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The Three Dimensional Viscous Camassa–Holm Equations, and Their Relation to the Navier–Stokes Equations and Turbulence Theory[1]

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We show here the global, in time, regularity of the three dimensional viscous Camassa–Holm (Navier–Stokes-alpha) (NS-α) equations. We also provide estimates, in terms of the physical parameters of the equations, for the Hausdorff and fractal dimensions of their global attractor. In analogy with the Kolmogorov theory of turbulence, we define a small spatial scale, ℓ_ϵ, as the scale at which the balance occurs in the mean rates of nonlinear transport of energy and viscous dissipation of energy. Furthermore, we show that the number of degrees of freedom in the long-time behavior of the solutions to these equations is bounded from above by $(L/\ell_\epsilon)^3$, where L is a typical large spatial scale (e.g., the size of the domain). This estimate suggests that the Landau–Lifshitz classical theory of turbulence is suitable for interpreting the solutions of the NS-α equations. Hence, one may consider these equations as a closure model for the Reynolds averaged Navier–Stokes equations (NSE). We study this approach, further, in other related papers. Finally, we discuss the relation of the NS-α model to the NSE by proving a convergence theorem, that as the length scale α_1 tends to

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zero a subsequence of solutions of the NS-α equations converges to a weak solution of the three dimensional NSE.

KEY WORDS: Navier–Stokes-α model; Camassa–Holm equations; Lagrangian averaged Navier–Stokes equations; second grad fluid; attractors.

1. INTRODUCTION

Proving global regularity for the 3D Navier–Stokes equations (NSE) is one of the most challenging outstanding problems in nonlinear analysis. The main difficulty in establishing this result lies in controlling certain norms of vorticity. More specifically, the vorticity stretching term in the 3D vorticity equation forms the main obstacle to achieving this control.

In this paper we consider a similar partial differential equation, the so-called viscous Camassa–Holm, or Navier–Stokes-alpha (NS-α) equations. The inviscid NS-α equations (Euler-α) were introduced in [25] as a natural mathematical generalization of the integrable inviscid 1D Camassa–Holm equation discovered in [3] through a variational formulation. Our studies in [5]–[7] indicated that there is a connection between the solutions of the NS-α and turbulence. Specifically, the explicit steady analytical solution of the NS-α equations were found to compare successfully with empirical and numerical experimental data for mean velocity and Reynolds stresses for turbulent flows in pipes and channels. These comparisons led us to identify the NS-α equations with the Reynolds averaged Navier–Stokes equations. These comparisons also led us to suggest the NS-α equations could be used as a closure model for the mean effects of subgrid excitations. Numerical tests that tend to justify this intuition were reported in [8].

An alternative more "physical" derivation for the inviscid NS-α equations (Euler-α), was introduced in [26] and [27] (see also [6]). This alternative derivation was based on substituting in Hamilton's principle the decomposition of the Lagrangian fluid-parcel trajectory into its mean and fluctuating components. This was followed by truncating a Taylor series approximation and averaging at constant Lagrangian coordinate, before taking variations. A variant of this approach was also elaborated considerably in [32]. See also [33] for the geometry and analysis of the Euler-α equations. For more information and a brief guide to the previous literature specifically about the NS-α model, see paper [20]. The latter paper also discusses connections to standard concepts and scaling laws in turbulence modeling, including the relationship of the NS-α model to large eddy simulation (LES) models. Results interpreting the NS-α model as an extension of scale similarity LES models of turbulence are reported in [17].

It is worth mentioning that another approach connecting the Lagrangian and Eulerian formulations for the Navier–Stokes equations was recently presented in [11]. This exact connection between the Lagrangian and Eulerian formulations adds perspective to the relationship between the Navier–Stokes equations and the Navier–Stokes-α model.

Equations similar to the NS-α equation, but with different dissipative terms, were considered previously in the theory of second grade fluids [18] and were treated recently in the mathematical literature [9, 10]. Second grade fluid models are derived from continuum mechanical principles of objectivity and material frame indifference, after which thermodynamic principles such as the Clausius–Duhem relation and stability of stationary equilibrium states are imposed that restrict the allowed values of the parameters in these models. In contrast, as mentioned earlier, the NS-α equation is derived by applying asymptotic expansions, Lagrangian means, and an assumption of isotropy of fluctuations in Hamilton's principle for an ideal incompressible fluid. Their different derivations also provide the different interpretations of the parameter α_1, namely, as a flow regime quantity for the NS-α equation, and as a fixed material property for the second grade fluid.

The aim of this paper is to establish the global regularity of solutions of the NS-α, subject to periodic boundary conditions. We also provide estimates of the fractal and Hausdorff dimensions of their global attractors. In particular, we identify the dimension of the attractor with the number of degrees of freedom governing the permanent regime of these equations and find a remarkable compatibility between these estimates and the number of degrees of freedom in turbulence a la Landau and Lifshitz [30]. This leads us to regard the NS-α equations as a suitable closure model for turbulence, thought of as an averaged theory rather than an individual realization, cf. [5]–[7], [26] and [27]. Finally, we relate the solutions of the viscous Camassa–Holm (NS-α) equations to those of the 3D NSE as the length scale α_1 tends to zero. Specifically, we prove that a subsequence of solutions to the NS-α model converges as $\alpha_1 \to 0$ to a weak solution of the 3D NSE.

2. FUNCTIONAL SETTING AND PRELIMINARIES

We consider the following viscous version of the three dimensional Camassa–Holm equations in the periodic box $\Omega = [0, L]^3$:

$$\frac{\partial}{\partial t}(\alpha_0^2 u - \alpha_1^2 \Delta u) - \nu \Delta(\alpha_0^2 u - \alpha_1^2 \Delta u) - u \times (\nabla \times (\alpha_0^2 u - \alpha_1^2 \Delta u)) + \frac{1}{\rho_0} \nabla p = f \tag{1a}$$

$$\nabla \cdot u = 0 \tag{1b}$$

$$u(x, 0) = u^{in}(x) \tag{1c}$$

where $\frac{p}{\rho_0} = \frac{\pi}{\rho_0} + \alpha_0^2 \, |u|^2 - \alpha_1^2 (u \cdot \Delta u)$ is the modified pressure, while π is the pressure, $\nu > 0$ is the constant viscosity and $\rho_0 > 0$ is a constant density. The function f is a given body forcing $\alpha_0 > 0$ and $\alpha_1 \geqslant 0$ are scale parameters. Notice α_0 is dimensionless while α_1 has units of length. Also observe that at the limit $\alpha_0 = 1$, $\alpha_1 = 0$ we obtain the three dimensional Navier–Stokes equations with periodic boundary conditions.

For simplicity we will assume the forcing term to be time independent, i.e., $f(x, t) \equiv f(x)$.

From (1) one can easily see, after integration by parts, that

$$\frac{d}{dt} \int_\Omega (\alpha_0^2 u - \alpha_1^2 \, \Delta u) \, dx = \int_\Omega f \, dx$$

On the other hand, because of the spatial periodicity of the solution, we have $\int_\Omega \Delta u \, dx = 0$. As a result, we have $\frac{d}{dt} \int_\Omega \alpha_0^2 u \, dx = \int_\Omega f \, dx$; that is, the mean of the solution is invariant provided the mean of the forcing term is zero. In this paper we will consider forcing terms and initial values with spatial means that are zero; i.e., we will assume $\int_\Omega u^{in} \, dx = \int f \, dx = 0$ and hence $\int_\Omega u \, dx = 0$.

Next, let us introduce some notation and background.

(i) Let X be a linear subspace of integrable functions defined on the domain Ω, we denote

$$\dot{X} := \left\{ \varphi \in X : \int_\Omega \varphi(x) \, dx = 0 \right\}$$

(ii) We denote $\mathcal{V} = \{\varphi : \varphi$ is a vector valued trigonometric polynomial defined on Ω, such that $\nabla \cdot \varphi = 0$ and $\int_\Omega \varphi(x) \, dx = 0\}$, and let H and V be the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and in $H^1(\Omega)^3$ respectively; observe that $H^\perp$, the orthogonal complement of H in $L^2(\Omega)^3$ is $\{\nabla p : p \in H^1(\Omega)\}$ (cf. [13] or [35]).

(iii) We denote $P_\sigma : \dot{L}^2(\Omega)^3 \to H$ the L^2 orthogonal projection, usually referred as Helmholtz–Leray projector, and by $A = -P_\sigma \Delta$ the Stokes operator with domain $D(A) = (H^2(\Omega))^3 \cap V$. Notice that in the case of periodic boundary condition $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j\}_{j=1}^\infty$ of eigenfunctions of A, i.e.,

$Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to \infty$; in fact these eigenvalues have the form $|k|^2 \frac{4\pi^2}{L^2}$ with $k \in \mathbf{Z}^3 \setminus \{0\}$.

(iv) We denote $(\cdot, \cdot)$ the L^2-inner product and by $|\cdot|$ the corresponding L^2-norm. By virtue of Poincaré inequality one can show that there is a constant $c > 0$, such that

$$C\,|Aw| \leqslant \|w\|_{H^2} \leqslant c^{-1}\,|Aw| \qquad \text{for every} \quad w \in D(A)$$

and that

$$c\,|A^{1/2}w| \leqslant \|w\|_{H^1} \leqslant c^{-1}\,|A^{1/2}w| \qquad \text{for every} \quad w \in V$$

Moreover, one can show that $V = D(A^{1/2})$, (cf. [13] and [35]). We denote $((\cdot, \cdot)) = (A^{1/2}\cdot, A^{1/2}\cdot)$ and $\|\cdot\| = |A^{1/2}\cdot|$ the inner product and norm on V, respectively. Notice that, based on the above, the inner product $((\cdot, \cdot))$, restricted to V, is equivalent to the H^1 inner product

$$[u, v] = \alpha_0^2 (u, v) + \alpha_1^2 ((u, v)) \qquad \text{for} \quad u, v \in V \tag{2}$$

provided $\alpha_1 > 0$.

Hereafter c will denote a generic scale invariant positive constant which is independent of the physical parameters in the equation.

(v) Following the notation for the Navier–Stokes equations we denote $B(u, v) = P_\sigma[(u \cdot \nabla)\, v]$, and we set $B(v)\, u = B(u, v)$ for every $u, v \in V$. That is, for ever fixed $v \in V$, $B(v)$ is a linear operator acting on u. Notice that

$$(B(u, v), w) = -(B(u, w), v) \qquad \text{for every} \quad u, v, w \in V \tag{3}$$

We also denote $\tilde{B}(u, v) = -P_\sigma (u \times (\nabla \times v))$ for every $u, v \in V$. Using the identity

$$(b \cdot \nabla)\, a + \sum_{j=1}^{3} a_j\, \nabla b_j = -b \times (\nabla \times a) + \nabla (a \cdot b)$$

one can easily show that

$$\begin{aligned}(\tilde{B}(u, v), w) &= (B(u, v), w) - (B(w, v), u)\\ &= (B(v)\, u - B^*(v)\, u, w)\end{aligned} \tag{4}$$

for every $u, v, w \in V$, where $B^*(v)$ denotes the adjoint operator of the linear operator $B(v)$ defined above. As a result we have

$$\tilde{B}(u, v) = (B(v) - B^*(v))\, u \qquad \text{for every} \quad u, v \in V \tag{5}$$

In the next lemma, we show that the bilinear operator $\tilde{B}$ can be extended continuously to a larger class of functions.

Lemma 1.

(i) *The operator A can be extended continuously to be defined on $V = D(A^{1/2})$ with values in $V' = D(A^{-1/2})$ such that*

$$\langle Au, v\rangle_{V'} = (A^{1/2}u, A^{1/2}v) = \int_\Omega (\nabla u : \nabla v)\, dx$$

for every $u, v \in V$.

(ii) *Similarly, the operator A^2 can be extended continuously to be defined on $D(A)$ with valves in $D(A)'$, the dual space of the Hilbert space $D(A)$, such that*

$$\langle A^2 u, v\rangle_{D(A)'} = (Au, Av), \qquad \textit{for every} \quad u, v \in D(A)$$

(iii) *The operator $\tilde{B}$ can be extended continuously from $V \times V$ with valves in V', and in particular it satisfies*

$$|\langle \tilde{B}(u, v), w\rangle_{V'}| \leqslant c\, |u|^{1/2}\, \|u\|^{1/2}\, \|v\|\, \|w\|$$

$$|\langle \tilde{B}(u, v), w\rangle_{V'}| \leqslant c\, \|u\|\, \|v\|\, |w|^{1/2}\, \|w\|^{1/2}$$

for every $u, v, w \in V$. Moreover,

$$\langle \tilde{B}(u, v), w\rangle_{V'} = -\langle \tilde{B}(w, v), u\rangle_{V'}, \qquad \textit{for every} \quad u, v, w \in V$$

and in particular,

$$\langle \tilde{B}(u, v), u\rangle_{V'} \equiv 0 \qquad \textit{for every} \quad u, v \in V$$

(iv) *Furthermore, we have*

$$|\langle \tilde{B}(u, v), w\rangle_{D(A)'}| \leqslant c\, |u|\, \|v\|\, \|w\|^{1/2}\, |Aw|^{1/2}$$

for every $u \in H$, $v \in V$ *and* $w \in D(A)$, *and by symmetry we have*

$$|(\tilde{B}(u,v),w)| \leqslant c\, \|u\|^{1/2}\, |Au|^{1/2}\, \|v\|\, |w|$$

for every $u \in D(A)$, $v \in V$ *and* $w \in H$.

(v) *Also,*

$$|\langle \tilde{B}(u,v),w\rangle_{D(A)'}| \leqslant c(|u|^{1/2}\, \|u\|^{1/2}\, |v|\, |Aw| + |v|\, \|u\|\, \|w\|^{1/2}\, |Aw|^{1/2})$$

for every $u \in V$, $v \in H$, $w \in D(A)$.

(vi) *In addition,*

$$|\langle \tilde{B}(u,v),w\rangle_{V'}| \leqslant c(\|u\|^{1/2}\, \|Au\|^{1/2}\, |v|\, \|w\| + |Au|\, |v|\, |w|^{1/2}\, \|w\|^{1/2})$$

for every $u \in D(A)$, $v \in H$, $w \in V$.

Proof. The proof of (i) can be found in [13] or in [35]. The proof of (ii) is a straight forward extension of that of (i).

To prove (iii), let us first consider the case when $u, v, w \in \mathcal{V}$. Then we have

$$\begin{aligned}|\langle \tilde{B}(u,v),w\rangle_{V'}| &= \left|\int_\Omega u \times (\nabla \times v)\cdot w\, dx\right| \\ &\leqslant c\, \|u\|_{L^3}\, \|\nabla v\|_{L^2}\, \|w\|_{L^6}\end{aligned}$$

Recall the following Sobolev inequalities in $\mathbb{R}^3$

$$\|\varphi\|_{L^4} \leqslant c\, \|\varphi\|_{L^2}^{1/4}\, \|\varphi\|_{H^1}^{3/4} \tag{6a}$$

$$\|\varphi\|_{L^3} \leqslant c\, \|\varphi\|_{L^2}^{1/2}\, \|\varphi\|_{H^1}^{1/2} \quad \text{and} \tag{6b}$$

$$\|\varphi\|_{L^6} \leqslant \|\varphi\|_{H^1}, \qquad \text{for every} \quad \varphi \in H^1(\Omega) \tag{6c}$$

Then by the above inequalities we have:

$$|\langle \tilde{B}(u,v),w\rangle_{V'}| \leqslant c\, |u|^{1/2}\, \|u\|^{1/2}\, \|v\|\, \|w\|$$

Moreover, it is clear that for $u, v, w \in \mathcal{V}$

$$\langle \tilde{B}(u,v),w\rangle_{V'} = -\langle \tilde{B}(w,v),u\rangle_{V'}$$

Since $\mathcal{V}$ is dense in V we conclude the proof of (iii).

Let us now prove (iv). Again we consider first the case where $u, v, w \in \mathscr{V}$

$$|\langle \tilde{B}(u,v), w\rangle_{D(A)'}| = \left|\int_\Omega [u\times(\nabla\times v)]\cdot w\,dx\right|$$
$$\leqslant c\,\|u\|_{L^2}\,\|\nabla v\|_{L^2}\,\|w\|_{L^\infty}$$

Recall Agmon's inequality in $\mathbb{R}^3$:

$$\|\varphi\|_{L^\infty} \leqslant c\,\|\varphi\|_{H^1}^{1/2}\,\|\varphi\|_{H^2}^{1/2} \tag{7}$$

The above gives

$$|\langle \tilde{B}(u,v), w\rangle_{D(A)'}| \leqslant c\,|u|\,\|v\|\,\|w\|^{1/2}\,\|Aw\|^{1/2}$$

To prove (v) we again take $u, v, w \in \mathscr{V}$ and we use (4) to find

$$|\langle \tilde{B}(u,v), w\rangle_{D(A)'}| \leqslant \left|\int_\Omega ((u\cdot\nabla)\,v)\cdot w\,dx\right| + \left|\int_\Omega ((w\cdot\nabla)\,u)\cdot v\,dx\right|$$
$$\leqslant \left|\int_\Omega ((u\cdot\nabla)\,w\cdot v)\,dx\right| + \|v\|_{L^2}\,\|\nabla u\|_{L^2}\,\|w\|_{L^\infty}$$
$$\leqslant c\,\|u\|_{L^3}\,\|\nabla w\|_{L^6}\,|v| + c\,|v|\,\|u\|\,\|w\|_{L^\infty}$$

By (6b–c) and (7) inequalities we finish our proof.

The proof of (vi) is similar to (v). From (4) we have

$$|\langle \tilde{B}(u,v), w\rangle_{V'}| \leqslant \left|\int_\Omega ((u\cdot\nabla)\,v)\,w\,dx\right| + \left|\int_\Omega ((w\cdot\nabla)\,u)\cdot v\,dx\right|$$
$$\leqslant \left|\int_\Omega ((u\cdot\nabla)\,w)\cdot v\,dx\right| + c\,\|w\|_{L^3}\,\|\nabla u\|_{L^6}\,|v|$$
$$\leqslant c(\|u\|_{L^\infty}\,\|w\|\,|v| + \|w\|_{L^3}\,\|\nabla u\|_{L^6}\,|v|)$$

By (6a) and (7) inequalities we finish our proof. □

We apply P_σ to (1) and use the above notation to obtain the equivalent system of equations

$$\frac{d}{dt}(\alpha_0^2 u + \alpha_1^2 Au) + \nu A(\alpha_0^2 + \alpha_1^2 A)\,u + \tilde{B}(u, \alpha_0^2 + \alpha_1^2 Au) = P_\sigma f \tag{8a}$$

$$u(0) = u^{in} \tag{8b}$$

Alternatively, if we denote

$$v = \alpha_0^2 u + \alpha_1^2 Au \tag{9}$$

the system (8) can be written as

$$\frac{dv}{dt} + \nu Av + B(v)\, u - B^*(v)\, u = P_\sigma f \tag{10a}$$

$$u(0) = u^{in} \tag{10b}$$

We will assume that $P_\sigma f = f$, otherwise we add the gradient part of f to the modified pressure and rename $P_\sigma f$ by f.

Definition 2 (Regular Solution). Let $f \in H$, and let $T > 0$. A function $u \in C([0,T); V) \cap L^2([0,T); D(A))$ with $\frac{du}{dt} \in L^2([0,T); H)$ is said to be a regular solution to (8) in the interval $[0,T)$ if it satisfies

$$\left\langle \frac{d}{dt}(\alpha_0^2 u + \alpha_1^2 Au), w \right\rangle_{D(A)'} + \nu \langle A(\alpha_0^2 u + \alpha_1^2 Au), w \rangle_{D(A)'} + \langle \tilde{B}(u, \alpha_0^2 u + \alpha_1^2 Au), w \rangle_{D(A)'} = (f, w) \tag{11}$$

for every $w \in D(A)$ and for almost every $t \in [0,T)$. Moreover, $u(0) = u^{in}$ in V. Here, the equation (11) is understood in the following sense: For every $t_0, t \in [0,T)$ we have

$$(\alpha_0^2 u(t) + \alpha_1^2 Au(t), w) - (\alpha_0^2 u(t_0) + \alpha_1^2 Au(t_0), w) + \nu \int_{t_0}^{t} (\alpha_0^2 u(s) + \alpha_1^2 Au(s), w)\, ds + \int_{t_0}^{t} \langle \tilde{B}(u(s), \alpha_0^2 u(s) + \alpha_1^2 Au(s)), w \rangle_{D(A)'}\, ds = \int_{t_0}^{t} (f, w)\, ds \tag{12}$$

3. GLOBAL EXISTENCE AND UNIQUENESS

In this section we prove global existence and uniqueness of regular solutions to Eq. (8), provided $\alpha_1 > 0$. In fact, from now on we will always assume that $\alpha_1 > 0$.

Theorem 3 (Global existence and uniqueness). *Let $f \in H$ and $u^{in} \in V$. Then for any $T > 0$, Eq. (8) has a unique regular solution u on $[0,T)$. Moreover, this solution satisfies:*

(i) $u \in L^\infty_{\text{loc}}((0,T]; H^3(\Omega))$.

(ii) *There are constants* R_k, *for* $k = 0, 1, 2, 3$, *which depend only on* ν, α_0, α_1 *and* f, *but not on* u^{in}, *such that*

$$\limsup_{t\to\infty} \left(\alpha_0^2 \, |A^{\frac{k}{2}} u|^2 + \alpha_1^2 \, |A^{\frac{k+1}{2}} u|^2\right) = R_k^2$$

for $k = 0, 1, 2, 3$. *In particular, we have*

$$R_0^2 = \frac{1}{\nu\lambda_1} \min\left\{\frac{|A^{-1/2} f|^2}{\nu\alpha_0^2}, \frac{|A^{-1/2} f|^2}{\nu\alpha_1^2}\right\} \leqslant \min\left\{\frac{|f|^2}{\nu^2\lambda_1^2\alpha_0^2}, \frac{|f|^2}{\nu^2\lambda_1^3\alpha_0^2}\right\} \tag{13}$$

that is:

$$R_0^2 \leqslant \frac{G^2\nu^2}{\lambda_1^{1/2}} \min\left\{\frac{1}{\alpha_0^2}, \frac{1}{\alpha_1^2\lambda_1}\right\} = \frac{G^2\nu^2}{\gamma\lambda_1^{1/2}}$$

where $G = \frac{|f|}{\nu^2\lambda_1^{3/4}}$ *is the Grashoff number, and* $\gamma^{-1} = \min\{\frac{1}{\alpha_0^2}, \frac{1}{\alpha_1^2\lambda_1}\}$. *Furthermore,*

$$\limsup_{T\to\infty} \frac{\nu}{T} \int_t^{t+T} \left(\alpha_0^2 \, \|u(s)\|^2 + \alpha_1^2 \, |Au(s)|^2\right) ds \leqslant \nu\lambda_1 R_0^2 \leqslant \frac{G^2\nu\lambda_1^{1/2}}{\gamma} \tag{14}$$

for all $t \geqslant 0$.

Proof. We use the Galerkin procedure to prove global existence and to establish the necessary a priori estimates.

Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of H consisting of eigenfunctions of the operator A. Denote $H_m = \text{span}\{w_1, \ldots, w_m\}$ and let P_m be the L^2-orthogonal projection from H onto H_m. The Galerkin procedure for Eq. (8) is the ordinary differential system

$$\frac{d}{dt}(\alpha_0^2 u_m + \alpha_1^2 A u_m) + \nu A(\alpha_0^2 u_m + \alpha_1^2 A u_m) + P_m \tilde{B}(u_m, \alpha_0^2 u_m + \alpha_1^2 A u_m) = P_m f \tag{15a}$$

$$u_m(0) = P_m u^{in} \tag{15b}$$

Since the nonlinear term is quadratic in u_m, then by the classical theory of ordinary differential equations, the system (15) has a unique solution for a short interval of time $(-\tau_m, T_m)$. Our goal is to show that the solutions of (15) remains finite for all positive times which implies that $T_m = \infty$.

H^1-Estimates

We take the inner product of (15) with u_m and use (4) to obtain

$$\frac{1}{2}\frac{d}{dt}(\alpha_0^2\,|u_m|^2+\alpha_1^2\,\|u_m\|^2)+\nu(\alpha_0^2\,\|u_m\|^2+\alpha_1^2\,|Au_m|^2)=(P_m f, u_m)$$

Notice that

$$|(P_m f, u_m)| \leqslant \begin{cases} |A^{-1}f|\,|Au_m| \\ |A^{-1/2}f|\,\|u_m\| \end{cases}$$

and by Young's inequality we have

$$|(P_m f, u_m)| \leqslant \begin{cases} \dfrac{|A^{-1}f|^2}{2\nu\alpha_1^2}+\dfrac{\nu}{2}\alpha_1^2\,|Au_m|^2 \\ \dfrac{|A^{-1/2}f|^2}{2\nu\alpha_0^2}+\dfrac{\nu}{2}\alpha_0^2\,\|u_m\|^2 \end{cases}$$

Denoting by $K_1 = \min\{\frac{|A^{-1/2}f|^2}{\nu\alpha_0^2}, \frac{|A^{-1}f|^2}{\nu\alpha_1^2}\}$, from the above inequalities we get:

$$\frac{d}{dt}(\alpha_0^2\,|u_m|^2+\alpha_1^2\,\|u_m\|^2)+\nu(\alpha_0^2\,\|u_m\|^2+\alpha_1^2\,|Au_m|^2)\leqslant K_1 \tag{16}$$

By Poincaré's inequality we obtain

$$\frac{d}{dt}(\alpha_0^2\,|u_m|^2+\alpha_1^2\,\|u_m\|^2)+\nu\lambda_1(\alpha_0^2\,|u_m|^2+\alpha_1^2\,\|u_m\|^2)\leqslant K_1$$

and then by Gronwall's inequality we reach

$$\alpha_0^2\,|u_m(t)|^2+\alpha_1^2\,\|u_m(t)\|^2\leqslant e^{-\nu\lambda_1 t}(\alpha_0^2\,|u_m(0)|^2+\alpha_1^2\,\|u_m(0)\|^2)+\frac{K_1}{\nu\lambda_1}(1-e^{-\nu\lambda_1 t})$$

That is

$$\alpha_0^2\,|u_m(t)|^2+\alpha_1^2\,\|u_m(t)\|^2\leqslant k_1 := \alpha_0^2\,|u^{in}|^2+\alpha_1^2\,\|u^{in}\|^2+\frac{K_1}{\nu\lambda_1} \tag{17}$$

H^2-Estimates

Integrating (16) over the interval $(t, t+\tau)$

$$\nu \int_t^{t+\tau} (\alpha_0^2 \, \|u_m(s)\|^2 + \alpha_1^2 \, |Au_m(s)|^2) \, ds \leqslant \tau K_1 + (\alpha_0^2 \, |u_m(t)|^2 + \alpha_1^2 \, \|u_m(t)\|^2)$$
$$\leqslant \tau K_1 + k_1 =: \bar{k}_2(\tau) \tag{18}$$

Now, take the inner product of (15) with Au_m to obtain

$$\frac{1}{2}\frac{d}{dt}(\alpha_0^2 \, \|u_m\|^2 + \alpha_1^2 \, |Au_m|^2) + \nu(\alpha_0^2 \, |Au_m|^2 + \alpha_1^2 \, |A^{3/2}u_m|^2)$$
$$+ (\tilde{B}(u_m, \alpha_0^2 u_m + \alpha_1^2 Au_m), Au_m) = (P_m f, Au_m)$$

Notice that

$$|(P_m f, Au_m)| \leqslant \begin{cases} |A^{-1/2}f| \, |A^{3/2}u_m| \\ |f| \, |Au_m| \end{cases} \leqslant \begin{cases} \dfrac{|A^{-1/2}f|^2}{\nu\alpha_1^2} + \dfrac{\nu}{4}\alpha_1^2 \, |A^{3/2}u_m|^2 \\ \dfrac{|f|^2}{\nu\alpha_0^2} + \dfrac{\nu}{4}\alpha_0^2 \, |Au_m|^2 \end{cases}$$

We denote $K_2 = \min\{\frac{|A^{-1/2}f|^2}{\nu\alpha_1^2}, \frac{|f|^2}{\nu\alpha_0^2}\}$. Then we have

$$\frac{1}{2}\frac{d}{dt}(\alpha_0^2 \, \|u_m\|^2 + \alpha_1^2 \, |Au_m|^2) + \frac{3\nu}{4}(\alpha_0^2 \, |Au_m|^2 + \alpha_1^2 \, |A^{3/4}u_m|^2)$$
$$+ (\tilde{B}(u_m, \alpha_0^2 u_m + \alpha_1^2 Au_m), Au_m) \leqslant K_2$$

We use part (iii) of Lemma 1 to obtain

$$\frac{1}{2}\frac{d}{dt}(\alpha_0^2 \, \|u_m\|^2 + \alpha_1^2 \, |Au_m|^2) + \frac{3}{4}\nu(\alpha_0^2 \, |Au_m|^2 + \alpha_1^2 \, |A^{3/2}u_m|^2)$$
$$\leqslant c \, \|u_m\|(\alpha_0^2 \, \|u_m\| + \alpha_1^2 \, |A^{3/2}u_m|) \, |Au_m|^{1/2} \, |A^{3/2}u_m|^{1/2} + K_2$$
$$\leqslant c \, \|u_m\|(\alpha_0^2 \lambda_1^{-1} + \alpha_1^2) \, |A^{3/2}u_m|^{3/2} \, |Au_m|^{1/2} + K_2$$

By Young's inequality we have

$$\frac{1}{2}\frac{d}{dt}(\alpha_0^2 \, \|u_m\|^2 + \alpha_1^2 \, |Au_m|^2) + \frac{\nu}{2}(\alpha_0^2 \, |Au_m|^2 + \alpha_1^2 \, |A^{3/2}u_m|^2)$$
$$\leqslant c \, \|u_m\|^4 \, (\alpha_0^2 \lambda_1^{-1} + \alpha_1^2)^4 \, (\nu\alpha_1^2)^{-3} \, |Au_m|^2 + K_2 \tag{19}$$

We integrate the above equation over (s, t) and use (17) and (18) to obtain:

$$\alpha_0^2 \|u_m(t)\|^2 + \alpha_1^2 |Au_m(t)|^2 \leqslant \alpha_0^2 \|u_m(s)\|^2 + \alpha_1^2 |Au_m(s)|^2 + 2(t-s) K_2 + \frac{2ck_1 2}{(\nu\alpha_1^2)^4 \alpha_1^4} (\alpha_0^2 \lambda_1^{-1} + \alpha_1^2)^4 [(t-s) K_1 + k_1]$$

Now, we integrate with respect to s over $(0, t)$ and use (18) to get

$$t(\alpha_0^2 \|u_m(t)\|^2 + \alpha_1^2 |Au_m(t)|^2) \leqslant \frac{1}{\nu} (tK_1 + k_1) + t^2 K_2 + \frac{2ck_1^2}{(\nu\alpha_1^2)^4 \alpha_1^4} (\alpha_0^2 \lambda_1^{-1} + \alpha_1^2)^4 \left[\frac{t^2 K_1}{2} + tk_1\right] \tag{20}$$

for all $t \geqslant 0$.

For $t \geqslant \frac{1}{\nu\lambda_1}$ we integrate with respect to s over the interval $(t - \frac{1}{\nu\lambda_1}, t)$

$$\frac{1}{\nu\lambda_1} (\alpha_0^2 \|u_m(t)\|^2 + \alpha_1^2 |Au_m(t)|^2) \leqslant \frac{1}{\nu} \left(\frac{1}{\nu\lambda_1} K_1 + k_1\right) + \left(\frac{1}{\nu\lambda_1}\right)^2 K_2 + \frac{2ck_1^2}{(\nu\alpha_1^2)^4 \alpha_1^4} (\alpha_0^2 \lambda_1^{-1} + \alpha_1^2)^4 \left[\left(\frac{1}{\nu\lambda_1}\right)^2 \frac{K_1}{2} + \frac{k_1}{\nu\lambda_1}\right] \tag{21}$$

From (20) and (21) we conclude:

$$\alpha_0^2 \|u_m(t)\|^2 + \alpha_1^2 |Au_m(t)|^2 \leqslant k_2(t) \tag{22}$$

for all $t > 0$, where $k_2(t)$ enjoys the following properties:

(i) $k_2(t)$ is finite for all $t > 0$.

(ii) $k_2(t)$ is independent of m.

(iii) If $u^{in} \in V$, but $u^{in} \notin D(A)$, then $k_2(t)$ depends on ν, f, α_0 and α_1. Moreover, in this case $\lim_{t \to 0^+} k_2(t) = \infty$.

(iv) $\limsup_{t \to \infty} k_2(t) = R_2^2 < \infty$.

Returning to (19) and integrating over the interval $(t, t+\tau)$, for $t > 0$ and $\tau \geqslant 0$ and using (22) we get

$$\int_t^{t+\tau} (\alpha_0^2 |Au_m(s)|^2 + \alpha_1^2 |A^{3/2} u_m(s)|^2) \, ds \leqslant \bar{k}_3(t, \tau) \tag{23}$$

where $\bar{k}_3(t, \tau)$ as a function of t satisfies properties (i)–(iii) as $k_2(t)$ above. Also, there exists T_1 large enough, depends on $(\alpha_0^2\,|u^{in}|^2 + \alpha_1^2\,\|u^{in}\|^2)$, but independent of m, such that

$$\frac{1}{t}\int_0^t (\alpha_0^2\,|Au_m(s)|^2 + \alpha_1^2\,|A^{3/2}u_m(s)|^2)\,ds \leqslant 2R_2^2 \qquad \text{for all} \quad t > T_1$$

H^3-Estimate (via the vorticity)

Let us denote $v_m = \alpha_0 u_m + \alpha_1 Au_m$ and $q_m = \nabla\times v_m$. The Galerkin system (15) is equivalent to

$$\frac{dv_m}{dt} + \nu A v_m - P_m(u_m\times q_m) = P_m f$$

Let us take **Curl** of the above equation, keeping in mind that we have periodic boundary conditions, to obtain

$$\frac{dq_m}{dt} + \nu A q_m - \nabla\times(P_m(u_m\times q_m)) = \nabla\times P_m f$$

Notice that $\nabla\cdot q_m = 0$ and that $P_m q_m = q_m$. Let us take the inner product of the above equation with q_m

$$\frac{1}{2}\frac{d}{dt}\,|q_m|^2 + \nu\,\|q_m\|^2 - (\nabla\times(P_m(u_m\times q_m)), q_m) = (\nabla\times P_m f, q_m)$$

We use the identity

$$\int_\Omega (\nabla\times\phi)\cdot\psi\,dx = \int_\Omega \phi\cdot(\nabla\times\psi)\,dx \tag{24}$$

to reach

$$\frac{1}{2}\frac{d}{dt}\,|q_m|^2 + \nu\,\|q_m\|^2 - (P_m(u_m\times q_m), \nabla\times q_m) = (P_m f, \nabla\times q_m)$$

Notice that $P_m(\nabla\times q_m) = \nabla\times q_m$, therefore

$$\frac{1}{2}\frac{d}{dt}\,|q_m|^2 + \nu\,\|q_m\|^2 = (u_m\times q_m, \nabla\times q_m) + (f, \nabla\times q_m)$$

and upon applying (24)

$$\frac{1}{2}\frac{d}{dt}|q_m|^2+\nu\,\|q_m\|^2=(\nabla\times(u_m\times q_m),q_m)+(f,\nabla\times q_m)$$

For every divergence-free function ϕ, and for every ψ we have the identity

$$\nabla\times(\phi\times\psi)=-(\phi\cdot\nabla)\,\psi+(\psi\cdot\nabla)\,\phi$$

As a result, we have

$$\frac{1}{2}\frac{d}{dt}|q_m|^2+\nu\,\|q_m\|^2=-((u_m\cdot\nabla)\,q_m,q_m)+(q_m\cdot\nabla u_m,q_m)+(f,\nabla\times q_m)$$

Thanks to the identity (3) we have $((u_m\cdot\nabla)\,q_m,q_m)=0$. Now, we estimate the right hand side of the above to get:

$$\frac{1}{2}\frac{d}{dt}|q_m|^2+\nu\,\|q_m\|^2\leqslant c\,\|q_m\|_{L^4}\,\|u_m\|+|f|\,\|q_m\|$$

We use the Sobolev inequality (6a) and Young's inequality to find

$$\frac{1}{2}\frac{d}{dt}|q_m|^2+\nu\,\|q_m\|^2\leqslant c\,\|q_m\|^{3/4}\,|q_m|^{1/4}\,\|u_m\|+\frac{1}{\nu}\,|f|^2+\frac{\nu}{4}\,\|q_m\|^2$$

and we use Young's inequality again to obtain

$$\frac{1}{2}\frac{d}{dt}|q_m|^2+\frac{\nu}{2}\,\|q_m\|^2\leqslant\frac{c}{\nu^3}\,|q_m|^2\,\|u_m\|^4+\frac{1}{\nu}\,|f|^2$$

Let us denote $z_m(t)=\nu^2\lambda_1^{1/2}+|q_m(t)|^2$, then

$$\frac{dz_m}{dt}\leqslant z_m(t)\left(\frac{c\,\|u_m(t)\|^4}{\nu^3}+\frac{|f|^2}{\nu^3\lambda_1^{1/2}}\right)$$

We use (17) to obtain

$$z_m(t)\leqslant z_m(s)\,e^{\int_0^t((ck_1^2/\nu^3\alpha_1^4)+(|f|^2/\nu^3\lambda_1^{1/2}))\,d\tau}$$

for every $0\leqslant s\leqslant t$. From the definition of z_m we observe

$$z_m(s)\leqslant c(\alpha_0^2\,|Au_m(s)|^2+\alpha_1^2\,|A^{3/2}u_m(s)|^2+\nu^2\lambda_1^{1/2})$$

Now, we integrate with respect to s over $(\frac{t}{2}, t)$ and use (23) to get

$$z_m(t) \leqslant \left[\frac{2}{t}\bar{k}_3\left(\frac{t}{2}, \frac{t}{2}\right) + \nu^2\lambda_1^{1/2}\right] e^{\int_0^t ((ck_1^2/\nu^3\alpha_1^4)+(|f|^3/\nu^3\lambda_1^{1/2}))\, d\tau} =: k_3(t) \tag{25}$$

Here again $k_3(t)$ enjoys the properties (i)–(iii) of $k_2(t)$, mentioned above. □

Remark 1. Notice that by establishing the estimate (25) for $|q_m|$ one indeed is providing an upper bound for the H^3-norm of u_m. Similar estimates for the H^3-norm of u_m can be also obtained by considering first the Galerkin system (15)

$$\frac{dv_m}{dt} + \nu A v_m + P_m \tilde{B}(u_m, v_m) = P_m f$$

taking the inner product with Av_m, and then following a sequence of inequalities and estimates to achieve an upper bound for $\|v_m\|$.

Let us now summarize our estimates. For any $T > 0$ we have

(i) From (17):

$$\|u_m\|^2_{L^\infty([0,T];V)} \leqslant \frac{k_1}{\alpha_1^2} \qquad \text{or} \qquad \|v_m\|^2_{L^\infty([0,T];V')} \leqslant k_1$$

(ii) From (18) we have

$$\|u_m\|^2_{L^2([0,T];D(A))} \leqslant \frac{\bar{k}_2(T)}{\nu\alpha_1^2} \qquad \text{or} \qquad \|v_m\|^2_{L^2([0,T],H)} \frac{\bar{k}_2(T)}{\nu}$$

(iii) From (22)

$$\|u_m\|^2_{L^\infty([\tau,T];D(A))} \leqslant \frac{\tilde{k}_2(\tau)}{\alpha_1^2} \qquad \text{or} \qquad \|v_m\|^2_{L^\infty([\tau,T];H)} \leqslant \tilde{k}_2(\tau)$$

for any $\tau \in (0, T]$, where $\tilde{k}_2(\tau) \to \infty$ as $\tau \to 0^+$.

Next, we establish uniform estimates, in m, for $\frac{du_m}{dt}$ and $\frac{dv_m}{dt}$.
Recall (15)

$$\frac{d}{dt} v_m(t) = -P_m \tilde{B}(u_m, v_m) - \nu A v_m + P_m f$$

From the above estimates and part (v) of Lemma 1 we have

$$\|Av_m\|^2_{L^2([0,T],\,D(A)')} \leqslant \frac{c\bar{k}_2(T)}{\nu}$$

and

$$\|P_m\tilde{B}(u_m, v_m)\|_{D(A)'} \leqslant c\,|u_m|^{1/2}\,\|u_m\|^{1/2}\,|v_m| + \frac{c}{\lambda_1^{1/4}}\,|v_m|\,\|u_m\|$$

Consequently

$$\|P_m\tilde{B}(u_m, v_m)\|^2_{L^2([0,T],\,D(A)')} \leqslant \frac{ck_1\bar{k}_2(T)}{\nu\lambda_1^{1/2}\alpha_1^2}$$

Therefore

$$\left\|\frac{dv_m}{dt}\right\|^2_{L^2([0,T];\,D(A)')} \leqslant \tilde{k}(T)$$

and in particular

$$\left\|\frac{du_m}{dt}\right\|^2_{L^2([0,T],\,H)} \leqslant \frac{\tilde{k}(T)}{\alpha_1^2}$$

where $\tilde{k}(T)$ is a constant which depends on ν, λ_1, f, α_0, α_1 and T.

By Aubin's Compactness Theorem (see, e.g., [13] or [31]) we conclude that there is a subsequence $u_{m'}(t)$ such that

$$\begin{aligned} u_{m'} &\to u(t) && \text{weakly in } L^2([0,T], D(A)) \\ u_{m'} &\to u(t) && \text{strongly in } L^2([0,T], V), \quad \text{and} \\ u_{m'} &\to u && \text{in } C([0,T], H) \end{aligned}$$

or equivalently

$$\begin{aligned} v_{m'} &\to v && \text{weakly in } L^2([0,T], H) \\ v_{m'} &\to v && \text{strongly in } L^2([0,T], V'), \quad \text{and} \\ v_{m'} &\to v && \text{in } C([0,T], D(A)') \end{aligned}$$

where v is given in (9).

Let us relabel $u_{m'}$ and $v_{m'}$ by u_m and v_m respectively. Let $w \in D(A)$, then from (15) we have

$$(v_m(t), w) + \nu \int_{t_0}^{t} (v_m(s), Aw)\, ds + \int_{t_0}^{t} (\tilde{B}(u_m(s), v_m(s), P_m w)\, ds$$

$$= (v_m(t_0), w) + (f, P_m w)(t - t_0)$$

for all t_0, $t \in [0, T]$. Since $v_m \to v$ weakly in $L^2([0, T]; H)$ then $v_m(s) \to v(s)$ weakly in H, for every $s \in [0, T] \setminus E$, where $|E| = 0$. In particular, there is a subsequence of v_m, which we will also denote v_m, such that $v_m(s) \to v(s)$ strongly in V' and $D(A)'$ for every $s \notin E$.

Now, it is clear that

$$\lim_{m \to \infty} \int_{t_0}^{t} (v_m(s), Aw)\, ds = \int_{t_0}^{t} (v(s), Aw)\, ds$$

also that $\lim_{m \to \infty} |P_m Aw - Aw| = \lim_{m \to \infty} |w - w_m| = 0$. On the other hand

$$\left| \int_{t_0}^{t} (\tilde{B}(u_m(s), v_m(s)), P_m w) - \langle \tilde{B}(u(s), v(s), w(s) \rangle_{D(A)'}\, ds \right| \leqslant I_m^{(1)} + I_m^{(2)} + I_m^{(3)}$$

$$I_m^{(1)} = \left| \int_{t_0}^{t} \langle \tilde{B}(u_m(s), v_m(s)), P_m w(s) - w(s) \rangle_{D(A)'}\, ds \right|$$

by part (v) of Lemma 1 we have

$$I_m^{(1)} \leqslant \frac{c}{\lambda_1^{1/4}} \int_{t_0}^{t} (\|u_m(s)\|\, |v_m(s)|\, |P_m Aw - Aw|)\, ds$$

applying Cauchy–Schwarz inequality

$$I_m^{(1)} \leqslant \frac{c}{\lambda_1^{1/4}} \left(\int_0^T \|u_m(s)\|^2\, ds \right)^{1/2} \left(\int_0^T \|v_m(s)\|^2\, ds \right)^{1/2} |P_m Aw - Aw|$$

and hence $\lim_{m \to \infty} I_m^{(1)} = 0$.

$$I_m^{(2)} = \left| \int_{t_0}^{t} \langle \tilde{B}(u_m(s) - u_m(s), v_m(s), w \rangle_{D(A)'}\, ds \right|$$

Again thanks to part (v) of Lemma 1

$$I_m^{(2)} \leqslant \frac{c}{\lambda_1^{1/4}} \int_{t_0}^{t} \|u_m(s)-u(s)\| \, |v_m(s)| \, |Aw| \, ds$$

and by Cauchy–Schwarz

$$I_m^{(2)} \leqslant \frac{c}{\lambda_1^{1/4}} \left(\int_0^T \|u_m(s)-u(s)\|^2 \, ds \right)^{1/2} \left(\int_0^T |v_m(s)|^2 \, ds \right)^{1/2} |Aw|$$

Since v_m bounded in $L^2([0,T]; H)$ and $u_m \to u$ in $L^2([0,T], V)$ we conclude that

$$\lim_{m\to\infty} I_m^{(2)} = 0$$

Finally,

$$I_m^{(3)} = \left| \int_{t_0}^{t} \langle \tilde{B}(u, v - v_m), w \rangle_{V'} \, ds \right|$$

by virtue of part (v) in Lemma 1, and since $v_m \to v$ weakly in $L^2([0,T]; H)$, we obtain

$$\lim_{m\to\infty} I_m^{(3)} = 0$$

As a result of the above we have for every $t_0, t \in [0,T] \setminus E$

$$\begin{aligned}(v(t), w) + \nu \int_{t_0}^{t} (v(s), Aw) \, ds + \int_{t_0}^{t} \langle \tilde{B}(u(s), v(s), w \rangle_{D(A)'} \, ds \\ = (v(t_0), w) + (f, w)(t - t_0) \qquad (26)\end{aligned}$$

for every $w \in D(A)$. Notice that since $\|v_m(t)\|_{L^\infty([0,T],V')} \leqslant k_1$, and since $v_m(t) \to v(t)$ strongly in V' for every $t \in [0,T] \setminus E$, we have $\|v(t)\|_{L^\infty([0,T],V')} \leqslant k_1$. Moreover, because $D(A)$ is dense in V', (26) implies that $v(t) \in C([0,T]; V')$ or equivalently $u(t) \in C([0,T], V)$.

In particular, from (26) we conclude the existence of a regular solution for the system (8).

Uniqueness of Regular Solutions

Next we will show the continuous dependence of regular solutions on the initial data and, in particular, we show the uniqueness of regular solutions.

Let u and $\bar{u}$ be any two solutions of Eq. (8) on the interval $[0, T]$, with initial values $u(0) = u^{in}$ and $\bar{u}(0) = \bar{u}^{in}$ respectively. Let us denote $v = (\alpha_0^2 u + \alpha_1^2 Au)$, $\bar{v} = (\alpha_0^2 \bar{u} + \alpha_1^2 A\bar{u})$, $\delta u = u - \bar{u}$, and by $\delta v = v - \bar{v}$. Then from Eq. (8) we get:

$$\frac{d}{dt} v + \nu A v + \tilde{B}(\delta u, v) + \tilde{B}(\bar{u}, \delta v) = 0$$

The above equation holds in $L^2([0, T], D(A)')$, since δu belongs to $L^2([0, T], D(A))$, the dual space of $L^2([0, T], D(A)')$, we use Lemma 1 to obtain

$$\left\langle \frac{d}{dt} v, \delta u \right\rangle_{D(A)'} + \nu(\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2) + \langle \tilde{B}(\bar{u}, \delta v), \delta u \rangle_{D(A)'} = 0$$

Notice that $\langle \frac{dv}{dt}, \delta u \rangle_{D(A)'} = \frac{1}{2} \frac{d}{dt} (\alpha_0^2 |\delta u|^2 + \alpha_1^2 \|\delta u\|^2)$, (see, e.g., [35], Chap. III, Lemma 1.2). As a result we have:

$$\frac{1}{2} \frac{d}{dt} (\alpha_0^2 |\delta u|^2 + \alpha_1^2 \|\delta u\|^2) + \nu(\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2) + \langle \tilde{B}(\bar{u}, \delta v), \delta u \rangle_{D(A)'} = 0$$

Now we use part (vi) of Lemma 1 to get

$$\frac{1}{2} \frac{d}{dt} (\alpha_0^2 |\delta u|^2 + \alpha_1^2 \|\delta u\|^2) + \nu(\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2)$$
$$\leqslant c(\|\bar{u}\|^{1/2} |A\bar{u}|^{1/2} |\delta v| \, \|\delta u\| + |A\bar{u}| \, |\delta v| \, |\delta u|^{1/2} \|\delta u\|^{1/2})$$

and by Young's inequality we have:

$$\frac{1}{2} \frac{d}{dt} (\alpha_0^2 |\delta u|^2 + \alpha_1^2 \|\delta u\|^2) + \nu(\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2)$$
$$\leqslant \frac{c}{\nu} (\|\bar{u}\| \, |A\bar{u}| \, \|\delta u\|^2 + |A\bar{u}|^2 \, |\delta u| \, \|\delta u\|) + \frac{\nu}{2} (\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2)$$
$$\leqslant \frac{c}{2\nu\alpha_1^2 \lambda_1^{1/2}} |A\bar{u}|^2 \, (\alpha_0^2 |\delta u|^2 + \alpha_1^2 \|\delta u\|^2) + \frac{\nu}{2} (\alpha_0^2 \|\delta u\|^2 + \alpha_1^2 |A\,\delta u|^2)$$

Hence,

$$(\alpha_0^2 |\delta u(t)|^2 + \alpha_1^2 \|\delta u(t)\|^2) \leqslant (\alpha_0^2 |\delta u(0)|^2 + \alpha_1^2 \|\delta u(0)\|^2) \exp\left(\int_0^t \frac{c\, |A\bar{u}(s)|^2}{\nu \alpha_1^2 \lambda_1^{1/2}} \, ds \right)$$

Since $\bar{u} \in L^2([0,T], D(A))$ we conclude the continuous dependence of the solutions of (8) on the initial data on any bounded interval $[0,T]$. In particular, we conclude the uniqueness of regular solutions.

Remark 2. Following the techniques introduced in [22] (see also [19] and [29]) we can easily show that if the forcing term, f, in Eq. (8) belongs to a certain Gevrey class of regularity then the solutions of (8) will instantaneously belong to a similar Gevrey class of regularity. Specifically, in this situation the solution will become analytic in space and time. In particular, one can also provide uniform lower bounds for the radii of analyticity (in space and in time) for the solutions that lie in the global attractor (see Section 4 for the existence of a compact finite dimensional global attractor.) As a result of this Gevrey regularity one can also show that the Galerkin approximating solutions, introduced earlier, converge exponentially fast in the wave number m, as $m \to \infty$ (see, e.g., [15], [23], and [28]). Furthermore, one can use this Gevrey result to establish rigorous estimates for the dissipative small scales in Eq. (8) (see, e.g., [16]).

4. ESTIMATING THE DIMENSION OF THE GLOBAL ATTRACTOR

Let $S(t)$ denote the semi-group of the solution operator to Eq. (8), i.e., $u(t) = S(t)\, u^{in}$. It can be easily shown, from the proof of Theorem 3 and Rellich's Lemma (see [1]), that $S(t)$ is a compact semi-group. Let us recall (see (13)) that the ball $B_1 = \{u \in V : \|u\| \leqslant \frac{R_0}{\alpha_1}\}$ is an absorbing ball, in the space V. Consequently, the Eq. (8) has a nonempty compact global attractor

$$\mathscr{A} = \bigcap_{s>0} \left(\bigcup_{t \geqslant s} S(t)\, B_1 \right)$$

(see, e.g., [2], [13], [24] and [36]).

In this section we employ the trace formula (see, e.g., [12], [13], and [36]) to estimate the Hausdorff and fractal (box counting) dimensions of the global attractor $\mathscr{A}$ in terms of the physical parameters of the Eq. (1). First, let us recall the Lieb–Thirring inequality

Lemma 4 (The Lieb–Thirring inequality). *Let $\{\psi_j\}_{j=1}^N$ be an orthonormal set of functions in $(H)^k = \underbrace{H \oplus \cdots \oplus H}_{k\text{-times}}$. Then there exists a constant C_{LT}, which depends on k, but independent of N such that*

$$\int_\Omega \left(\sum_{j=1}^N \psi_j(x) \cdot \psi_j(x) \right)^{5/3} dx \leqslant C_{LT} \sum_{j=1}^N \int_\Omega (\nabla \psi_j(x) : \nabla \psi_j(x))\, dx \qquad (27)$$

Next we will present a new technical lemma which we will use in estimating the dimension of the global attractor.

Lemma 5. *Let $\{\varphi_j\}_{j=1}^N \subset V$ be an orthonormal set with respect to the inner product $[\cdot,\cdot]$ which is defined in (2), i.e.,*

$$[\varphi_i, \varphi_j] = \alpha_0^2(\varphi_i, \varphi_j) + \alpha_1^2((\varphi_i, \varphi_j)) = \delta_{ij}$$

Let $\psi_j(x) = (\alpha_0 \varphi_j(x),\ \alpha_1 \frac{\partial}{\partial x_1}\varphi_j(x),\ \alpha_1 \frac{\partial}{\partial x_2}\varphi_j(x),\ \alpha_1 \frac{\partial}{\partial x_3}\varphi_j(x))^T$, and $\varphi^2(x) = \sum_{j=1}^N (\varphi_j(x)\cdot\varphi_j(x))$ it . Then, there exists a constant C_F, which is independent of N, such that

$$\|\varphi(x)\|_{L^\infty}^2 \leqslant \frac{C_F}{\alpha_1^2}\left(\sum_{j=1}^N \int_\Omega (\nabla\psi_j(x) \colon \nabla\psi_j(x))\,dx\right)^{1/2} \tag{28}$$

Proof. Let $\xi_j \in \mathbb{R}$, $j = 1,\dots, N$, to be chosen later. By Agmon's inequality (7) we have

$$\begin{aligned}
&\alpha_0^2\left|\sum_{j=1}^N \xi_j (A^{-1/2}\varphi_j)(x)\right|^2 + \alpha_1^2\left|\sum_{j=1}^N \xi_j\varphi_j(x)\right|^2 \\
&\quad\leqslant c\alpha_0^2\left|\sum_{j=1}^N \xi_j\varphi_j\right|\left\|\sum_{j=1}^N \xi_j\varphi_j\right\| + c\alpha_1^2\left\|\sum_{j=1}^N \xi_j\varphi_j\right\|\left|\sum_{j=1}^N \xi_j A\varphi_j\right|
\end{aligned}$$

and by Cauchy–Schwarz

$$\begin{aligned}
&\alpha_0^2\left|\sum_{j=1}^N \xi_j (A^{-1/2}\varphi_j)(x)\right|^2 + \alpha_1^2\left|\sum_{j=1}^N \xi_j\varphi_j(x)\right|^2 \\
&\leqslant c\left(\alpha_0^2\left|\sum_{j=1}^N \xi_j\varphi_j\right|^2 + \alpha_1^2\left\|\sum_{j=1}^N \xi_j\varphi_j\right\|^2\right)^{1/2}\left(\alpha_0^2\left\|\sum_{j=1}^N \xi_j\varphi_j\right\|^2 + \alpha_1^2\left|\sum_{j=1}^N \xi_j A\varphi_j\right|^2\right)^{1/2} \\
&\leqslant c\left[\sum_{j=1}^N \xi_j\varphi_j, \sum_{j=1}^N \xi_j\varphi_j\right]^{1/2}\left(\sum_{j=1}^N \xi_j^2\right)^{1/2}\left(\alpha_0^2\sum_{j=1}^N \|\xi_j\|^2 + \alpha_1^2\sum_{j=1}^N |A\varphi_j|^2\right)^{1/2}
\end{aligned}$$

Since $[\varphi_i, \varphi_j] = \delta_{ij}$ we have

$$\begin{aligned}
&\alpha_0^2\left|\sum_{j=1}^N \xi_j (A^{-1/2}\varphi_j)(x)\right|^2 + \alpha_1^2\left|\sum_{j=1}^N \xi_j\varphi_j(x)\right|^2 \\
&\quad\leqslant c\left(\sum_{j=1}^N \xi_j^2\right)\left(\alpha_0^2\sum_{j=1}^N \|\varphi_j\|^2 + \alpha_1^2\sum_{j=1}^N |A\varphi_j|^2\right)^{1/2} \\
&\quad\leqslant c\left(\sum_{j=1}^N \xi_j^2\right)\left(\sum_{j=1}^N \int_\Omega (\nabla\psi_j : \nabla\psi_j)\,dx\right)^{1/2}
\end{aligned}$$

Let $i \in \{1, 2, 3\}$ be fixed, we choose $\xi_j = \varphi_{ji}(x)$, from the above we have

$$\alpha_1^2 \left(\sum_{j=1}^{N} \varphi_{ji}^2(x) \right)^2 \leqslant c \left(\sum_{j=1}^{N} \varphi_{ji}^2(x) \right) \left(\sum_{j=1}^{N} \int_\Omega (\nabla \psi_j(x) : \nabla \psi_j(x)) \, dx \right)^{1/2}$$

Now we sum over i, $i = 1, 2, 3$, to reach

$$\alpha_1^2 \varphi^2(x) \leqslant \left(\sum_{j=1}^{N} \int_\Omega (\nabla \psi_j(x) : \nabla \psi_j(x)) \, dx \right)^{1/2}$$

which concludes our proof. □

Theorem 6. *The Hausdorff and fractal dimensions of the global attractor of the viscous Camassa–Holm (NS-α) equations, $d_H(\mathscr{A})$ and $D_F(\mathscr{A})$, respectively, satisfy:*

$$d_H(\mathscr{A}) \leqslant d_F(\mathscr{A}) \leqslant c \max \left\{ G^{4/3} \left(\frac{1}{\gamma \alpha_1^2 \lambda_1} \right)^{2/3}, G^{3/2} \left(\frac{1}{\alpha_0^4 \gamma^2 \lambda_1 \alpha_1^2} \right)^{3/8} \right\}$$

where $G = \frac{|f|}{\nu^2 \lambda_1^{3/4}}$ is the Grashoff number and, as before, $\frac{1}{\gamma} = \min \left\{ \frac{1}{\alpha_0^2}, \frac{1}{\alpha_1^2 \lambda_1} \right\}$.

Proof. The linearized equation (8) about a regular solution $u(t)$ takes the form

$$\frac{d}{dt} \delta v + \nu A \, \delta v + \tilde{B}(\delta u, v) + \tilde{B}(u, \delta v) = 0 \tag{29}$$

where $v(t) = \alpha_0^2 u + \alpha_1^2 A u$ and $\delta v = \alpha_0^2 \delta u + \alpha_1^2 A \, \delta u$. Notice that δu evolves according to the equation

$$\frac{d}{dt} \delta u + \nu A \, \delta u + (\alpha_0^2 I + \alpha_1^2 A)^{-1} \left[\tilde{B}(\delta u, \alpha_0^2 u + \alpha_1^2 A u) + \tilde{B}(u, \alpha_0^2 \, \delta u + \alpha_1^2 A \, \delta u) = 0 \right. \tag{30}$$

which we write symbolically as

$$\frac{d}{dt} \delta u + T(t) \, \delta u = 0$$

Let $\delta u_j(0)$, for $j=1,\dots,N$, be a set of linearly independent vectors in V, and let $\delta u_j(t)$ be the corresponding solutions of (30) with initial value $\delta u_j(0)$, for $j=1,\dots,N$. We denote

$$\mathscr{T}_N(t) = \operatorname{Trace}(P_N(t)\, T(t)|_{P_N(t)\, V}) \tag{31}$$

where $P_N(t)\, V = \mathbb{R}\, \delta v_1(t) + \mathbb{R}\, \delta v_2(t) + \cdots + \mathbb{R}\, \delta v_N(t)$, and $P_N(t)$ is the orthogonal projector of V onto $P_N(t)\, V$ with respect to the inner product $[\cdot,\cdot]$ given in (2).

Let $\{\varphi_j(t)\}_{j=1}^N$ be an orthonormal basis, with respect to inner product $[\cdot,\cdot]$ of the space $P_N V$, i.e., $[\varphi_i, \varphi_j] = \delta_{ij}$, $i, j = 1,\dots, N$. We set

$$\psi_j = \left(\alpha_0 \varphi_j, \alpha_1 \frac{\partial}{\partial x_1} \varphi_j, \alpha_1 \frac{\partial}{\partial x_2} \varphi_j, \alpha_1 \frac{\partial}{\partial x_3} \varphi_j \right)^T$$

Notice that $(\psi_j, \psi_k) = \delta_{jk}$, $j, k = 1,\dots, N$. We set

$$\begin{aligned}\psi^2(x,t) &= \sum_{j=1}^N (\psi_j(x,t)\cdot\psi_j(x,t))\\ &= \alpha_0^2 \sum_{j=1}^N \varphi_j(x,t)\cdot\varphi_j(x,t) + \alpha_1^2 \sum_{j=1}^N (\nabla\varphi_j(x,t) : \nabla\varphi_j(x,t))\end{aligned}$$

Notice that by the Lieb–Thirring inequality (27)

$$\int_\Omega (\psi(x,t))^{10/3}\, dx \leqslant C_{LT} Q_N(t)$$

where $Q_N(t) := \sum_{j=1}^N \int_\Omega (\nabla\psi_j(x,t) : \nabla\psi_j(x,t))\, dx$.

Let us denote $\theta_j(x,t) = \alpha_0^2 \varphi_j(x,t) + \alpha_1^2 A\varphi_j(x,t)$, for $j=1,\dots,N$. From (31) we have

$$\begin{aligned}\mathscr{T}_N(t) &= \sum_{j=1}^N [T(t)\, \varphi_j(\cdot,t), \varphi_j(\cdot,t)]\\ &= \nu \sum_{j=1}^N [A\varphi_j, \varphi_j] + \sum_{j=1}^N (\tilde{B}(\varphi_j, v), \varphi_j) + \sum_{j=1}^N (\tilde{B}(u, \theta_j), \varphi_j)\end{aligned}$$

and by virtue of (4) we have

$$\mathscr{T}_n(t) = \nu \sum_{j=1}^N [A\varphi_j, \varphi_j] + \sum_{j=1}^N (\tilde{B}(u, \theta_j), \varphi_j)$$

Observe that

$$\sum_{j=1}^{N} [A\varphi_j, \varphi_j] = \alpha_0^2 \sum_{j=1}^{N} (A\varphi_j, \varphi_j) + \alpha_1^2 \sum_{j=1}^{N} (A\varphi_j, A\varphi_j)$$
$$= \sum_{j=1}^{N} \int_\Omega (\nabla\psi_j(x,t) : \nabla\psi_j(x,t))\, dx = Q_N(t)$$

Thus

$$\mathscr{T}_N(t) = \nu Q_N(t) + \mathscr{I}_N(t) \tag{32}$$

where $\mathscr{I}_N(t) := \sum_{j=1}^{N} (\tilde{B}(u, \theta_j), \varphi_j)$. Let us now estimate $\mathscr{I}_N(t)$. Using (4) and (3) we have

$$\mathscr{I}_N(t) = \sum_{j=1}^{N} [((u\cdot\nabla)\,\theta_j, \varphi_j) + ((\varphi_j\cdot\nabla)\,u, \theta_j)]$$
$$= \sum_{j=1}^{N} [-((u\cdot\nabla)\,\varphi_j, \theta_j) + ((\varphi_j\cdot\nabla)\,u, \theta_j)]$$

again by (3)

$$\mathscr{I}_N(t) = \sum_{j=1}^{N} [-\alpha_1^2((u\cdot\nabla)\,\varphi_j, A\varphi_j) + \alpha_0^2((\varphi_j\cdot\nabla)\,u, \varphi_j) + \alpha_1^2((\varphi_j\cdot\nabla)\,u, A\varphi_j)$$

integrating by parts and using (3)

$$\mathscr{I}_N(t) = \sum_{j=1}^{N} \sum_{k=1}^{3} \alpha_1^2 \left(\left(\frac{\partial}{\partial x_k} u\cdot\nabla\right)\varphi_j, \frac{\partial}{\partial x_k}\varphi_j\right) + \alpha_0^2 \sum_{j=1}^{N} ((\varphi_j\cdot\nabla)\,u, \varphi_j)$$
$$-\alpha_1^2 \sum_{j=1}^{N} \sum_{k=1}^{3} \left(\left(\frac{\partial}{\partial x_k} u\cdot\nabla\right) u, \frac{\partial}{\partial x_k}\varphi_j\right)$$
$$-\alpha_1^2 \sum_{j=1}^{N} \sum_{k=1}^{3} \left((\varphi_j\cdot\nabla)\frac{\partial}{\partial x_k} u, \frac{\partial}{\partial x_k}\varphi_j\right)$$

Therefore,

$$|\mathscr{I}_N(t)| \leqslant c \int_\Omega (\nabla u(x,t) : \nabla u(x,t))^{1/2}\, \psi^2(x,t)\, dx$$
$$+ c\alpha_1^2 \int_\Omega \left[\left(\sum_{i,k=1}^{3} \left(\frac{\partial^2 u}{\partial x_i\, \partial x_k}(x,t)\right)^2\right) \varphi^2(x,t)\right.$$
$$\left.\times \left(\sum_{j=1}^{N} (\nabla\varphi_j(x,t) : \nabla\varphi_j(x,t))\right)\right]^{1/2} dx$$

where $\varphi^2(x, t) = \sum_{j=1}^{N} (\varphi_j(x, t) \cdot \varphi_j(x, t))$. As a result we have

$$|\mathscr{I}_N(t)| \leqslant c \int_\Omega (\nabla u(x, t) : \nabla u(x, t))^{1/2} \, \psi^2(x, t) \, dx$$

$$+ \alpha_1 \int_\Omega \psi(x, t) \, \varphi(x, t) \left(\sum_{k, i=1}^{3} \left(\frac{\partial^2}{\partial x_i \, \partial x_k} u(x, t) \right)^2 \right)^{1/2} dx$$

Thanks to (28) we have

$$|\mathscr{I}_N(t)| \leqslant c \int_\Omega (\nabla u(x, t) : \nabla u(x, t))^{1/2} \, \psi^2(x, t) \, dx$$

$$+ C_F^{1/2} Q_N^{1/4}(t) \int_\Omega \left(\sum_{k, i=1}^{3} \left(\frac{\partial^2}{\partial x_i \, \partial x_k} u(x, t) \right)^2 \right)^{1/2} \psi(x, t) \, dx \qquad (33)$$

and by the Hölder inequality we get

$$|\mathscr{I}_n(t)| \leqslant c \, \|\nabla u\|_{L^{5/2}} \left(\int_\Omega (\psi(x, t))^{10/3} \, dx \right)^{3/5} + c Q_N^{1/4}(t) \, |Au| \left(\int_\Omega \psi^2(x, t) \, dx \right)^{1/2}$$

Since $[\varphi_i, \varphi_j] = \delta_{ij}$ we have $\int_\Omega \psi^2(x, t) \, dx = N$. Therefore, the above gives

$$|\mathscr{I}_N(t)| \leqslant c \, \|\nabla u\|_{L^{5/2}} \left(\int_\Omega (\psi(x, t))^{10/3} \, dx \right)^{3/5} + c Q_N^{1/4}(t) \, |Au| \, N^{1/2}$$

Applying the Lieb–Thirring inequality (27) we obtain

$$|\mathscr{I}_N(t)| \leqslant c c_{LT} \, \|\nabla u\|_{L^{5/2}} \left(\sum_{j=1}^{N} \int_\Omega (\nabla \psi_j(x, t) : \nabla \psi_j(x, t)) \, dx \right)^{3/5}$$

$$+ c Q_N^{1/4}(t) \, |Au| \, N^{1/2}$$

that is

$$|\mathscr{I}_N(t)| \leqslant c \, \|\nabla u\|_{L^{5/2}} \, Q_N^{3/5}(t) + c Q_N^{1/4}(t) \, |Au| \, N^{1/2}$$

Applying Young's inequality

$$|\mathscr{I}_N(t)| \leqslant \frac{c}{\nu^{3/2}} \, \|\nabla u\|_{L^{5/2}}^{5/2} + \frac{\nu}{2} \, Q_N(t) + c \, |Au|^{4/3} \left(\frac{N^2}{\nu} \right)^{1/3}$$

then using Hölder's inequality

$$|\mathscr{I}_N(t)| \leqslant \frac{c}{\nu^{3/2}} \|\nabla u\|_{L^2}^{7/4} \|\nabla u\|_{L^6}^{3/4} + \frac{\nu}{2} Q_N(t) + c\,|Au|^{4/3} \left(\frac{N^2}{\nu}\right)^{1/3}$$

and by virtue of the Sobolev inequality (6c) we obtain

$$|\mathscr{I}_N(t)| \leqslant \frac{c}{\nu^{3/2}} \|u\|^{1/2} \|u\|^{5/4} |Au|^{3/4} + \frac{\nu}{2} Q_N(t) + c\,|Au|^{4/3} \left(\frac{N^2}{\nu}\right)^{1/3}$$

using Young's inequality again we reach

$$|\mathscr{I}_N(t)| \leqslant \frac{c}{\nu^{3/2}} \frac{\|u\|^{1/2}}{\alpha_0^{5/4}\alpha_1^{3/4}} (\alpha_0^2 \|u\|^2 + \alpha_1^2 |Au|^2) + \frac{\nu}{2} Q_N(t) + c\,|Au|^{4/3} \left(\frac{N^2}{\nu}\right)^{1/3}$$

Substituting in (32) we get:

$$\mathscr{T}_N(t) \leqslant \frac{\nu}{2} Q_N(t) - \frac{c}{\nu^{3/2}} \frac{\|u\|^{1/2}}{\alpha_0^{5/4}\alpha_1^{3/4}} (\alpha_0^2 \|u\|^2 + \alpha_1^2 |Au|^2) - c\,|Au|^{4/3} \left(\frac{N^2}{\nu}\right)^{1/3}$$

Now, we require N to be large enough such that

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathscr{T}_N(s)\, ds > 0 \tag{34}$$

According to the trace formula (see [12], [13] or [36]) such an N will be an upper bound for the fractal and Hausdorff dimensions of the global attractor. Observe that from the asymptotic behavior of the eigenvalues of the operator A there is a constant c_0 such that

$$\lambda_j \geqslant c_0 \lambda_1 j^{2/3} \qquad \text{for} \quad j = 1, 2, \ldots$$

Therefore, since $Q_N(t)$ is the trace of the operator A restricted to some subspace of dimension N, we have

$$Q_N(t) \geqslant \sum_{j=1}^{N} \lambda_j \geqslant c\lambda_1 N^{5/3} \tag{35}$$

Let us require N to be large enough so that

$$\nu \lambda_1 N^{5/3} \geqslant c \limsup_{T \to \infty} \left(\frac{1}{T} \int_0^T |Au(s)|^{4/3}\, ds\right) \left(\frac{N^2}{\nu}\right)^{1/3}$$

and

$$\nu\lambda_1 N^{5/3} \geqslant \frac{c}{\nu^{3/2}\alpha_0^{5/4}\alpha_1^{3/4}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \|u(s)\|^{1/2} \left(\alpha_0^2\,\|u(s)\|^2+\alpha_1^2\,|Au(s)|^2\right) ds$$

For such an N the inequality (34) is satisfied, and thus N provides an upper bound for the fractal and Hausdorff dimensions of the global attractor.

By Hölder's inequality we have

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{4/3}\, ds \leqslant \limsup_{T\to\infty} \left(\frac{1}{T}\int_0^T |Au(s)|^2\right)^{2/3}$$

and thanks to (14) we get

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{4/3}\, ds \leqslant \left(\frac{G^2\nu^2\lambda_1^{1/2}}{\gamma\alpha_1^2}\right)^{2/3}$$

Therefore, from the above, (13) and (14) we have

$$d_H(\mathscr{A}) \leqslant d_F(\mathscr{A}) \leqslant c \max\left\{G^{4/3}\left(\frac{1}{\gamma\alpha_1^2\lambda_1}\right)^{2/3}, G^{3/2}\left(\frac{1}{\alpha_0^2\gamma^2\lambda_1\alpha_1^2}\right)^{3/8}\right\}$$

which concludes our proof. □

5. NUMBERS OF DEGREES OF FREEDOM IN TURBULENT FLOWS

An argument from the classical theory of turbulence [30] suggests that there are finitely many degrees of freedom in turbulent flows. Heuristic physical arguments are used to justify this assertion and to provide an estimate for this number of degrees of freedom by dividing a typical length scale of the flow, L, by the Kolmogorov dissipation length scale and taking the third power in 3D. The resulting formula is usually expressed explicitly in terms of the mean rate of dissipation of energy and the kinematic viscosity. In analogy with this heuristic approach we will derive here an estimate for the "dissipation" length scale (i.e., what would correspond to the Kolmogorov length scale) for the viscous Camassa–Holm (NS-α) equations in terms of the mean rate of dissipation of "energy" and the kinematic viscosity. We will also show that the corresponding number of degrees of freedom is proportional to the dimension of the global attractor. This, in a sense, suggests that in the absence of boundary effects (e.g., in the case of periodic boundary conditions) the viscous Camassa–Holm equations represent, very well, the averaged equation of motion of turbulent flows. Hence, one is tempted to use the viscous Camassa–Holm equations as a closure model for the Reynolds equations, which represent the ensemble-averaged Navier–Stokes equations. Indeed, this idea motivated our studies in [5], [6] and [7], and it also led to a physical derivation in [26] (see also [6]) of the viscous Camassa–Holm (NS-α) equations, in the inviscid case, as averaged equations.

As before, we denote $v = \alpha_0^2 u + \alpha_1^2 Au$, hence Eq. (8) and Eq. (10) take the form

$$\begin{gathered} \frac{dv}{dt} + \nu Av + \tilde{B}(u, v) = f \\ u(0) = u^{in} \end{gathered} \tag{36}$$

In analogy with Kolmogorov's mean rate of dissipation of energy in turbulent flow [30] we define

$$\epsilon(u^{in}) = \lambda_1^{3/2} \nu \left[\limsup_{T \to \infty} \frac{1}{T} \int_0^T (\alpha_0^2 \, \|u(s)\|^2 + \alpha_1^2 \, |Au(s)|^2) \, ds \right] \tag{37}$$

the mean rate of dissipation of "energy," and

$$\bar{\epsilon} = \sup_{u^{in} \in \mathscr{A}} \epsilon(u^{in})$$

the maximal mean rate of dissipation of energy on the attractor. From equation (33), and since $\int \psi^2(x, t) \, dx = N$, we have:

$$\begin{aligned} |\mathscr{I}_N(t)| \leqslant c \int_\Omega (\nabla u(x, t) : \nabla u(x, t))^{1/2} \, \psi^2(x, t) \, dx \\ + c C_F^{1/2} Q_N^{1/4}(t) \, |Au| \, N^{1/2} \end{aligned}$$

and by Hölder's inequality we have

$$|\mathscr{I}_N(t)| \leqslant c \, \|\nabla u\|_{L^6} \, \|\psi^2\|_{L^{6/5}} + c Q_N^{1/4}(t) \, |Au| \, N^{1/2}$$

Again by Hölder's inequality and (6) we get

$$\begin{aligned} |\mathscr{I}_N(t)| \leqslant c \, |Au(t)| \left(\int_\Omega \psi^2(x, t) \, dx \right)^{7/12} \left(\int_\Omega (\psi^2(x, t))^{5/3} \, dx \right)^{1/4} \\ + c Q_N^{1/4}(t) \, |Au(t)| \, N^{1/2} \end{aligned}$$

Using the Lieb–Thirring inequality (27) and $\int \psi^2(x,t)\,dx = N$ we obtain

$$|\mathscr{I}_N(t)| \leqslant c\,|Au(t)|\,N^{7/12} Q_N^{1/4}(t) + c Q_n^{1/4}(t)\,|Au(t)|\,N^{1/2}$$

and hence

$$|\mathscr{I}_N(t)| \leqslant c\,|Au(t)|\,N^{7/12} Q_N^{1/4}(t)$$

After applying Young's inequality to the above and substituting in Eq. (32) we obtain

$$\mathscr{T}_N(t) \geqslant \frac{\nu}{2}\, Q_N(t) - c\nu^{-1/3} N^{7/9}\,|Au(t)|^{4/3}$$

Therefore, in order to satisfy (34), and based on the above, it suffices to choose N large enough so that for every trajectory $u(t)$ on the global attractor $\mathscr{A}$ we have

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \left[\frac{\nu}{2}\, Q_N(t) - c\nu^{-1/3} N^{7/9}\,|Au(t)|^{4/3}\right] dt > 0$$

Therefore, such a large N is an upper bound for the dimension of the global attractor. Based on (35) it suffices to require

$$\nu^{4/3}\lambda_1 N^{5/3} \cdot N^{-7/9} > c \limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{4/3}\,ds$$

for every solution $u^{in} \in \mathscr{A}$, i.e.,

$$\nu^{4/3}\lambda_1 N^{8/9} > c \sup_{u^{in}\in\mathscr{A}} \left(\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{4/3}\,ds\right)$$

On the other hand, using Hölder's inequality and (37) we have

$$\begin{aligned}
&\sup_{u^{in}\in\mathscr{A}} \left(\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{4/3}\,ds\right)\\
&\quad\leqslant \sup_{u^{in}\in\mathscr{A}} \left(\limsup_{T\to\infty} \frac{1}{T}\int_0^T |Au(s)|^{2}\,ds\right)^{2/3}\\
&\quad\leqslant \sup_{u^{in}\in\mathscr{A}} \left(\limsup_{T\to\infty} \frac{1}{T\alpha_1^2}\int_0^T (\alpha_0^2\,\|u(s)\|^2 + \alpha_1^2\,|Au(s)|^2)\,ds\right)^{2/3}\\
&\quad\leqslant \sup_{u^{in}\in\mathscr{A}} \left(\frac{\epsilon(u^{in})}{\nu\lambda_1^{1/2}(\alpha_1^2\lambda_1)}\right)^{2/3} \leqslant \left(\frac{\bar{\epsilon}}{\nu\lambda_1^{1/2}(\alpha_1^2\lambda_1)}\right)^{2/3}
\end{aligned}$$

Therefore, every N large enough such that

$$N \geqslant c \left(\frac{\bar{\epsilon}}{\nu^3 \lambda_1^2 (\alpha_1^2 \lambda_1)} \right)^{3/4} \tag{38}$$

is an upper bound for the fractal dimension of the global attractor, and hence is an upper bound for the number of degrees of freedom in turbulent flows.

We set the dissipation length scale, in analogy with the Kolmogorov dissipation length scale in the classical theory of turbulence, to be

$$\ell_\epsilon = \left(\frac{\nu^3}{\bar{\epsilon}} \right)^{1/4}$$

Then Eq. (38) leads to the following

Theorem 7. *The Hausdorff and fractal dimensions of the global attractor of the viscous Camassa–Holm (NS-α) equations,* $d_H(\mathscr{A})$ *and* $d_F(\mathscr{A})$*, respectively, satisfy:*

$$d_H(\mathscr{A}) \leqslant d_F(\mathscr{A}) \leqslant \frac{c}{(\alpha_1^2 \lambda_1)^{3/4}} \left(\frac{1}{\ell_\epsilon \lambda_1^{1/2}} \right)^3$$

This estimate for the number of degrees of freedom is consistent with the conventional estimate a la Kolmogorov–Landau–Lifshitz [30]. In particular, the number of degrees of freedom scales as the cube of the ratio of the domain size divided by the Kolmogorov dissipation length scale (times a factor involving the fixed α_1).

6. CONVERGENCE TO THE NAVIER–STOKES EQUATIONS

We observed earlier that the system (8) reduces to the Navier–Stokes for $\alpha_0 = 1$ and $\alpha_1 = 0$. In this section we will fix $\alpha_0 = 1$ and investigate the convergence of the solutions of the system (8) as $\alpha_1 \to 0^+$, and relate the limit to the Navier–Stokes equations. We will be studying further the relation between solutions of the system (8) and the 3D Navier–Stokes equation in a subsequent work. In particular, we will investigate the convergence (in a suitable sense) of the global attractor of the system (8) to the global attractor of the 3D Navier–Stokes equations, as it was defined by Sell in [34], and to the "universal attracting" set introduced in [21].

Theorem 8. *Let $f \in H$, $u^{in} \in V$ and $\alpha_0 = 1$. Let u_{α_1} and $v_{\alpha_1} = u_{\alpha_1} + \alpha_1^2 A u_{\alpha_1}$, denote the solution of the initial-value problem (8) (or equivalently (36)). Then there are subsequences $u_{\alpha_1^j}$, $v_{\alpha_1^j}$, and a function u such that as $\alpha_1^j \to 0^+$ we have:*

(i) $u_{\alpha_1^j} \to u$, strongly in $L^2_{\mathrm{loc}}([0,\infty); H)$;

(ii) $u_{\alpha_1^j} \to u$, weakly in $L^2_{\mathrm{loc}}([0,\infty); V)$;

(iii) for every $T \in (0,\infty)$ and every $w \in H$ we have $(u_{\alpha_1^j}(t), w) \to (u(t), w)$ uniformly on $[0, T]$;

(iv) $v_{\alpha_1^j} \to u$ weakly in $L^2_{\mathrm{loc}}([0,\infty); H)$ and strongly in $L^2_{\mathrm{loc}}([0,\infty); V')$.

Furthermore, u is a weak solution of the 3D Navier–Stokes equations with the initial data $u(0) = u^{in}$ (for the definition of weak solutions to the 3D Navier–Stokes equations see [13] and [35].)

Proof. Let $T > 0$ be fixed. From the proof of Theorem 3 and by passing to the limit one can show that the estimates (17) and (18) also hold for the exact solution of the system (8). That is

$$|u_{\alpha_1}(t)|^2 + \alpha_1^2 \, \|u_{\alpha_1}(t)\|^2 \leqslant k_1$$

and

$$\nu \int_0^T (\|u_{\alpha_1}(s)\|^2 + \alpha_1^2 \, |Au_{\alpha_1}(s)|^2) \, ds \leqslant \bar{k}_2(T)$$

This implies that there are subsequences $\{u_{\alpha_1^j}\}$ and $\{v_{\alpha_1^j}\}$, and corresponding functions u and v such that:

$$\{u_{\alpha_1^j}\} \to u \qquad \text{weakly in } L^2([0,T]; V)$$

and

$$\{v_{\alpha_1^j}\} \to v \qquad \text{weakly in } L^2([0,T]; H)$$

as $\alpha_1^j \to 0^+$.

Next we will use the above estimates and Eq. (8) to show that

$$\int_0^T \left| A^{-1} \frac{du_{\alpha_1}(t)}{dt} \right|^2 dt = \int_0^T \left\| \frac{du_{\alpha_1}(t)}{dt} \right\|^2_{D(A)'} dt \leqslant K(T) \tag{39}$$

for some constant K which depends on T, but is independent of α_1. Indeed, from Eq. (8) (or equivalently (36)) we have

$$\frac{du_{\alpha_1}}{dt}+\nu Au_{\alpha_1}+(I+\alpha_1^2 A)^{-1}\,\tilde{B}(u_{\alpha_1}, v_{\alpha_1})=(I+\alpha_1^2 A)^{-1}\,f$$

Thus

$$\left|A^{-1}\frac{du_{\alpha_1}(t)}{dt}\right|\leqslant \nu\,|u_{\alpha_1}|+|A^{-1}(I+\alpha_1^2 A)^{-1}\,\tilde{B}(u_{\alpha_1}, v_{\alpha_1})|+|A^{-1}f|$$

In order to prove (39) we only need to find the proper estimate for

$$|A^{-1}(I+\alpha_1^2 A)^{-1}\,\tilde{B}(u_{\alpha_1}, v_{\alpha_1})|\leqslant |A^{-1}\tilde{B}(u_{\alpha_1}, v_{\alpha_1})|$$

Applying part (v) of Lemma 1 we obtain

$$\begin{aligned}|A^{-1}\tilde{B}(u_{\alpha_1}, v_{\alpha_1})| &\leqslant c(|u_{\alpha_1}|^{1/2}\,\|u_{\alpha_1}\|^{1/2}\,|v_{\alpha_1}|+\lambda_1^{-1/4}\,|v_{\alpha_1}|\,\|u_{\alpha_1}\|)\\ &\leqslant 2c\lambda_1^{-1/4}\,|v_{\alpha_1}|\,\|u_{\alpha_1}\|\\ &\leqslant 2c\lambda_1^{-1/4}\,\|u_{\alpha_1}\|(|u_{\alpha_1}|+\alpha_1^2\,|Au_{\alpha_1}|)\end{aligned}$$

As a result of the above estimates we have

$$\begin{aligned}|A^{-1}\tilde{B}(u_{\alpha_1}, v_{\alpha_1})|^2 &\leqslant 8c^2\lambda_1^{-1/2}(\|u_{\alpha_1}\|^2\,|u_{\alpha_1}|^2+(\alpha_1^2\,\|u_{\alpha_1}\|^2)(\alpha_1^2\,|Au_{\alpha_1}|^2))\\ &\leqslant 8c^2\lambda_1^{-1/2}k_1(\|u_{\alpha_1}\|^2+\alpha_1^2\,|Au_{\alpha_1}|^2)\end{aligned}$$

and by integrating the above estimate over the interval $[0, T]$ we have

$$\int_0^T |A^{-1}\tilde{B}(u_{\alpha_1}(t), v_{\alpha_1}(t))|^2\,dt\leqslant \frac{\bar{k}_2(T)}{\nu}\,8c^2\lambda_1^{-1/2}k_1$$

From all the above we conclude (39).

By virtue of the above estimates and Aubin's compactness Theorem (see, e.g., [13], [31], or [35]) there exists a subsequence, which will also be labeled by $\{u_{\alpha_1^j}\}$, that converges to u strongly in $L^2([0, T]; H)$. Furthermore, since

$$\int_0^T |A^{-1/2}(v_{\alpha_1^j}(t)-u_{\alpha_1^j}(t))|^2\,dt=(\alpha_1^j)^2\int_0^T \|u_{\alpha_1^j}(t)\|^2\,dt\leqslant(\alpha_1^j)^2\,\frac{\bar{k}_2(T)}{\nu}$$

we have that $v_{\alpha_1^j}\to u$ strongly in $L^2([0, T]; V')$, as $\alpha_1^j\to 0^+$; and that $v(t)=u(t)$ a.e. in $[0, T]$.

As a result of these estimates one can extract subsequences, which will be also labeled by $\{u_{\alpha_1^j}\}$ and $\{v_{\alpha_1^j}\}$, respectively, and show that as $\alpha_1^j \to 0^+$

$$\tilde{B}(u_{\alpha_1^j}, v_{\alpha_1^j}) \to \tilde{B}(u, u) = B(u, u) \qquad \text{weakly in } L^2([0, T]; D(A)')$$

by following an approach similar to that used in the proof of Theorem 3. This finishes the proof of the theorem. □

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REFERENCES

1. Adams, R. A. (1975). Sobolev spaces. In *Pure and Applied Mathematics*, Vol. 65, Academic Press.
2. Billotti, J. E., and LaSalle, J. P. (1971). Dissipative periodic processes. *Bull. Amer. Math. Soc.* **77**, 1082–1088.
3. Camassa, R., and Holm, D. D. (1993). An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71**, 1661–1664.
4. Camassa, R., Holm, D. D., and Hyman, J. M. (1994). A new integrable shallow water equation. *Adv. Appl. Mech.* **31**, 1–33.
5. Chen, S., Foias, C., Holm, D. D., Olson, E., Titi S., and Wynne, S. (1198). The Camassa–Holm equations as a closure model for turbulent channel flow. *Phys. Rev. Lett.* **81**, 5338–5341.
6. Chen, S., Foias, C., Holm, D. D., Olson, E., Titi E. S., and Wynne, S. (1999). A connection between Camassa–Holm equations and turbulent flows in channels and pipes. *Phys. Fluids* **11**, 2343–2353.
7. Chen, S., Foias, C., Holm, D. D., Olson, E., Titi E. S., and Wynne, S. (1999). The Camassa–Holm equations and turbulence. *Physica D* **133**, 49–65.
8. Chen, S. Y., Holm, D. D., Margolin, L. G., and Zhang, R. (1999). Direct numerical simulations of the Navier–Stokes alpha model. *Physica D* **133**, 66–83.
9. Cioranescu, D., and Girault, V. (1996). Solutions variationelles et classiques d'une famille de fluides de grade deux. *C. R. Acad. Sci. Paris Sér. I* **322**, 1163–1168.
10. Cioranescu, D., and Girault, V. (1997). Weak and classical solutions of a family of second grade fluids. *Int. J. Non-Lin. Mech.* **32**, 317–335.
11. Constantin, P. An Eulerian–Lagrangian approach to the Navier–Stokes equations. *Commun. Math. Phys* (in press) adds perspective to the relationship between the NS and CH equation in the presence of viscosity and that it would not be inappropriate to refer to it.
12. Constantin, P., and Foias, C. (1985). Global Lyaponuv exponents, Kaplan–York formulas and the dimension of the attractor for the 2*D* Navier–Stokes equations. *Comm. Pure Appl. Math.* **38**, 1–27.
13. Constantin, P., and Foias, C. (1988). *Navier–Stokes Equations*, University of Chicago Press, Chicago.

14. Constantin, P., Foias, C., Manley, O. P., and Temam, R. (1985). Determining modes and fractal dimension of turbulent flows. *J. Fluid Mech.* **150**, 427–440.
15. Doelman, A., and Titi, E. S. (1993). Regularity of solutions and the convergence of the Galerkin method in the Ginzburg–Landau equation. *Numer. Funct. Anal. Optim.* **14**, 299–321.
16. Doering, C. R., and Titi, E. S. (1995). Exponential decay rate of the power spectrum for solutions of the Navier–Stokes equations. *Phys. Fluids* **7**, 1384–1390.
17. Domaradzki, J. A., and Holm, D. D. (2001). *Navier–Stokes-alpha model: LES equations with nonlinear dispersion*, Special LES volume of ERCOFTAC Bulletin, Modern Simulations Strategies for turbulent flow. B. J. Geurts, editor, Edwards Publishing.
18. Dunn, J. E., and Fosdick, R. L. (1974). Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade. *Arch. Rational Mech. Anal.* **56**, 191–252.
19. Ferrari, A. B., and Titi, E. S. (1998). Gevrey regularity for nonlinear analytic parabolic equations. *Comm. Partial Differential Equations* **23**, 1–16.
20. Foias, C., Holm, D. D., and Titi, E. S. (2001). The Navier–Stokes-alpha model of fluid turbulence. *Physica D* **152**, 505–519.
21. Foias, C., and Temam, R. (1987). *The Connection between the Navier–Stokes Equations, Dynamical Systems and Turbulence Theory*, Directions in Partial Differential Equations, Academic Press, New York, 55–73.
22. Foias, C., and Temam, R. (1989). Gevrey class regularity for the solutions of the Navier–Stokes equations. *J. Funct. Anal.* **87**, 359–369.
23. Graham, M., Steen, P., and Titi, E. S. (1993). Computational efciency and approximate inertial manifolds for a Bénard convection system. *J. Nonlinear Sci.* **3**, 153–167.
24. Hale, J. (1988). Asymptotic behavior of dissipative systems. In *Mathematical Surveys and Monographs*, Vol. 25, AMS, Providence.
25. Holm, D. D., Marsden, J. E., and Ratiu, T. S. (1998). Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. in Math.* **137**, 1–81.
26. Holm, D. D., Marsden, J. E., Ratiu, T. S. (1998). Euler–Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **80**, 4173–4176.
27. Holm, D. D. (1999). Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion. *Physica D* **133**, 215–269.
28. Jones, D., Margolin, L., and Titi, E. S. (1995). On the effectiveness of the approximate inertial manifolds—computational study. *Theoret. Comput. Fluid Dynam.* **7**, 243–260.
29. Kreiss, H. O. (1988). Fourier expansions of the Navier–Stokes equations and their exponential decay rate. In *Analyse Mathématique et Applications*, Gauthier–Villars, Paris, 245–262.
30. Landau, L. D., and Lifshitz, E. M. (1959). *Fluid Mechanics* **6** of *Course of Theoretical Physics*, Pergamon Press Ltd..
31. Lions, J.-L. (1969). *Quelque Méthodes de Résolutions des Problèmes aux Limites Non-Linéares*, Dunod, Paris.
32. Marsden, J. E., and Shkoller, S. (2001). The anisotropic averaged Euler equations. *Arch. Rational Mechan. Anal.* (in press).
33. Marsden, J. E., Ratiu, T., and Shkoller, S. (2000). The geometry and analysis of the averaged Euler equations and a new diffeomorphism group. *Geom. Funct. Anal.* **10**, 582–599.
34. Sell, G. R. (1996). Global attractors for the three dimensional Navier–Stokes equations. *J. Dynam. Differential Equations* **8**, 1–33.
35. Temam, R. (1984). *Navier–Stokes Equations, Theory and Numerical Analysis*, 3rd rev. ed., North–Holand, Amsterdam.
36. Temam, R. (1988). Infinite dimensional dynamical systems in mechanics and physics. *Applied Mathematical Sciences*, Vol. 68, Springer-Verlag, Berlin.