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The topological sphere theorem for complete submanifolds*

KATSUHIRO SHIOHAMA and HONGWEI XU
Graduate School of Mathematics, Kyushu University, Fukuoka, 812-Japan
Institute of Mathematics, Zhejiang University, Hangzhou, 310027-China

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Abstract. A topological sphere theorem is obtained from the point of view of submanifold geometry. An important scalar is defined by the mean curvature and the squared norm of the second fundamental form of an oriented complete submanifold M^n in a space form of nonnegative sectional curvature. If the infimum of this scalar is negative, we then prove that the Ricci curvature of M^n has a positive lower bound. Making use of the Lawson–Simons formula for the nonexistence of stable k-currents, we eliminate $H_k(M^n, \mathbb{Z})$ for all $1 < k < n-1$. We then observe that the fundamental group of M^n is trivial. It should be emphasized that our result is optimal.

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1. Introduction

The diameter sphere theorem due to Grove–Shiohama (see [GS]) states that a complete and connected Riemannian n-manifold M with sectional curvature $K_M \geqslant 1$ is homeomorphic to a sphere if the diameter $d(M)$ of M satisfies $d(M) > \pi/2$. Since the compact rank one symmetric spaces have sectional curvature bounded below by 1 and diameter $\pi/2$, this result is optimal for complete manifolds having sectional curvature bounded below by 1. The rigidity theorem by Gromoll and Grove (see [GG1], [GG2]) provides the determination of almost all compact rank one symmetric spaces under the hypotheses that $K_M \geqslant 1$ and $d(M) = \pi/2$.

The purpose of the present article is to establish the optimal sphere theorem for a completely new class of Riemannian manifolds from the point of view of submanifold theory. Here the squared norm S of the second fundamental form $\mathbf{h}$ of a submanifold M in the $(n+p)$-dimensional space form $F^{n+p}(c)$ of constant sectional curvature c plays an essential role. We deal with this S instead of K_M. Our result is obtained without assuming compactness by eliminating the Clifford tori, cylinders and Euclidean spaces. We refer for the basic tools in Riemannian geometry and topology to [CE] and [HW].

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To begin, we introduce the rigidity theorem for a minimal submanifold M in the standard unit $(n+p)$-sphere $\mathbf{S}^{n+p}(1)$, investigated by Simons [Si], Lawson [La], Chern-do Carmo–Kobayashi [CDK] and Li–Li [LL], summarized as follows.

THEOREM A. *Let M be an n-dimensional oriented compact minimal submanifold in $\mathbf{S}^{n+p}(1)$. If*

$$S \leqslant \max\left\{\frac{n}{2-1/p}, \frac{2n}{3}\right\},$$

then M is congruent to one of the following

(1) $\mathbf{S}^n(1)$

(2) $\mathbf{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$, *for* $k = 1, \ldots, n-1$

(3) *the Veronese surface in* $\mathbf{S}^4(1)$.

Remark 1. In cases where $p = 1$ and $n = 2$, the condition for S in Theorem A is optimal. In the case that $p = 1$, then $S = n$ and M is a Clifford torus, while if $n = 2$, then $S = \frac{4}{3}$ and it is a Veronese surface.

Leung [L] first applied the Lawson–Simons Theorem on minimal submanifolds in spheres to obtain a topological sphere theorem, as stated

THEOREM B ([L]). *Let $M \subset \mathbf{S}^{n+p}(1)$ be an n-dimensional oriented compact minimal submanifold with $n \neq 3$. Then M is homeomorphic to a sphere if $S < n$.*

Next, the minimality assumption in Theorem A is replaced by that of a parallel mean curvature normal field. For given integers $n \geqslant 2, p \geqslant 1$ and for a number H, we define the constants $C(n,p,H)$ and $\lambda = \lambda(n,H)$ by

$$C(n,p,H) := \begin{cases} \alpha(n,H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\{\alpha(n,H), & \frac{1}{3}(2n + 5nH^2)\}, \text{ otherwise,} \end{cases}$$

and

$$\alpha(n,H) := n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

$$\lambda := H + \sqrt{\frac{\alpha(n,H) - nH^2}{n(n-1)}}.$$

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THEOREM C ([Xu]). *Let $M \subset \mathbf{S}^{n+p}(1)$ be an n-dimensional compact submanifold with a parallel mean curvature normal field. If H is the mean curvature of M and if $S \leqslant C(n,p,H)$, then M is congruent to one of the following*

(1) $\mathbf{S}^n\left(\frac{1}{\sqrt{1+H^2}}\right)$
(2) $\mathbf{S}^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbf{S}^1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$
(3) $\mathbf{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$, *for* $k = 1, \ldots, n-1$
(4) *the Clifford torus* $\mathbf{S}^1(r_1) \times \mathbf{S}^1(r_2)$ *in* $\mathbf{S}^3(r)$ *with constant mean curvature* H_0, *where* $r_1, r_2 = [2(1+H^2) \pm 2H_0(1+H^2)^{1/2}]^{-1/2}$, $r = (1+H^2-H_0^2)^{-1/2}$, *and* $0 \leqslant H_0 \leqslant H$
(5) *the Veronese surface in* $\mathbf{S}^4\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Remark 2. If M is complete and if $\sup_M S < \alpha(n,H)$, then M is congruent to (1) or (5). This fact was proved in [SX2].

For a given integer $n \geqslant 2$ and constants c and H, we define a number $\alpha(n,H,c)$ by

$$\alpha(n,H,c) := nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2}.$$

A recent result by the second author [Xu2] is stated as follows.

THEOREM D. *Let $M \subset F^{n+p}(c)$ be an n-dimensional complete submanifold with parallel mean curvature normal field such that $c \leqslant 0$ and $c + H^2 > 0$. If $n \geqslant 3$ and if $\sup_M S < \alpha(n,H,c)$, then M is a totally umbilical sphere $\mathbf{S}^n(1/\sqrt{c+H^2})$.*

The following examples show that the condition for S in Theorem D is optimal.

EXAMPLE 1. In the case where $c = 0$, consider $M := \mathbf{S}^{n-1}((n-1)/nH) \times \mathbf{R}^1 \subset \mathbf{R}^{n+1}$. Also set $M := F^{n-1}(c+\lambda^2) \times F^1(c+\mu^2)$, where $c < 0$. In both cases we have $S = \alpha(n,H,c)$. The first case of the proof of $S = \alpha(n,H,c)$ is trivial. If λ and μ are principal curvatures in the second case, then $(n-1)\lambda + \mu = nH$ and $S = (n-1)\lambda^2 + \mu^2 = \alpha(n,H,c)$.

In this article we relax the assumption for the parallel mean curvature normal. Thus the $\alpha(n,H,c)$ may be considered as a scalar on M, for so too is the mean curvature H. We shall prove

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MAIN THEOREM. *Let M be an oriented complete submanifold in $F^{n+p}(c)$ with $c \geqslant 0$. If $n \neq 3$ and if*

$$\Lambda(M) := \sup_M (S - \alpha(n, H, c)) < 0,$$

then M is homeomorphic to a sphere. Moreover, M is diffeomorphic to a spherical space form if $n = 3$.

Remark 3. The orientability assumption in Main Theorem is needed for the use of the Poincaré duality. The solution to the Poincaré conjecture is also employed. If the orientability of M is not assumed, we need only employ the orientable double cover. Then $M \subset F^{n+p}(c)$ is an immersed sphere. The class $\{M; \Lambda(M) < 0\}$ of all Riemannian n-manifolds in the above theorem is nonempty. In fact, $\mathbf{S}^n(1)$ is a totally geodesic great sphere in $\mathbf{S}^{n+1}(1)$, and $\Lambda(\mathbf{S}^n(1)) = -n$ for $c = 1$. Also, $\mathbf{S}^n(1) \subset \mathbf{R}^{n+1}$ is totally umbilic, and $\Lambda(\mathbf{S}^n(1)) = -\frac{n}{n-1}$ for $c = 0$.

The following examples show that the assumption in Main Theorem is optimal.

EXAMPLE 2. Clearly $\mathbf{R}^n \subset \mathbf{R}^{n+1}$ satisfies $\Lambda(\mathbf{R}^n) = 0$. Let $M := \mathbf{S}^{n-1}(1/\sqrt{1+\lambda^2}) \times \mathbf{S}^1(\lambda/\sqrt{1+\lambda^2}) \subset \mathbf{S}^{n+1}$ for $c = 1$, and also $M := \mathbf{S}^{n-1}((n-1)/nH) \times \mathbf{R}^1 \subset \mathbf{R}^{n+1}$ for $c = 0$. Here λ is as in Theorem C, and H is a nonnegative constant for the case $c = 1$, and H a positive constant for the case $c = 0$. In both cases H is the mean curvature of M and $\Lambda(M) = 0$.

Remark 4. In the case $n = 2$, clearly $\Lambda(M) < 0$ is equivalent to $\inf_M K_M > 0$, and hence the conclusion of our theorem is trivial.

Remark 5. The method of proof for our theorem demonstrates the following fact. In the case where $n = 3$, we can replace the ambient space by a general Riemannian $(3 + p)$-manifold N with $K_N \geqslant c$ for a constant c. If $\wedge(M) < 0$ and if the mean curvature H of $M \subset N$ satisfies $c + H^2 > 0$, then M is diffeomorphic to a spherical space form.

2. The lower bound for Ricci curvature

The following convention on the range of indices will be used throughout.

$$1 \leqslant A, B, C, \ldots \leqslant n + p, \qquad 1 \leqslant i, j, k, \ldots \leqslant n,$$

$$n + 1 \leqslant \alpha, \beta, \gamma, \ldots \leqslant n + p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N such that $\{e_i\}$ is tangent to M. Let K and R be the Riemannian curvature tensors of N and M respectively and **h** the second fundamental form of M. Let W_A be the dual frame field of e_A. Then,

$$\mathbf{h} = \sum_{\alpha,i,j} h_{ij}^{\alpha}\omega_i \otimes \omega_j \otimes e_{\alpha},$$

$$R_{ijk\ell} = K_{ijk\ell} + \sum_{\alpha}(h_{ik}^{\alpha}h_{j\ell}^{\alpha} - h_{i\ell}^{\alpha}h_{jk}^{\alpha}).$$

The squared norm S of the second fundamental form and the mean curvature H of M are given as

$$S := \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2, \qquad H := \left|\frac{1}{n}\sum_{\alpha,i} h_{ii}^{\alpha}e_{\alpha}\right|.$$

THEOREM 1. *Let $M \subset N$ be a complete submanifold in an $(n+p)$-dimensional Riemannian manifold N with $K_N \geqslant c$. Here c is a constant satisfying $c + H^2 > 0$. If $\Lambda(M) < 0$, then M is compact. Moreover the fundamental group $\Pi_1(M)$ of M is finite.*

Proof of Theorem 1. Note that if $\{e_i\}$ is an orthonormal basis for the tangent space M_x to M at a point x, then

$$\sum_i K_{inin} \geqslant (n-1)c.$$

For a unit vector X tangent to M at x, if we choose an orthonormal frame such that $X = e_n$, then the Ricci curvature $\mathrm{Ric}_M(X)$ of M with respect to X is expressed as

$$\begin{aligned}\mathrm{Ric}_M(X) &= \sum_i R_{inin} = \sum_i K_{inin} + \sum_{\alpha,i}[h_{ii}^{\alpha}h_{nn}^{\alpha} - (h_{in}^{\alpha})^2] \\ &\geqslant (n-1)c + \sum_{\alpha}\left[\mathrm{tr}\ H_{\alpha}\cdot h_{nn}^{\alpha} - \sum_i (h_{in}^{\alpha})^2\right], \qquad (1)\end{aligned}$$

where H_{α} is the $(n\times n)$-matrix whose i,j-entry is h_{ij}^{α}. Setting

$$Y_{\alpha} := \sum_i (h_{in}^{\alpha})^2 - \mathrm{tr}\ H_{\alpha}\cdot h_{nn}^{\alpha}, \qquad S_{\alpha} := \mathrm{tr}\ H_{\alpha}^2 = \sum_{i,j}(h_{ij}^{\alpha})^2,$$

we observe

$$S = \sum_{\alpha} S_{\alpha}.$$

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Furthermore, setting

$$T_\alpha := \text{tr } H_\alpha, \qquad \tilde{S}_\alpha := \sum_i (h^\alpha_{ii})^2,$$

we have

$$n^2 H^2 = \sum_\alpha T^2_\alpha.$$

By definition,

$$\begin{aligned} Y_\alpha &= \sum_i (h^\alpha_{in})^2 - T_\alpha h^\alpha_{nn} \\ &= \frac{1}{2}\sum_{i\neq n}(h^\alpha_{in})^2 + \frac{1}{2}\sum_{i\neq n}(h^\alpha_{ni})^2 + (h^\alpha_{nn})^2 - T_\alpha h^\alpha_{nn}, \end{aligned} \tag{2}$$

and

$$(T_\alpha - h^\alpha_{nn})^2 = \left(\sum_{i=1}^{n-1} h^\alpha_{ii}\right)^2 \leqslant (n-1)\sum_{i=1}^{n-1}(h^\alpha_{ii})^2 = (n-1)[\tilde{S}_\alpha - (h^\alpha_{nn})^2].$$

Setting $y_\alpha := (h^\alpha_{nn})^2 - T_\alpha h^\alpha_{nn}$, the above relation reduces to

$$\begin{aligned} 0 &\geqslant n(h^\alpha_{nn})^2 - 2T_\alpha h^\alpha_{nn} + T^2_\alpha - (n-1)\tilde{S}_\alpha \\ &= ny_\alpha + (n-2)T_\alpha h^\alpha_{nn} + T^2_\alpha - (n-1)\tilde{S}_\alpha \\ &= ny_\alpha + (n-2)T_\alpha\left(h^\alpha_{nn} - \frac{T_\alpha}{n}\right) + \frac{2(n-1)}{n}T^2_\alpha - (n-1)\tilde{S}_\alpha. \end{aligned} \tag{3}$$

From the relations

$$\sum_i \left(h^\alpha_{ii} - \frac{T_\alpha}{n}\right) = 0, \qquad \sum_i \left(h^\alpha_{ii} - \frac{T_\alpha}{n}\right)^2 = \tilde{S}_\alpha - \frac{T^2_\alpha}{n},$$

we get

$$\left(h^\alpha_{ii} - \frac{T_\alpha}{n}\right)^2 \leqslant \frac{n-1}{n}\left(\tilde{S}_\alpha - \frac{T^2_\alpha}{n}\right). \tag{4}$$

It follows from (3) and (4) that

$$0 \geqslant ny_\alpha - (n-2)|T_\alpha|\sqrt{\frac{n-1}{n}\left(\tilde{S}_\alpha - \frac{T^2_\alpha}{n}\right)} + \frac{2(n-1)}{n}T^2_\alpha - (n-1)\tilde{S}_\alpha,$$

and hence

$$y_\alpha \leqslant \frac{n-1}{n}\tilde{S}_\alpha + \frac{n-2}{n}|T_\alpha|\sqrt{\frac{n-1}{n}\left(\tilde{S}_\alpha - \frac{T_\alpha^2}{n}\right)} - \frac{2(n-1)}{n^2}T_\alpha^2. \tag{5}$$

The relations (2) and (5) yield

$$Y_\alpha \leqslant \frac{n-1}{n}S_\alpha + \frac{n-2}{n}|T_\alpha|\sqrt{\frac{n-1}{n}\left(S_\alpha - \frac{T_\alpha^2}{n}\right)} - \frac{2(n-1)}{n^2}T_\alpha^2. \tag{6}$$

It then follows from (1) and (6) that

$$\begin{aligned}
\mathrm{Ric}_M(X) &\geqslant (n-1)c - \sum_\alpha Y_\alpha \\
&\geqslant (n-1)c - \sum_\alpha \left[\frac{n-1}{n}S_\alpha\right. \\
&\quad \left. + \frac{n-2}{n}|T_\alpha|\sqrt{\frac{n-1}{n}\left(S_\alpha - \frac{T_\alpha^2}{n}\right)} - \frac{2(n-1)}{n^2}T_\alpha^2\right] \\
&\geqslant (n-1)c - \frac{n-1}{n}S \\
&\quad - \frac{n-2}{n}\sqrt{\frac{n-1}{n}\left(\sum_\alpha T_\alpha^2\right)\left[\sum_\alpha\left(S_\alpha - \frac{T_\alpha^2}{n}\right)\right]} \\
&\quad + 2(n-1)H^2 \\
&= (n-1)c - \frac{n-1}{n}S - \frac{n-2}{n}\sqrt{\frac{n-1}{n}}nH\sqrt{S-nH^2} \\
&\quad + 2(n-1)H^2 \\
&= \frac{n-1}{n}\left[nc + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2}\right].
\end{aligned}$$

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The bracketed factor in the final line above can be expressed as a product of two terms, and we obtain

$$\begin{aligned}\mathrm{Ric}_M(X) \geqslant -\frac{n-1}{n}&\left[\sqrt{S-nH^2}+\frac{n(n-2)}{2\sqrt{n(n-1)}}H\right.\\&\left.+\frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2+4n(n-1)^2c}\right]\\&\times\left[\sqrt{S-nH^2}+\frac{n(n-2)}{2\sqrt{n(n-1)}}H\right.\\&\left.-\frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2+4n(n-1)^2c}\right].\end{aligned}$$

Thus we observe that

$$S < \alpha(n, H, c),$$

is equivalent to

$$\begin{aligned}&\sqrt{S-nH^2}+\frac{n(n-2)}{2\sqrt{n(n-1)}}H\\&\quad+\frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2+4n(n-1)^2c}<0.\end{aligned}$$

Therefore $\Lambda(M) < 0$ implies that there exists an $\varepsilon > 0$ such that $\mathrm{Ric}_M \geqslant \varepsilon$. Application of the classical Myers Theorem then concludes the proof of Theorem 1. □

The following Proposition 2 is immediate from the proof of Theorem 1.

PROPOSITION 2. *Let $M^n \subset N^{n+p}$ be a submanifold and let the curvature tensor of N satisfy*

$$\sum_i K_{inin} \geqslant (n-1)c, \quad \textit{for any orthonormal basis } \{e_i\}$$
$$\textit{for } M_x \textit{ at any point } x \in M.$$

Here c is a constant. Then,

$$\mathrm{Ric}_M(X) \geqslant \frac{n-1}{n}\left[nc+2nH^2-S-\frac{n(n-2)}{\sqrt{n(n-1)}}H(S-nH^2)^{1/2}\right]$$

holds for any unit vector $X \in M_x$.

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3. Proof of Main Theorem

The proof of Main Theorem in the case where $n = 3$ follows directly from Theorem 1 and the Hamilton Theorem [H] which states that a compact and connected Riemannian 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form.

We discuss only the case where $n \geqslant 4$. The non-existence theorem (see [LS], [Xi]) for stable currents in a compact Riemannian manifold M isometrically immersed into $F^{n+p}(c)$ is employed to eliminate the homology groups $H_k(M;Z)$ for $1 < k < n-1$. We then employ the universal coefficient theorem to obtain the homology sphere.

LEMMA 1 ([LS], [Xi]). *Let $M \subset F^{n+p}(c)$ for $c \geqslant 0$ be a compact submanifold of dimension $n \geqslant 2$. Assume that*

$$\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|\mathbf{h}(e_i,e_k)|^2 - < \mathbf{h}(e_i,e_i), \mathbf{h}(e_k,e_k) >] < q(n-q)c \tag{7}$$

holds for any orthonormal basis $\{e_i\}$ of M_x at any point $x \in M$. Here, q is an integer satisfying $0 < q < n$. We then have

$$H_q(M;Z) = H_{n-q}(M;Z) = 0,$$

where $H_i(M;Z)$ is the i-th homology group of M with integer coefficients.

The above Lemma 1 is combined with our assumption for S and we prove.

LEMMA 2. *Let $M \subset F^{n+p}(c)$ for $c \geqslant 0$ be a compact submanifold of dimension n. Assume that $n \geqslant 4$ and $S < \alpha(n,H,c)$. We then have*

$$H_q(M;Z) = 0, \quad \textit{for all } 1 < q < n-1.$$

Proof. We observe that

$$\begin{aligned}
&\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|\mathbf{h}(e_i,e_k)|^2 - \langle \mathbf{h}(e_i,e_i), \mathbf{h}(e_k,e_k)\rangle] \\
&\quad = 2\sum_{\alpha}\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^{\alpha})^2 - \sum_{\alpha}\sum_{k=q+1}^{n}\sum_{i=1}^{q} h_{ii}^{\alpha}h_{kk}^{\alpha} \\
&\quad = \sum_{\alpha}\left[2\sum_{k=q+1}^{n}\sum_{i=1}^{q} h_{ik}^{\alpha})^2 - \left(\sum_{i=1}^{q} h_{ii}^{\alpha}\right)\left(\mathrm{tr}\ H_{\alpha} - \sum_{i=1}^{q} h_{ii}^{\alpha}\right)\right].
\end{aligned}$$

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By setting

$$Z_\alpha := -\left(\sum_{i=1}^{q} h_{ii}^\alpha\right)\left(\operatorname{tr} H_\alpha - \sum_{i=1}^{q} h_{ii}^\alpha\right) \tag{8}$$

and $r := n - q$, we have

$$\begin{aligned} qr\tilde{S}_\alpha &= qr\sum_{i=1}^{q}(h_{ii}^\alpha)^2 + qr\sum_{k=q+1}^{n}(h_{kk}^\alpha)^2 \\ &\geqslant r\left(\sum_{i=1}^{q} h_{ii}^\alpha\right)^2 + q\left(\sum_{k=q+1}^{n} h_{kk}^\alpha\right)^2. \end{aligned}$$

Inserting $T_\alpha - \sum_{i=1}^{q} h_{ii}^\alpha = \sum_{k=q+1}^{n} h_{kk}^\alpha$ into the right-hand side of the above inequality, we get

$$(r+q)\left(\sum_{i=1}^{q} h_{ii}^\alpha\right)^2 - 2q\,T_\alpha\sum_{i=1}^{q} h_{ii}^\alpha + q\,T_\alpha^2 - qr\tilde{S}_\alpha \leqslant 0$$

and using (8), this expression is rewritten as

$$nZ_\alpha + (r-q)T_\alpha\sum_{i=1}^{q} h_{ii}^\alpha + q\,T_\alpha^2 - qr\tilde{S}_\alpha \leqslant 0. \tag{9}$$

Making use of the relations

$$\begin{aligned} &\sum_{i=1}^{n}\left(h_{ii}^\alpha - \frac{T_\alpha}{n}\right)^2 = \tilde{S}_\alpha - \frac{T_\alpha^2}{n}, \sum_{i=1}^{n}\left(h_{ii}^\alpha - \frac{T_\alpha}{n}\right) = 0, \\ &\sum_{i=1}^{q}\left(h_{ii}^\alpha - \frac{T_\alpha}{n}\right) + \frac{q}{n}T_\alpha = \sum_{i=1}^{q} h_{ii}^\alpha, \end{aligned}$$

we have, by letting $\tilde{h}_{ii}^\alpha = h_{ii}^\alpha - T_\alpha/n$,

$$\begin{aligned} \tilde{S}_\alpha - \frac{T_\alpha^2}{n} &\geqslant \frac{1}{q}\left(\sum_{i=1}^{q}\tilde{h}_{ii}^\alpha\right)^2 + \frac{1}{r}\left(\sum_{k=q+1}^{n}\tilde{h}_{kk}^\alpha\right)^2 \\ &= \left(\frac{1}{q} + \frac{1}{r}\right)\left[\sum_{i=1}^{q}\left(h_{ii}^\alpha - \frac{T_\alpha}{n}\right)\right]^2. \end{aligned}$$

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Therefore we find

$$\left|\sum_{i=1}^{q}\left(h_{ii}^{\alpha}-\frac{T_\alpha}{n}\right)\right| \leqslant \sqrt{\frac{qr}{n}\left(\tilde{S}_\alpha-\frac{T_\alpha^2}{n}\right)}. \tag{10}$$

From (9) and (10) it follows that

$$Z_\alpha \leqslant \frac{qr}{n}\tilde{S}_\alpha - \left[\frac{q(r-q)}{n^2}+\frac{q}{n}\right]T_\alpha^2 + \frac{|r-q|}{n}|T_\alpha|\sqrt{\frac{qr}{n}\left(\tilde{S}_\alpha-\frac{T_\alpha^2}{n}\right)}. \tag{11}$$

From (8), (11), and the fact $qr > n$ we obtain

$$\begin{aligned}
&\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|\mathbf{h}(e_i,e_k)|^2 - \langle \mathbf{h}(e_i,e_i),\mathbf{h}(e_k,e_k)\rangle] - qrc \\
&\leqslant \sum_{\alpha}\left[\frac{qr}{n}S_\alpha - \frac{2qr}{n^2}T_\alpha^2 + \frac{|r-q|}{n}|T_\alpha|\sqrt{\frac{qr}{n}\left(S_\alpha-\frac{T_\alpha^2}{n}\right)}\right] - qrc \\
&= \frac{qr}{n}S - 2qrH^2 + \frac{|r-q|}{n}\sqrt{\frac{qr}{n}\sum_\alpha T_\alpha^2\sum_\beta\left(S_\beta-\frac{T_\beta^2}{n}\right)} - qrc \\
&= \frac{qr}{n}\left\{S - 2nH^2 - nc + \frac{\sqrt{n}|q-r|}{\sqrt{qr}}H\sqrt{S-nH^2}\right\} \\
&\leqslant \frac{qr}{n}\left\{S - 2nH^2 - nc + \frac{\sqrt{n}(n-2)}{\sqrt{n-1}}H\sqrt{S-nH^2}\right\}.
\end{aligned} \tag{12}$$

Note that $\sum_\alpha T_\alpha^2\sum_\beta(S_\beta - T_\beta^2/n) = n^2H^2(S-nH^2)$ is employed here. Since $S < \alpha(n,H,c)$ is equivalent to
late

$$S - 2nH^2 - nc + \frac{\sqrt{n}(n-2)}{\sqrt{n-1}}H\sqrt{S-nH^2} < 0,$$

the left hand side of (12) is negative. It follows from Lemma 1 that $H_q(M;Z) = H_r(M;Z) = 0$ for all $1 < q, r < n-1$ such that $q+r=n$. This completes the proof of Lemma 2. □

Proof of Main Theorem. It follows from Lemma 2 and the universal coefficient theorem that $H^{n-1}(M;Z)$ has no torsion, and hence neither does $H_1(M;Z)$ by the Poincaré duality. The Myers Theorem then implies that $\Pi_1(M)$ is finite, and hence

$H_1(M; Z) = 0$. Therefore we have $H_{n-1}(M; Z) = 0$. Let $\tilde{M}$ be the universal Riemannian covering of M. We may consider $\tilde{M}$ to be a submanifold of $F^{n+p}(c)$, and hence $\tilde{M}$ is a homology sphere. Since $\tilde{M}$ is simply connected, it is also a topological sphere. Because M is covered by a topological sphere, a result by Sjerve [S] states that M is simply connected. This proves our Theorem. $\square$

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