hat{\eta}}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}(D\hat{p}) - \frac{\partial}{\partial \zeta}\left(\zeta\frac{\partial D}{\partial x_\alpha}\hat{p}\right) = -\frac{\partial S_{\alpha\beta}}{\partial x_\beta} + \frac{\partial \overline{\tilde{s}_\alpha \tilde{p}}}{\partial \zeta} - \frac{\partial}{\partial \zeta}\overline{\langle \acute{w}\acute{u}_\alpha\rangle}, \tag{34a}$$

wherein we invoke the boundary layer approximation and neglect horizontal gradients of the Reynolds stresses. From the hydrostatic relation $\partial\hat{p}/\partial\zeta = -Dg\hat{\rho}/\rho_o$, we obtain

$$\hat{p} = -gD\zeta - \int_\zeta^0 \frac{\hat{\rho} - \rho_o}{\rho_o}\, d\zeta. \tag{34b}$$

The first term on the right of (34a) is

$$S_{\alpha\beta} \equiv D\overline{\tilde{u}_\alpha \tilde{u}_\beta} + \delta_{\alpha\beta}\overline{\tilde{s}_\zeta \tilde{p}}, \tag{34c}$$

where

$$D\overline{\tilde{u}_\alpha \tilde{u}_\beta} = D\frac{k_\alpha k_\beta}{k^2}\frac{k^2 a^2 c^2}{2}\frac{\cosh^2 kD(1+\zeta)}{\sinh^2 kD} = DkE\frac{k_\alpha k_\beta}{k^2}\frac{\cosh^2 kD(1+\zeta)}{\sinh kD \cosh kD} \tag{34d}$$

using (25a) and (27b). From (24b) and (29b), we obtain

$$\overline{\tilde{s}_\zeta \tilde{p}} = kDE\left[\frac{\cosh^2 kD(1+\zeta)}{\sinh kD \cosh kD} - \frac{\sinh kD(1+\zeta)\cosh kD(1+\zeta)}{\sinh^2 kD}\right]. \tag{34e}$$

In a similar way from (23b) and (29b),

$$\overline{\tilde{s}_\alpha \tilde{p}} = E^{1/2}\left[\frac{\cosh kD(1+\zeta)}{\cosh kD} - \frac{\sinh kD(1+\zeta)}{\sinh kD}\right] \times \frac{\partial}{\partial x_\alpha}\left[E^{1/2}\frac{\sinh kD(1+\zeta)}{\sinh kD}\right]. \tag{34f}$$

Equations (34d) and (34e) after insertion into (34c) are the so-called stress radiation terms derived in their vertically integrated form by Longuet-Higgins and Stewart (1961). The term $\partial\overline{\tilde{s}_\alpha\tilde{p}}/\partial\zeta$, using (34f), is an additional radiation term that vertically integrates to zero.

6. The wave energy equation

To close the momentum equation derived in section 5, one needs to know $E(x_\alpha, t)$ which is the vertically integrated or two-dimensional wave energy. Obtaining an equation for $E(x_\alpha, t)$ involves complicated algebra, and I have followed Phillips (1977) but also utilized the method developed above. Thus, multiply (14) by u_i to form a kinetic energy equation:

$$\frac{\partial}{\partial t}\left(\frac{u_i^2}{2} + gz\right) + \frac{\partial}{\partial x_\beta}u_\beta\left(\frac{u_i^2}{2} + p + gz\right) + \frac{\partial}{\partial z}w\left(\frac{u_i^2}{2} + p + gz\right) = -u_i\frac{\partial}{\partial x_j}\langle \acute{u}_j\acute{u}_i\rangle. \tag{35}$$

The buoyancy term is excluded from (14) because I assume that buoyancy does not affect the near surface wave motion. Now transform (35) according to (17), (19), and (20) and phase average so that

$$\frac{\partial}{\partial t}\overline{s_\zeta\left(\frac{u_i^2}{2} + gs\right)} + \frac{\partial}{\partial x_\beta}\overline{s_\zeta u_\beta\left(\frac{u_i^2}{2} + p + gs\right)} + \frac{\partial}{\partial \zeta}\overline{\mathring{\omega}\left(\frac{u_i^2}{2} + p + gs\right)} + \frac{\partial}{\partial \zeta}\overline{s_t p} = -\overline{u_i\frac{\partial}{\partial \zeta}\langle \acute{w}\acute{u}_i\rangle}. \tag{36a}$$

Because it is known that the Reynolds stress tensor components are of the same order, we have used the boundary layer approximation in the last term in (36a). Next, expand

$$\overline{s_\zeta u_i^2/2} = \overline{s_\zeta(\hat{u}_\alpha^2/2 + \tilde{u}_\alpha\hat{u}_\alpha + \tilde{u}_i^2/2)} = D(\hat{u}_\alpha^2/2 + u_{s\alpha}\hat{u}_\alpha + \overline{\tilde{u}_i^2}/2)$$

and use $\hat{u}_\alpha = U_\alpha - \tilde{u}_{s\alpha}$ to obtain

$$\overline{s_\zeta u_i^2/2} = D(U_\alpha^2/2 - u_{s\alpha}^2/2 + \overline{\tilde{u}_i^2}/2) = D(U_\alpha^2/2 + \overline{\tilde{u}_i^2}/2). \tag{36b}$$

The deleted term is of order $(ka)^4$. In a similar way,

$$\overline{s_\zeta u_\beta u_i^2/2} = DU_\beta U_\alpha^2/2 + DU_\beta\overline{\tilde{u}_i^2}/2 + DU_\alpha\overline{\tilde{u}_\beta\tilde{u}_\alpha} \tag{36c}$$

and

$$\overline{\mathring{\omega} u_i^2/2} = \Omega(U_i^2/2 + \overline{\tilde{u}_i^2}/2). \tag{36d}$$

Neglecting buoyancy terms, one has $p = \hat{p} + \tilde{p} = -\zeta gD + \tilde{p}$ and, of course, $s = \hat{\eta} + \zeta D + \tilde{s}$ so that $p + gs = \tilde{p} + g\tilde{s} + g\hat{\eta}$. Completing the operations involving these terms in (36a) yields

$$g\overline{s_\zeta s} = gD(\hat{\eta} + \zeta D) + g\overline{\tilde{s}_\zeta\tilde{s}}, \tag{36e}$$

$$\overline{s_\zeta u_\beta(p + gs)} = DU_\beta g\hat{\eta} + U_\beta\overline{\tilde{s}_\zeta(\tilde{p} + g\tilde{s})} + D\overline{\tilde{u}_\beta(\tilde{p} + g\tilde{s})}, \quad \text{and} \tag{36f}$$

$$\overline{\mathring{\omega}(p + gs)} = \Omega[g\hat{\eta} + \overline{\tilde{s}_\zeta(\tilde{p} + g\tilde{s})}/D]. \tag{36g}$$

Inserting (36b–g) into (36a) yields

$$\frac{\partial}{\partial t}\left[\frac{DU_\alpha^2}{2} + \frac{\overline{\tilde{u}_i^2}}{2} + gD(\hat{\eta} + \zeta D) + g\overline{\tilde{s}_\zeta\tilde{s}}\right]$$

$$+ \frac{\partial}{\partial x_\beta}\left[\frac{DU_\beta U_\alpha^2}{2} + DU_\alpha\overline{\tilde{u}_\beta\tilde{u}_\alpha} + \frac{U_\beta D\overline{\tilde{u}_i^2}}{2} + \hat{u}_\beta\overline{\tilde{s}_\zeta(\tilde{p} + g\tilde{s})} + U_\alpha g\hat{\eta} + D\overline{\tilde{u}_\beta(\tilde{p} + g\tilde{s})}\right] + \frac{\partial}{\partial\zeta}\left\{\Omega\left[\frac{U_\alpha^2}{2} + \frac{\overline{\tilde{u}_i^2}}{2} + g\hat{\eta} + \frac{\overline{\tilde{s}_\zeta(\tilde{p} + g\tilde{s})}}{D}\right]\right\} + \frac{\partial}{\partial\zeta}\overline{s_t p} = -\overline{u_i\frac{\partial}{\partial\zeta}\langle\acute{w}\acute{u}_i\rangle}. \quad (37)$$

Now from (37), subtract U_α times (34a) and add $U_\alpha^2/2 - g\hat{\eta}$ times (33). After complicated but straightforward algebra, one obtains

$$\frac{\partial}{\partial t}\left[\frac{\overline{\tilde{u}_i^2}}{2} + gD(\hat{\eta} + \zeta D) + g\overline{\tilde{s}_\zeta\tilde{s}} - \frac{g\hat{\eta}^2}{2}\right] + \frac{\partial}{\partial x_\beta}\left[\frac{DU_\beta\overline{\tilde{u}_i^2}}{2} + U_\beta\overline{\tilde{s}_\zeta\tilde{s}} + D\overline{\tilde{u}_\beta(\tilde{p} + g\tilde{s})}\right] + S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial}{\partial\zeta}\left\{\Omega\left[\frac{\overline{\tilde{u}_i^2}}{2} + \frac{\overline{\tilde{s}_\zeta(\tilde{p} + g\tilde{s})}}{D}\right]\right\} + \frac{\partial}{\partial\zeta}\overline{s_t p} = -\hat{u}_\alpha\frac{\partial}{\partial\zeta}\overline{\tilde{s}_\alpha\tilde{p}} - \overline{u_i\frac{\partial}{\partial\zeta}\langle\acute{w}\acute{u}_i\rangle} + \hat{u}_\alpha\frac{\partial}{\partial\zeta}\overline{\langle\acute{w}\acute{u}_\alpha\rangle}.$$

A term, $-u_{s\alpha}\partial\overline{\tilde{s}_\alpha\tilde{p}}/\partial\zeta$, on the right is of order $(ka)^4$ and has been discarded.

Next integrate from $\zeta = -1$ to 0 and obtain

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\beta}\left[c_{g\beta}E + \int_{-1}^{0}\hat{u}_\beta\left(\frac{D\overline{\tilde{u}_i^2}}{2} + g\overline{\tilde{s}_\zeta\tilde{s}}\right)d\zeta\right] = -\int_{-1}^{0}S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}d\zeta - \int_{-1}^{0}\hat{u}_\alpha\frac{\partial}{\partial\zeta}\overline{\tilde{s}_\alpha\tilde{p}}\,d\zeta - |\overline{s_t p}|_{-1}^{0} + \int_{-1}^{0}u_{S\alpha}\frac{\partial}{\partial\zeta}\overline{\langle\acute{w}\acute{u}_\alpha\rangle}\,d\zeta - \int_{-1}^{0}\overline{\tilde{u}_i\frac{\partial}{\partial\zeta}\langle\acute{w}\acute{u}_i\rangle}\,d\zeta, \quad (38)$$

where the wave energy is defined according to

$$E \equiv \int_{-1}^{0}[D(\overline{\tilde{u}_i^2}/2) + g\overline{\tilde{s}_\zeta\tilde{s}}]\,d\zeta = \int_{-1}^{0}D(\overline{\tilde{u}_i^2}/2)\,d\zeta + g\overline{\tilde{\eta}^2}/2. \quad (39)$$

Using (25) and (29), both terms on the right of (39) may be shown to be equal to $ga^2/4$ and their sum, $E = ga^2/2$. The term $|\overline{s_t p}|_{-1}^{0} = 0$, but it is left as a place holder, useful in section 8. To obtain (38), I have also used

$$\frac{\partial}{\partial x_\beta}\int_{-}^{0}D\overline{\tilde{u}_\beta(\tilde{p} + g\tilde{s})}\,d\zeta = \frac{\partial}{\partial x_\beta}\left(c_{g\beta}\frac{ga^2}{2}\right) = \frac{\partial}{\partial x_\beta}(c_{g\beta}E),$$

where $c_{g\beta} = k_\beta c_g/k$ is the group velocity and

$$\frac{\partial}{\partial t}\left[gD\int_{-1}^{0}(\hat{\eta} + \zeta D)\,d\zeta - \frac{g}{2}\frac{\partial\hat{\eta}^2}{\partial t}\right] = -\frac{g}{2}\frac{\partial h^2}{\partial t} = 0.$$

7. Correspondence with conventional equations

To compare with formulas in Phillips (1977) and others, neglect baroclinicity, so that $\hat{p} = -gD\zeta$, and turbulence terms; then integrate (33) and (34a) from $\zeta = -1$ to $\zeta = 0$ and obtain

$$\frac{\partial\hat{\eta}}{\partial t} + \frac{\partial M_\beta}{\partial x_\beta} = 0 \quad \text{and} \quad (40)$$

$$\frac{\partial}{\partial t}(M_\alpha) + \frac{\partial}{\partial x_\beta}\left(M_\alpha\frac{M_\beta}{D} + \overline{S}_{\alpha\beta}\right) + \epsilon_{\alpha\beta z}f_z M_\beta + gD\frac{\partial\hat{\eta}}{\partial x_\alpha} = 0, \quad (41)$$

where the transport is

$$M \equiv D\int_{-1}^{0}U_\alpha\,d\zeta \quad \text{and} \quad (42)$$

$$\overline{S}_{\alpha\beta} \equiv E\frac{c_g}{c}\frac{k_\alpha k_\beta}{k^2} + \delta_{\alpha\beta}E\left(\frac{c_g}{c} - \frac{1}{2}\right). \quad (43)$$

The conventional simplification of (38) requires that $\hat{u}_\alpha$ be independent of ζ, in which case (38) reduces to

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x_\alpha}[(c_{g\alpha} + \hat{u}_\alpha)E] = -\overline{S}_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}. \quad (44)$$

Phillips had $\hat{u}_\alpha$ in place of U_α, but the difference is a term of order $(ak)^4$.

Except for the fact that I include the Coriolis terms in (41), I recover the depth-independent results obtained by Phillips (1977) and others.

8. Wind pressure forcing

Heretofore, to simplify and to check some of the derivations, I have assumed that the atmospheric pressure is nil; that is, $\hat{p}(0) = \tilde{p}(0) = 0$. Waves can be driven by wind pressure fluctuations acting on the sloping sea surface, however (Miles 1954; Phillips 1954; Donelan 1999; etc.).

I will continue to represent the free wave pressure field by $\tilde{p}(x, \zeta, t)$ to which I now add the pressure field driven by atmospheric pressure forcing. This pressure field may be written

$$p_w = \hat{p}_{\text{atm}} + \tilde{p}_w; \qquad \tilde{p}_w = a_w\frac{\cosh kD(1 + \zeta)}{\cosh kD}\sin\psi,$$

where $\hat{p}_{\text{atm}}$ is the slowly varying atmospheric imposed pressure and $\tilde{p}_w$ is the solution for imposed surface fluctuation for the Fourier constituent in phase with $\partial\tilde{\eta}/\partial x_\alpha$. Therefore,

$$\overline{\tilde{s}_\alpha \tilde{p}_w} = \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}}\,\frac{\cosh kD(1+\zeta)\,\sinh kD(1+\zeta)}{\cosh kD\,\sinh kD},$$

where $\tilde{p}_{w\eta} = a_w \sin\psi$ is the wind pressure at the surface. Differentiation yields

$$\frac{\partial \overline{\tilde{s}_\alpha \tilde{p}_w}}{\partial \zeta} = \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}}\,\frac{2kD\,\cosh 2kD(1+\zeta)}{\sinh 2kD}, \qquad (45)$$

which in the next section is a term that will be added to $\partial\overline{\tilde{s}_\alpha \tilde{p}}/\partial\zeta$ in (34a).

In a similar way,

$$\int_{-1}^{0} \hat{u}_\alpha \frac{\partial \overline{\tilde{s}_\alpha \tilde{p}_w}}{\partial \zeta}\, d\zeta + |\overline{s_t p_w}|_{-1}^{0} = \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x}} \int_{-1}^{0} \hat{u}_\alpha \frac{2kD\,\cosh 2kD(1+\zeta)}{\sinh 2kD}\, d\zeta + \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}} = \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x}}\hat{u}_{A\alpha} + \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial t}}. \qquad (46)$$

Last, because $\overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial t} = (k_\alpha \omega/k^2)\overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha}$ and using (27a)

$$\int_{-1}^{0} \hat{u}_\alpha \frac{\partial \overline{\tilde{s}_\alpha \tilde{p}_w}}{\partial \zeta}\, d\zeta + |\overline{s_t p_w}|_{-1}^{0} = -\frac{k_\alpha}{k} c\,\overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}}, \qquad (47)$$

which is a term that will be added to the corresponding free wave pressure terms in (38).

In potential flow over a solid wavy wall, Donelan (1999) reminds us that the pressure and wall slope are in antiphase so that, at the wave surface, the existence of a correlation between the two quantities depends on the existence of a nonzero relative phase shift. He advances specific laboratory-derived relations for $\overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x_\alpha}$ as a function of the "wave age" defined as the ratio of wind speed to dominant phase speed, which, when considering only monochromatic waves, is equal to $c = \sigma/k$.

9. Summary and final comments

Here I summarize the final equations for which it is useful to define

$$F_{SS} = \frac{\sinh kD(1+\zeta)}{\sinh kD}, \qquad (48a)$$

$$F_{CS} = \frac{\cosh kD(1+\zeta)}{\sinh kD}, \qquad (48b)$$

$$F_{SC} = \frac{\sinh kD(1+\zeta)}{\cosh kD}, \quad \text{and} \qquad (48c)$$

$$F_{CC} = \frac{\cosh kD(1+\zeta)}{\cosh kD}, \qquad (48d)$$

after which (30b) may be written

$$u_{S\alpha} = \frac{k_\alpha}{k}\frac{E}{cD}\frac{\partial F_{SS}F_{CC}}{\partial \zeta}. \qquad (49)$$

The functions in (48) are plotted in Fig. 3. (The product kDF_{CS} is plotted because it is finite in the shallow water limit, $kD \to 0$.) The factor $\partial F_{SS}F_{CC}/\partial\zeta$ is plotted in Fig. 4.

In summary, I first repeat (33):

$$\frac{\partial DU_\alpha}{\partial x_\alpha} + \frac{\partial \Omega}{\partial \zeta} + \frac{\partial \hat{\eta}}{\partial t} = 0. \qquad (50)$$

Equation (34a), after addition of (45) may be written

$$\frac{\partial}{\partial t}(DU_\alpha) + \frac{\partial}{\partial x_\beta}(DU_\alpha U_\beta) + \frac{\partial}{\partial \zeta}(\Omega U_\alpha) + \epsilon_{\alpha\beta z} f_z DU_\beta + D\frac{\partial}{\partial x_\alpha}(g\hat{\eta} + \hat{p}_{\rm atm}) + D^2 \int_\zeta^0 \left(\frac{\partial b}{\partial x_\alpha} - \zeta\frac{\partial D}{\partial x_\alpha}\frac{\partial b}{\partial \zeta}\right) d\zeta$$
$$= -\frac{\partial S_{\alpha\beta}}{\partial x_\beta} + \frac{\partial \overline{\tilde{s}_\alpha \tilde{p}}}{\partial \zeta} + \overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}}\,\frac{\partial F_{SS}F_{CC}}{\partial \zeta} - \frac{\partial}{\partial \zeta}\overline{\langle \hat{w}\hat{u}_\alpha\rangle}, \qquad (51a)$$

and, from (34c,d,e),

$$S_{\alpha\beta} \equiv kDE\left[\frac{k_\alpha k_\beta}{k^2}F_{CS}F_{CC} + \delta_{\alpha\beta}(F_{CS}F_{CC} - F_{SS}F_{CS})\right]. \qquad (51b)$$

From (34f),

$$\overline{\tilde{s}_\alpha \tilde{p}} = (F_{CC} - F_{SS})E^{1/2}\frac{\partial}{\partial x_\alpha}(E^{1/2}F_{SS}). \qquad (51c)$$

In (51a) I have used (34b) and defined the buoyancy

$$b \equiv -g\frac{\hat{\rho} - \rho_o}{\rho_o}.$$

An important result is that the third term on the right of (51a), the wind pressure forcing term, has the same vertical structure as the Stokes drift velocity in (49). Also, the third and fourth terms on the right of (51a) have regularly been lumped together in mixed layer modeling, which would seem to be a questionable strategy in view of the above results.

Upon insertion of (47), the surface wind pressure terms, the wave energy equation (38) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}[(c_{g\alpha} + \hat{u}_{A2\alpha})E] = -\int_{-1}^{0} S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}\, d\zeta - \int_{-1}^{0} \overline{\tilde{s}_\alpha \tilde{p}}\frac{\partial U_\alpha}{\partial \zeta}\, d\zeta + \frac{k_\alpha}{k}c\,\overline{\tilde{p}_{w\eta}\frac{\partial \tilde{\eta}}{\partial x_\alpha}} - \overline{\tilde{u}_{i\eta}\langle \hat{w}\hat{u}_i\rangle_\eta} + (\tilde{u}_{S\alpha}\overline{\langle \hat{w}\hat{u}_\alpha\rangle})_\eta + \int_{-1}^{0} \overline{\langle \hat{w}\hat{u}_i\rangle\frac{\partial \tilde{u}_i}{\partial \zeta}}\, d\zeta - \int_{-1}^{0} \overline{\langle \hat{w}\hat{u}_\alpha\rangle}\frac{\partial u_{S\alpha}}{\partial \zeta}\, d\zeta. \qquad (52a)$$

In (52a) we have defined

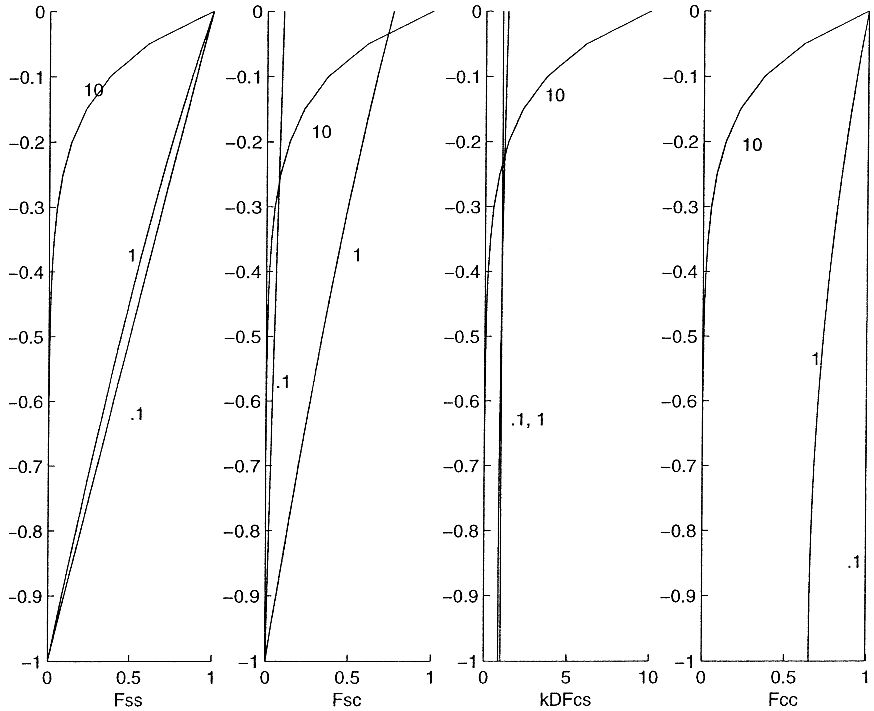

FIG. 3. Plots of the functions (48a–d). The labels on each curve are kD values. As $kD \to 0$, $F_{SS} \sim 1 + \zeta$, $F_{SC} \sim 0$, $kDF_{CS} \sim 1$, and $F_{CC} \sim 1$, and, as $kD \to \infty$, F_{SS}, F_{SC}, F_{CS}, and $F_{CC} \sim \exp(kD\zeta)$.

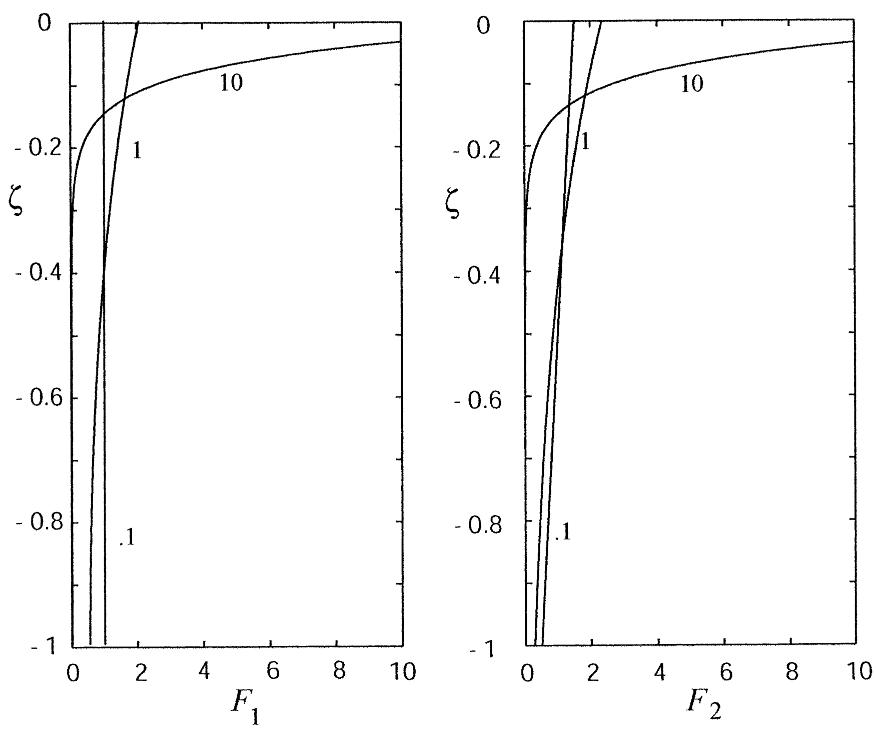

FIG. 4. Plots of the functions $F_1 \equiv \partial F_{SS}F_{CC}/\partial\zeta = kD(F_{CS}F_{CC} + F_{SS}F_{SC})$ in (49) and (53) and $F_2 \equiv kD[(F_{CS}F_{CC} + F_{SS}F_{SC})/2 + F_{CS}F_{SS}]$ in (52b). Note that $\tilde{u}_{s\alpha} \propto F_1$. The labels on each curve are kD values. As $kD \to 0$, $F_1 \sim 1$ and $F_2 \sim 2(1 + \zeta)$. As $kD \to \infty$, F_1 and $F_2 \sim kD\exp(2kD\zeta)$.

$$\hat{u}_{A2\alpha} \equiv \int_{-1}^{0} \hat{u}_\alpha E^{-1}\overline{(D\tilde{u}_i^2/2 + g\tilde{s}\tilde{s}_\zeta)}\, d\zeta$$
$$= kD \int_{-1}^{0} \hat{u}_\alpha[(F_{CS}F_{CC} + F_{SS}F_{SC})/2 + F_{CS}F_{SS}]\, d\zeta. \quad (52b)$$

The phase and group speeds are as in (28a,b) and the Doppler velocity may be written

$$\hat{u}_{A\alpha} = kD \int_{-1}^{0} \hat{u}_\alpha(F_{CS}F_{CC} + F_{SS}F_{SC})\, d\zeta. \quad (53)$$

The weighting factors in (52b) and (53) are similar. They both integrate to unity cum sole and are identical in the short-wave limit but differ in the long-wave limit as shown in Fig. 4. Nevertheless, in the interest of simplicity, the approximation $\hat{u}_{A2\alpha} \simeq \hat{u}_{A\alpha}$ is suggested. Furthermore, $\hat{u}_{A2\alpha} \simeq \hat{u}_{A\alpha} \simeq U_\alpha$ is also an acceptable approximation because the difference $u_{s\alpha}$ is of order $(ka)^2$ relative to the group velocity in (52a).

In (52a), the third, fourth, and fifth terms on the right are surface wind work terms, one from pressure and the others from turbulence. The sixth term is wave dissipation; any eddy viscosity model will yield a negative value.

Appendix A provides further discussion, useful for understanding interaction of the momentum and wave energy equations.

To model an ocean with interacting waves and currents, one needs (50), (51), and (52). To solve for any scalar Θ (such as temperature and salinity after which an equation of state will yield b), one can write

$$\frac{\partial}{\partial t}(D\Theta) + \frac{\partial}{\partial x_\beta}(DU_\alpha\Theta) + \frac{\partial}{\partial \zeta}(\omega\Theta) = \frac{\partial}{\partial \zeta}(-\overline{\hat{w}\hat{\theta}}), \quad (54)$$

which neglects interaction between the scalar properties and surface waves.

To complete the set of equations initiated in section 2, I write the turbulence kinetic energy equation

$$\frac{\partial}{\partial t}\left(D\frac{q^2}{2}\right) + \frac{\partial}{\partial x_\alpha}\left(DU_\alpha\frac{q^2}{2}\right) + \frac{\partial}{\partial \zeta}\left(\Omega\frac{q^2}{2}\right)$$
$$= -\frac{\partial}{\partial \zeta}(\overline{\langle\acute{w}\acute{u}_i^2\rangle} + \overline{\langle\acute{w}\acute{p}\rangle}) - \overline{\langle\acute{w}\acute{u}_i\rangle\frac{\partial u_i}{\partial \zeta}} - Dg\frac{\overline{\langle\acute{\rho}\acute{w}\rangle}}{\rho_o} - D\epsilon, \quad (55)$$

which is derived from (16) after multiplication of every term by $\acute{u}_i$; then, the terms are operated on by $\langle\ \rangle$, transformed to sigma coordinates, and phase averaged. Thus, $q^2 \equiv \overline{\langle\acute{u}_i^2\rangle}$ is 2 times the turbulence kinetic energy, and the terms on the right are turbulence diffusion, shear production, buoyancy production, and dissipation; the first and last terms need to be modeled. The turbulence energy equation is the common basis of many current turbulence closure models. The turbulence dissipation represents the final conversion of wind work into thermal (or internal) energy.

Left for the future are prescriptions for the Reynolds flux, wind pressure forcing, and turbulence energy production. There is a large literature on empirical ways of determining the Reynolds stresses and fluxes where surface waves are neglected. And, utilizing the turbulence kinetic equation, there are beginning attempts to include the effects of wave breaking on current structure (e.g., Craig and Banner 1994; Terray et al. 1996). Now, however, there is a need to scrutinize the existing empirical knowledge base and to enter into new research framed by the continuity, momentum, wave energy, and turbulence energy equations derived in this paper.

Acknowledgments. I made much use of the book by Owen Phillips; if my equations, when vertically integrated, did not initially agree with with those in his book, then I knew I was wrong and renewed effort to find the mistake. I profited from discussions with Mark Donelan, Gene Terray, and Jurjen Battjes. The research was funded by the NOPP surf-zone project and by ONR Grant N00014-01-1-0170.

APPENDIX A

Some Applications

Associated with a given wave energy E is a Stokes drift velocity given by (49b). However, how does a given velocity field develop and do (51) and (52) produce compatible results?

Consider a thought experiment in which, initially, $\hat{u}_\alpha = 0$, turbulence and Coriolis terms are nil, and the wave field is horizontally homogeneous (infinite fetch). Let the wave field be given by $k_\alpha = (k_x, 0)$ and define $P_x \equiv \overline{\tilde{p}_{w\eta}\partial\tilde{\eta}/\partial x}$ so that (51) and (52) reduce to

$$\frac{\partial}{\partial t}(DU_x) = P_x\frac{\partial F_{SS}F_{CC}}{\partial \zeta} \quad \text{and} \quad (A1a)$$

$$\frac{\partial E}{\partial t} = cP_x. \quad (A1b)$$

Notice that the U_x profile has the same vertical distribution as the Stokes profile in (30b). Let $P_x = 0$ for $t < 0$ and $P_x > 0$ for $t \geq 0$. Now, define

$$M_x \equiv D\int_{-1}^{0} U_x\, d\zeta = \hat{M}_x + M_{Sx},$$

and because, from (49), $M_{Sx} \equiv \int_{-1}^{0} u_{Sx}\, d\zeta = E/c$, the momentum and energy equations result in the same vertically integrated Stokes transport such that $M_{Sx} > 0$ and $\hat{M}_x = 0$. We conclude that the wind pressure forcing creates a pure Stokes velocity response. In particular, if $P_x =$ constant for $t \geq 0$, then

$$\hat{M}_x = 0, \quad \hat{M}_y = 0, \quad M_{Sx} = P_x t, \quad \text{and} \quad M_{Sy} = 0.$$

The situation changes if one includes Coriolis terms. The same wind forcing is invoked, so that

$$\frac{\partial(M_x, M_y)}{\partial t} + f(-M_y, M_x) = [P_x(t), 0] \quad \text{and} \tag{A2a}$$

$$\frac{\partial(M_{Sx}, M_{Sy})}{\partial t} = [P_x(t), 0]. \tag{A2b}$$

For zero forcing and steady state, $(M_x, M_y) = (0, 0)$, and, if a Stokes drift exists, the current transport $\hat{M}_x$ just cancels the Stokes drift; this was first demonstrated by Ursell (1950) using a vorticity/circulation argument. This has been said to be a paradox (Huang 1970; Xu and Bowen 1994) because there seems to be a discontinuity in solution form between $f = 0$ and $f > 0$ no matter how small the value f. But let us now set up a specific example flow that is steady and with a Stokes drift. Set $M \equiv M_x + iM_y$; then (A2a,b) may be written

$$\frac{\partial M}{\partial t} + ifM = P_x \quad \text{and} \quad \frac{\partial M_{Sx}}{\partial t} = P_x.$$

For $P_x = 0$ when $t \leq 0$ and $P_x =$ constant for $t > 0$, the solutions for M and M_{Sx} are

$$M = \int_0^t P_x(t')e^{if(t'-t)}\,dt' = -\frac{iP_x}{f}(1 - e^{-ift}), \tag{A3a}$$

$$M_{Sx} = P_x t, \quad \text{and} \tag{A3b}$$

$$M_{Sy} = 0. \tag{A3c}$$

If we create a top hat forcing such that $P_x = 0$ for $t > T$, where $T = 2\pi/f$ is the inertial period, then $M = 0$ or

$$\hat{M}_x = -M_{Sx}, \qquad \hat{M}_y = 0,$$

$$M_{Sx} = P_x T, \quad \text{and} \quad M_{Sy} = 0.$$

This is just one of many ways to generate a steady state with a Stokes drift and an equal and opposite current. Similar results have been obtained by Hasselman (1970) and Xu and Bowen (1994) but from a specialized and different analytical route, whereas here the same results are imbedded in (51) and (52).

For small ft, (A3a) yields $M = P_x t$ so that now

$$\hat{M}_x = 0, \qquad \hat{M}_y = 0, \qquad M_{Sx} = P_x t, \quad \text{and} \quad M_{Sy} = 0$$

as in the case of no Coriolis term, thus resolving the aforementioned paradox because the solution evolves smoothly as ft increases from small to large values.

In all of the above, it is evident that all components $\hat{u}_\alpha$ and $u_{S\alpha}$ have the same z-dependent, Stokes-like profile shape.

These examples and others (e.g., the progression of dissipating waves on a shore) also illustrate how the current and Stokes drift may be coupled, as discussed in section 2.

APPENDIX B

Nomenclature

$\hat{u}_i$ = current velocity, slowly varying in space and time, $= (\hat{u}, \hat{v}, \hat{w}) = (\hat{u}_\alpha, \hat{w})$

$\hat{u}_\alpha = (\hat{u}, \hat{v})$

$\hat{p}$ = pressure, slowly varying

$\hat{\eta}$ = surface elevation, slowly varying

$\tilde{u}_i$ = wave velocity $= (\tilde{u}, \tilde{v}, \tilde{w})$

$\tilde{p}$ = wave pressure

$\tilde{\eta}$ = surface wave elevation

$u_i = \hat{u}_i + \tilde{u}_i$

$p = \hat{p} + \tilde{p}$

$\acute{u}_i$ = turbulence velocity

$u_{S\alpha}$ = Stokes drift velocity, derived from $\tilde{u}_i$

$U_\alpha = \hat{u}_\alpha + \tilde{u}_{S\alpha}$

ζ = transformed vertical coordinate such that $\zeta = -1$ when $z = -h$ and $\zeta = 0$ when $z = \hat{\eta} + \tilde{\eta}$

$z = -h$, the ocean bottom; $D \equiv h + \hat{\eta}$

$\hat{M}_\alpha \equiv D \int_{-1}^{0} \hat{u}_\alpha \, d\zeta$

$M_{S\alpha} \equiv D \int_{-1}^{0} u_{S\alpha} \, d\zeta$

$M_\alpha \equiv \hat{M}_\alpha + M_{S\alpha}$

$\hat{u}_{A\alpha}$ = weighted vertical average of $\hat{u}_\alpha$ equal to $\int_{-1}^{0} r(\zeta)\hat{u}_\alpha \, d\zeta$, where $\int_{-1}^{0} r(\zeta)\, d\zeta = 1$

$E \equiv g\overline{\tilde{\eta}^2} = ga^2/2$, where g is the gravity constant and a is wave elevation amplitude

REFERENCES

Bleck, R. C., and D. Boudra, 1986: Wind-driven spin-up in eddy resolving ocean models formulated in isopynic and isobaric coordinates. *J. Geophys. Res.*, **91,** 7611–7621.

Blumberg, A. F., and G. L. Mellor, 1987: A description of a three-dimensional coastal ocean model. *Three Dimensional Coastal Ocean Models*, N. S. Heaps, Ed., American Geophysical Union, 1–16.

Bryan, K., and M. D. Cox, 1968: A nonlinear model of an ocean driven by wind and differential heating: Part I. Description of the three-dimensional velocity and density field. *J. Atmos. Sci.*, **25,** 945–967.

Cheung, T. K., and R. L. Street, 1988: The turbulent layer in the water at an air–water interface. *J. Fluid Mech.*, **194,** 133–151.

Craig, P. D., and M. L. Banner, 1994: Modeling wave-enhanced turbulence in the ocean surface layer. *J. Phys. Oceanogr.*, **24,** 2513–2529.

Dolata, L. F., and W. Rosenthal, 1984: Wave setup and wave-induced currents in coastal zones. *J. Geophys. Res.*, **89,** 1973–1982.

Donelan, M. A., 1999: Wind-induced, growth and attenuation of laboratory waves. *Wind-Over-Wave Couplings*, S. G. Sajjadi et al., Eds., Clarendon Press, 183–194.

Hasselmann, K., 1970: Wave-driven inertial oscillations. *Geophys. Fluid Dyn.*, **1,** 433–472.

Huang, N. E., 1970: Mass transport induced by wave motion. *J. Mar. Res.*, **29,** 35–50.

Jensen, B. L., B. M. Summer, and J. Fredsoe, 1989: Turbulent oscillatory boundary layers at high Reynolds numbers. *J. Fluid Mech.*, **206,** 265–297.

Kirby, J. T., and T.-M. Chen, 1989: Surface waves on vertically sheared flows—Approximate dispersion relations. *J. Geophys. Res.*, **94,** 1013–1027.

Komen, G. J., L. Cavaleri, M. Donelan, K. Hasselman, S. Hasselman, and P. A. E. M. Janssen, Eds.,1994: *Dynamics and Modelling of Ocean Waves,* Cambridge University Press, 502 pp.

Longuet-Higgins, M. S., 1953: Mass transport in water waves. *Philos. Trans. Roy. Soc. London,* **245A,** 535–581.

——, 1969: A nonlinear mechanism for the generaion of sea waves. *Proc. Roy. Soc. London,* **311A,** 371–389.

——, and R. W. Stewart, 1961: The changes in amplitude of short gravity waves on steady non-uniform currents. *J. Fluid Mech.,* **10,** 529–549.

Mellor, G. L., and T. Yamada, 1982: Development of a turbulence closure model for geophysical fluid problems. *Rev. Geophys. Space Phys.,* **20,** 851–875.

Miles, J. W., 1954: On the generation of surface waves by shear flows. *J. Fluid Mech.,* **3,** 185–204.

Phillips, O. M., 1954: On the generation of surface waves by turbulent wind. *J. Fluid Mech.,* **2,** 417–452.

——, 1977: *The Dynamics of the Upper Ocean.* Cambridge University Press, 336 pp.

Pierson, W. J., and L. Moskowitz, 1964: A proposed spectral form for fully developed wind seas based on the similarity theory of S. A. Kitaigorodski. *J. Geophys. Res.,* **69,** 4881–4890.

Terray, E. A., M. A. Donelan, Y. C. Agrawal, W. M. Drennan, K. K. Kahma, A. J. Williams III, P. A. Hwang, and S. A. Kitaigorodskii, 1996: Estimates of kinetic energy dissipation under breaking waves. *J. Phys. Oceanogr.,* **26,** 792–807.

Ursell, F., 1950: On the theoretical form of ocean swell on a rotating earth. *Mon. Not. Roy. Astron. Soc.,* **6** (geophys. suppl.), 1–8.

WAMDI Group, The 1988: The WAM Model—A third-generation ocean wave prediction model. *J. Phys. Oceanogr.,* **18,** 1775–1810.

Xie, L., K. Wu, L. Pietrafesa, and C. Zhang, 2001: A numerical study of wave–current interaction through surface and bottom stresses: Wind driven circulation in the South Atlantic Bight under uniform winds. *J. Geophys. Res.,* **106,** 16 841–16 852.

Xu, Z., and A. J. Bowen, 1994: Wave- and wind-driven flow in water of finite depth. *J. Phys. Oceanogr.,* **24,** 1850–1866.