The Tidal Oscillations in an Elliptic Basin of Variable Depth.

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§ 1. *Introduction.*

The problem of the "long" waves in a circular basin of uniform depth involves in its solution a transcendental function*—the Bessel function, and the determination of the free periods requires a knowledge of the zeros of this function or an allied function. On the other hand, when the basin, still circular, has a certain variable depth,† it was shown by Lamb that the solution is expressed in terms of simple algebraic polynomials and the free periods of oscillation are expressed by an extremely simple formula.

In similar fashion, the solution of the problem of the "long" waves in an elliptic basin of uniform depth involves the use of elliptic cylinder functions, and the free periods are only obtained as the result of lengthy numerical approximations.‡

The present paper examines the oscillations in an elliptic basin having a paraboloidal law of depth such that, when the eccentricity vanishes, the basin has the same form as that discussed by Lamb. It is found in this case that the problem is again soluble in terms of polynomials and the free periods of any type of motion can be found with comparative ease. The simple form of the results makes it possible to examine the general character of the oscillations without great complication.

The analysis is also interesting in that it affords a physical interpretation of certain theorems concerning the nature of the solutions of differential equations.

The results obtained should have an application in the theory of *seiches* in lakes where the motion cannot be accurately taken as longitudinal.

* Lamb, "Hydrodynamics," 4th ed., p. 277.

† *Ibid.*, p. 283.

‡ Jeffreys, 'Proc. Lond. Math. Soc.,' vol. 23, p. 455 (1923); Goldstein, *ibid.*, vol. 28, p. 91 (1928).

§ 2. *Statement of the Equation* .

Consider the long waves in an elliptic basin of variable depth h. We form the equations in elliptic co-ordinates ξ, η, by means of the transformation

$$x = c \cosh \xi \cos \eta,$$

$$y = c \sinh \xi \sin \eta.$$

Then, if u, v are the velocity components in the new system, we may write for simple-harmonic vibrations of period $2\pi/\lambda$,

$$\left.\begin{aligned} \mathrm{H}u &= \frac{\iota g}{\lambda}\frac{\partial \zeta}{\partial \xi}, \\ \mathrm{H}v &= \frac{\iota g}{\lambda}\frac{\partial \zeta}{\partial \eta}, \end{aligned}\right\}, \tag{1}$$

where ζ is the displacement of the free surface at the point (ξ, η) and $\mathrm{H}^2 = c^2 (\cosh^2 \xi - \cos^2 \eta)$. The corresponding equation of continuity is

$$\frac{\partial}{\partial \xi}(h\mathrm{H}u) + \frac{\partial}{\partial \eta}(h\mathrm{H}v) + \mathrm{H}^2 \iota \lambda \zeta = 0. \tag{2}$$

On eliminating u and v from (1) and (2) we have the equation

$$\frac{\partial}{\partial \xi}\left(h\frac{\partial \zeta}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(h\frac{\partial \zeta}{\partial \eta}\right) + \mathrm{H}^2\lambda^2 \zeta/g = 0. \tag{3}$$

We shall choose as the law of depth

$$h = h_0 (a - \cosh^2 \xi)(a - \cos^2 \eta), \tag{4}$$

where $a = \cosh^2 \xi_0$ and $\xi < \xi_0$.

The surface of the lake is then bounded by the ellipse $\xi = \xi_0$ which has axes $2c \cosh \xi_0$, $2c \sinh \xi_0$. Further, the law of depth (4) reduced to rectangular co-ordinates gives

$$h/h_0 = a^2 - a - \frac{x^2}{c^2}(a-1) - y^2\frac{a}{c^2}. \tag{5}$$

Hence the contour of the lake is paraboloidal in form and the greatest depth is $h_0 (a^2 - a)$, or $h_0 \cosh^2 \xi_0 \sinh^2 \xi_0$.

On substituting (4) in (3) we have the resulting equation

$$(a - \cos^2 \eta)\frac{\partial}{\partial \xi}(a - \cosh^2 \xi)\frac{\partial \zeta}{\partial \xi} + (a - \cosh^2 \xi)\frac{\partial}{\partial \eta}(a - \cos^2 \eta)\frac{\partial \zeta}{\partial \eta}$$
$$+ \frac{\lambda^2 c^2}{g h_0}(\cosh^2 \xi - \cos^2 \eta)\zeta = 0. \tag{6}$$

Then, if ζ may be expressed in the form XY, where X is independent of η and Y is independent of ξ, equation (6) breaks up in the usual way into the two ordinary equations

$$\left.\begin{aligned}\frac{d}{d\xi}(a-\cosh^2\xi)\frac{dX}{d\xi}+\tfrac{1}{4}\kappa^2 X\{1-k^2(a-\cosh^2\xi)\}=0,\\ \frac{d}{d\eta}(a-\cos^2\eta)\frac{dY}{d\eta}+\tfrac{1}{4}\kappa^2 Y\{-1+k^2(a-\cos^2\eta)\}=0,\end{aligned}\right\} \quad (7)$$

where k^2 is an undetermined constant and κ^2 is written for λ^2c^2/gh_0.

If in the first equation of (7) we put $\xi=\iota\eta$ it reduces to the second. Hence from the solution of one equation we may easily infer that of the other.

The only other conditions to be fulfilled are that hu, $hv\to 0$ at the edge of the lake $\xi=\xi_0$. Since $\cosh\xi_0>1$, H does not vanish anywhere at the boundary. These conditions are then satisfied if ζ has its first derivatives finite at $\xi=\xi_0$, for all values of η. The forms of the solutions adopted in the succeeding sections are in accordance with this requirement.

§ 3. *Discussion of Equation* (7).

In the second of equations (7) put $z=\cos^2\eta$, $Y=w$, $\tfrac{1}{4}\kappa^2(1-k^2a)=\sigma\tau q$ and $\tfrac{1}{4}\kappa^2k^2=-\sigma\tau$. The equation then becomes

$$\frac{d^2w}{dz^2}+\frac{dw}{dz}\left(\frac{\frac{1}{2}}{z}+\frac{\frac{1}{2}}{z-1}+\frac{1}{z-a}\right)+\frac{w\sigma\tau(z-q)}{z(z-1)(z-a)}=0. \quad (8)$$

This equation has four regular singularities which with their corresponding exponents are given by the following scheme :

$$\begin{Bmatrix}0 & 1 & a & \infty\\ 0 & 0 & 0 & \sigma\\ \frac{1}{2} & \frac{1}{2} & 0 & \tau.\end{Bmatrix}$$

And since the sum of the exponents for this equation is 2 we have $\tau=1-\sigma$.

Equation (8) is a particular case of the equation discussed by Heun.* It is obtainable from the general equation of the second order with five singularities (one at infinity) by the confluence of one of the finite singularities with that at infinity. For instance, the general equation of the second order with five regular singularities at a_1, a_2, a_3, a_4 and ∞ respectively may be written†

$$\frac{d^2w}{dz^2}+\left\{\sum_{r=1}^{4}\frac{1-\alpha_r-\beta_r}{z-a_r}\right\}\frac{dw}{dz}+\left\{\sum_{r=1}^{4}\frac{\alpha_r\beta_r}{(z-a_r)^2}+\frac{Az^2+2Bz+C}{\prod_{r=1}^{4}(z-a_r)}\right\}w=0. \quad (9)$$

* 'Math. Annalen,' vol. 33, p. 161 (1889).

† Whittaker and Watson, "Modern Analysis," 3rd ed., p. 203.

In this equation, the exponents at a_r are α_r, β_r and A, B, C are constants.

If now in equation (9) we put

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = a,$$
$$\alpha_1 = \alpha_2 = \tfrac{1}{2}, \quad \alpha_3 = 0, \quad \beta_r = 0 \ (r = 1, 2, 3, 4)$$
$$A = \sigma\tau, \quad 2B = -\sigma\tau\,(q - a_4). \quad C = -a_4 q\sigma\tau,$$

and let $a_4 \to \infty$, equation (9) reduces at once to equation (7).

Equation (8) is similar to Lamé's equation. The only difference lies in the fact that (8) has a double zero exponent at the singularity a. The method of procedure used in dealing with Lamé's equation in its algebraic form may be applied at once to (8).

If the solution of (8) valid in the vicinity of $z = 0$ be expressed as

$$\zeta = F\,(a, q\,;\ \sigma, 1 - \sigma\,;\ \tfrac{1}{2}, \tfrac{1}{2}, 0\,;\ z), \tag{10}$$

then it is readily shown in the usual way that

$$\zeta = z^{\frac{1}{2}}F\,(a, q\,;\ \sigma + \tfrac{1}{2}, \tfrac{3}{2} - \sigma\,;\ \tfrac{3}{2}, \tfrac{1}{2}, 0\,;\ z), \tag{11}$$

$$\zeta = z^{\frac{1}{2}}\,(z - 1)^{\frac{1}{2}}\,F\,(a, q\,;\ \sigma + 1, 2 - \sigma\,;\ \tfrac{3}{2}, \tfrac{3}{2}, 0\,;\ z), \tag{12}$$

and

$$\zeta = (z - 1)^{\frac{1}{2}}\,F(a, q\,;\ \sigma + \tfrac{1}{2}, \tfrac{3}{2} - \sigma\,;\ \tfrac{1}{2}, \tfrac{3}{2}, 0\,;\ z) \tag{13}$$

are also solutions.

These forms give the different types of motion.

§ 4. *Solution of Equation* (8).

It is known that an equation of type (8) has a solution in the form of a polynominal for appropriate values of σ and q. To determine these, assume a solution in the vicinity of the point $z = 0$ in the form

$$w = \Sigma C_n z^{p+n},$$

where the exponent $p = 0$ or $\frac{1}{2}$. On substituting in equation (8), we have the following relation between the constants C where n is any integer.

$$\begin{aligned} C_{n+1}a\,\{(p+n+1)(p+n+\tfrac{1}{2})\} \\ + C_n\,\{-\sigma(1-\sigma)q - (p+n)^2\,(1+a) - \tfrac{1}{2}\,(p+n)\} \\ + C_{n-1}\,\{(p+n)\,(p+n-1) + \sigma\,(1-\sigma)\} = 0. \end{aligned} \tag{14}$$

If we choose $\sigma = p + m$, m being an integer, it may be shown* that q is then the root of an algebraic equation of degree m and also that the corresponding polynominal solution is of order $(m-1)$. Further, with the conditions as in the present problem, the roots q are all real and distinct.

By taking $p = 0, \frac{1}{2}$ we arrive at polynomial solutions corresponding to (10) and (11).

Since the solutions have this form, the boundary condition is at once fulfilled.

For the forms (12) and (13) the sequence-relation between the constants C is given by putting $p = 0, \frac{1}{2}$ in the equation

$$\begin{aligned} &C_{n-1}\{(p+n-1)(p+n+1)+\sigma(1-\sigma)+\tfrac{3}{4}\} \\ &\quad + C_n\{-(a+1)(p+n)^2-(a+\tfrac{1}{2})(p+n)-(a/4)-\sigma(1-\sigma)q\} \\ &\qquad + C_{n+1}\,a\{(p+n+1)(p+n+\tfrac{1}{2})\} = 0. \quad (15) \end{aligned}$$

The polynomial solution in this case is obtained by putting $\sigma = p + m + \frac{1}{2}$, m being integral.

The second solution associated with any one of the above polynomial solutions is, in general, a transcendental function involving a logarithm.

It has been shown by Stieltjes† that if the equation

$$\frac{d^2w}{dz^2} + \frac{dw}{dz}\sum_{r=1}^{n}\frac{a_r}{x-e_r} + \frac{\psi(z)}{\prod_{r=1}^{n}(x-e_r)}w = 0, \quad (16)$$

where $\psi(z)$ is a polynomial of suitable form, the constants e_r are real and distinct from one another and the constants a_r are real and positive, has a polynomial solution $\phi(z)$, then all the roots of $\phi(z) = 0$, considered as a function of z, are real and lie between the greatest and least of the quantities e_r. In the case of equation (8), these necessary conditions are fulfilled. If, therefore, w_{m-1} is a polynomial solution of (8) of order $(m-1)$ then w_{m-1} vanishes for $(m-1)$ values of z lying between 0 and a, and we may write

$$w_{m-1} = (z-z_0)(z-z_1)\ldots(z-z_{m-1}).$$

The complete solution of (6) corresponding to this will be

$$\begin{aligned} \zeta = &(\cosh^2\xi - z_0)(\cosh^2\xi - z_1)\ldots(\cosh^2\xi - z_{m-1}) \\ &\times(\cos^2\eta - z_0)(\cos^2\eta - z_1)\ldots(\cos^2\eta - z_{m-1}). \end{aligned}$$

If

$$z_r > 1, \qquad \zeta = 0 \text{ for } \cosh^2\xi = z_r;$$

and if

$$z_r < 1, \qquad \zeta = 0 \text{ for } \cos^2\eta = z_r.$$

* *Cf.* Whittaker and Watson, "Modern Analysis," p. 556.

† 'Acta Math.,' vol. 6, p. 121 (1885).

The former case corresponds to a nodal ellipse and the latter to a pair of nodal hyperbolas. The polynomial solution of order $(m-1)$ therefore gives $(m-1)$ nodal lines of one kind or the other, a pair of hyperbolas being counted as one nodal line.

Stieltjes has further shown that each distribution of the quantities z_r in the intervals 0 to 1 and 1 to a corresponds to a unique set of coefficients in the polynomial $\psi(z)$, or in the case of (8) to a unique determination of q. Now it has just been shown that there are m independent values of q which give a polynomial solution of order $(m-1)$. To each of these values of q there is a different distribution of the zeros of w_{m-1}, and it is clear that there are m different distributions of them corresponding to the m different values of q. It follows that each mode of vibration involving a polynomial solution of a certain order has the same number of nodal lines, but that the distribution of the lines between the ellipses and hyperbolas is different for each separate frequency of that type.

This result is illustrated in the next section.

§ 5. *The Modes of Oscillation.*

It is of some interest to examine in detail the earlier modes of vibration of the various types.

(i) In (14) put $p=0$. The relation then becomes

$$C_{n+1}\,a\,(n+1)\,(n+\tfrac{1}{2})+C_n\,\{\sigma(\sigma-1)\,q-n^2(1+a)-\tfrac{1}{2}n\}$$
$$+\,C_{n-1}\,\{n\,(n-1)-\sigma(\sigma-1)\}=0. \quad (17)$$

For the lowest mode of this type take $\sigma=2$ and $C_2=C_3=\ldots=0$. The equations then become

$$C_0\,.\,2\,.\,1\,.\,q+C_1 a\,.\,\tfrac{1}{2} \qquad =0,$$
$$-\,C_0\,.\,2 \qquad +C_1\,(2\,.\,1\,.\,q-a-\tfrac{3}{2})=0.$$

From these

$$q=a/4+\tfrac{3}{8}\pm\tfrac{1}{4}\sqrt{(a^2-a+\tfrac{9}{4})}, \quad (18)$$

and therefore

$$\kappa^2=4\sigma\,(\sigma-1)\,(a-q)$$
$$=6a-3\mp 2\sqrt{(a^2-a+\tfrac{9}{4})}. \quad (19)$$

The surface level for this type is given by

$$\zeta\propto\left(\frac{c_0}{c_1}+\cosh^2\xi\right)\left(\frac{c_0}{c_1}+\cos^2\eta\right).$$

The mode of lower frequency exhibits a pair of nodal hyperbolas and the higher frequency a nodal ellipse.

(ii) Next put $p = \frac{1}{2}$ in (14). The equation then becomes

$$C_{n+1}\, a\,(n + \tfrac{3}{2}\,(n+1) + C_n\,\{-\sigma(1-\sigma)\,q - (n+\tfrac{1}{2})^2\,(1+a) - \tfrac{1}{2}\,(n+\tfrac{1}{2})\} + C_{n-1}\,\{n^2 - \tfrac{1}{4} - \sigma(\sigma-1)\} = 0. \quad (20)$$

The lowest mode of this type is given by

$$m = 1, \quad \text{or} \quad \sigma = 3/2 \quad \text{and} \quad C_1 = C_2 = \ldots = 0.$$

The solution is then

$$q = \tfrac{2}{3} + \tfrac{1}{3}a,$$

or

$$\left.\begin{aligned} \kappa^2 &= 2\,(a-1), \\ \zeta &\propto \cosh\xi\cos\eta. \end{aligned}\right\} \quad (21)$$

and

The surface remains plane and oscillates about the minor axis.

(iii) The next higher mode of the same type is obtained by putting $\sigma = 5/2$, $C_2 = C_3 = \ldots = 0$.

Then

$$\left.\begin{aligned} \kappa^2 &= 10a - 7 \pm 4\sqrt{(a^2 - 2a + \tfrac{25}{16})}, \\ \zeta &\propto \cosh\xi\cos\eta\left(\frac{c_0}{c_1} + \cosh^2\xi\right)\left(\frac{c_0}{c_1} + \cos^2\eta\right). \end{aligned}\right\} \quad (22)$$

In this mode, the minor axis is always a nodal line. With the lower frequency there is also a pair of nodal hyperbolas, while with the higher frequency there is a nodal ellipse.

(iv) For modes involving symmetry about the major axis put $p = 0$ in (15). We have then

$$\begin{aligned} &C_{n-1}\,\{n^2 - \tfrac{1}{4} + \sigma(1-\sigma)\} \\ &\quad + C_n\,\{-n^2\,(a+1) - n\,(a+\tfrac{1}{2}) - (a/4) - \sigma(1-\sigma)\,q\} \\ &\quad + C_{n+1}\,a\,(n+1)\,(n+\tfrac{1}{2}) = 0. \end{aligned} \quad (23)$$

For the lowest mode of this type take $\sigma = 3/2$, $C_1 = C_2 = \ldots = 0$.

Then

$$\sigma(\sigma-1)\,q = a/4,$$

or

$$\left.\begin{aligned} \kappa^2 &= 2a, \\ \xi &\propto \sinh\xi\sin\eta. \end{aligned}\right\} \quad (24)$$

and

The surface is plane and oscillates about the major axis.

(v) The next mode of the same type is obtained from (23) by putting $\sigma = 5/2$, $C_2 = C_3 = \ldots = 0$.

The surface is

$$\zeta \propto \sinh \xi \sin \eta \left(\frac{c_0}{c_1} + \cosh^2 \xi\right)\left(\frac{c_0}{c_1} + \cos^2 \eta\right).$$

For the lower frequency the nodal lines are the major axis and a pair of hyperbolas, while for the higher frequency the nodal lines are the major axis and an ellipse.

(vi) For modes involving symmetry about both axes we put $p = \frac{1}{2}$ in (15). We have then

$$\begin{aligned} C_{n-1} &\{(n - \tfrac{1}{2})(n + \tfrac{3}{2}) + \sigma(1 - \sigma) + \tfrac{3}{4}\} \\ &+ C_n \{-(a + 1)(n + \tfrac{1}{2})^2 - (a + \tfrac{1}{2})(\tfrac{1}{2} + n) - (a/4) - \sigma(1 - \sigma) q\} \\ &+ C_{n+1}\, a (n + \tfrac{3}{2})(n + 1) = 0. \end{aligned} \tag{25}$$

The lowest mode of this type is obtained by taking $\sigma = 2$, $c_1 = c_2 = \ldots = 0$. Whence

$$\left.\begin{aligned} &\zeta \propto \cosh \xi \sinh \xi \cos \eta \sin \eta, \\ &\kappa^2 = 4a - 2. \end{aligned}\right\} \tag{26}$$

The nodal lines of this type are the major and minor axes.

(vii) The next higher mode of this type will have the form

$$\zeta \propto \cosh \xi \sinh \xi \cos \eta \sin \eta \left(\frac{c_0}{c_1} + \cosh^2 \xi\right)\left(\frac{c_0}{c_1} + \cos^2 \eta\right),$$

involving as nodal lines the two axes and either a pair of hyperbolas or an ellipse according to the frequency.

§ 7. *Comparison with the Case of a Circular Basin.*

In the limiting case, when $\xi_0 \to \infty$ and $c \to 0$ in such a way that $c \cosh \xi_0$ remains finite, the results just obtained should correspond to those for a circular basin. Lamb has shown* that the natural frequencies for a basin of radius r_0 and greatest depth h_0', the law of depth being $h = h_0'(1 - r^2/r_0^2)$, are given by the formula

$$\frac{\lambda^2 r_0^2}{g h_0'} = s(4j - 2) + 2j(2j - 2), \tag{27}$$

s being the number of nodal diameters and $(j - 1)$ the number of nodal circles in the mode.

* "Hydrodynamics," 4th ed., p. 283.

For the limiting form of the elliptical case we have

$$r_0 = \tfrac{1}{2} c e^{\xi_0}, \qquad a = \tfrac{1}{4} e^{2\xi_0},$$

and therefore from (5)

$$h/h_0 = \frac{1}{16} e^{4\xi_0} \left(1 - \frac{4r^2}{c^2 e^{2\xi_0}}\right)$$

$$= \frac{1}{16} e^{4\xi_0} \left(1 - \frac{r^2}{r_0^2}\right),$$

showing that the law of depth reduces to Lamb's form, provided we take $h_0' = h_0 \cdot \dfrac{1}{16} e^{4\xi_0}$.

From these

$$\frac{\lambda^2 r_0^2}{g h_0'} = \kappa^2/a.$$

Taking large values of a, we have from (19) for case (i),

$$\frac{\lambda^2 r_0^2}{g h_0'} = 4 \text{ or } 8. \tag{28}$$

The lower frequency corresponds to $j = 1$, $s = 2$—a pair of nodal diameters in place of a pair of nodal hyperbolas. The higher frequency corresponds to $s = 0$, $j = 2$—a nodal circle in place of a nodal ellipse.

Cases (ii) and (iv) are indistinguishable from one another in the limit, giving $\lambda^2 r_0^2/g h_0' = 2$. This is the mode $s = 1$, $j = 1$, with a single nodal diameter.

Case (iii) gives $\lambda^2 r_0^2/g h_0' = 6$ or 14. In the circular case, these arise from $j = 1$, $s = 3$ and $j = 2$, $s = 1$ respectively.

Case (vi) is interesting. From (26) we find in the limit that $\kappa^2 = 4a$, or $\lambda^2 r_0^2/g h_0' = 4$. This case degenerates to the same form as the lower frequency of (28) with a pair of nodal diameters. These diameters are the limiting forms on the one hand of a pair of hyperbolas and on the other hand of the major and minor axes.

§ 8. *Comparison with Longitudinal Oscillations.*

The theories of *seiche* motions in lakes worked out by Chrystal* and Proudman† depend upon the assumption that the oscillations are purely longitudinal. It is of some interest, therefore, to work out the earlier frequencies for the basin of this paper on the longitudinal theory and to compare the results.

* 'Trans. Roy. Soc. Edin.,' vol. 41, Part 3, p. 599 (1905).

† 'Proc. Lond. Math. Soc.,' vol. 14, p. 240 (1915).

If S be the area of the section of the lake normal to the length at a distance x from the origin and b be the breadth of the free surface there, the equation of the oscillations is*

$$\frac{g}{b}\frac{d}{dx}\left(\mathrm{S}\,\frac{d\zeta}{dx}\right) + \lambda^2\zeta = 0. \tag{29}$$

From (5) we find

$$b = \frac{2c}{\sqrt{a}}\left\{a^2 - a - \frac{x^2}{c^2}(a-1)\right\}^{1/2},$$

$$\mathrm{S} = \frac{4ch_0}{3\sqrt{a}}\left\{a^2 - a - \frac{x^2}{c^2}(a-1)\right\}^{3/2}.$$

If we write

$$\beta = a^2 - a,$$

$$\gamma = (a-1)/c^2,$$

equation (29) becomes

$$(\beta - \gamma x^2)\frac{d^2\zeta}{dx^2} - 3\gamma x\frac{d\zeta}{dx} + \frac{3\lambda^2\zeta}{2gh_0} = 0. \tag{30}$$

Assuming a solution $\Sigma \mathrm{A}_n x^n$ in the vicinity of $x = 0$, we find the indices are 0, 1 and the relation is

$$\mathrm{A}_{n+2}\,\beta(n+1)(n+2) = \left\{\gamma n\,(n+2) - \frac{3\lambda^2}{2gh_0}\right\}\mathrm{A}_n. \tag{31}$$

The series is not convergent at the point β/γ, or the extremity of the lake, unless it terminate. Hence we must have

$$\begin{aligned}\frac{3\lambda^2}{2gh_0} &= \gamma n\,(n+2) \\ &= n\,(n+2)\,(a-1)/c^2,\end{aligned} \tag{32}$$

for some integral value of n.

The lowest mode is obtained by putting $n = 1$.

Then $\lambda^2c^2/gh_0 = 2\,(a-1)$, exactly as in (21). This result might have been anticipated as, since

$$\begin{aligned}\zeta &\propto \cosh\xi\cos\eta \\ &\propto x,\end{aligned}$$

the transverse velocity, which is proportional to $\partial\zeta/\partial y$, is zero.

For $n = 2$, a two-noded vibration, (32) gives

$$\lambda^2c^2/gh_0 = \tfrac{16}{3}\,(a-1). \tag{33}$$

* Lamb, "Hydrodynamics," p. 267.

The corresponding result is the smaller value in (19). If we assume $a - 1 = \delta$ = a small quantity, (19) gives

$$\frac{\lambda^2 c^2}{gh_0} = 6a - 3 - 2\sqrt{(a^2 - a + \tfrac{9}{4})}$$
$$= \tfrac{16}{3}\delta + O(\delta^2).$$

This exhibits the effect of the departure from purely longitudinal motion in a narrow lake of this type.

Similarly, taking the three-noded vibration we have from (32)

$$\frac{\lambda^2 c^2}{gh_0} = 10(a - 1) = 10\delta.$$

The corresponding result (22) gives

$$\frac{\lambda^2 c^2}{gh_0} = 10a - 7 - 4\sqrt{(a^2 - 2a + \tfrac{25}{16})}$$
$$= 10\delta + O(\delta^2).$$

The Direction of Magnetisation of Single Ferromagnetic Crystals.

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§ 1. *Introduction.*

The study of the magnetisation of single crystals of ferromagnetic substances has shown that there are definite relations between magnetic properties and crystalline structure. One of the most important, the relation between the crystalline structure and the direction of magnetisation, will be studied in this paper. It is well known that on applying a magnetic field to a crystal the direction of the magnetisation produced does not, in general, coincide with that of the field (here and elsewhere, unless the external field is specifically mentioned, the effective field is to be understood). The phenomenon has been extensively studied experimentally, but hitherto there has been no entirely satisfactory theory.

A theory, at least partially successful, was proposed by Mahajani,* who assumed that the elementary magnets are electron orbits. Owing to their

* G. S. Mahajani, 'Phil. Trans.,' A, vol. 228, p. 63 (1929).