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# THE TILTED CARATHÉODORY CLASS AND ITS APPLICATIONS

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Abstract. This paper mainly deals with the tilted Carathéodory class by angle $\lambda \in (-\pi/2, \pi/2)$ (denoted by $\mathcal{P}_\lambda$) an element of which maps the unit disc into the tilted right half-plane $\{w : \operatorname{Re} e^{i\lambda} w > 0\}$. Firstly we will characterize $\mathcal{P}_\lambda$ from different aspects, for example by subordination and convolution. Then various estimates of functionals over $\mathcal{P}_\lambda$ are deduced by considering these over the extreme points of $\mathcal{P}_\lambda$ or the knowledge of functional analysis. Finally some subsets of analytic functions related to $\mathcal{P}_\lambda$ including close-to-convex functions with argument $\lambda$, $\lambda$-spirallike functions and analytic functions whose derivative is in $\mathcal{P}_\lambda$ are also considered as applications.

## 1. Introduction

Let $\mathcal{A}$ be the family of functions $f$ analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}_1$ be the subset of $\mathcal{A}$ consisting of functions $f$ which are normalized by $f(0) = f'(0) - 1 = 0$ while $\mathcal{A}_0$ with normalization $f(0) = 1$. A function $f \in \mathcal{A}$ is said to be subordinate to a function $F \in \mathcal{A}$ (in symbols $f \prec F$ or $f(z) \prec F(z)$) in $\mathbb{D}$ if there exists an analytic function $\omega$ on $\mathbb{D}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$, such that

$$f(z) = F(\omega(z))$$

in $\mathbb{D}$. When $F$ is a univalent function, the condition $f \prec F$ is equivalent to $f(\mathbb{D}) \subseteq F(\mathbb{D})$ and $f(0) = F(0)$.

Let

$$\mathcal{P}_\lambda = \left\{ p \in \mathcal{A}_0 : \operatorname{Re} e^{i\lambda} p(z) > 0 \right\}.$$

Here and hereafter we always suppose $-\pi/2 < \lambda < \pi/2$. Note that $\mathcal{P}_\lambda$ is a convex and compact subset of $\mathcal{A}$ which is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. Since $\mathcal{P}_0$ is the well-known Carathéodory

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class, we call $\mathcal{P}_\lambda$ *the tilted Carathéodory class by angle* $\lambda$. Also we let

$$\mathcal{P} = \bigcup_{-\pi/2<\lambda<\pi/2} \mathcal{P}_\lambda.$$

In this note, we always let

$$p_\lambda(z) = \frac{1+e^{-2i\lambda}z}{1-z}.$$

It is easy to see that $p_\lambda$ univalently maps the unit disc $\mathbb{D}$ onto $\mathbb{H}_\lambda = \{w \in \mathbb{C} : \operatorname{Re} e^{i\lambda}w > 0\}$ which is called *the tilted right half-plane by angle* $\lambda$. Later we will see that this function plays an important role while investigating the properties of $\mathcal{P}_\lambda$.

The Carathéodory class $\mathcal{P}_0$ occupies an extremely important place in the theory of functions and has been studied by many authors ([5, Chapter 7, p. 77], [6], [16], [19], [20], [23], [24], [29], [30]). The tilted Carathéodory class $\mathcal{P}_\lambda$ scatters in some papers (see [13], [14], [16], [21]), although the name was not given in the literature. In the geometric function theory, there are some functions defined by using the tilted Carathéodory class $\mathcal{P}_\lambda$, such as the class of close-to-convex functions. When researching this class, some authors restrict to the special case $\lambda = 0$ because of a difficulty lying in $\mathcal{P}_\lambda$. Therefore it is worthwhile investigating the class $\mathcal{P}_\lambda$ in order to understand its relating geometric functions well.

Section 2 is devoted to characterizations of the functions belonging to the class $\mathcal{P}_\lambda$ from different aspects. A linear relation between the elements of $\mathcal{P}_\lambda$ and $\mathcal{P}_0$ implies that the functions in $\mathcal{P}_\lambda$ can be described in terms of integral and subordination. We show also that $\mathcal{P}_\lambda$ can be regarded as the dual and the second dual sets of some families of analytic functions.

In Section 3, the extreme points of $\mathcal{P}_\lambda$ are deduced directly from those of $\mathcal{P}_0$. With the aid of these extreme points, the sharp estimates of some functionals over $\mathcal{P}_\lambda$, for instance, the $n$-th coefficient functional, the distortion and growth functionals, have been obtained. For the cases of other functionals, we summarize some other methods to deal with the extremal problems. The estimate of $|zp'(z)/(p(z)+i\tan\lambda)|$ for $p \in \mathcal{P}_0$ was considered by some authors, see Bernardi [2] and Robertson [20]. The sharp estimate was obtained in a paper of Ruscheweyh and Singh [23] by a variational method but the extremal functions were not given there. In Theorem 6 of the present paper, we estimate the functional $|zp'(z)/p(z)|$ with $p \in \mathcal{P}_\lambda$ which is actually equivalent to the above problem. By using fundamental functional analysis, we obtain the sharp upper bound of $|zp'(z)/p(z)|$ for $p \in \mathcal{P}_\lambda$ and give all the extremal functions which make the estimate sharp.

The last section is concerned with some applications of our results to Geometric Function Theory. We will consider $\lambda$-spirallike functions, close-to-convex functions with argument $\lambda$ and analytic functions whose derivative is in $\mathcal{P}_\lambda$.

## 2. Characterizations of $\mathcal{P}_\lambda$

In this section, we list some characterizations of $\mathcal{P}_\lambda$ for later use. Before proceeding to it, we shall first introduce some notations. The convolution (or Hadamard product) $f * g$ of two functions $f$, $g \in \mathcal{A}$ with series expansions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined by

$$f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Obviously, we have $f*g \in \mathcal{A}$. For an introduction to the theory of convolutions in the present context we refer to [22]. For an analytic function $h \in \mathcal{A}_0$, we note that

$$\begin{aligned} h(z) * \frac{1+Az}{1-z} &= h(z) * \left(\frac{1+A}{1-z} - A\right) \\ &= (1+A)h(z) - A \end{aligned} \tag{1}$$

for a complex number $A$ which will be used several times. For a set $V \subset \mathcal{A}_0$, define the dual set

$$V^* = \{g \in \mathcal{A}_0 : (f * g)(z) \neq 0 \text{ in } \mathbb{D} \text{ for any } f \in V\},$$

and $V^{**} = (V^*)^*$, the second dual.

The next lemma can be deduced from the proof of Theorem 1.3 in [22].

**Lemma 1.** *Let $h \in \mathcal{A}_0$. If*

$$th(xz) + (1-t)h(yz) \neq 0$$

*for any $|x| = |y| = 1$, $0 \leq t \leq 1$ and $z \in \mathbb{D}$, then $h(\mathbb{D})$ is contained in a half-plane $H$ with $0 \in \partial H$.*

**Theorem 1.** *Let $\lambda \in (-\pi/2, \pi/2)$ be a real constant. Then the following conditions are equivalent for a function $p \in \mathcal{A}_0$*

(i) $p \in \mathcal{P}_\lambda$;

(ii) $\frac{e^{i\lambda}p - i\sin\lambda}{\cos\lambda} \in \mathcal{P}_0$;

(iii) *There exists a Borel probability measure $\mu$ on $\partial\mathbb{D}$ such that $p(z)$ can be represented by*

$$p(z) = \int_{\partial\mathbb{D}} \frac{1 + e^{-2i\lambda}xz}{1 - xz} d\mu(x);$$

(iv) $p \prec p_\lambda$ *in* $\mathbb{D}$;

(v) $p \in V_\lambda^*$, *where*

$$V_\lambda = \left\{ \frac{1}{1-z} \left(1 + \frac{1 - e^{-2i\lambda}x}{(1 + e^{-2i\lambda})x} z\right) : |x| = 1 \right\};$$

(vi) $p \in W_\lambda^{**}$, *where*

$$W_\lambda = \left\{ t\frac{1+xe^{-2i\lambda}z}{1-xz} + (1-t)\frac{1+ye^{-2i\lambda}z}{1-yz} : |x| = |y| = 1,\ 0 \le t \le 1 \right\}.$$

*Remark* 1. Theorem 1 implies that

$$\mathcal{P}_\lambda = V_\lambda^*$$

and

$$\mathcal{P}_\lambda = W_\lambda^{**}.$$

For $\mathcal{P}$, we have (see [22, Theorem 1.6])

$$\mathcal{P} = \{f \in \mathcal{A}_0 : \operatorname{Re} f(z) > 1/2\}^*$$

and

$$\mathcal{P} = \left\{ \frac{1+xz}{1+yz} : |x| = |y| = 1 \right\}^{**}.$$

*Proof of Theorem 1.* The equivalence of (i), (ii) and (iv) can be obtained immediately from the definition of $\mathcal{P}_\lambda$. The condition (iii) is reduced to the Herglotz integral representation of $\mathcal{P}_0$ ([9], see also [8]) when $\lambda = 0$. Thus by the equivalence of (i) and (ii), we can easily get (i) $\Leftrightarrow$ (iii). Hence we only need to prove the equivalence of (i), (v) and (vi).

Firstly we will show (i) $\Leftrightarrow$ (v). We have to prove that

$$p(z) * \frac{1}{1-z}\left(1 + \frac{1-e^{-2i\lambda}x}{(1+e^{-2i\lambda})x}z\right) \ne 0,\ |x| = 1,\ z \in \mathbb{D} \tag{2}$$

if and only if $p \in \mathcal{P}_\lambda$.

By making use of relation (1), we see that (2) is equivalent to

$$p(z) * \left(\frac{1}{1-z} - \frac{1-e^{-2i\lambda}x}{1+x}\right) \ne 0,$$

namely,

$$p(z) \ne \frac{1-e^{-2i\lambda}x}{1+x}$$

for $|x| = 1$, $x \ne -1$ and $z \in \mathbb{D}$. Since the set

$$\left\{(1-e^{-2i\lambda}x)/(1+x) : |x| = 1,\ x \ne -1\right\}$$

is just the line $\{w : \operatorname{Re} e^{i\lambda}w = 0\}$, the above condition implies that $p(\mathbb{D})$ lies in the tilted right half-plane by angle $\lambda$, which is our assertion.

Next we will show (i)$\Leftrightarrow$(vi). To this end, we need to prove that an analytic function $h \in \mathcal{A}_0$ satisfies

$$h(z) * \left(t\frac{1+xe^{-2i\lambda}z}{1-xz} + (1-t)\frac{1+ye^{-2i\lambda}z}{1-yz}\right) \ne 0 \tag{3}$$

for any $|x| = |y| = 1$, $0 \le t \le 1$ and $z \in \mathbb{D}$, if and only if

$$h * p(z) \ne 0$$

for $z \in \mathbb{D}$ and $p \in \mathcal{P}_\lambda$.

If an analytic function $h$ with $h(0) = 1$ satisfies (3) for any $|x| = |y| = 1$, $0 \le t \le 1$ and $z \in \mathbb{D}$, then by relation (1), we have

$$th(xz) + (1-t)h(yz) \neq \frac{e^{-2i\lambda}}{1+e^{-2i\lambda}}$$

for any $|x| = |y| = 1$, $0 \le t \le 1$ and $z \in \mathbb{D}$. Thus by Lemma 1, there exists a real constant $\alpha \in [0, 2\pi)$ such that

$$\mathrm{Re}\left(e^{i\alpha}\left(h(z) - \frac{e^{-2i\lambda}}{1+e^{-2i\lambda}}\right)\right) > 0,$$

which implies that for any Borel probability measure $\mu$ on $\partial\mathbb{D}$ we have

$$\int_{\partial\mathbb{D}} \left(h(xz) - \frac{e^{-2i\lambda}}{1+e^{-2i\lambda}}\right) d\mu(x) \neq 0,$$

equivalently,

$$h(z) * \int_{\partial\mathbb{D}} \frac{1+e^{-2i\lambda}xz}{1-xz} d\mu(x) \neq 0.$$

Then for any function $p \in \mathcal{P}_\lambda$ we have

$$p * h(z) \neq 0$$

for $z \in \mathbb{D}$ by the equivalence of (i) and (iii). Since the above process is invertible, we thus arrive at our conclusions. □

By making use of the following Schur's lemma, we can get a convolution property of $\mathcal{P}_\lambda$.

**Lemma 2** (see [25] or [15])**.** *Let* $p_1(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{P}_0$ *and* $p_2(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{P}_0$. *Then*

$$1 + \sum_{n=1}^{\infty} a_n b_n z^n/2 \in \mathcal{P}_0.$$

**Theorem 2.** *Let* $p_1 \in \mathcal{P}_{\lambda_1}$ *and* $p_2 \in \mathcal{P}_{\lambda_2}$. *Then*

$$\mathrm{Re}\left(e^{i(\lambda_1+\lambda_2)}(p_1 * p_2)\right) > -\cos(\lambda_1 - \lambda_2). \tag{4}$$

*In particular, if* $\cos(\lambda_1 - \lambda_2) < 0$, *then*

$$p_1 * p_2 \in \mathcal{P}_{\lambda_1+\lambda_2}.$$

*Proof.* Since $p_1(z) \in \mathcal{P}_{\lambda_1}$ and $p_2(z) \in \mathcal{P}_{\lambda_2}$, by the equivalence (i) and (ii) in Theorem 1 we have

$$\frac{e^{i\lambda_1}p_1(z) - i\sin\lambda_1}{\cos\lambda_1} \in \mathcal{P}_0$$

and

$$\frac{e^{i\lambda_2}p_2(z) - i\sin\lambda_2}{\cos\lambda_2} \in \mathcal{P}_0.$$

Hence Schur's lemma implies that

$$\frac{1}{2}\left(\frac{e^{i\lambda_1}p_1(z) - i\sin\lambda_1}{\cos\lambda_1} * \frac{e^{i\lambda_2}p_2(z) - i\sin\lambda_2}{\cos\lambda_2}\right) + \frac{1}{2} \in \mathcal{P}_0$$

which is equivalent to (4). □

## 3. Basic estimates of $\mathcal{P}_\lambda$

Let $K$ be a subset of a vector space $X$. A point $s$ in $K$ is called an extreme point if it is not an internal point of any line interval whose endpoints are in $K$, except when both endpoints are $s$. We denote the set of all extreme points of $K$ by $\operatorname{Ext} K$. The extremal points of $\mathcal{P}_0$ can be obtained by the Herglotz integral representation formula. A truly beautiful derivation of $\operatorname{Ext}\mathcal{P}_0$ was given by Holland [10], while Kortram [15] obtained it by elementary functional analysis. We state it as a lemma in order to get the corresponding result of $\mathcal{P}_\lambda$.

**Lemma 3.**

$$\operatorname{Ext}\mathcal{P}_0 = \left\{\frac{1+xz}{1-xz},\ |x| = 1\right\}.$$

**Theorem 3.**

$$\operatorname{Ext}\mathcal{P}_\lambda = \{p_\lambda(xz),\ |x| = 1\}.$$

*Proof.* A combination of Lemma 3 and the equivalence of (i) and (ii) in Theorem 1 implies our assertion. □

The next result which can be found in [8, p. 45] gives a useful technique to solve extremal problems. If $\mathcal{F}$ is a convex subset of $\mathcal{A}$ and $J : \mathcal{A} \to \mathbb{R}$, then $J$ is called *convex* on $\mathcal{F}$ provided that $J(tf + (1-t)g) \le tJ(f) + (1-t)J(g)$ whenever $f,\ g \in \mathcal{F}$ and $0 \le t \le 1$.

**Lemma 4.** *Let $\mathcal{F}$ be a compact and convex subset of $\mathcal{A}$ and let $J$ be a real-valued, continuous, convex functional on $\mathcal{F}$. Then*

$$\max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in \operatorname{Ext}\mathcal{F}\}.$$

**Theorem 4.** *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}_\lambda$ with $\lambda \in (-\pi/2, \pi/2)$. Then*

$$|p_n| \le 2\cos\lambda,$$

*and*

$$|p'(z)| \le \frac{2\cos\lambda}{(1-r)^2},$$

*where $r = |z| < 1$. The inequalities are sharp with the extremal functions $p_\lambda(xz)$, where $|x| = 1$.*

*Proof.* It is easy to check that the above two functionals are real-valued, continuous and convex. Thus by applying Lemma 4 the maxima of them are obtained over the reduced subset $\operatorname{Ext}\mathcal{P}_\lambda$, which consists of $p_\lambda(xz)$ with $|x| = 1$ by Theorem 3. Since

$$p_\lambda(xz) = 1 + (1 + e^{2i\lambda}) \sum_{n=1}^{\infty} x^n z^n$$

and

$$p'_\lambda(xz) = \frac{1 + e^{2i\lambda}}{(1 - xz)^2},$$

we complete the proof. □

**Theorem 5.** *Let $p \in \mathcal{P}_\lambda$ with $\lambda \in (-\pi/2, \pi/2)$. Then*

$$\left| p(z) - \frac{1 + r^2 e^{-2i\lambda}}{1 - r^2} \right| \le \frac{2r\cos\lambda}{1 - r^2},$$

*where $r = |z| < 1$. In particular, we have*

$$\frac{1 + r^2\cos 2\lambda - 2r\cos\lambda}{1 - r^2} \le \operatorname{Re} p(z) \le \frac{1 + r^2\cos 2\lambda + 2r\cos\lambda}{1 - r^2}$$

*and*

$$1/A(\lambda, r) \le |p(z)| \le A(\lambda, r),$$

*where $A(\lambda, r)$ is given by*

$$A(\lambda, r) = \frac{\sqrt{(1 - r^2)^2 + 4r^2\cos^2\lambda} + 2r\cos\lambda}{1 - r^2}. \tag{5}$$

*These inequalities are sharp with the extremal functions $p_\lambda(xz)$, where $|x| = 1$.*

*Proof.* By using the same arguments as in Theorem 4, the maximum of the first functional over $\mathcal{P}_\lambda$ is obtained over the set of $p_\lambda(xz)$ with $|x| = 1$. Therefore

$$\begin{aligned} \left| p_\lambda(xz) - \frac{1 + r^2 e^{-2i\lambda}}{1 - r^2} \right| &= \left| \frac{(1 + e^{2i\lambda})(xz - r^2)}{(1 - xz)(1 - r^2)} \right| \\ &= \frac{2\cos\lambda}{1 - r^2} \left| \frac{xz - r^2}{1 - xz} \right| \\ &= \frac{2r\cos\lambda}{1 - r^2}, \end{aligned}$$

where $r = |z| < 1$ and $|x| = 1$, which is what we want. The other estimates can be deduced directly from the first one. □

*Remark* 2. The first inequality in Theorem 5 implies that the function $p \in \mathcal{P}_\lambda$ maps the disc $|z| < r$ into $U(\lambda, r)$ the hyperbolic disc in the tilted half plane $\mathbb{H}_\lambda$ centered at 1 with radius $\operatorname{arctanh} r$.

Although the extremal problems of real-valued continuous convex functionals over $\mathcal{P}_\lambda$ can be solved within the set $p_\lambda(xz)$, $|x| = 1$, it is not applicable for general functionals. Robertson [19], [20] and Sakaguchi [24] obtained variational formulae for $\mathcal{P}_0$ and showed that, for fixed $z \in \mathbb{D}$, the extremal values of

$$\operatorname{Re} F(p(z), zp'(z)),\ p \in \mathcal{P}_0,$$

where $F(u, v)$ is analytic in $(u, v) \in \mathbb{C}^2$, $\operatorname{Re} u > 0$, are always attained by the functions

$$t\left(\frac{1+xz}{1-xz}\right) + (1-t)\left(\frac{1+\bar{x}z}{1-\bar{x}z}\right),\ 0 \le t \le 1,\ |x| = 1.$$

Since there exists a linear relation between $\mathcal{P}_0$ and $\mathcal{P}_\lambda$, we can see that for a fixed $z \in \mathbb{D}$, the extremal values of

$$\operatorname{Re} F(p(z), zp'(z)),\ p \in \mathcal{P}_\lambda,$$

where $F(u, v)$ is analytic in $(u, v) \in \mathbb{C}^2$, $\operatorname{Re} e^{i\lambda} u > 0$, are attained by the functions

$$t\left(\frac{1+xe^{-2i\lambda}z}{1-xz}\right) + (1-t)\left(\frac{1+\bar{x}e^{-2i\lambda}z}{1-\bar{x}z}\right),\ 0 \le t \le 1,\ |x| = 1.$$

For another functional

$$\frac{F_1(p)}{F_2(p)},$$

where $F_1$ and $F_2$ are real-valued continuous linear functionals over $\mathcal{P}_\lambda$ with $F_2(p) \neq 0$ for $p \in \mathcal{P}_\lambda$, it follows from the Duality Principle [18, Corollary 1.1] and Remark 1 that the extremal value of it is attained by a function in $W_\lambda$. However, for many cases of interest, it is not easy to obtain extremal values even for those restricted classes of functions, for instance the functional $zp'(z)/p(z)$ over $\mathcal{P}_\lambda$. Our next theorem solves this problem by using elementary functional analysis.

**Theorem 6.** *Let $p \in \mathcal{P}_\lambda$ with $\lambda \in (-\pi/2, \pi/2)$. Then*

$$\left|\frac{zp'(z)}{p(z)}\right| \le M(\lambda, |z|),$$

*where*

$$M(\lambda, r) = \begin{cases} \dfrac{2r\cos\lambda}{1+r^2-2r|\sin\lambda|}, & r < |\tan(\lambda/2)|, \\ \dfrac{2r}{1-r^2}, & r \ge |\tan(\lambda/2)|. \end{cases} \tag{6}$$

*Equality holds for some point $z_0 = re^{i\theta}$, $0 < r < 1$, if and only if $p(z) = p_\lambda(xz)$ where $x = e^{i(\alpha-\theta)}$ with $\alpha$ satisfying*

$$\begin{cases} \alpha = \pi/2 + \lambda, & r < -\tan(\lambda/2), \\ \alpha = -\pi/2 + \lambda, & r < \tan(\lambda/2), \\ \sin(\alpha - \lambda) = -\frac{1+r^2}{1-r^2}\sin\lambda, & r \ge |\tan(\lambda/2)|. \end{cases} \tag{7}$$

*Remark* 3. For a fixed $0 < r < 1$, $M(\lambda, r)$ is a symmetric function in $\lambda$ with respect to the origin and it is also decreasing in $0 \leq \lambda < \pi/2$. We thus have $M(\lambda, r) \leq M(0, r) = 2r/(1-r^2)$ for any $\lambda \in (-\pi/2, \pi/2)$ which is a known result for Gelfer functions ([5, p. 73], see also [28], [4] or [12]).

In order to prove the above theorem, the following lemma is needed.

**Lemma 5.**
$$N(\lambda, r) \leq \left|\frac{zp'_\lambda(z)}{p_\lambda(z)}\right| \leq M(\lambda, r)$$
*for $r = |z| < 1$, where $M(\lambda, r)$ is defined by* (6) *and*

$$N(\lambda, r) = \frac{2r\cos\lambda}{1 + r^2 + 2r|\sin\lambda|}. \tag{8}$$

*The equality in the right-hand side holds at $z_0 = re^{i\theta}$ with $\theta$ in place of $\alpha$ satisfying* (7) *while the other side holds at $z_0 = re^{i\theta}$ with $\theta$ satisfying*
$$\begin{cases} \theta - \lambda = \pi/2, & \lambda > 0, \\ \theta - \lambda = -\pi/2, & \lambda \leq 0. \end{cases}$$

*Proof.* By observing that
$$\left|\frac{\overline{z}p'_\lambda(\overline{z})}{p_\lambda(\overline{z})}\right| = \left|\frac{zp'_{-\lambda}(z)}{p_{-\lambda}(z)}\right|,$$
we can restrict ourselves to the case $\lambda \geq 0$.

Since $p'_\lambda(z)/p_\lambda(z) = (1+e^{2i\lambda})/(1-z)(e^{2i\lambda}+z)$, after letting $z = re^{i(\alpha+\lambda+\pi/2)}$ and $h(\alpha) = |(1-z)(e^{2i\lambda}+z)|^2 = |(1-re^{i(\alpha+\lambda+\pi/2)})(e^{2i\lambda}+re^{i(\alpha+\lambda+\pi/2)})|^2$, we obtain
$$h(\alpha) = (1 + r^2 + 2r\sin(\alpha+\lambda))(1 + r^2 + 2r\sin(\lambda - \alpha))$$
and

$$\frac{2r\cos\lambda}{\max_{-\pi<\alpha\leq\pi}\sqrt{h(\alpha)}} \leq \left|\frac{zp'_\lambda(z)}{p_\lambda(z)}\right| \leq \frac{2r\cos\lambda}{\min_{-\pi<\alpha\leq\pi}\sqrt{h(\alpha)}}. \tag{9}$$

It is thus sufficient to search for the maximum and minimum of $h(\alpha)$ over $-\pi < \alpha \leq \pi$. A simple calculation yields
$$h'(\alpha) = -4r\sin\alpha[(1+r^2)\sin\lambda + 2r\cos\alpha].$$

Since $h(\alpha)$ is smooth and periodic, the candidate minimum points of $h(\alpha)$ are the zero points of $h'(\alpha)$ which are $\alpha_1 = 0$, $\alpha_2 = \pi$ and $\alpha_3 = \pm\arccos(-(1+r^2)\sin\lambda/2r)$. Here $\alpha_3$ is meaningful only when $\sin\lambda \leq 2r/(1+r^2)$, namely $r \geq \tan(\lambda/2)$. A calculation gives
$$h(0) = (1 + r^2 + 2r\sin\lambda)^2,$$
$$h(\pi) = (1 + r^2 - 2r\sin\lambda)^2,$$
and
$$h(\alpha_3) = \cos^2\lambda(1 - r^2)^2.$$

$\lambda \geq 0$ implies that $h(\pi) \leq h(0)$. Since $h(\pi) \geq h(\alpha_3)$, we can get the minimum of $h(\alpha)$ is

$$\begin{cases} (1+r^2-2r\sin\lambda)^2, & r < \tan(\lambda/2), \\ \cos^2\lambda(1-r^2)^2, & r \geq \tan(\lambda/2), \end{cases} \tag{10}$$

and the maximum of $h(\alpha)$ is

$$(1+r^2+2r\sin\lambda)^2. \tag{11}$$

Finally by using (9), (10) and (11), we can obtain our claims immediately. □

*Proof of Theorem 6.* The equivalence of (i) and (iv) in Theorem 1 implies that if $p \in \mathcal{P}_\lambda$, then there exists a function $\omega \in \mathcal{A}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ such that

$$p(z) = p_\lambda(\omega(z))$$

in $\mathbb{D}$. Then by making use of Lemma 5 and the Schwarz-Pick lemma, we have

$$\begin{aligned} \left|\frac{zp'(z)}{p(z)}\right| &= \left|\frac{z\omega'(z)p'_\lambda(\omega(z))}{p_\lambda(\omega(z))}\right| = \left|\frac{z\omega'}{\omega}\right|\left|\frac{\omega' p'_\lambda(\omega)}{p_\lambda(\omega)}\right| \\ &\leq \begin{cases} \dfrac{1-|\omega|^2}{1-|z|^2}\dfrac{2|z|\cos\lambda}{1+|\omega|^2-2|\omega||\sin\lambda|}, & |\omega| < \tan(\lambda/2), \\ \dfrac{1-|\omega|^2}{1-|z|^2}\dfrac{2|z|}{1-|\omega|^2}, & |\omega| \geq \tan(\lambda/2). \end{cases} \end{aligned} \tag{12}$$

Since $|\omega(z)| \leq |z| = r$, the function $2|\omega|/(1+|\omega|^2)$ is increasing in $|\omega| \in [0, r]$ and attains its maximum value $2r/(1+r^2)$ if and only if $\omega(z) \equiv xz$ with $|x| = 1$. On the other hand,

$$\frac{1-|\omega|^2}{1-r^2}\frac{2r\cos\lambda}{1+|\omega|^2-2|\omega||\sin\lambda|}$$

is also increasing in $|\omega|$ provided $|\omega| < |\tan(\lambda/2)|$. Therefore inequality (12) implies

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \begin{cases} \dfrac{2r\cos\lambda}{1+r^2-2r|\sin\lambda|}, & r < |\tan(\lambda/2)|, \\ \dfrac{2r}{1-r^2}, & r \geq |\tan(\lambda/2)|. \end{cases}$$

Hence the proof of Theorem 6 is completed. □

The sharp estimate of the entity in Theorem 6 first appears in the following form in a paper [23] by Ruscheweyh and Singh. Their proof was based on a variational method.

**Theorem A.** *For $p \in \mathcal{P}_0$ and $\lambda \in (-\pi/2, \pi/2)$ the estimate*

$$\left|\frac{zp'(z)}{p(z)+i\tan\lambda}\right| \leq \begin{cases} \dfrac{(1-|z|^2)\cos\lambda}{1-2|z||\sin\lambda|+|z|^2}, & |z| < |\tan\frac{\lambda}{2}|, \\ 1, & |z| \geq |\tan\frac{\lambda}{2}|, \end{cases}$$

*is valid and sharp. Equality holds for certain functions in* $\mathcal{P}_0$*.*

Note that Theorem 6 improves Theorem A since it shows that the extremal functions are only $p_\lambda(xz)$ with $|x| = 1$.

**Proposition 1.** *Let* $p \in \mathcal{P}_\lambda$ *with* $\lambda \in (-\pi/2, \pi/2)$*. Then*

$$\left|\operatorname{Im}\frac{zp'(z)}{p(z)}\right| \le M(\lambda, r)$$

*and*

$$\left|\operatorname{Re}\frac{zp'(z)}{p(z)}\right| \le M(\lambda, r),$$

*where* $r = |z| < 1$ *and* $M(\lambda, r)$ *is given in* (6)*. Equality occurs at point* $z_0 = re^{i\theta}$ *in the first inequality if and only if* $p(z) = p_\lambda(xz)$ *and* $r \le |\tan(\lambda/2)|$*, where* $x = e^{i(\alpha-\theta)}$ *with* $\alpha$ *satisfying* (7)*.*

*Proof.* Since the above inequalities are straightforward consequences of Theorem 6, we only need to verify the sharpness. By a simple calculation, it is easy to see that if $\lambda < 0$ for any fixed $r \le -\tan(\lambda/2)$

$$\frac{zp_\lambda'(z)}{p_\lambda(z)} = \frac{z(1+e^{2i\lambda})}{(1-z)(e^{2i\lambda}+z)} = \frac{-2ri\cos\lambda}{1+r^2-2r\sin\lambda} = -iM(\lambda, r)$$

when $z = ire^{i\lambda}$. Similarly, we can get if $\lambda \ge 0$ for any fixed $r \le \tan(\lambda/2)$

$$\frac{zp_\lambda'(z)}{p_\lambda(z)} = \frac{2ri\cos\lambda}{1+r^2-2r\sin\lambda} = iM(\lambda, r)$$

when $z = -ire^{i\lambda}$. Our proof is completed. □

We shall conclude this section with a result due to Kim and Sugawa [14] which gives a sufficient condition for membership of $\mathcal{P}_\lambda$. Note that the function $zp_\lambda'(z)/p_\lambda(z)$ maps $\mathbb{D}$ univalently onto $U_\lambda$, where $U_\lambda$ is the slit domain defined by

$$U_\lambda = \mathbb{C} \setminus \{iy : \ y \ge A_\lambda \text{ or } y \le -1/A_\lambda\}, \ A_\lambda = \cos\lambda/(1+\sin\lambda).$$

Since $U_\lambda$ is a starlike domain, the function $zp_\lambda'(z)/p_\lambda(z)$ is starlike. Hence by using Lemma 3 of [14], we can obtain:

**Theorem 7.** *Let* $p \in \mathcal{A}_0$ *satisfy the subordination*

$$\frac{zp'(z)}{p(z)} \prec \frac{zp_\lambda'(z)}{p_\lambda(z)}$$

*in* $\mathbb{D}$*. Then* $p \in \mathcal{P}_\lambda$*.*

## 4. Applications

### 4.1. λ-spirallike functions

**Definition 1** ([3], see also [1])**.** A function $f \in \mathcal{A}_1$ is called $\lambda$-*spirallike* (denoted by $f \in \mathcal{SP}(\lambda)$) for a real number $\lambda \in (-\pi/2, \pi/2)$ if

$$\frac{zf'}{f} \in \mathcal{P}_\lambda.$$

Spirallike functions are shown to be univalent by Špaček [26]. Note that $\mathcal{SP}(0)$ is precisely the set of starlike functions normally denoted by $\mathcal{S}^*$.

By the definition of $\lambda$-spirallike function, we can easily deduce the following corollary from Theorem 5:

**Corollary 1.** *Let* $f \in \mathcal{SP}(\lambda)$*. Then*

$$\left|\frac{zf'(z)}{f(z)} - \frac{1 + r^2 e^{-2i\lambda}}{1 - r^2}\right| \le \frac{2r\cos\lambda}{1 - r^2},$$

*where* $r = |z| < 1$*. In particular, we have*

$$\frac{1 + r^2\cos 2\lambda - 2r\cos\lambda}{1 - r^2} \le \operatorname{Re}\frac{zf'(z)}{f(z)} \le \frac{1 + r^2\cos 2\lambda + 2r\cos\lambda}{1 - r^2}$$

*and*

$$1/A(\lambda, r) \le \left|\frac{zf'(z)}{f(z)}\right| \le A(\lambda, r),$$

*where* $A(\lambda, r)$ *is given by* (5)*. Those inequalities are sharp with the extremal function given by*

$$f_\lambda(z) = \frac{z}{(1 - z)^{1 + e^{-2i\lambda}}}.$$

Note that the lower bound of the second estimate was proved by Robertson [21], but the others are not given in the literature as far as the author knows.

### 4.2. Close-to-convex functions with argument λ

**Definition 2.** A function $f \in \mathcal{A}_1$ is said to be *close-to-convex* (denoted by $f \in \mathcal{CL}$) if there exist a starlike function $g$ and a real number $\lambda \in (-\pi/2, \pi/2)$ such that

$$\frac{zf'}{g} \in \mathcal{P}_\lambda.$$

If we specify the real number $\lambda$ in the above definition, the corresponding function is called a *close-to-convex function with argument* $\lambda$ and we denote the class of these functions by $\mathcal{CL}(\lambda)$ (see [5, II, Definition 11.4]). Note that the union of class $\mathcal{CL}(\lambda)$ over $\lambda \in (-\pi/2, \pi/2)$ is precisely $\mathcal{CL}$. The sharp coefficient bounds of the class $\mathcal{CL}(\lambda)$ are known (see [27]).

**Lemma 6** (See [3]). *Let $f \in \mathcal{S}^*$. Then*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2},$$

*where $r = |z| \in (0,1)$. Equalities occur if and only if $f$ is a suitable rotation of the Koebe function $k(z) = z/(1-z)^2$.*

By applying Theorem 5 and Lemma 6, we can get a sharp distortion theorem for $\mathcal{CL}(\lambda)$.

**Theorem 8.** *Let $f(z) \in \mathcal{CL}(\lambda)$ for a real constant $\lambda \in (-\pi/2, \pi/2)$. Then*

$$\frac{1}{A(\lambda, r)(1+r)^2} \le |f'(z)| \le \frac{A(\lambda, r)}{(1-r)^2},$$

*where $A(\lambda, r)$ is given in (5) and $r = |z| < 1$. The inequalities are sharp with the extremal functions $f(z)$ satisfying*

$$f'(z) = \frac{1 + e^{-2i\lambda} xz}{(1-yz)^2(1-xz)}$$

*for $|x| = |y| = 1$.*

*Remark* 4. Theorem 8 improves the distortion theorem of close-to-convex functions (see [3]) since the real-valued function $A(\lambda, r)$ is symmetric in $\lambda$ with respect to the origin and

$$\frac{1-r}{1+r} \le A(\lambda, r) \le \frac{1+r}{1-r}$$

for any $\lambda \in (-\pi/2, \pi/2)$.

Though it is easy to deduce the growth theorem of close-to-convex functions with argument $\lambda$ from Theorem 8, we omit it here since the form is not very esthetics.

### 4.3. Analytic functions whose derivative is in $\mathcal{P}_\lambda$

Let $\mathcal{D}(\lambda) = \{f \in \mathcal{A}_1 : f' \in \mathcal{P}_\lambda\}$ for $-\pi/2 < \lambda < \pi/2$. It is easy to see that $\mathcal{D}(\lambda) \subset \mathcal{CL}(\lambda)$, thus $\mathcal{D}(\lambda) \subset \mathcal{S}$. Some properties of $\mathcal{D}(\lambda)$ can be deduced from those of $\mathcal{D}(0)$ which have been studied in [7], [17] and so on. We shall only present a distortion theorem which is a direct consequence of Theorem 5.

**Theorem 9.** *Let $f \in \mathcal{D}(\lambda)$. Then*

$$1/A(\lambda, r) \le |f'(z)| \le A(\lambda, r),$$

*where $r = |z| < 1$ and $A(\lambda, r)$ is given by (5). These inequalities are sharp with the extremal function*

$$f(z) = -(1 + e^{-2i\lambda}) \log(1-z) - e^{-2i\lambda} z. \tag{13}$$

For a locally univalent function $f$ on $\mathbb{D}$, the hyperbolic norm of the pre-Schwarzian derivative $T_f = f''/f'$ is defined by

$$||T_f|| = \sup_{|z|<1} (1 - |z|^2)|T_f(z)|.$$

Since each function in $\mathcal{P}_\lambda$ is a Gelfer function, by Gelfer's theorem (see [12, Theorem 2.4]), we have for each function $f \in \mathcal{D}(\lambda)$,

$$||T_f|| \leq 2.$$

Our next result shows that this estimate is sharp for the class $\mathcal{D}(\lambda)$, and the extremal functions are also given.

**Theorem 10.** *Let $f \in \mathcal{D}(\lambda)$. Then*

$$||T_f|| \leq 2.$$

*This bound is sharp for each $\lambda \in (-\pi/2, \pi/2)$ with the extremal function $f$ given in* (13).

*Proof.* For $f \in \mathcal{D}(\lambda)$, we have $f' \in \mathcal{P}_\lambda$, thus in view of Theorem 6,

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq M(\lambda, |z|),$$

where $M(\lambda, r)$ is given in (6). Remark 3 gives that $M(\lambda, r) \leq M(0, \lambda) = 2r/(1 - r^2)$, therefore we have $||T_f|| \leq 2$. The sharpness can be obtained by observing that $M(\lambda, r) = M(0, \lambda)$ if $r \geq |\tan(\lambda/2)|$. □

Note that the hyperbolic norm of $f \in \mathcal{D}(0)$ was obtained by Nunokawa [18] as well. It is known that (cf. [11]) $f$ is bounded if $||T_f|| < 2$ and the bound depends only on the value of $||T_f||$.

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