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THE TOPOLOGICAL STRUCTURE OF 4-MANIFOLDS WITH EFFECTIVE TORUS ACTIONS (II)

BY

PETER SIE PAO[1]

A closed orientable 4-manifold is called a T^2-manifold if it supports an effective T^2 $(= SO(2) \times SO(2))$ action. Given a T^2 action on a 4-manifold M, denote the totality of fixed points and c-orbits (orbits with isotropy group $SO(2)$) by F and C respectively. The topological classification of T^2-manifolds is studied in the following three cases:

(1) $F \cup C = \emptyset$;
(2) $F \neq \emptyset$;
(3) $F = \emptyset$, $C \neq \emptyset$.

Cases (1) and (2) were treated in [3] and [5]. In this article we will continue this program and study the third case. In Section 2 we give a geometric construction involving the weighted orbit space. Using this construction we are able to compute the fundamental groups, and they in turn determine the homology and cohomology groups of these manifolds. Let M be a T^2-manifold corresponding to the orbit data

$$\{O;\, g;\, s;\, (m_1, n_1), \ldots, (m_s, n_s);\, (\alpha_1;\, p_1, q_1;\, \beta_1), \ldots, (\alpha_t;\, p_t, q_t;\, \beta_t)\}$$

in the sense of [2]. Analyzing $\pi_1(M)$, in Section 3 we prove the following:

THEOREM 2. *Let M be the above T^2-manifold. Then the integers $2g + s$, t, $\alpha_1, \ldots, \alpha_t$ and $m = \gcd(m_1, m_2, \ldots, m_s)$ are topologically invariant.*

In Section 4, we apply some elementary surgery theory to compute the second Stiefel–Whitney class of these manifolds. For example:

THEOREM 3. *Let M be a T^2-manifold with orbit invariants*

$$\{O;\, g;\, s;\, (0, 1), (m_2, n_2), \ldots, (m_s, n_s)\}.$$

Then $\omega_2(M) \neq 0$ if and only if there are integers i and j, $2 \leq i, j \leq s$, such that $m_i \equiv m_j \equiv n_i \equiv 1$ (2) and $n_j \equiv 0$ (2).

Unfortunately, $\pi_1(M)$ and $\omega_2(M)$ do not completely classify these manifolds. Some additional results are given in Section 5: If two of the c-orbits have

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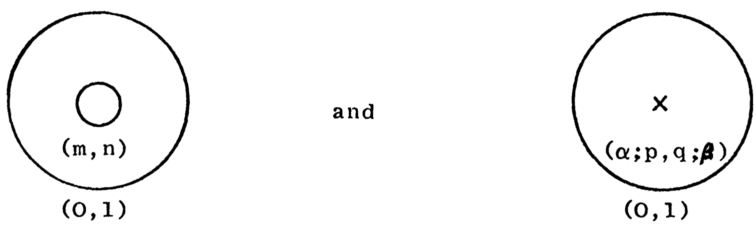
mutually orthogonal isotropy groups, then this manifold is a connected sum of copies of $S^2 \times S^2$, CP^2, $\overline{CP}^2$, $S^1 \times S^3$, L_n and L'_n ($n = 2, 3, \ldots$); if $\pi_1(M)$ has trivial center then M is stably homeomorphic to a connected sum of $S^2 \times S^2$, $\overline{CP}^2$, CP^2, $S^1 \times S^3$, L_n, and L'_n ($n = 3, 2, \ldots$). It is known that the products of suitable 3-dimensional lens spaces with a circle yield examples of stably diffeomorphic 4-manifolds that are not diffeomorphic. It would be interesting to know whether the stable homeomorphic classification of the above T^2-manifolds is indeed a homeomorphic classification.

Part of the material of this paper appeared first in the author's doctoral dissertation.

1. Two types of basic T^2-manifolds

Recall from [5, Section 3 and Section 5] that $Q(m, n)$ and $S(\alpha; p, q; \beta)$ are the T^2-manifolds corresponding to the weighted orbit spaces



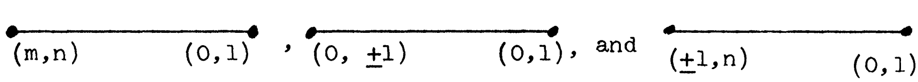
respectively. In this section we will classify these two types of manifolds:

PROPOSITION 1.

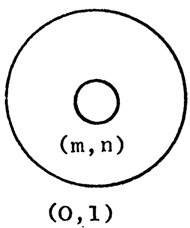
$$Q(m, n) = \begin{cases} S^1 \times L(m, n) & \text{if } m \neq 0, \pm 1, \\ S^1 \times (S^1 \times S^2) & \text{if } m = 0, \\ S^1 \times S^3 & \text{if } m = \pm 1. \end{cases}$$

PROPOSITION 2. *$S(\alpha; p, q; \beta)$ is homeomorphic to $S^1 \times L$, where L stands for a 3-dimensional lens spaces or $S^1 \times S^2$ or S^3.*

Proof of Proposition 1. Recall from [2, Section 2.5] that there are effective T^2 actions on $L(m, n)$, $S^1 \times S^2$, and S^3 with weighted orbit spaces



respectively. Using these actions, define T^2 actions on $S^1 \times L(m, n)$, $S^1 \times (S^1 \times S^2)$ and $S^1 \times S^3$ by letting T^2 act trivially on their S^1 factors. Then the orbit spaces of these manifolds are clearly



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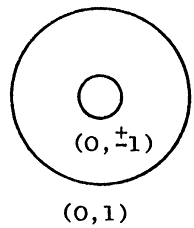

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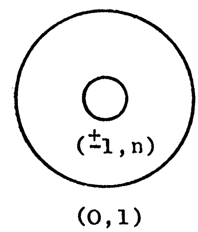


respectively. The proposition follows immediately from the Equivarient Classification Theorem of [3, Section 1.2].

To prove Proposition 2, we shall construct an effective T^3 action on the manifold $S(\alpha, p, q; \beta)$. The assertion will then follow from a theorem of J. Pak [4] which says when $n \geq 4$, T^n and $T^{n-3} \times$ (lens space) are the only closed orientable n-manifolds having effective T^{n-1} action. (It will be clear that $S(\alpha; p, q; \beta) \neq T^4$ by a fundamental group argument.)

Let N_0 and N_1 be two copies of $D^2 \times T^2$, and let T^2 act on N_0 and N_1 as follows:

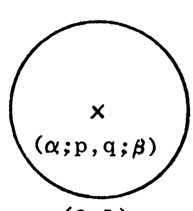
$$(\theta, \phi) \times (\rho, x; y, z) \mapsto (\rho, x + \theta; y + \phi, z) \quad \text{on } N_0,$$

$$(\theta, \phi) \times (\rho, x; y, z) \mapsto (\rho, x + \beta a_0 \theta + \beta b_0 \phi; y + \alpha a_0 \theta + \alpha b_0 \phi, z + (\alpha a_0 + q)\theta + (\alpha b_0 - p)\phi) \quad \text{on } N_1,$$

where a_0 and b_0 are integers such that $a_0 p + b_0 q = 1$. (By [5, Lemma V.5], we can always assume p and q to be relatively prime. For notation see [5, Section 2].) Let f: Bd $(N_0) \mapsto$ Bd (N_1) be the linear homeomorphism with the associated matrix A, where

$$A = \begin{pmatrix} \beta a_0 & \beta b_0 & n \\ \alpha a_0 & \alpha b_0 & m \\ \alpha a_0 + q & \alpha b_0 - p & m \end{pmatrix} \quad \text{and} \quad \alpha n - \beta m = 1.$$

(Since det $A = -1$, f is indeed a homeomorphism.) Clearly f is equivariant with respect to the above T^2 actions on N_0 and N_1. So the manifold $N_0 \cup_f N_1$ has an effective T^2 action. Observe that the weighted orbit space of this action is



Hence $N_0 \cup_f N_1 = S(\alpha; p, q; \beta)$. We can define our effective T^3 action on $S(\alpha; p, q; \beta)$ according to f. Namely, let T^3 act on N_0 and N_1 as follows:

$$(\theta, \phi, \psi) \times (\rho, x; y, z) \mapsto (\rho, x + \theta; y + \phi, z + \psi) \quad \text{on } N_0.$$

$$\begin{aligned}(\theta, \phi, \psi) &\times (\rho, x; y, z)\\ &\mapsto (\rho, x + \beta a_0\theta + \beta b_0\phi + n\psi; y + \alpha a_0\theta + \alpha b_0\phi + m\psi;\\ &\qquad z + (\alpha a_0 + q)\theta + (\alpha b_0 - p)\phi + m\psi) \quad \text{on } N_1.\end{aligned}$$

The proposition is proved.

By studying the orbit spaces of the actions of appropriate T^2 subgroups of the above defined T^3 action and Proposition 1, we conclude the following:

(1) $S(\alpha; 1, 0; \beta) = S^1 \times L(\alpha, \beta)$;
(2) $S(\alpha; 0, 1; \beta) = S^1 \times S^3$;
(3) $S(\alpha; p, q; 1) = S^1 \times L(q, a)$ if $\alpha = p \cdot q$, where $ap \equiv 1 \pmod q$.

In general it is a rather complicated number theoretic problem to determine the manifold L of Proposition 2. To compensate for this undesirable solution, we will compute the fundamental groups of these S-type manifolds. By the geometric construction of $S(\alpha; p, q; \beta)$ given above and the Van Kampen theorem, it follows easily that

$$\pi_1(S(\alpha; p, q; \beta)) = \{u_1 u_2 \mid u_1 u_2 = u_2 u_1;\ 1 = u_1^{\alpha a_0} u_2^{(\alpha a_0 + q)}\}.$$

Notice that $(\alpha a_0 + q)/\gcd(\alpha, q)$ and $(\alpha a_0)/\gcd(\alpha, q)$ are relatively prime. Choose integers s and t such that

$$s(\alpha a_0 + q)/\gcd(\alpha, q) - t(\alpha a_0)/\gcd(\alpha, q) = 1.$$

It is easily seen that $\pi_1(S(\alpha; p, q; \beta))$ is generated by the elements

$$u_1^s \cdot u_2^t \quad \text{and} \quad u_1^{\alpha a_0/\gcd(\alpha,q)} u_2^{(\alpha a_0 + q)/\gcd(\alpha,q)}.$$

In conclusion we have the following:

PROPOSITION 3. $\pi_1(S(\alpha; p, q; \beta)) = Z \oplus Z_{\gcd(\alpha,q)}$.

COROLLARY. *$S(\alpha; p, q; \beta) = S^1 \times L(n, m)$ for some integer m, where $n = \gcd(\alpha, q)$.*

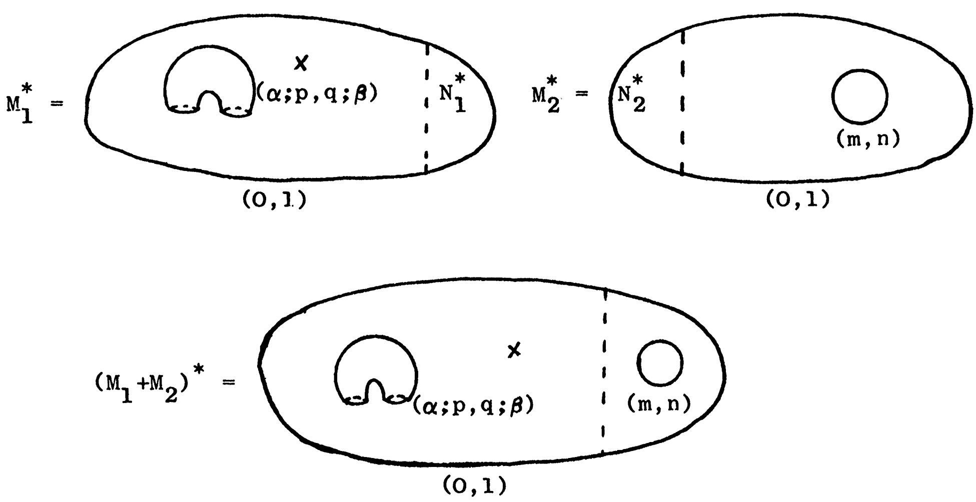
As an application, let us consider the 4-manifolds obtained by attaching two copies of $D^2 \times T^2$ along their boundaries. It is known that every self-homeomorphism of $S^1 \times S^1 \times S^1$ is pseudo-isotopic to a linear automorphism of T^3. Thus we need only to consider those 4-manifolds obtained by attaching two copies of $D^2 \times T^2$ through a linear automorphism of T^3. However, it is clear that these manifolds have effective T^3 actions, namely, the action defined according to the attaching maps. Hence they must be $S^1 \times L$, where L is some lens space or $S^1 \times S^2$ or S^3.

2. A geometric construction

As before, we will identify a T^2-manifold by the weighted orbit space of one of its T^2 actions, or equivalently, by its entire collection of orbit data (cf. [2, Section 4.4]). Let M be a T^2-manifold with $C \neq \emptyset$ and $F = \emptyset$. Represent M by its orbit data. By the standard T^2 action on M, we mean the unique T^2 action which gives this orbit data. By a suitable automorphism of T^2, we can change this standard action so that the stability group of at least one c-orbit is $G(0, 1)$, which we shall assume henceforth. Let M_1 and M_2 be two such T^2-manifolds. Let N_1 and N_2 be invariant tubular neighborhoods of a c-orbits of type $G(0, 1)$ in M_1 and M_2 respectively. Notice that the T^2 actions on Bd (N_1) and Bd (N_2) $(= S^1 \times S^2)$ are equivalent. Let f: Bd $(N_1) \to$ Bd (N_2) be an equivariant homeomorphism. Define

$$M_1 + M_2 = (M_1 - \mathrm{Int}\,(N_1)) \cup_f (M_2 - \mathrm{Int}\,(N_2)).$$

Then $M_1 + M_2$ is again a T^2-manifold. Furthermore, $M_1 + M_2$ is independent of the choices of N_1, N_2 and the choices of the equivariant homeomorphism f. So $+$ is a well-defined operation in this category. On the orbit spaces, the operation $+$ can be pictured as follows:



Let M be a T^2-manifold with orbit data

$$\{o;\, g;\, s;\, (0,\, 1),\, (m_2,\, n_2),\, \ldots,\, (m_s,\, n_s);\, (\alpha_1;\, p_1,\, q_1;\, \beta_1),\, \ldots,\, (\alpha_t;\, p_t,\, q_t;\, \beta_t)\}.$$

(As shown in [5, Lemma V.5], we may assume gcd $(p_i, q_i) = 1$ for all $i = 1, 2, \ldots, t$.) The following is an immediate consequence of the Equivariant Classification Theorem of Raymond and Orlik:

PROPOSITION 4.

$$M = M' + Q(m_2, n_2) + \cdots + Q(m_s, n_s) + S(\alpha_1; p_1, q_1; \beta_1) + \cdots + S(\alpha_t; p_t, q_t; \beta_t),$$

where M' is the T^2-manifold with orbit data $\{o; g; 1; (0, 1)\}$.

Recall from [6, Theorem 1] that $S^3\#(S^1 \times S^2)_1 \#\cdots\#(S^1 \times S^2)_{2g}$ has an effective S^1 action such that the orbit space is an orientable surface of genus g with connected boundary. The interior points are the images of the principal orbits, and the boundary points are fixed points. Using this S^1 action, define a T^2 action on

$$S^1 \times (S^3\#(S^1 \times S^2)_1 \#\cdots\#(S^1 \times S^2)_{2g})$$

by letting the complementary circle act on S^1 by multiplication. It is clear that the orbit data of this action is $\{o; g; 1; (0, 1)\}$. Hence the manifold M' above is just $S^1 \times (S^3\#(S^1 \times S^2)_1 \#\cdots\#(S^1 \times S^2)_{2g})$.

The operation $+$ is rather undesirable, for it depends on the T^2 actions. But we can use it to compute some algebraic invariants of these manifolds, for example the fundamental groups. By a theorem of Conner on torus actions, $\chi(F) = \chi(M)$, and since $F = \emptyset$, $\chi(M) = 0$. The reader should notice that the homology (cohomology) groups of M are determined by $\pi_1(M)$.

3. The fundamental groups and their implications

Let M and M' be the T^2-manifolds described in Proposition 4. The following facts can be seen easily.

(1) $\pi_1(M' - \mathrm{Int}\,(N)) \cong \pi_1(M') \cong Z \oplus (Z * \cdots * Z)$ ($2g$ times). Represent $\pi_1(M' - \mathrm{Int}\,(N))$ in terms of generators and relations as

$$\{h_1, h_2, \ldots, h_{2g+1} \,|\, h_1 h_i = h_i h_1 \text{ for } i = 2, 3, \ldots, 2g + 1\}.$$

Then $\phi_{M'}(\imath) = h_1$, where $\phi_{M'}\colon \pi_1(\mathrm{Bd}\,(N)) \to \pi_1(M' - \mathrm{Int}\,(N))$ is the homomorphism induced by the inclusion $\mathrm{Bd}\,(N) \rightsquigarrow M' - \mathrm{Int}\,(N)$, and

$$\imath \in \pi_1(\mathrm{Bd}\,(N_1)) \cong \pi_1(S^1 \times S^2) \cong Z$$

is a generator.

(2) $\pi_1(Q(m_i, n_i) - \mathrm{Int}\,(N)) \cong \pi_1(Q(m_i, n_i)) \cong Z \oplus Z_{m_i}$. Represent

$$\pi_1(Q(m_i, n_i))$$

as

$$\{k_{i,1}, k_{i,2} \,|\, k_{i,1} \cdot k_{i,2} = k_{i,2} \cdot k_{i,1};\ k_{i,2}^{m_i} = 1\}.$$

Then $\phi_{Q(m_i,n_i)}(\imath) = k_{i,2}$.

(3) $\pi_1(S(\alpha_i; p_i, q_i; \beta_i) - \mathrm{Int}\,(N)) \cong \pi_1(S(\alpha_i; p_i, q_i; \beta_i)) \cong Z \oplus Z_{\gcd(\alpha_i, q_i)}$. Represent

$$\pi_1(S(\alpha_i; p_i, q_i; \beta_i))$$

as

$$\{U_{i,1}, U_{i,2} \mid U_{i,1}U_{i,2} = U_{i,2}U_{i,1};\ 1 = U_{i,1}^{\alpha_i a_i} U_{i,2}^{(\alpha_i a_i + q_i)}\},$$

where a_i, b_i are integers such that $a_i p_i + b_i q_i = 1$. Then $\phi_{S(\alpha_i; p_i, q_i; \beta_i)}(\iota) = U_{i,1}^{\alpha_i b_i} U_{i,2}^{(\alpha_i b_i - p_i)}$.

((3) follows immediately from the geometric construction of $S(\alpha_i; p_i, q_i; \beta_i)$ in Proposition 2. (2) follows from [5, Corollary IV.5] which says $R(m_i; n_i)$ can be obtained from $Q(m_i, n_i)$ by surgering along N. Recall $R(m_i, n_i)$ is either $S^2 \times S^1 \# S^1 \times S^3$ or $CP^2 \# \overline{CP}^2 \# S^1 \times S^3$. $\pi_1(R(m_i, n_i)) = Z$. So $\phi_{Q(m_i, n_i)}(\iota)$ must be a generator of the torsion subgroup of $\pi_1(Q(m_i, n_i))$.)

The following is an easy consequence of the Van Kampen's theorem and the facts (1), (2), and (3) above.

THEOREM 1. *Let M be a T^2-manifold. If M has an effective T^2 action with orbit invariant*

$$\{o; g; s; (0, 1), (m_2, n_2), \ldots, (m_s, n_s); (\alpha_1; p_1, q_1; \beta_1), \ldots, (\alpha_t; p_t, q_t; \beta_t)\},$$

where $g + s + t \geq 2$, then $\pi_1(M)$ can be presented as

$$\begin{aligned}\{h_1, h_2, \ldots, h_{2g+1}, k_2, k_3, \ldots, k_s, U_{1,1}U_{1,2}, \ldots, U_{t,1}, U_{t,2}:&\\ h_1 h_i = h_i h_1\ (i = 1, 2, \ldots, 2g+1),&\\ h_1 k_i = k_i h_1\ (i = 2, \ldots, s),&\\ h_1 = U_{i,1}^{\alpha_i b_i} U_{i,2}^{(\alpha_i b_i - p_i)}\ (i = 1, \ldots, t),&\\ h_1^{m_2} = h_1^{m_3} = \cdots = h_1^{m_s} = 1,&\\ U_{i,1}^{\alpha_i a_i} U_{i,2}^{(\alpha_i a_i + q_2)} = 1\ (i = 1, \ldots, t),&\\ U_{i,1}U_{i,2} = U_{i,2}U_{i,1}\ (i = 1, \ldots, t)\}.&\end{aligned}$$

Remark. Notice that h_1 belongs to the center of $\pi_1(M)$. Consider the quotient group of $\pi_1(M)$ by the normal subgroup $\langle h_1 \rangle$. It is

$$\begin{aligned}\{h_2, \ldots, h_{2g+1}, k_2, \ldots, k_s, U_{1,1}U_{1,2}, \ldots, U_{t,1}U_{t,2}:&\\ U_{i,1}U_{i,2} = U_{i,2}U_{i,1},&\\ U_i^{\alpha_i a_i} U_{i,2}^{(\alpha_i a_i + q_i)} = 1,&\\ U_{i,1}^{\alpha_i b_i} U_{i,2}^{(\alpha_i b_i - p_i)} = 1\ (i = 1, 2, \ldots, t)\}&\end{aligned}$$

$$\begin{aligned}\cong \underbrace{Z * \cdots * Z}_{2g+s-1} * \{U_{1,1}, U_{1,2}: U_{1,1}U_{1,2} = U_{1,2}U_{1,1}, U_{1,1}^{\alpha_1 a_1} U_{1,2}^{(\alpha_1 a_1 + q_1)} = 1,&\\ U_{1,1}^{\alpha_1 b_1} U_{1,2}^{(\alpha_1 b_1 - p_1)} = 1\}&\\ * \cdots * \{U_{t,1}, U_{t,2}: U_{t,1}U_{t,2} = U_{t,2}U_{t,1}, U_{t,1}^{\alpha_t a_t} U_{t,2}^{(\alpha_t a_t + q_t)} = 1,&\\ U_{t,1}^{\alpha_t b_t} U_{t,2}^{(\alpha_t b_t - p_t)} = 1\}&\end{aligned}$$

$$\cong \underbrace{Z * Z * \cdots * Z}_{2g+s-1} * Z_{\alpha_1} * \cdots * Z_{\alpha_t}.$$

The above two equivalences are rather clear. The assertion, that the group

$$\{U_{i,1}U_{i,2} \mid U_{i,1}U_{i,2} = U_{i,2}U_{i,1};\ U_{i,1}^{\alpha_i a_i}U_{i,2}^{(\alpha_i a_i + q_i)} = 1; U_{i,1}^{\alpha_i b_i}U_{i,2}^{(\alpha_i b_i - p_i)} = 1\} = Z_{\alpha_i}$$

follows from elementary algebra. It is easily seen that the center of $\pi_1(M)$ is the cyclic group generated by h_1. Since h_1 has order gcd $(m_2, m_3, \ldots, m_s)$, gcd $(m_2, m_3, \ldots, m_s)$ is a topological invariant. Obviously, the group $\pi_1(M)$/center is topologically invariant. By the uniqueness of the indecomposable free product decomposition of $\pi_1(M)$/center, it follows that the numbers $2g + s, \alpha_1, \ldots, \alpha_t$ are all topological invariants. In conclusion:

THEOREM 2. *Let M be the T^2-manifold as above. Then the integers $2g + s$, t, $\alpha_1, \alpha_2, \ldots, \alpha_t$ and $m = \gcd(m_2, m_3, \ldots, m_s)$ are topological invariants.*

4. The second Stiefel–Whitney class

Let M be an orientable closed 4-manifold with second Stiefel–Whitney class $\omega_2(M)$. Let $S \subset M$ be an embedded circle. The following are known:

(1) There are exactly two ways to do surgery on S; they are usually called the framings of the surgery. Let

$$f_1, f_2 \colon \mathrm{Bd}\,(D^2 \times S^2) \to \mathrm{Bd}\,(M - \nu(S))$$

be the attaching maps of two surgeries, where $\nu(S) = S^1 \times D^3$ is the tubular neighborhood of S. Then these two surgeries are the same if

$$D^2 \times S^2 \cup_{f_2^{-1} \circ f_1} D^2 \times S^2 = S^2 \times S^2;$$

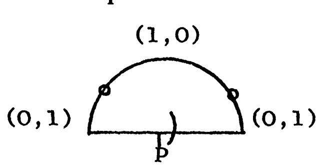
they are different if

$$D^2 \times S^2 \cup_{f_2^{-1} \circ f_1} D^2 \times S^2 = CP^2 \# \overline{CP}^2.$$

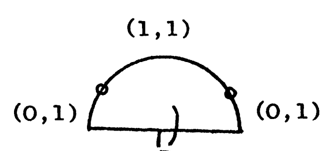
(2) If $\omega_2(M) \neq 0$, then by surgery on S, the second Stiefel–Whitney class of the resulting manifold must also be none zero.

(3) If $\omega_2(M) = 0$, then we can always choose a framing so that the manifold resulting from surgery on S has $\omega_2 = 0$. It follows from (2) and (3) immediately, that $\omega_2(M) = 0$ if and only if there exists a manifold W such that $\omega_2(W) = 0$ and W can be obtained from M by a surgery on S. In this section we shall use these techniques to determine the second Stiefel–Whitney class of the T^2-manifolds.

Recall from [2, Section 3.8] that there are effective T^2 actions on $D^2 \times S^2$ whose orbit spaces are



and



Denote these two T^2-manifolds by $(D^2 \times S^2)_1$ and $(D^2 \times S^2)_2$ respectively. Let M be a T^2-manifold with $C \neq \emptyset$ and $F = \emptyset$. Let $N \subset M$ be an invariant tubular neighborhood of a c-orbit of type $G(0, 1)$ in M. Notice that on Bd $(D^2 \times S^2)_1$, Bd $(D^2 \times S^2)_2$, and Bd $(M - \text{Int}\,(N))$ the T^2 actions are all equivalent. Let

$$f_1\colon \text{Bd}\,(D^2 \times S^2)_1 \to \text{Bd}\,(M - \text{Int}\,(N))$$

and

$$f_2\colon \text{Bd}\,(D^2 \times S^2)_2 \to \text{Bd}\,(M - \text{Int}\,(N))$$

be some T^2 equivariant homeomorphisms. Let

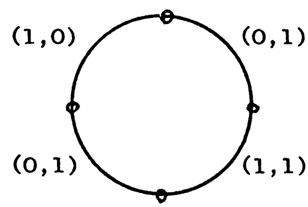
$$W_1 = (M - \text{Int}\,(N)) \cup_{f_1} (D^2 \times S^2)_1$$

and

$$W_2 = (M - \text{Int}\,(N)) \cup_{f_2} (D^2 \times S^2)_2.$$

Observe that the above geometric constructions are just equivariant surgeries. W_1 and W_2 are obtained from M by surgeries on the same circle.

Moreover, $(D^2 \times S^2)_1 \cup_{f_2^{-1} \circ f_1} (D^2 \times S^2)_2$ is a T^2-manifold with fixed points whose orbit space is



By the table in [2, Section 5.4], $(D^2 \times S^2)_1 \cup_{f_2^{-1} \circ f_1} (D^2 \times S^2)_2 = CP^2 \# \overline{CP}^2$. Hence by (1) above, these two surgeries are different. It follows from (2) and (3) that $\omega_2(M) \neq 0$ if and only if $\omega_2(W_1) \neq 0$ and $\omega_2(W_2) \neq 0$. But both W_1 and W_2 are T^2-manifolds with fixed points. They can be identified by the method of [5] and their second Stiefel–Whitney class can be computed very easily. Therefore $\omega_2(M)$ is determined. In the case that no exceptional orbit is present, we have the following:

THEOREM 3. *Let M be a T^2-manifold with orbit invariants*

$$\{o;\, g;\, s;\, (0, 1), (m_2, n_2), \ldots, (m_s, n_s)\}.$$

Then $\omega_2(M) = 0$ if and only if there are integers i and j, $2 \leq i, j \leq s$, such that $m_i \equiv m_j \equiv n_i \equiv 1$ (2) and $n_j \equiv 0$ (2).

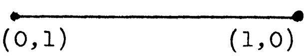
(Deriving a formula in general is a rather complicated number theoretic problem. The author found no practical reason to do so.)

5. Some additional results

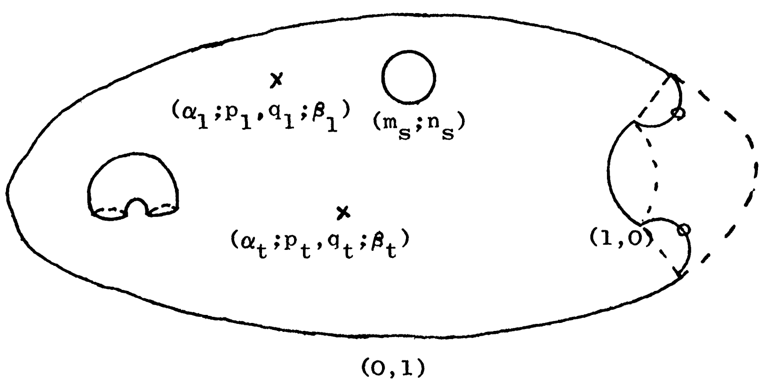
Two circle subgroups of T^2, $G(m_1, n_1)$ and $G(m_2, n_2)$ are said to be mutually orthogonal if $m_1 n_2 - m_2 n_1 = \pm 1$.

(1) Let M be a T^2-manifold with $C \neq \emptyset$ and $F = \emptyset$. If two of its c-orbits have mutually orthogonal stability groups, then M is a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, $\overline{CP}^2$, $S^1 \times S^3$, L_n and L'_n $(n = 2, 3, \ldots)$.

Proof. Without loss of generality, we may assume that the mutually orthogonal c-stability groups are $G(0, 1)$ and $G(1, 0)$. Consider the invariant S^3 whose orbit space is an arc



connecting the two boundary components of M^* whose stability groups are $G(0, 1)$ and $G(1, 0)$. Cutting M along this S^3 and equivariantly attaching two 4-cells, we obtain a T^2-manifold M' with two fixed points:



By [5, Theorem V.1], M' can be identified as a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, $\overline{CP}^2$, $S^1 \times S^3$, L_n, and L'_n $(n = 2, 3, \ldots)$. However, removing two 4-cells from M' and identifying the boundaries corresponds to taking connected sum with $S^1 \times S^3$. The assertion is proved.

(2) Suppose the T^2-manifold M has orbit invariants

$$\{o; g; s; (0, 1), (0, 1)_2, (0, 1)_3, \ldots, (0, 1)_s; (\alpha_1; 0, 1, \beta_1), \ldots, (\alpha_t; 0, 1; \beta_t)\}.$$

(Notice that all the stability groups of the standard T^2 action on M are contained in the circle group $G(0, 1)$.) Then

$$M = S^1 \times [S^3 \# (S^1 \times S^2)_1 \# \cdots \# (S^1 \times S^2)_{2g+s-1} \# L(\alpha_1, \beta_1) \# \cdots \# L(\alpha_t, \beta_t)].$$

Proof. Recall from [6, Section 7] that the 3-manifold

$$M' = S^3 \# (S^1 \times S^2)_1 \# \cdots \# (S^1 \times S^2)_{2g+s-1}$$
$$\# L(\alpha_1, \beta_1) \# \cdots \# L(\alpha_t, \beta_t)$$

has an effective S^1 action, and its orbit data is

$$\{o; g; s; O; (\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}.$$

Use this S^1 action to define an effective T^2 action on $S^1 \times M'$ by letting the other circle act on S^1 by multiplication. It can be seen easily that this action on $S^1 \times M'$ gives the desired orbit space.

(3) Let us look at the surgery construction on the T^2-manifold M in Section 4 again. It is clear that the circle which we do surgery on generates the center of $\pi_1(M)$. In the case that $m = \gcd(m_2, m_3, \ldots, m_s) = 1$, $\pi_1(M)$ has trivial center. So we were doing surgery on a trivial circle, which is equivalent to taking connected sum with $S^2 \times S^2$, or $CP^2 \# \overline{CP}^2$. The resulting manifolds are T^2-manifolds with fixed points. They can be identified. Therefore in the case $m = 1$, we have obtained a stable homeomorphism classification. For example:

THEOREM 4. *Let M be a T^2-manifold with orbit invariants*

$$\{o; g; s; (0, 1), (m_2, n_2), \ldots, (m_s, n_s)\}.$$

If $m = \gcd(m_2, \ldots, m_s) = 1$, then

$$M \# S^2 \times S^2 = \begin{cases} \underbrace{S^2 \times S^2 \# \cdots \# S^2 \times S^2}_{2g+s-1} \# \underbrace{S^1 \times S^3 \# \cdots \# S^1 \times S^3}_{2g+s-1} & \text{if } \omega_2(M) = 0, \\ CP^2 \# \overline{CP}^2 \# \underbrace{S^2 \times S^2 \# \cdots \# S^2 \times S^2}_{2g+s-2} \# \underbrace{S^1 \times S^3 \# \cdots \# S^1 \times S^3}_{2g+s-1} & \text{if } \omega_2(M) \neq 0, \end{cases}$$

and

$$M \# CP^2 \# \overline{CP}^2 = CP^2 \# \overline{CP}^2 \# \underbrace{S^2 \times S^2 \# \cdots \# S^2 \times S^2}_{2g+s-2} \# \underbrace{S^1 \times S^3 \# \cdots \# S^1 \times S^3}_{2g+s+1}.$$

We are as yet unable to determine these T^2-manifolds. In general the T^2-manifolds with $C \neq \emptyset$, $F = \emptyset$ are not classified by the manifolds resulting from surgery. For example, by doing surgery on a generator of the torsion subgroup of their fundamental groups, $S^1 \times L(9, 2)$ and $S^1 \times L(9, 8)$ yield the same manifold [5, Section 5]. $S^1 \times L(9, 2)$ and $S^1 \times L(9, 8)$ *have the same homotopy type*, but they are not homeomorphic. It would be very interesting if the stable homeomorphism classification of Theorem 4 failed to be the homeomorphism classification.

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The University of Georgia
Athens, Georgia