THE TOPOLOGY OF SPACES OF COPRIME POLYNOMIALS

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§1 Introduction.

Let E_d^n be the set of n-tuples $(p_1,\ldots,p_n)$ of *mutually coprime* monic polynomials of degree d. Equivalently (in terms of the roots of the polynomials) E_d^n is the set of n-tuples $(\xi_1,\ldots,\xi_n)$ of positive divisors in $\mathbb{C}$ of degree d, such that $\xi_i\cap\xi_j=\emptyset$ whenever $i\neq j$. A positive divisor ξ in $\mathbb{C}$ of degree d is simply an element $\sum_{i=1}^d\alpha_i$ of the symmetric product $Sp^d(\mathbb{C})\cong\mathbb{C}^d$, so the set E_d^n acquires a natural topology as an open subspace of the n-fold product of $S^d(\mathbb{C})$. It is therefore a complex manifold of dimension dn.

Example 1.1 E_1^n is the space $F(\mathbb{C},n)$ of n-tuples of distinct points in $\mathbb{C}$.

Example 1.2 E_d^1 is the space $Sp^d(\mathbb{C})$.

Example 1.3 E_d^2 may be identified with the space of rational functions $\mathbb{C}\cup\infty\to\mathbb{C}\cup\infty$ of degree d which take the value 1 at ∞.

The fundamental group of E_d^n was computed by Epshtein:

Proposition 1.4 [Ep]. *If $d\geq 2$, then $\pi_1E_d^n\cong\mathbb{Z}^{n(n-1)/2}$.*

If $d=1$, the group $\pi_1E_d^n$ is by definition the group of pure braids on n strings. It is non-abelian if $n>2$. Our first main result permits the computation of some of the higher homotopy and homology groups of E_d^n.

Theorem 1.5. *There is a map $E_d^n\to\Omega_0^2(\bigvee^n\mathbb{C}P^\infty)$ which is a homotopy equivalence up to dimension d.*

(A homotopy equivalence up to dimension d is a map which induces isomorphisms on homotopy groups π_i for $i<d$, and an epimorphism for $i=d$.)

The space E_d^2 of rational functions was investigated by Segal ([Se]), who obtained theorem 1.5 in the case $n=2$. As in the case $n=2$, the space E_d^n admits a description in terms of "rational maps". Let X_n be the subspace of $\mathbb{C}P^{n-1}$ consisting of points $[z_1;\ldots;z_n]$ such that at most one homogeneous coordinate is zero. In other words,

$$X_n = \mathbb{C}P^{n-1} - \bigcup_{i\neq j} H_{ij}$$

where $H_{ij} = \{[z_1;\ldots;z_n] \mid z_i = z_j = 0\}$. We may identify E^n_d with the space $\mathrm{Hol}^*_d(S^2, X_n)$ consisting of holomorphic maps $f : S^2 \to X_n$ which satisfy the conditions $f(\infty) = [1;\ldots;1]$ and $[f] = d \in \mathbb{Z} \cong \pi_2 X_n$. The space X_n is an example of a "subspace arrangement", i.e. the complement of a finite number of linear subspaces (see [Bj]). As a consequence of theorem 1.5 we shall obtain the following closely related result:

Theorem 1.6. *The inclusion* $\mathrm{Hol}^*_d(S^2, X_n) \to \mathrm{Map}^*_d(S^2, X_n)$ *is a homotopy equivalence up to dimension* d*.*

Here, Map indicates the space of smooth (or continuous) maps. For $n = 2$ we have $X_n = \mathbb{C}P^1$, and theorem 1.6 reduces to the main theorem of [Se].

Our approach is similar to that of [Se] in that we show first that the homotopy groups of E^n_d "stabilize" as $d \to \infty$, and then that "E^n_∞" (suitably defined) is homotopy equivalent to $\Omega^2_0(\bigvee^n \mathbb{C}P^\infty)$. The major new feature of our approach lies in the first step. In [Se] the homotopy stabilization was obtained by first proving a homology stabilization theorem, and then by showing that the action of the fundamental group was nilpotent on the homology of the universal covering space. In addition, the homology stabilization theorem itself involved a rather mysterious application of Poincaré Duality. Our method has the virtue that the passage from homology to homotopy is much simpler and perhaps more natural. In §2 we show that for each $d > 1$ there is an inclusion $j_d : E^n_d \to E^n_{d+1}$ which is an acyclic map up to dimension d, that is, the homomorphism $H_i(E^n_d; j^*_d L) \to H_i(E^n_{d+1}; L)$ is an isomorphism for $i < d$ and an epimorphism for $i = d$, for any local coefficient system L on E^n_{d+1}. Since an acyclic map between two spaces with abelian fundamental groups is a homotopy equivalence, this, together with Epshtein's result (proposition 1.4) implies that j_d is in fact a homotopy equivalence up to dimension d. For the sake of completeness and because Epshtein's proof lacks detail, we give a proof of proposition 1.4 in an Appendix. In §3 we observe that the method of [Se] shows that the space "E^n_∞" is actually homotopy equivalent to a component of $\Omega^2(\bigvee^n \mathbb{C}P^\infty)$. As this part of the argument closely follows the lines of [Se] we shall omit most of the details.

In §4, we shall indicate the following generalizations of theorems 1.5 and 1.6 to the case of n-tuples of monic polynomials with "arbitrary coprime conditions". Let I be any collection of subsets of $\{1,\ldots,n\}$ with $|\Lambda| \geq 2$ for all $\Lambda \in I$. Let

$$E^I_d = \{(\xi_1,\ldots,\xi_n) \in Sp^d(\mathbb{C})^n \mid \cap_{i\in\Lambda}\, \xi_i = \emptyset \text{ for all } \Lambda \in I\}.$$

Define the generalized wedge product $\bigvee^I \mathbb{C}P^\infty$ to be the subset of the n-fold product $(\mathbb{C}P^\infty)^n$ consisting of points $(x_1,\ldots,x_n)$ such that, for each $\Lambda \in I$, x_i is equal to the basepoint of $\mathbb{C}P^\infty$ for at least some $i \in \Lambda$. Then we have the following generalization of theorem 1.5:

Theorem 1.7. *There is a map* $E_d^I \to \Omega_0^2(\bigvee^I \mathbb{C}P^\infty)$ *which is a homotopy equivalence up to dimension* d*.*

Similarly, if we define X_I by

$$X_I = \mathbb{C}P^{n-1} - \bigcup_{\Lambda \in I} H_\Lambda,$$

where $H_\Lambda = \{[z_1; \ldots; z_n] \mid z_i = 0 \text{ for all } i \in \Lambda\}$, then theorem 1.6 generalizes as follows:

Theorem 1.8. *The inclusion* $\mathrm{Hol}_d^*(S^2, X_I) \to \mathrm{Map}_d^*(S^2, X_I)$ *is a homotopy equivalence up to dimension* d*.*

We conclude with some comments on the wider significance of these results. Theorem 1.7 can be regarded as giving a simple "homotopy model" (namely E_d^I) for a double loop space (i.e. $\Omega^2(\bigvee^I \mathbb{C}P^\infty)$). In a future article [GKY], our method will be used to give a model for the space $\Omega^3 S^3$. The problem of finding such a model for $\Omega^n S^n$ was posed in [Mc].

The significance of theorem 1.8 is that it produces a non-trivial family of complex manifolds X_I for which the natural inclusion $\mathrm{Hol}(S^2, X) \to \mathrm{Map}(S^2, X)$ is a homology equivalence up to some dimension depending on the component. Several other isolated instances of this phenomenon are known; a survey appears in [Gu1]. The explanation for the phenomenon is thought to be Morse theoretic in nature, i.e. the fact that the holomorphic maps form the set of absolute minima of the energy functional on (a fixed component of) the space of smooth maps, when X is a Kähler manifold. At present, however, no proof along these lines is known, so it is of interest to construct new families of examples or counter-examples.

In [Gu2] the stabilization method developed here is an essential step in proving a version of theorem 1.8 for compact "toric varieties"; such varieties provide important connections between algebraic geometry and combinatorics (see [Od],[Fu]). The variety X_I is actually an example of a non-compact toric variety; it seems likely that by using the method of [Gu2] one can extend theorem 1.8 further to non-compact toric varieties.

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§2 Stabilization.

Let U be an open subset of $\mathbb{C}$. Let $d_1, \ldots, d_n$ be positive integers.

Definition: Let $E_{d_1,\dots,d_n}(U)$ be the set of n-tuples $(p_1,\dots,p_n)$ of mutually coprime monic polynomials such that deg $p_i = d_i$ and the roots of p_i lie in U, for $i = 1,\dots,n$. If $U = \mathbb{C}$ we write $E_{d_1,\dots,d_n}(U) = E_{d_1,\dots,d_n}$. If $d_1 = \cdots = d_n = d$ we write $E_{d_1,\dots,d_n}(U) = E^n_d(U)$.

Thus, $E_{d_1,\dots,d_n}(U)$ is a natural generalization of the space E^n_d.

For $d = 0, 1, 2, \dots,$ choose distinct points $x_{d+1,1},\dots,x_{d+1,n}$ in the region $\{d \le |z| < d+1\}$. Then we have an inclusion map

$$j_d : E^n_d(|z| < d) \to E^n_{d+1}(|z| < d+1)$$
$$(\xi_1,\dots,\xi_n) \mapsto (\xi_1 + x_{d+1,1},\dots,\xi_n + x_{d+1,n}).$$

Since $E^n_d(|z| < d)$ is homeomorphic to E^n_d, j_d defines (up to homotopy) a "stabilization map" $j_d : E^n_d \to E^n_{d+1}$. The main theorem of this section is:

Theorem 2.1. *The stabilization map $j_d : E^n_d \to E^n_{d+1}$ is a homotopy equivalence up to dimension d.*

The case $n = 2$ of this theorem was proved in [Se] (see proposition 5.1 and corollary 6.3). Our proof of theorem 2.1 will in particular give a new proof of Segal's theorem.

Let $f : X \to Y$ be a continuous map between path connected spaces. Recall that f is said to be *acyclic* if for any coefficient system L on Y the induced map

$$f_* : H_*(X; f^*L) \to H_*(Y; L)$$

is an isomorphism, where f^*L is the induced local system on X. If f_* is an isomorphism for $i < d$ and an epimorphism for $i = d$, we shall say that f is acyclic up to dimension d. We shall prove

Theorem 2.2. *The stabilization map $j_d : E^n_d \to E^n_{d+1}$ is acyclic up to dimension d.*

If $d > 1$, theorem 2.1 follows from theorem 2.2, proposition 1.4, and the next well known result (proposition 1.4 of [HH]).

Proposition 2.3. *Let $d > 1$. Let $f : X \to Y$ be a map between path connected spaces. Then f is a homotopy equivalence up to dimension d if and only if f is acyclic up to dimension d and $\pi_1 f : \pi_1 X \to \pi_1 Y$ is an isomorphism.*

If $d = 1$, theorem 2.1 follows directly from theorem 2.2 and the fact (which we shall prove in the Appendix) that $\pi_1 E^n_2$ is abelian.

We turn now to the proof of theorem 2.2. It will use the following lemma:

Lemma 2.4. *Let X be the subspace of $\mathbb{C}$ obtained by removing k distinct points. Let $i_d : Sp^d(X) \to Sp^{d+1}(X)$ be the inclusion given by "adjoining a basepoint". Then i_d is a homotopy equivalence up to dimension d.*

Proof. Since X is homotopy equivalent to the bouquet of circles $\bigvee^k S^1$, it suffices to prove the lemma for this space. From the known homology of $\bigvee^k S^1$, it may be shown (by induction on d) that i_d is acyclic up to dimension d. Alternatively, a proof may be given using the method of Proposition (A.2) of [Se]. For $d \geq 2$, the fundamental group of $Sp^d(\bigvee^k S^1)$ is abelian, so the lemma follows from proposition 2.3. We omit the details. □

We shall deduce theorem 2.2 from a slightly more general theorem, which involves stabilizing with respect to just one of the variables. Let x be a point in the region $\{1 \leq |z| < 2\}$. Then we have an inclusion

$$\begin{gathered} j : E_{d_1,\dots,d_i,\dots,d_n}(|z| < 1) \to E_{d_1,\dots,d_i+1,\dots,d_n}(|z| < 2) \\ (\xi_1, \dots, \xi_i, \dots, \xi_n) \mapsto (\xi_1, \dots, \xi_i + x, \dots, \xi_n). \end{gathered}$$

Up to homotopy, j defines a map $j : E_{d_1,\dots,d_i,\dots,d_n} \to E_{d_1,\dots,d_i+1,\dots,d_n}$. Theorem 2.2 is a direct consequence of:

Theorem 2.5. *The stabilization map $j : E_{d_1,\dots,d_i,\dots,d_n} \to E_{d_1,\dots,d_i+1,\dots,d_n}$ is acyclic up to dimension d_i.*

Proof. It suffices to give the proof in the case $i = 1$. Let L be a local coefficient system on the space $E_{d_1+1,d_2,\dots,d_n}$. We shall also use the letter L to denote its restriction to any (open or closed) subspace. Consider the projection map $p_1 : E_{d_1,\dots,d_n} \to E_{d_2,\dots,d_n}$ which sends $(q_1, \dots, q_n)$ to $(q_2, \dots, q_n)$. We have a commutative diagram:

$$\begin{array}{ccc} E_{d_1,d_2,\dots,d_n} & \xrightarrow{j} & E_{d_1+1,d_2,\dots,d_n} \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_1}\downarrow \\ E_{d_2,\dots,d_n} & \xrightarrow{=} & E_{d_2,\dots,d_n}. \end{array}$$

If the vertical maps were fibrations, we could attempt to prove the theorem by applying lemma 2.4 to the restriction of j to the fibres. The long exact sequence of homotopy groups would then give us a homotopy equivalence of total spaces up to dimension d_1. However, the vertical maps are fibrations only over certain subspaces, as we shall now explain.

Let $E^k_{d_1,\dots,d_n}$ be the subset of $E_{d_1,\dots,d_n}$ consisting of n-tuples $(q_1, \dots, q_n)$ such that the polynomial $q_2 \dots q_n$ has at most k distinct roots. Let

$$X^k_{d_1,\dots,d_n} = E^k_{d_1,\dots,d_n} - E^{k-1}_{d_1,\dots,d_n},$$

i.e. the subset where $q_2 \dots q_n$ has exactly k distinct roots. Let

$$Y^k_{d_2,\dots,d_n} = p_1(X^k_{d_1,\dots,d_n}) = E^k_{d_2,\dots,d_n} - E^{k-1}_{d_2,\dots,d_n}.$$

The map p_1 restricts to a map $p_1 : X^k_{d_1,\dots,d_n} \to Y^k_{d_2,\dots,d_n}$ and we have a commutative diagram:

$$\begin{array}{ccc} X^k_{d_1,d_2,\dots,d_n} & \xrightarrow{j} & X^k_{d_1+1,d_2,\dots,d_n} \\ p_1 \downarrow & & p_1 \downarrow \\ Y^k_{d_2,\dots,d_n} & \xrightarrow{=} & Y^k_{d_2,\dots,d_n}. \end{array}$$

In this diagram the vertical maps are fibrations with fibres

$$Sp^{d_1}(\mathbb{C} - \{k \text{ distinct points}\}), \qquad Sp^{d_1+1}(\mathbb{C} - \{k \text{ distinct points}\})$$

respectively. The map j restricts on fibres to the map of lemma 2.4. Hence, from the long exact sequence of homotopy groups we see that $j : X^k_{d_1,d_2,\dots,d_n} \to X^k_{d_1+1,d_2,\dots,d_n}$ is a homotopy equivalence up to dimension d_1, and hence that

$(*)$ $\qquad j : X^k_{d_1,d_2,\dots,d_n} \to X^k_{d_1+1,d_2,\dots,d_n}$ is acyclic up to dimension d_1.

We shall use this to prove, by induction on k, that the following statement holds:

$(\dagger)_k$ $\qquad j : E^k_{d_1,d_2,\dots,d_n} \to E^k_{d_1+1,d_2,\dots,d_n}$ is acyclic up to dimension d_1.

(Observe that $(\dagger)_{d_2+\dots+d_n}$ gives the statement of the theorem.) The induction begins with $k = n-1$, because $E^{n-1}_{d_1,\dots,d_n} = X^{n-1}_{d_1,\dots,d_n}$. Let us assume the truth of $(\dagger)_{k'}$ for all k' with $k' < k$. To show that $E^k_{d_1,d_2,\dots,d_n} \to E^k_{d_1+1,d_2,\dots,d_n}$ is acyclic up to dimension d_1, we shall use a Mayer-Vietoris argument based on the diagram

$$\begin{array}{ccc} E^k_{d_1,d_2,\dots,d_n} & = & E^{k-1}_{d_1,d_2,\dots,d_n} \cup X^k_{d_1,d_2,\dots,d_n} \\ j \downarrow & & j \downarrow \\ E^k_{d_1+1,d_2,\dots,d_n} & = & E^{k-1}_{d_1+1,d_2,\dots,d_n} \cup X^k_{d_1+1,d_2,\dots,d_n}, \end{array}$$

together with the facts (from $(*)$ and $(\dagger)_{k-1}$) that the vertical map(s) on the right hand side are acyclic up to dimension d_1.

To do this, we construct an open neighbourhood $U^k_{d_1,\dots,d_n}$ of $E^{k-1}_{d_1,\dots,d_n}$ in $E^k_{d_1,\dots,d_n}$, homotopy equivalent to $E^{k-1}_{d_1,\dots,d_n}$, such that j restricts to a map

$$j : U^k_{d_1,d_2,\dots,d_n} \to U^k_{d_1+1,d_2,\dots,d_n},$$

and such that the restriction

$$j : U^k_{d_1,d_2,\dots,d_n} \cap X^k_{d_1,d_2,\dots,d_n} \to U^k_{d_1+1,d_2,\dots,d_n} \cap X^k_{d_1+1,d_2,\dots,d_n}$$

is acyclic up to dimension d_1. We then obtain $(\dagger)_k$ by applying the Five Lemma to the Mayer-Vietoris exact sequences of the open coverings

$$E^k_{d_1,\ldots,d_n} = U_{d_1} \cup X^k_{d_1,\ldots,d_n}, \quad E^k_{d_1+1,d_2,\ldots,d_n} = U_{d_1+1} \cup X^k_{d_1+1,d_2,\ldots,d_n}.$$

Let $U^k_{d_1,\ldots,d_n} = E^{k-1}_{d_1,\ldots,d_n} \cup F^k_{d_1,\ldots,d_n}$, where $F^k_{d_1,\ldots,d_n}$ consists of all n-tuples $(q_1,\ldots,q_n)$ in $X^k_{d_1,\ldots,d_n}$ such that, amongst the k distinct roots of $q_2\ldots q_n$, there exists a "cluster" of roots $z_1,\ldots,z_l$ (with $l > 1$) which are "far" from all other roots of $q_1\ldots q_n$. More precisely, the condition to be satisfied by $z_1,\ldots,z_l$ here is that there exists some number ε, with $0 < \varepsilon < 1$, such that

(C1) For $1 \le i,j \le l$, $|z_i - z_j| < \varepsilon$,

(C2) if z is a root of $q_1\ldots q_n$ and $z \notin \{z_1,\ldots,z_l\}$, then $|z - z_i| \ge 100\varepsilon$ for $i = 1,\ldots,l$, and

(C3) $\{z_1,\ldots,z_l\}$ is maximal with respect to properties (C1),(C2).

It is clear that the stabilization map restricts to a map

$$j : U^k_{d_1,d_2,\ldots,d_n} \to U^k_{d_1+1,d_2,\ldots,d_n}.$$

By following a suitable "gravitational flow" obtained by thinking of roots of $q_2\ldots q_n$ as heavy particles, we see that $U^k_{d_1,\ldots,d_n}$ is homotopy equivalent to $E^{k-1}_{d_1,\ldots,d_n}$. Similar considerations show that $U^k_{d_1,\ldots,d_n}$ is an open neighbourhood of $E^{k-1}_{d_1,\ldots,d_n}$.

It remains to prove that the stabilization map

$$j : V^k_{d_1,d_2,\ldots,d_n} \to V^k_{d_1+1,d_2,\ldots,d_n}$$

is acyclic up to dimension d_1, where $V^k_{d_1,\ldots,d_n} = U^k_{d_1,\ldots,d_n} \cap X^k_{d_1,\ldots,d_n}$. To do this we observe that $V^k_{d_1,\ldots,d_n}$ is a space of the same type as $E^{k-1}_{d_1,\ldots,d_n}$, hence the result follows by another application of the inductive hypothesis $(\dagger)_{k'}$. $\Box$

Remark 1: The Mayer-Vietoris argument used above may in fact be replaced by an argument based on Poincaré Duality (see [Ko]).

Remark 2: It is a corollary of the proof of theorem 2.5 that if $d_1 \ge d_2 + \cdots + d_n$ then the stabilization map $j : E_{d_1,d_2,\ldots,d_n} \to E_{d_1+1,d_2,\ldots,d_n}$ is a homotopy equivalence. This is because the stabilization map

$$Sp^d(\mathbb{C} - \{k \text{ distinct points}\}) \to Sp^{d+1}(\mathbb{C} - \{k \text{ distinct points}\})$$

is a homotopy equivalence when $d \ge k$, as $Sp^d(\mathbb{C} - \{k \text{ distinct points}\}) \simeq (S^1)^k$ and the stabilization map is, up to homotopy, the identity map on $(S^1)^k$.

§3 The stabilized space.

The inclusions

$$E_1^n(|z|<1) \xrightarrow{f_1} E_2^n(|z|<2) \xrightarrow{f_2} E_3^n(|z|<3) \xrightarrow{f_3} \dots$$

defined in the last section allow us to construct the stabilized space

$$E_\infty^n = \bigcup_{d\geq 1} E_d^n(|z|<d).$$

In this section we shall sketch the proof of the following result:

Theorem 3.1. *There is a homotopy equivalence* $E_\infty^n \to \Omega_0^2(\bigvee^n \mathbb{C}P^\infty)$.

Together with theorem 2.1, this gives theorem 1.5. The argument is quite analogous to the proof in [Se] for the case $n=2$. Segal's method, based on earlier ideas of Gromov, has been used in other similar situations (cf. [Mc],[Gu1]), and so we shall comment on the proof only very briefly.

The map in question arises from a "scanning map"

$$S_d : E_d^n \to \Omega_d^2 E^n(S^2,\infty),$$

where $E^n(S^2,\infty)$ denotes the space of n-tuples of positive divisors in $S^2 = \mathbb{C}\cup\infty$ modulo the equivalence relation which identifies two n-tuples if they agree on $\mathbb{C}$. If $D(z)$ denotes the open unit disc with centre z, then one defines a map $S_d^* : \mathbb{C}\times E_d^n \to E^n(S^2,\infty)$ by

$$S_d^*(z,(\xi_1,\dots,\xi_n)) = (\xi_1\cap D(z),\dots,\xi_n\cap D(z)).$$

Since one can identify the pair $(\overline{D(z)},\partial\overline{D(z)})$ canonically with (S^2,∞), the formula does define such a map, and one obtains a continuous extension $S_d^* : S^2\times E_d^n \to E^n(S^2,\infty)$ with $S_d^*(\infty,\quad) = (\emptyset,\dots,\emptyset)$. The adjoint map is the required map $S_d : E_d^n \to \Omega^2 E^n(S^2,\infty)$. This maps into a component of $\Omega^2 E^n(S^2,\infty)$ as E_d^n is connected, and we denote this component by $\Omega_d^2 E^n(S^2,\infty)$.

On taking the limit as $d\to\infty$ one obtains (up to homotopy) a map

$$S : E_\infty^n \to \Omega_0^2 E^n(S^2,\infty).$$

Theorem 3.2. *S is a homotopy equivalence.*

Proof. This is similar to the proof in §3 of [Se] (cf. also [Mc],[Gu1]). Indeed, the proof of proposition 2 in §3 of [Gu1] carries over word for word to the present case, on replacing "$(n+1)$-tuples of coprime polynomials" by "n-tuples of mutually coprime polynomials". □

The proof of theorem 3.1 is completed by the following result:

Proposition 3.3. *$E^n(S^2, \infty)$ is homotopy equivalent to $\bigvee^n \mathbb{C}P^\infty$.*

Proof. The case $n = 2$ is proposition 3.1 of [Se]. For $n > 2$, the same method may be used. $\square$

This also completes the proof of theorem 1.5. To prove theorem 1.6, it suffices to show that the inclusion map $\mathrm{Hol}_d^*(S^2, X_n) \to \mathrm{Map}_d^*(S^2, X_n)$ agrees up to homotopy with the scanning map $S_d : E_d^n \to \Omega_d^2 E^n(S^2, \infty)$. This can be done following the idea of [Se]; an explicit proof may be obtained from §3 of [Gu1] by replacing "$(n + 1)$-tuples of coprime polynomials" by "n-tuples of mutually coprime polynomials".

§4 Generalizations.

To prove theorem 1.7 it suffices to prove the generalizations of theorem 2.5 and theorem 3.1. We begin with the generalization of theorem 2.5:

Theorem 4.1. *The stabilization map $j : E^I_{d_1,\ldots,d_i,\ldots,d_n} \to E^I_{d_1,\ldots,d_i+1,\ldots,d_n}$ is acyclic up to dimension d_i.*

Proof. It suffices to give the proof in the case $i = 1$. For any n-tuple $(q_1, \ldots, q_n)$ of monic polynomials, let q be the monic polynomial whose roots are $\bigcup_{\Lambda \in I} \{\cap \xi_i \mid i \in \Lambda, i \neq 1\}$, where ξ_i denotes the roots of q_i. Let L be a local coefficient system on the space $E^I_{d_1+1,d_2,\ldots,d_n}$ (and hence on any subspace). Consider the projection map $p_1 : E^I_{d_1,\ldots,d_n} \to E^I_{d_2,\ldots,d_n}$ which sends $(q_1, \ldots, q_n)$ to $(q_2, \ldots, q_n)$. We have a commutative diagram:

$$\begin{array}{ccc} E^I_{d_1,d_2,\ldots,d_n} & \xrightarrow{\ j\ } & E^I_{d_1+1,d_2,\ldots,d_n} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ E^I_{d_2,\ldots,d_n} & \xrightarrow{\ =\ } & E^I_{d_2,\ldots,d_n}. \end{array}$$

Let $E^k_{d_1,\ldots,d_n}$ be the subset of $E^I_{d_1,\ldots,d_n}$ consisting of n-tuples $(q_1, \ldots, q_n)$ such that q has at most k distinct roots. Let

$$X^k_{d_1,\ldots,d_n} = E^k_{d_1,\ldots,d_n} - E^{k-1}_{d_1,\ldots,d_n},$$

i.e. the subset where the polynomial q has exactly k distinct roots. Let

$$Y^k_{d_2,\ldots,d_n} = p_1(X^k_{d_1,\ldots,d_n}) = E^k_{d_2,\ldots,d_n} - E^{k-1}_{d_2,\ldots,d_n}.$$

The map p_1 restricts to a map $p_1 : X^k_{d_1,\ldots,d_n} \to Y^k_{d_2,\ldots,d_n}$ and we have a commutative diagram:

$$\begin{array}{ccc} X^k_{d_1,d_2,\ldots,d_n} & \xrightarrow{j} & X^k_{d_1+1,d_2,\ldots,d_n} \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_1}\downarrow \\ Y^k_{d_2,\ldots,d_n} & \xrightarrow{=} & Y^k_{d_2,\ldots,d_n}. \end{array}$$

In this diagram the vertical maps are fibrations with fibres

$$Sp^{d_1}(\mathbb{C} - \{k \text{ distinct points}\}), \qquad Sp^{d_1+1}(\mathbb{C} - \{k \text{ distinct points}\})$$

respectively. The map j restricts on fibres to the map of lemma 2.4. Hence, by applying the long exact sequence of homotopy groups we see that $j : X^k_{d_1,d_2,\ldots,d_n} \to X^k_{d_1+1,d_2,\ldots,d_n}$ is a homotopy equivalence up to dimension d_1, and hence that

$(*)$ $\qquad j : X^k_{d_1,d_2,\ldots,d_n} \to X^k_{d_1+1,d_2,\ldots,d_n}$ is acyclic up to dimension d_1.

As in the proof of theorem 2.5 we can use this to prove, by induction on k, that the following statement holds:

$(\dagger)_k$ $\qquad j : E^k_{d_1,d_2,\ldots,d_n} \to E^k_{d_1+1,d_2,\ldots,d_n}$ is acyclic up to dimension d_1

The statement of the theorem is given by $(\dagger)_k$ for k sufficiently large. $\square$

In order to deduce that the stabilization map $j : E^I_d \to E^I_{d+1}$ is a homotopy equivalence up to dimension d, we need to know in addition that $\pi_1 E^I_d$ is abelian (for $d \geq 2$). Let I_2 be the subset $\{\Lambda \in I \mid |\Lambda| = 2\}$. (Recall that we are assuming $|\Lambda| \geq 2$ for all $\Lambda \in I$.) Then there is an inclusion $E^I_d \to E^{I_2}_d$, and this induces an isomorphism on fundamental groups, since $E^{I_2}_d$ is a manifold and $E^{I_2}_d - E^I_d$ has real codimension 4. Similarly there is an inclusion $E^n_d \to E^{I_2}_d$, and this induces an epimorphism on fundamental groups, since $E^{I_2}_d - E^n_d$ has real codimension 2. Since $\pi_1 E^n_d$ is abelian, it follows that $\pi_1 E^I_d$ is abelian.

We obtain a stabilized space E^n_∞ as in the case of E^n_d, and the method of the proof of theorem 3.1 carries over to give the following result.

Theorem 4.2. *There is a homotopy equivalence* $E^I_\infty \to \Omega^2_0(\bigvee^I \mathbb{C}P^\infty)$. $\square$

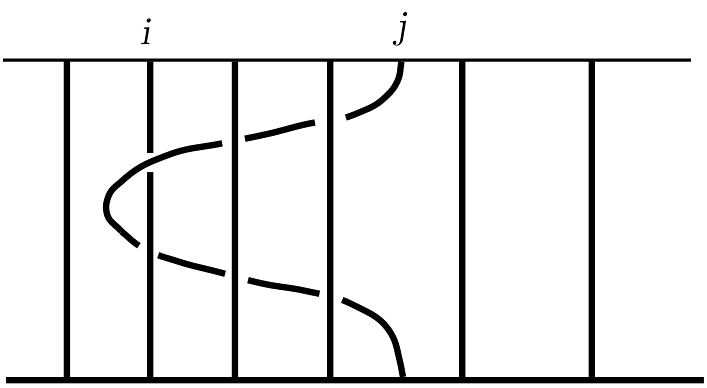
Thus we obtain theorem 1.7. The proof of theorem 1.8 presents no new difficulties, and we omit it.

Appendix: The fundamental group.

We shall re-prove Epshtein's result on the fundamental group of the space E_d^n. For our main theorem we only need to know that $\pi_1 E_d^n$ is abelian (for $d \geq 2$), but in view of the brevity of the arguments in [Ep] we shall give a complete proof here.

Proof of proposition 1.4. It is well known that $\pi_1 E_1^n$ may be identified with the group $H(n)$ of pure braids on n strings, i.e. braids on n strings which give the trivial permutation of the endpoints. (For a recent exposition of these matters, see [Ha], whose notation we shall follow.) In a similar way, elements of $\pi_1 E_d^n$ may be represented by braids on nd strings, but with certain identifications corresponding to the fact that the roots of the i-th polynomial (for $i = 1, \ldots, n$) are not required to be distinct. Indeed, it is easy to see that $\pi_1 E_d^n$ may be identified with a quotient group of $H(nd)$; the equivalence relation may be described geometrically by saying that two (or more) strings corresponding to roots of the i-th polynomial are allowed to collide, and then separate, in an arbitrary (continuous) fashion.

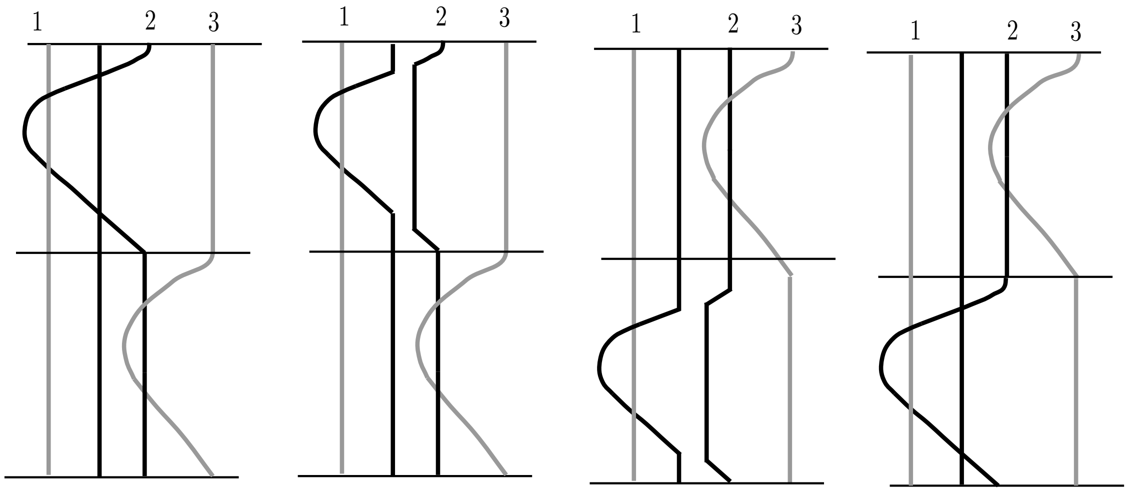
First, we shall show that $\pi_1 E_d^n$ is abelian if $d \geq 2$. The group $H(nd)$ is generated by elementary braids a_{ij}, where a_{ij} moves the j-th string once around the i-th string, as shown below:



$a_{i,j}$

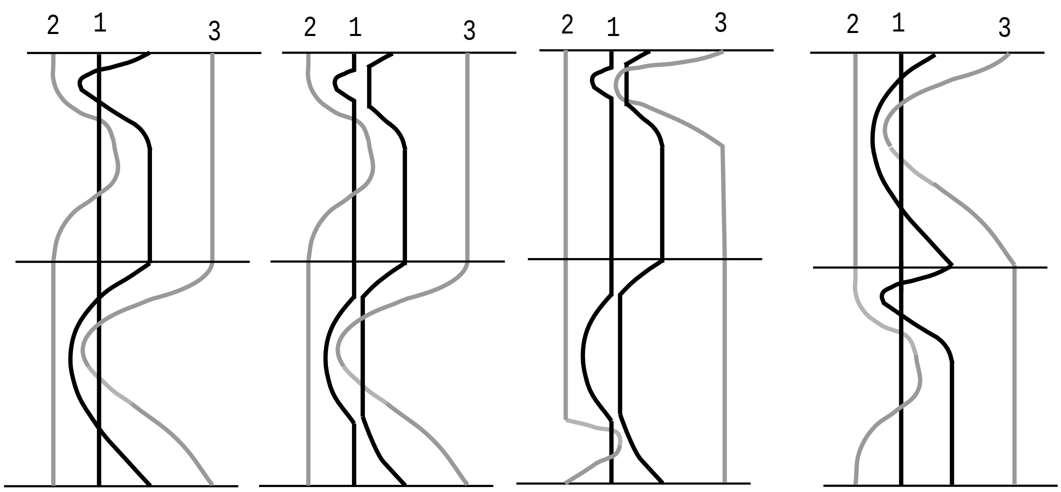
We shall show that the images $[a_{ij}]$ in $\pi_1 E_d^n$ of the elements a_{ij} commute. It suffices to show that the pairs (a) $[a_{12}], [a_{23}]$, (b) $[a_{12}], [a_{13}]$ commute, as all other cases either are trivial or are easily obtained from these two.

To see that $[a_{12}][a_{23}] = [a_{23}][a_{12}]$, we use the following sequence of homotopies:



$[a_{12}][a_{23}]$ $[a_{23}][a_{12}]$

To see that $[a_{12}][a_{13}] = [a_{13}][a_{12}]$, we use the following sequence of homotopies:



$[a_{13}][a_{12}]$ $[a_{13}][a_{12}]$

Next we shall show that $H_1 E_d^n \cong \mathbb{Z}^{n(n-1)/2}$, which will complete the proof of proposition 1.4. We shall make use of the stabilization map $j : F(\mathbb{C}, n) \cong E_1^n \to E_d^n$. It is known that $H_1 F(\mathbb{C}, n) \cong \bigoplus_{i<j} \mathbb{Z}\, a'_{ij}$ where a'_{ij} is the image of $a_{ij} \in \pi_1 F(\mathbb{C}, n)$ under the Hurewicz homomorphism. There is a transfer map $r : H_1 E_d^n \to H_1 E_1^n$ defined in a similar way to the map in proposition 5.3 of [Se] (which is the case $n = 2$). It follows from this that the map $j_* : H_1 F(\mathbb{C}, n) \to H_1 E_d^n$ is injective. But by theorem 2.2, j_* is surjective, so $H_1 E_d^n \cong H_1 F(\mathbb{C}, n) \cong \mathbb{Z}^{n(n-1)/2}$. □

As a special case, we obtain the theorem of Jones ([Se]) that $\pi_1 E_d^2 \cong \mathbb{Z}$.

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