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The topological center of the spectrum of some distal algebras

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Abstract. The topological center of the spectrum of the Weyl algebra W, i.e. the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T}$ and $i \in \mathbb{N}\}$, is characterized in a recent paper by Jabbari and Namioka (Ellis group and the topological center of the flow generated by the map $n \mapsto \lambda^{n^k}$, to appear in *Milan J. Math.*). By the techniques essentially used in the cited paper, the topological center of the spectrum of the subalgebra W_k, the norm closure of the algebra generated by the set of functions $\{n \mapsto \lambda^{n^i}; \lambda \in \mathbb{T}$ and $i = 0, 1, 2, ..., k\}$, will be characterized, for all $k \in \mathbb{N}$. Also an example of a non-minimal dynamical system, with the enveloping semigroup Σ, for which the set of all continuous elements of Σ is not equal to the topological center of Σ, is given.

1. Introduction

The history of characterizing the topological center of the spectrum of a left-invariant non-distal algebra on a group G goes back to a paper by Lau, Milnes and Pym [18] (see also [19]). They showed that the topological center of the spectrum of the largest compactification of any locally compact group G equals G itself. But, similar results for distal algebras are rather meager (see [11]). A distal algebra is a left-invariant conjugate-closed Banach algebra of distal functions. The spectrum of any distal subalgebra A of $\ell^\infty(\mathbb{Z})$, the Banach algebra of all bounded complex-valued functions on $\mathbb{Z}$, is an example of a compact admissible right-topological group. Namioka [22], and also Milnes and Pym in [20] and [21], have studied the structure of compact admissible right-topological groups, and Lau and Loy [17] have investigated harmonic analysis on these groups.

Distal functions on groups were first introduced by Auslander and Hahn in [1]. By definition, a bounded function on a group G is *distal* if whenever $\{g_n\}_{n=1}^{\infty}$,

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$\{h_i\}_{i=1}^{\infty}$ and $\{k_j\}_{j=1}^{\infty}$ are nets in G such that for all $g\in G$,

$$\lim_n \lim_i f(gg_nh_i) = \lim_n \lim_j f(gg_nk_j)$$

then $\lim_i f(gh_i)=\lim_j f(gk_j)$, for all g in G. Later, Knapp [14] gave an analysis and synthesis of distal functions on groups. He showed that the set of all distal functions on a group G is itself a distal algebra. For a general reference on distal functions on semigroups see [2].

Let $\mathbb{T}$ denote the unit circle in the complex plane. Then $\mathbb{T}$ is a compact topological group under the complex multiplication. Let W denote the norm closure of the algebra generated by the set of functions $\{n\mapsto\lambda^{n^i};\lambda\in\mathbb{T}$ and $i\in\mathbb{N}\}$, where, as usual, $\mathbb{N}$ is the set of all positive integers. The algebra W has been studied from different aspects: Knapp [14] showed that all of the elements of W are distal. Later, Namioka [22, Theorem 3.6] gave a simpler proof of this fact. By using a result of Furstenberg [6], Salehi [24] showed that all of the elements of W, called the Weyl algebra, are uniquely ergodic, and he derived that W does not exhaust all distal functions on $(\mathbb{Z},+)$ [24, Theorem 2.14]. In [12] (see also [10]), by giving a characterization of the Weyl algebra in a more general setting of semitopological semigroups, the authors showed that the algebra W is actually a distal algebra. In a recent work of Isaac Namioka joint with the author [11], they proved that, for each irrational member λ of the unit circle, the shift-orbit closure X_f of the function $f(n)=\lambda^{n^k}$ is homeomorphic to a k-torus. Using this homeomorphism, by generalizing an interesting result of Namioka [23], they also characterized the topological center of the dynamical system X_f, as well as the topological center of the spectrum of the Weyl algebra W.

Throughout this paper we fix $k\in\mathbb{N}$, $k>1$. Let W_k be the norm closure of the algebra generated by the set of functions $\{n\mapsto\lambda^{n^i};\lambda\in\mathbb{T}$ and $i=0,1,2,...,k\}$. In [12] (see also [10]), this algebra is generalized to arbitrary semitopological semigroups. It is proved that W_k is left invariant, and hence is a distal algebra [12, Theorem 3.5].

The aim of the present paper is to determine the topological center of the spectrum $M(W_k)$ of the distal algebra W_k (Theorem 1.1). To this end, we characterize $M(W_k)$ with the Ellis group $\Sigma(W_k,U)$ of the distal flow (W_k,U) [11, Proposition 5.3], where $U\colon W_k\to W_k$ is the shift operator defined by $U(g)(n)=g(n+1)$, for all $g\in W_k$ and all $n\in\mathbb{Z}$. Then we show that $\Sigma(W_k,U)$ is homeomorphically isomorphic to a subgroup of $E(\mathbb{T})^k$ (Theorem 3.1), where $E(\mathbb{T})$ is the set of all endomorphisms of the group $\mathbb{T}$.

Note that $M(W_k)$ is the set of all multiplicative means μ on W_k, that is, $\mu(1)=1$, $\mu(f)\geq 0$ whenever $f\in W_k$ and $f\geq 0$, and $\mu(fg)=\mu(f)\mu(g)$ for all $f,g\in W_k$. $M(W_k)$ is a weak* compact subsemigroup of W_k^*, with the product $\langle\mu*\nu,f\rangle=\langle\mu,T_\nu f\rangle$, where $T_\nu f(n)=\nu(U^nf)$ for all $\mu,\nu\in M(W_k)$, $f\in W_k$ and $n\in\mathbb{Z}$.

Let H be the torsion subgroup of $\mathbb{T}$, and let V denote the quotient space $\mathbb{T}/H$. Then V is a vector space over $\mathbb{Q}$. Let $\mathrm{Hom}(V,\mathbb{T})$ denote the set of all $\mathbb{Q}$-vector space homomorphisms from V into $\mathbb{T}$. The following is the main result of this paper.

Theorem 1.1. *Let $k\in\mathbb{N}$, with $k>1$. The topological center of $M(W_k)$ is isomorphic to the group $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$ with the group structure*

$$(p,\theta)(q,\theta')=(p+q,\theta\theta'(\,\cdot\,)^{(p+q)^k-p^k-q^k}).$$

We remark that the topological center of $M(W)$ is essentially characterized in [11].

Finally, we prove that the set of all continuous elements of $\Sigma(W_k,U)$ is isomorphic to $\mathbb{Z}$ (Theorem 3.10), which along with Theorem 1.1 shows that there exists a non-minimal dynamical system with the Ellis group Σ such that the topological center of Σ is different from Σ_c, the set of all continuous elements of Σ (see [11, Lemma 4.3]).

Remark. The notion of topological center began with the earlier works of Isik–Pym–Ülger [9], Lau [15] and Lau–Losert [16]. In recent years there has been significant interest in the subject. Interested readers are referred to the recent papers and references contained in [3] and [4].

2. Preliminaries

A *dynamical system* is a pair (X,T), where X is a Hausdorff space and T is a homeomorphism from X onto X. The dynamical system (X,T) is said to be *compact* if X is a compact Hausdorff space. We always assume that the space X^X is provided with the product topology. The closure $\Sigma(X,T)$ of the set $\{T^n;n\in\mathbb{Z}\}$ is a sub-semigroup of X^X called the *enveloping semigroup* of the dynamical system (X,T). With the relativization of the product topology from X^X, the mapping $\sigma\mapsto\sigma\circ\tau\colon\Sigma\to\Sigma$ is continuous for all $\tau\in\Sigma$, in other words, $\Sigma=\Sigma(X,T)$ is a Hausdorff right-topological semigroup with the topology of pointwise convergence; and it is not left-topological, in general. The *topological center* $\Lambda(\Sigma)$ of Σ is defined as follows,

$$\Lambda(\Sigma)=\{\tau\in\Sigma\,;\sigma\mapsto\tau\circ\sigma\colon\,\Sigma\to\Sigma\text{ is continuous}\}.$$

$\Lambda(\Sigma)$ is a sub-semigroup of Σ containing $\{T^n;n\in\mathbb{Z}\}$. Also, for each x in X, the orbit closure of x is $\Sigma(x)$ and the map $\sigma\mapsto\sigma(x)\colon\Sigma\to X$ is continuous. A compact dynamical system (X,T) is called *distal* if $\lim_\alpha T^{n_\alpha}x=\lim_\alpha T^{n_\alpha}y$ for some net $\{n_\alpha\}_\alpha$ in $\mathbb{Z}$ and $x,y\in X$ implies that $x=y$. It was Ellis [5, Proposition 5.3] who showed that a compact dynamical system is distal if and only if its enveloping semigroup is a group (whose identity is the identity mapping of X). The enveloping semigroup of a distal dynamical system is called the *Ellis group* of the dynamical system. A closed non-empty subset M of a dynamical system (X,T) is called *minimal* if M is invariant (i.e. $T^nM\subseteq M$ for all $n\in\mathbb{Z}$) and no proper closed subset of M is invariant. It is readily seen that M is minimal if and only if $\Sigma(x)=M$ for each x in M. A *minimal dynamical system* is a dynamical system (X,T) for which the phase space X is minimal. Two dynamical systems (X,T) and (X',T'), or briefly X and X', are *isomorphic* if there exists a homeomorphism $\Gamma\colon X\to X'$ such that $\Gamma\circ T=T'\circ\Gamma$.

By $\ell^\infty(\mathbb{Z})$ (or ℓ^∞), we mean the Banach space of all bounded complex-valued functions on $\mathbb{Z}$, with the supremum norm. The topology of ℓ^∞ is the weak* topology, where ℓ^∞ is regarded as the dual space of $\ell^1(\mathbb{Z})$. Recall that the weak* topology of ℓ^∞ coincides with the topology of pointwise convergence on norm-bounded subsets. Define the shift operator $U\colon \ell^\infty\to\ell^\infty$ by $U(g)(n)=g(n+1)$ for all $g\in\ell^\infty$ and all $n\in\mathbb{Z}$. It is clear that the shift operator U is a continuous map of ℓ^∞ into itself. Therefore the pair $(\ell^\infty(\mathbb{Z}),U)$ is a dynamical system. Namioka [22, Lemma 3.1] showed that the enveloping semigroup $\Sigma(\ell^\infty,U)$ of the flow (ℓ^∞,U) is compact. We observe here that each element of $\Sigma(\ell^\infty,U)$ is a multiplicative bounded linear transformation of norm 1 of the Banach space ℓ^∞ into itself. The object of the present paper is the dynamical system (W_k,U), where W_k is given the topology induced by that of ℓ^∞. With a proof similar to the proof of [11, Theorem 5.1], one can readily verify that $\Sigma(W_k,U)$ is a compact right-topological group. Also, it is easily seen that the topological center of $\Sigma(W_k,U)$ coincides with its "center" in the group theoretic sense (just because of the commutativity of the acting group $\mathbb{Z}$).

By giving a new characterization of Weyl algebras in a more general setting of semitopological semigroups S, it is shown in [12] that, for each $\sigma\in\Sigma(\ell^\infty,U)$, $\sigma(W_k)\subset W_k$, that is W_k is left-invariant and therefore a distal algebra. Hence the enveloping semigroup $\Sigma(W_k,U)$ of the dynamical system (W_k,U) is $\{\sigma|_{W_k};\ \sigma\in\Sigma(\ell^\infty,U)\}$, where $\sigma|_{W_k}$ denotes the restriction of σ to W_k.

For each $f\in\ell^\infty$, let X_f denote the orbit closure of f with respect to the shift mapping U. Then $U(X_f)\subseteq X_f$. Therefore (X_f,U) is a dynamical system as well. In fact $X_f=\{\sigma f;\sigma\in\Sigma(\ell^\infty,U)\}$. It follows that $X_f\subset W_k$ for each $f\in W_k$. Thus $\Sigma(X_f,U)=\{\sigma|_{X_f};\sigma\in\Sigma(W_k,U)\}$. Now, because of the continuity of the restriction mapping, the enveloping semigroups $\Sigma(W_k,U)$ and $\Sigma(X_f,U)$, for $f\in W_k$, are also compact. Finally, we remark that a function $f\in\ell^\infty$ is distal if and only if the dynamical system (X_f,U) is distal. The structure of distal flows is essentially studied in [7].

3. The Ellis group of the dynamical system (W_k, U)

Let us recall some results from [11]. The binomial coefficients are extended as follows: Let $j\in\mathbb{N}$ and $n\in\mathbb{Z}$, then

$$\binom{n}{0}=1 \quad \text{and} \quad \binom{n}{j}=\frac{1}{j!}n(n-1)...(n-j+1).$$

Fix an irrational (i.e. a non-root of unity) element $\lambda\in\mathbb{T}$. Define $f\in\mathbb{T}^{\mathbb{Z}}$ by $f(n)=\lambda^{n^k}$ for all $n\in\mathbb{Z}$. It is a result of [11] that there exists a continuous map $T\colon \mathbb{T}^k\to\mathbb{T}^k$ such that the mapping $\Gamma\colon \mathbb{T}^k\to X_f$ defined by

$$\Gamma(x_1,x_2,...,x_k)(n)=\lambda^{n^k}x_1^{Q_{k-1}(n)}x_2^{Q_{k-2}(n)}...x_{k-1}^{Q_1(n)}x_k \tag{1}$$

(for $(x_1,x_2,...,x_k)\in\mathbb{T}^k$ and $n\in\mathbb{Z}$) is an isomorphism between the dynamical systems $(\mathbb{T}^k,T)$ and (X_f,U) [11, Theorem A], where for each $n\in\mathbb{Z}$ and $j\in\{0\}\cup\mathbb{N}$,

$$Q_j(n)=\binom{n+[j/2]}{j}. \tag{2}$$

By using this isomorphism, it is also proved that the mapping $\Theta\colon \Sigma(X_f,U)\to E(\mathbb{T})^{k-1}\times\mathbb{T}$ defined by $\Theta(\sigma)=(\theta_1,\theta_2,...,\theta_{k-1},u)$ is a homeomorphic embedding into $E(\mathbb{T})^{k-1}\times\mathbb{T}$ [11, Theorem B], where $(\theta_1,\theta_2,...,\theta_{k-1},u)$ is associated with $\sigma=\lim_\alpha U^{m_\alpha}\in\Sigma$, that is for each $j\in\{1,2,...,k-1\}$, $\theta_j\in\mathbb{T}^{\mathbb{T}}$, is given by

$$\theta_j(x)=\lim_\alpha x^{Q_j(m_\alpha)}, \tag{3}$$

and

$$u=\lim_\alpha \lambda^{m_\alpha^k}\in\mathbb{T}. \tag{4}$$

Finally, if $\sigma\in\Lambda(\Sigma(X_f,U))$, then there exists an integer p such that $\theta_j=(\quad)^{Q_j(p)}$, for each $j\in\{1,2,...,k-1\}$ [11, Lemma 4.5].

To characterize the topological center of $\Sigma(W_k,U)$, we need some preliminaries. Let $\sigma\in\Sigma(W_k,U)$, and let $\{m_\alpha\}_\alpha$ be a net in $\mathbb{Z}$ such that $\sigma=\lim_\alpha U^{m_\alpha}$. By taking a subnet of m_α if necessary, we may assume that for each $j\in\{0,1,...,k-1\}$, $\lim_\alpha x^{Q_j(m_\alpha)}$ exists for each $x\in\mathbb{T}$ and $\lim_\alpha x^{m_\alpha^k}$ exists for each $x\in\mathbb{T}$. For each $j\in\{0,1,...,k-1\}$ define $\theta_j\in\mathbb{T}^{\mathbb{T}}$ by

$$\theta_j(x)=\lim_\alpha x^{Q_j(m_\alpha)} \tag{5}$$

for all $x\in\mathbb{T}$. Also define $\theta\colon\mathbb{T}\to\mathbb{T}$ by

$$\theta(x)=\lim_\alpha x^{m_\alpha^k}. \tag{6}$$

With this introduction, we can prove the next theorem.

Theorem 3.1. *Let* $\Sigma = \Sigma(W_k, U)$, *and define the mapping* $\Theta_k \colon \Sigma \to E(\mathbb{T})^k$ *by* $\Theta_k(\sigma) = (\theta_1, ..., \theta_{k-1}, \theta)$, *where* σ, $\theta_1, ..., \theta_{k-1}$ *and* θ *are defined as in the paragraph preceding the theorem. Then* Θ_k *is a homeomorphic embedding into* $E(\mathbb{T})^k$.

Proof. Let $\sigma = \lim_\alpha U^{m_\alpha} = \lim_\beta U^{n_\beta} \in \Sigma$. Let $y \in \mathbb{T}$, and let $j \in \{1, 2, ..., k-1\}$. Let $f \in \mathbb{T}^{\mathbb{Z}}$ be the function defined by $f(n) = y^{j! Q_j(n)}$. Since $j! Q_j(n)$ is a polynomial of degree j with integral coefficients, one has $f \in W_k$. Hence $\lim_\alpha U^{m_\alpha}(f)(0) = \lim_\beta U^{n_\beta}(f)(0)$. Thus $\lim_\alpha y^{j! Q_j(m_\alpha)} = \lim_\beta y^{j! Q_j(n_\beta)}$. Since each $x \in \mathbb{T}$ can be written of the form $y^{j!}$ for some $y \in \mathbb{T}$, it follows that

$$\lim_\alpha x^{Q_j(m_\alpha)} = \lim_\beta x^{Q_j(n_\beta)}. \tag{7}$$

Again let $x \in \mathbb{T}$. Then the function $g \in \mathbb{T}^{\mathbb{Z}}$ defined by $g(n) = x^{n^k}$, $n \in \mathbb{Z}$, is an element of W_k. Hence $\lim_\alpha U^{m_\alpha}(g)(0) = \lim_\beta U^{n_\beta}(g)(0)$. Therefore $\lim_\alpha x^{m_\alpha^k} = \lim_\beta x^{n_\beta^k}$. From this and (7), it follows that $\Theta_k(\sigma)$ depends only on σ, not on the choice of the net representing σ. Hence Θ_k is well-defined. To show that Θ_k is one-to-one, assume that $\sigma, \tau \in \Sigma$ are such that $\Theta_k(\sigma) = \Theta_k(\tau) = (\theta_1, ..., \theta_{k-1}, \theta)$. Then we must prove that $\sigma = \tau$. Since σ and τ are bounded linear transformations on W_k, and since W_k is the norm-closed subalgebra of ℓ^∞ generated by the set $A_k = \{n \mapsto x^{n^i}; 0 \le i \le k \text{ and } x \in \mathbb{T}\}$, it is enough to show that $\sigma(f) = \tau(f)$ for all $f \in A_k$. To this end, fix $x \in \mathbb{T}$. First, let f be the function $n \mapsto x^{n^k}$. Then for each $m \in \mathbb{Z}$, $(U^m f)(n) = x^{(n+m)^k} = f(n) x^{S(m)} x^{m^k}$ with $S \in \mathcal{S}(k-1)$, where $\mathcal{S}(k-1)$ is the set of all integral linear combinations of the functions $Q_1, Q_2, ..., Q_{k-1}$, as defined in Section 4 of [11]. Let $S(m) = \sum_{j=1}^{k-1} a_j Q_j(m)$ with $a_j \in \mathbb{Z}$. Let $\{m_\alpha\}_\alpha$ be a net in $\mathbb{Z}$ such that $\lim_\alpha U^{m_\alpha}(g) = \sigma(g)$ for each $g \in W_k$, as in the beginning of this proof, then

$$\sigma(f)(n) = f(n) \lim_\alpha x^{S(m_\alpha)} \lim_\alpha x^{m_\alpha^k} = f(n) \prod_{j=1}^{k-1} \theta_j(x^{a_j}) \theta(x).$$

Similarly $\tau(f)(n) = f(n) \prod_{j=1}^{k-1} \theta_j(x^{a_j}) \theta(x)$. Therefore by the hypothesis, $\sigma(f) = \tau(f)$. Now, for $i \in \{1, 2, ..., k-1\}$, let $f_i \in A_k$ be defined by $f_i(n) = x^{n^i}$. A similar, but simpler, proof applies to show that $\sigma(f_i) = \tau(f_i)$, for all $i = 1, 2, ..., k-1$. Hence $\sigma = \tau$. Finally, to show that Θ_k is continuous, we must prove that for each $x \in \mathbb{T}$ the map $\varphi \colon \sigma \mapsto \theta(x)$ and the maps $\varphi_j \colon \sigma \mapsto \theta_j(x)$, for $j = 1, 2, ..., k-1$, are continuous on Σ. For the latter, fix $j = 1, 2, ..., k-1$ and $x, y \in \mathbb{T}$ with $x = y^{j!}$. If $g \in W_k$ is defined by $g(n) = y^{j! Q_j(n)}$, then $\varphi_j(\sigma) = \theta_j(x) = \theta_j(y^{j!}) = \sigma(g)(0)$, hence the continuity of φ_j follows from the continuity of the map $\sigma \mapsto \sigma(g)(0)$ on Σ. It remains to prove that $\varphi \colon \sigma \mapsto \theta(x)$ is continuous for all $x \in \mathbb{T}$. To this end, fix $x \in \mathbb{T}$. Define $f \in W_k$ by $f(n) = x^{n^k}$. Then $\varphi(\sigma)(x) = \theta(x) = \sigma(f)(0)$. Hence the continuity of φ follows from the continuity of the map $\sigma\mapsto\sigma(f)(0)$ on $\Sigma(W_k,U)$. That is, Θ_k is continuous on Σ. $\square$

In the next lemma, we show that for an element σ, with $\Theta_k(\sigma)=(\theta_1,...,\theta_{k-1},\theta)$, to be in the topological center of Σ it is enough that the first $k-1$ components are continuous elements of $E(\mathbb{T})$.

Lemma 3.2. *Let Σ and $\Theta_k\colon\Sigma\to E(\mathbb{T})^k$ be as in Theorem 3.1. Let $\sigma\in\Sigma$ and let $\Theta_k(\sigma)=(\theta_1,...,\theta_{k-1},\theta)$. Then $\sigma\in\Lambda(\Sigma)$ if and only if θ_i is a continuous endomorphism of $\mathbb{T}$ for each $i\in\{1,2,...,k-1\}$.*

Proof. Let $\sigma\in\Lambda(\Sigma)$. An observation similar to the one at the beginning of the proof of [11, Theorem E], with W replaced by W_k, shows that $\sigma|_{X_f}\in\Lambda(\Sigma(X_f,U))$ for each $f\in W_k$. Let λ be an irrational member of $\mathbb{T}$, and let $f(n)=\lambda^{n^k}$. Then $\sigma|_{X_f}\in\Lambda(\Sigma(X_f,U))$. Now we can apply Lemma 4.4 of [11]. (In that lemma, $\sigma\in\Sigma(\mathbb{T}^k,T)$ is expressed as $\sigma=\lim_\alpha T^{m_\alpha}$, but since, by [11, Theorem A], the dynamical systems (X_f,U) and $(\mathbb{T}^k,T)$ are isomorphic, this is equivalent to writing $\sigma|_{X_f}=\lim_\alpha U^{m_\alpha}$ in this section. Recall that $U\colon\mathbb{T}^{\mathbb{Z}}\to\mathbb{T}^{\mathbb{Z}}$ is the shift map and $T\colon\mathbb{T}^k\to\mathbb{T}^k$ is the map satisfying $U\circ\Gamma=\Gamma\circ T$, where $\Gamma\colon\mathbb{T}^k\to X_f$ is the isomorphism of Theorem 1.1 in [11], as defined in (1) of the present paper.) Let $\Theta_k(\sigma|_{X_f})=(\theta_1,...,\theta_{k-1},u)$ as in Theorem B of [11] (or as illustrated at the beginning of this section). Then by [11, Lemma 4.4] $\theta_1,\theta_2,...,\theta_{k-1}$ are all continuous. Conversely, let $\sigma\in\Sigma$, let $\Theta_k(\sigma)=(\theta_1,...,\theta_{k-1},\theta)$, and assume that $\theta_i\in\mathbb{T}^{\mathbb{T}}$ is continuous for all $i=1,2,...,k-1$. Hence for each $i=1,2,...,k-1$, $\theta_i=(\,\cdot\,)^{n_i}$ for some $n_i\in\mathbb{Z}$. As we remarked in Section 2, $\Sigma=\Sigma(W_k,U)$ is a compact right-topological group, and the topological center of Σ coincides with its center in the group theoretic sense. Hence, to prove that $\sigma\in\Lambda(\Sigma)$ is to prove that $\sigma\circ\tau=\tau\circ\sigma$ for all $\tau\in\Sigma$. So fix $\tau=\lim_\beta U^{n_\beta}\in\Sigma$ and let $\Theta_k(\tau)=(\theta'_1,...,\theta'_{k-1},\theta')$. Since σ and τ are bounded linear transformations on W_k and since W_k is generated by $A_k=\{n\mapsto x^{n^i};x\in\mathbb{T}\text{ and }i=0,1,2,...,k\}$, it is enough to show that $\sigma\circ\tau(f)=\tau\circ\sigma(f)$ for all $f\in A_k$. First note that for each $i\in\{1,2,...,k\}$ the polynomial m^i is an element of $\mathcal{S}(i)$, the set of all integral linear combinations of $Q_1(m),...,Q_i(m)$ [11]. (For instance, $m=Q_1(m)\in\mathcal{S}(1)$, $m^2=2Q_2(m)-Q_1(m)\in\mathcal{S}(2),\dots$.) For simplicity, we shall confine the proof for the special case $k=3$, since it contains all the necessary ideas. So let $k=3$, fix $x\in\mathbb{T}$ and let $f\in W_k$ be defined by $f(n)=x^{n^3}$. Let σ and τ be as above. Then

$$\begin{aligned}\sigma\circ\tau(f)(n)&=\lim_\alpha\lim_\beta U^{(m_\alpha+n_\beta)}f(n)=\lim_\alpha\lim_\beta f(m_\alpha+n_\beta+n)\\ &=\lim_\alpha\lim_\beta x^{(m_\alpha+n_\beta+n)^3}.\end{aligned}\tag{8}$$

But, for each $n\in\mathbb{Z}$ and for all α,β one has

$$\begin{aligned}(m_\alpha+n_\beta+n)^3 &= n^3+3n^2(Q_1(m_\alpha)+Q_1(n_\beta))\\ &\quad+3n(2Q_2(m_\alpha)-Q_1(m_\alpha)+2Q_1(m_\alpha)Q_1(n_\beta)+2Q_2(n_\beta)-Q_1(n_\beta))\\ &\quad+3(2Q_2(m_\alpha)-Q_1(m_\alpha))Q_1(n_\beta)+3Q_1(m_\alpha)(2Q_2(n_\beta)-Q_1(n_\beta))\\ &\quad+m_\alpha^3+n_\beta^3.\end{aligned}$$

Hence

$$\sigma\circ\tau(f)(n)=f(n)[(\theta_1\theta_1')^{3n(n-1)}(\theta_2\theta_2')^{6n}(\theta_1\circ\theta_1')^{6(n-1)}(\theta_2\circ\theta_1')^6(\theta_1\circ\theta_2')^6\theta\theta'](x). \tag{9}$$

Similarly

$$\tau\circ\sigma(f)(n)=f(n)[(\theta_1\theta_1')^{3n(n-1)}(\theta_2\theta_2')^{6n}(\theta_1'\circ\theta_1)^{6(n-1)}(\theta_2'\circ\theta_1)^6(\theta_1'\circ\theta_2)^6\theta\theta'](x). \tag{10}$$

Now, since $\theta_i=(\,\cdot\,)^{n_i}$ one has $\theta_i\circ\theta_j'(x)=\theta_j'(x)^{n_i}=\theta_j'(x^{n_i})=\theta_j'\circ\theta_i(x)$. Therefore, it follows from (9) and (10) that $\sigma\circ\tau(f)=\tau\circ\sigma(f)$. A similar argument shows that $\sigma\circ\tau(g)=\tau\circ\sigma(g)$ for all $g\in A_k$ defined by $g(n)=x^{n^i}$, with $i=1,2,\dots,k-1$. That is $\sigma\circ\tau=\tau\circ\sigma$. Thus $\sigma\in\Lambda(\Sigma(W_k,U))$. □

The proof of the next lemma is similar to [11, Lemma 4.5], but we shall give it here for the sake of completeness.

Lemma 3.3. *Using the notation of Lemma 3.2, let σ be an element of Σ and let $\Theta_k(\sigma)=(\theta_1,\dots,\theta_{k-1},\theta)$. Then $\sigma\in\Lambda(\Sigma)$ if and only if there exists an $n\in\mathbb{Z}$ such that $\theta_j(x)=x^{Q_j(n)}$ for each $j\in\{1,2,\dots,k-1\}$ and each $x\in\mathbb{T}$.*

Proof. Assume that $\sigma\in\Lambda(\Sigma)$. Then by the previous lemma, each θ_j is continuous. This means that for some $n_j\in\mathbb{Z}$; $\theta_j(x)=x^{n_j}$ for each $x\in\mathbb{T}$. Hence it is enough to show that for each $j=1,2,\dots,k-1$, $(\,\cdot\,)^{n_j}=(\,\cdot\,)^{Q_j(n_1)}$. Let $\{m_\alpha\}_\alpha$ be a net in $\mathbb{Z}$ such that $\sigma=\lim_\alpha U^{m_\alpha}$. Fix a prime number $p>(k-1)!$. Then for $\eta=e(1/p)$, $\theta_1(\eta)=\lim_\alpha \eta^{m_\alpha}=\eta^{n_1}$, hence $m_\alpha=n_1 \pmod p$ eventually. Here recall that for each $t\in\mathbb{R}$, $e(t)=e^{2\pi i t}$. Note that $\mathbb{Z} \pmod p$ is a field. Since $p>j!$, the division by $j!$ is well defined in $\mathbb{Z} \pmod p$, and we see that $Q_j(m_\alpha)=Q_j(n_1) \pmod p$ eventually, for $j=2,\dots,k-1$. Therefore $\lim_\alpha e((1/p)Q_j(m_\alpha))=e((1/p)Q_j(n_1))$. It follows that for each integer q with $0<q<p$, one has $\lim_\alpha e((q/p)Q_j(m_\alpha))=e((q/p)Q_j(n_1))$. On the other hand $\lim_\alpha e((q/p)Q_j(m_\alpha))=\theta_j(e(q/p))=e(q/p)^{n_j}$. Thus $e(q/p)^{Q_j(n_1)}=e(q/p)^{n_j}$. Now since $x\mapsto x^{Q_j(n_1)}$ and $x\mapsto x^{n_j}$ are both continuous functions on $\mathbb{T}$ and the set $\{e(q/p);0<q<p,\ p>(k-1)! \text{ and } p \text{ is prime}\}$ is dense in $\mathbb{T}$, we have $x^{Q_j(n_1)}=x^{n_j}$ for each $x\in\mathbb{T}$. The converse is clear from the previous lemma and the fact that for each j the map $x\mapsto x^{Q_j(n_1)}$ is continuous on $\mathbb{T}$. □

Remark 3.4. Notice that the previous lemma implies that each element $\sigma \in \Lambda(\Sigma)$ determines a unique element (n,θ) of the space $\mathbb{Z}\times E(\mathbb{T})$, such that $\theta(x)=x^{n^k}$ for all $x\in H$, where H denotes the torsion subgroup of $\mathbb{T}$.

Let H be as above. Then $\mathbb{T}=H\times V$, where V is isomorphic to $\mathbb{T}/H$. Hence V is a divisible torsion-free subgroup. Consequently, it is a linear space over the rationals $\mathbb{Q}$ [8, Appendix A]. As remarked in [11], for $v,w\in V$ and $q\in\mathbb{Q}$, the linear space "addition" $v+w$ is the complex multiplication vw and the multiplication by scalar qv is actually the power v^q. For instance, $\frac{1}{2}v$ is actually $v^{1/2}$ which is the *unique member* $u\in V$ such that $u^2=v$.

Let $\mathrm{Hom}(V,\mathbb{T})$ denote the set of all homomorphisms from the vector space V (over $\mathbb{Q}$) into $\mathbb{T}$. Then $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$ is a group with the group structure defined as follows: for $(p,\theta),(q,\theta')\in\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$, $(p,\theta)(q,\theta')=\big(p+q,\theta\theta'(\,\cdot\,)^{(p+q)^k-(p^k+q^k)}\big)$.

We are going to show that $\Lambda(\Sigma)$ is isomorphic to $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$. To this end, let $\Phi\colon\Lambda(\Sigma)\to Z\times\mathrm{Hom}(V,\mathbb{T})$ be defined by $\Phi(\sigma)=(n,\theta)$, where σ, n and θ are as in the above remark. To show that the map Φ is an isomorphism of groups, we need some preliminaries.

Note that we shall confine to the case $k=3$, since this case exhibits all the necessary ideas (even the ideas for the case $k=2$). The general case is then derived similarly.

Now, as in [11], let $\{v_\gamma;\gamma\in\Gamma\}$ be a $\mathbb{Q}$-basis of V. Similar to Lemmas (i) and (ii) of [11] we have the following lemmas.

Lemma 3.5. *For each $\varepsilon>0$, each finite subset F of Γ, each $h\in\mathbb{N}$ and each $\theta\in\mathrm{Hom}(V,\mathbb{T})$ there is an $m\in\mathbb{N}$ such that*

(a) $m\equiv 0 \pmod d$ *for each integer d, $0<d\le h$;*

(b) $|(v_\gamma^{1/d})^m-1|<\varepsilon$ *for each $\gamma\in F$ and for each $d\in\{1,2,\dots,h\}$;*

(c) $|(v_\gamma^{1/d})^{m^2}-1|<\varepsilon$ *for each $\gamma\in F$ and for each $d\in\{1,2,\dots,h\}$;*

(d) $|(v_\gamma^{1/d})^{m^3}-\theta(v_\gamma^{1/d})|<\varepsilon$ *for each $\gamma\in F$ and for each $d\in\{1,2,\dots,h\}$.*

Proof. Consider the following statements:

(b′) $|(v_\gamma^{1/h!})^m-1|<\varepsilon/h!$ for each $\gamma\in F$;

(c′) $|(v_\gamma^{1/h!})^{m^2}-1|<\varepsilon/h!$ for each $\gamma\in F$;

(d′) $|(v_\gamma^{1/h!})^{m^3}-\theta(v_\gamma^{1/h!})|<\varepsilon/h!$ for each $\gamma\in F$.

It is proved in [11, Lemma (i)] that the statements (b) and (c) are implied by (b′) and (c′), respectively. We show that the statement (d) is also implied by (d′). First, recall that if $x,y\in\mathbb{T}$, then for each $n\in\mathbb{N}$, $|x^n-y^n|\le n|x-y|$. Now assume (d′). Then for each $d \in \{1,2,...,h\}$,

$$|(v_\gamma^{1/d})^{m^3} - \theta(v_\gamma^{1/d})| = |((v_\gamma^{1/h!})^{m^3})^{h!/d} - (\theta(v_\gamma^{1/h!}))^{h!/d}| < \frac{h!}{d}\frac{\varepsilon}{h!} = \frac{\varepsilon}{d} \leq \varepsilon.$$

Let $F = \{\gamma_j ; j \in \{1,2,...,p\}\}$ for some $p \in \mathbb{N}$. For each $j \in \{1,2,...,p\}$ choose $\xi_j \in [0,1)$ such that $v_{\gamma_j}^{1/h!} = e(\xi_j)$. For $j \in \{1,2,...,p\}$, let

$$\phi_j^1(t) = \xi_j(th!), \quad \phi_j^2(t) = \xi_j(th!)^2 \quad \text{and} \quad \phi_j^3(t) = \xi_j(th!)^3.$$

Then the polynomials $\phi_j^1(t), \phi_j^2(t)$ and $\phi_j^3(t)$, $1 \leq j \leq p$, satisfy the hypothesis of Satz 14 of H. Weyl [25] since $\xi_1, \xi_2, ..., \xi_p$ are independent (mod 1) over $\mathbb{Q}$. It follows that the sequence

$$\{(\phi_1^1(n), ..., \phi_p^1(n), \phi_1^2(n), ..., \phi_p^2(n), \phi_1^3(n), ..., \phi_p^3(n)) ; n \in \mathbb{N}\}$$

is dense in $(\mathbb{R}/\mathbb{Z})^{3p}$. Hence the image of this sequence under the map e is dense in $\mathbb{T}^{3p}$. Thus, for some n, $m = nh!$ satisfies conditions (a), (b′), (c′), and (d′). Hence the lemma is proved. □

Lemma 3.6. *Given $\theta \in \mathrm{Hom}(V, \mathbb{T})$, there is a net $\{n_\alpha\}_\alpha$ in $\mathbb{N}$ such that*
(a) $\lim_\alpha x^{n_\alpha} = 1$ *for each* $x \in \mathbb{T}$;
(b) $\lim_\alpha x^{n_\alpha^2} = 1$ *for each* $x \in \mathbb{T}$;
(c) $\lim_\alpha x^{n_\alpha^3} = \theta(x)$ *for each* $x \in V$.

Proof. As in the proof of [11, Lemma (ii)], let $\mathcal{F}$ be the family of all finite subsets F of Γ and let $D = \mathcal{F} \times (0, \infty) \times \mathbb{N}$. Partially order D as follows: for $\alpha = (F, \varepsilon, h)$ and $\alpha' = (F', \varepsilon', h') \in D$, $\alpha \leq \alpha'$ if and only if $F \subset F'$, $\varepsilon \geq \varepsilon'$ and $h \leq h'$. Then clearly D is a directed set. For each $\alpha = (F, \varepsilon, h)$, let n_α be an integer $m \in \mathbb{N}$ satisfying the conditions (a)–(d) of Lemma 3.5.

The fact that (a) and (b) are satisfied follows from a proof similar to the proof of Lemma (ii) of [11].

For each $\gamma \in \Gamma$ let V_γ be $\mathbb{Q}v_\gamma = \{v_\gamma^r ; r \in \mathbb{Q}\}$. Suppose $x \in V$. Then there is an $F \in \mathcal{F}$ such that $x \in \prod\{V_\gamma ; \gamma \in F\}$. It remains to prove (c) for the case $x \in V_\gamma$ for each $\gamma \in \Gamma$, so let $x \in V_\gamma$. Then $x = (v_\gamma^{1/d})^c$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{N}$. Hence in order to prove (c) for $x \in V_\gamma$, it is sufficient to show (c) for $x = v_\gamma^{1/d}$. Let $\varepsilon > 0$ and let $\alpha = (\{\gamma\}, \varepsilon, d) \in D$. If $\beta = (G, \delta, h) \geq \alpha$ in D, then $\gamma \in G$, $\delta \leq \varepsilon$ and $d \leq h$. Hence by Lemma 3.5(d), $|(v_\gamma^{1/d})^{n_\beta^3} - \theta(v_\gamma^{1/d})| < \delta \leq \varepsilon$ whenever $\beta \geq \alpha$. This completes the proof of (c). □

Corollary 3.7. *Let* $(q,\theta)\in\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$. *Then there is a net* $\{m_\alpha\}_\alpha$ *in* $\mathbb{N}$ *such that for all* $x\in\mathbb{T}$,

$$\lim_\alpha x^{m_\alpha}=x^q,\quad \lim_\alpha x^{m_\alpha^2}=x^{q^2}\quad \textit{and}\quad \lim_\alpha v^{m_\alpha^3}=\theta(v)\ \textit{for each}\ v\in V.$$

Proof. Let $\{n_\alpha\}_\alpha$ be the net given by Lemma 3.6 for $\theta'\in\mathrm{Hom}(V,\mathbb{T})$ instead of θ, in which $\theta'(v)=\theta(v)v^{-q^3}$ for all $v\in V$. For each α, let $m_\alpha=n_\alpha+q$. Then the first two equations follow from (a) and (b) of Lemma 3.6. Similarly

$$\lim_\alpha v^{m_\alpha^3}=\lim_\alpha v^{(n_\alpha^3+3n_\alpha^2q+3n_\alpha q^2+q^3)}=\theta'(v)v^{q^3}=\theta(v)v^{-q^3}v^{q^3}=\theta(v).\quad\square$$

Corollary 3.8. *For each* $(q,\theta)\in\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$, *there is a net* $\{m_\alpha\}_\alpha$ *in* $\mathbb{Z}$ *such that the following conditions are satisfied*:

(a) $\lim_\alpha x^{Q_1(m_\alpha)}=x^{Q_1(q)}$ *for each* $x\in\mathbb{T}$;
(b) $\lim_\alpha x^{Q_2(m_\alpha)}=x^{Q_2(q)}$ *for each* $x\in\mathbb{T}$;
(c) $\lim_\alpha x^{m_\alpha^3}=\theta(x)$ *for each* $x\in V$.

Proof. Let $\{m_\alpha\}_\alpha$ be the net given in the Corollary 3.7. Since $Q_1(n)=n$ for each $n\in\mathbb{Z}$, (a) is obvious. Now $2Q_2(n)=n^2+n$. Again by Corollary 3.7, we have $\lim_\alpha y^{2Q_2(m_\alpha)}=y^{2Q_2(q)}$ for each $y\in\mathbb{T}$. For each $x\in\mathbb{T}$, let $y\in\mathbb{T}$ be chosen so that $x=y^2$. Then (b) follows easily. The statement (c) is trivially part (c) of Corollary 3.7. $\square$

Proof of Theorem 1.1. It is enough to show that the map Φ is an isomorphism of groups. From Lemma 3.3 and the results preceding Lemma 3.5, Φ is well defined and is clearly one-to-one. In order to show that Φ is onto, it is clear from (5) and (6) that for each $(n,\theta)\in\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$ we must find a net $\{m_\alpha\}_\alpha$ such that for each $j\in\{1,...,k-1\}$ and for each $x\in\mathbb{T}$,

$$x^{Q_j(n)}=\lim_\alpha x^{Q_j(m_\alpha)}\quad\text{and}\quad \lim_\alpha v^{m_\alpha^k}=\theta(v)\ \text{for all}\ v\in V.$$

This follows from Corollary 3.8 above for the case $k=3$, and as we remarked, the general case is completely analogous. It remains to show the group structure on $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$ induced by the map Φ.

Let $\lambda\in\mathbb{T}$ be irrational, and let $f\in\mathbb{T}^{\mathbb{Z}}$ be defined by $f(n)=\lambda^{n^k}$. Recall that $U\colon\mathbb{T}^{\mathbb{Z}}\to\mathbb{T}^{\mathbb{Z}}$ is the shift map and $T\colon\mathbb{T}^k\to\mathbb{T}^k$ is the map satisfying $U\circ\Gamma=\Gamma\circ T$, where $\Gamma\colon\mathbb{T}^k\to X_f$ is the isomorphism of Theorem 1.1 in [11]. Let $g\in X_f$ be arbitrary. Then there is an $\mathbf{x}=(x_1,x_2,...,x_k)\in\mathbb{T}^k$ such that $\Gamma(\mathbf{x})=g$. Hence for each $m\in\mathbb{Z}$,

$$U^m(g)(0)=g(m)=\Gamma(x)(m)=\lambda^{m^k}x_1^{Q_{k-1}(m)}x_2^{Q_{k-2}(m)}...x_{k-1}^{Q_1(m)}x_k. \tag{11}$$

Now suppose $\sigma\in\Lambda(\Sigma(W_k,U))$ and $\Phi(\sigma)=(p,\theta)\in\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$. Then there is a net $\{m_\alpha\}_\alpha$ in $\mathbb{Z}$ such that $\lim_\alpha U^{m_\alpha}=\sigma$, $\lim_\alpha x^{Q_j(m_\alpha)}=x^{Q_j(p)}$ for each $j\in\{1,2,...,k-1\}$ and each $x\in\mathbb{T}$, and $\lim_\alpha \lambda^{m_\alpha^k}=\theta(\lambda)$. Suppose that $\sigma,\tau\in\Lambda(\Sigma)$ and that $\Phi(\sigma)=(p,\theta)$, $\Phi(\tau)=(q,\theta')$ and $\Phi(\sigma\circ\tau)=(r,\theta'')$. If $u=\theta(\lambda)$, $v=\theta'(\lambda)$ and $w=\theta''(\lambda)$, then it is not hard to verify that $w=uv\lambda^{(p+q)^k-(p^k+q^k)}$ and $r=p+q$. Therefore $\Phi(\sigma\circ\tau)=(p+q,\theta\theta'(\,\cdot\,)^{(p+q)^k-(p^k+q^k)})$. This completes the proof of the Theorem 1.1. $\square$

Remark 3.9. Let G_k denote the set $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$, given the group operation defined in the proof of Theorem 1.1. Then G_1 is $\mathbb{Z}\times\mathrm{Hom}(V,\mathbb{T})$ with the group operation $(p,\theta)(q,\theta')=(p+q,\theta\theta')$. The groups G_1 and G_k seem to be different, but they are actually isomorphic by the map ϕ: $G_k\to G_1$ defined by $\phi((p,\theta))=(p,\theta(\,\cdot\,)^{-p^k})$.

Finally, we are going to prove the next result, which, comparing with Theorem 1.1, shows that Lemma 4.3 of [11] (or [13, Proposition 2.1]), on the equivalence of the topological center of the enveloping semigroup S, corresponding to any minimal flow, with the set of all continuous elements of S, is not generally valid for non-minimal flows. Let $\Sigma_c(W_k,U)$ denote the set of all continuous elements of $\Sigma(W_k,U)$.

Theorem 3.10. *For each $k>1$, $\Sigma_c(W_k,U)=\{U^n;n\in\mathbb{Z}\}$.*

Proof. Assume that $\sigma=\lim_\alpha U^{m_\alpha}\in\Sigma_c(W_k,U)$. Clearly $\sigma\in\Lambda(\Sigma(W_k,U))$. Let $\Theta_k(\sigma)=(\theta_1,...,\theta_{k-1},\theta)$, as in Theorem 3.1. Then by Lemma 3.3 there exist an integer n such that $\theta_j=(\,\cdot\,)^{Q_j(n)}$ for each j, $1\le j\le k-1$. On the other hand, $\Theta_k(U^n)=((\,\cdot\,)^{Q_1(n)},...,(\,\cdot\,)^{Q_{k-1}(n)},(\,\cdot\,)^{n^k})$ and Θ_k is one-to-one, hence it remains to show that $\theta=(\,\cdot\,)^{n^k}$. First, we show that θ is continuous. To this end, let $\{x_\beta\}_\beta$ be an arbitrary net in $\mathbb{T}$ which converges to $x\in\mathbb{T}$. We show that $\theta(x_\beta)\to\theta(x)$. Let $f_\beta,f\in W_k$ be defined by $f_\beta(n)=x_\beta^{n^k}$ and $f(n)=x^{n^k}$ for all $n\in\mathbb{Z}$. Then $f_\beta\to f$ (pointwise) in W_k. Hence, by the continuity of σ,

$$\theta(x_\beta)=\lim_\alpha(x_\beta)^{m_\alpha{}^k}=\sigma(f_\beta)(0)\to\sigma(f)(0)=\lim_\alpha x^{m_\alpha^k}=\theta(x).$$

That is, θ is continuous. Therefore, $\theta=(\,\cdot\,)^m$ for some $m\in\mathbb{Z}$. Let p be any prime number. Since $\lim_\alpha x^{m_\alpha}=x^n$, for all $x\in\mathbb{T}$, we have $m_\alpha=n$ (mod p) eventually. Therefore $m_\alpha^k=n^k$ (mod p) eventually. Thus $\lim_\alpha e((1/p)m_\alpha^k)=e((1/p)n^k)$. It follows that for each integer q with $0<q<p$, one has $\lim_\alpha e((q/p)m_\alpha^k)=e((q/p)n^k)$. On the other hand, $\lim_\alpha e((q/p)m_\alpha^k)=\theta(e(q/p))=e(q/p)^m$. Hence $e(q/p)^{n^k}=e(q/p)^m$. Now since $x\mapsto x^{n^k}$ and $x\mapsto x^m$ are both continuous functions on $\mathbb{T}$ and the set $\{e(q/p);0<q<p,\ p>(k-1)!$ and p is prime$\}$ is dense in $\mathbb{T}$, we have $x^{n^k}=x^m=\theta(x)$ for each $x\in\mathbb{T}$. The theorem is now proved. $\square$

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