



**The Topology on the Primitive Ideal Space of Transformation Group C # -
Algebras and C.C.R. Transformation Group C # -Algebras**

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THE TOPOLOGY ON THE PRIMITIVE IDEAL SPACE OF
TRANSFORMATION GROUP C^* -ALGEBRAS AND
C.C.R. TRANSFORMATION GROUP C^* -ALGEBRAS

BY

DANA P. WILLIAMS

ABSTRACT. If (G, Ω) is a second countable transformation group and the stability groups are amenable then $C^*(G, \Omega)$ is C.C.R. if and only if the orbits are closed and the stability groups are C.C.R. In addition, partial results relating closed orbits to C.C.R. algebras are obtained in the nonseparable case.

In several cases, the topology of the primitive ideal space is calculated explicitly. In particular, if the stability groups are all contained in a fixed abelian subgroup H , then the topology is computed in terms of H and the orbit structure, provided $C^*(G, \Omega)$ and $C^*(H, \Omega)$ are EH -regular. These conditions are automatically met if G is abelian and (G, Ω) is second countable.

1. Introduction. This paper grew out of an attempt to determine when a locally compact transformation group C^* -algebra, $C^*(G, \Omega)$, is C.C.R. Motivated by a result of Elliot Gootman [15] which shows that, for second countable locally compact transformation groups, $C^*(G, \Omega)$ is G.C.R. if and only if every stability group is G.C.R. and the orbit space satisfies the T_0 axiom of separability, one is led to try and prove an analogous result for C.C.R. algebras. I. Schochetman [28] has proved that when G/S_x is compact for every stability group, S_x , then $C^*(G, \Omega)$ is C.C.R. if and only if every stability group is C.C.R. We remark that the hypotheses implies that the orbits are closed. In this paper, we show that if we require only at every point of discontinuity, y , of the map $x \mapsto S_x$ that the stability group S_y be amenable and that (G, Ω) is second countable, then $C^*(G, \Omega)$ is C.C.R. if and only if every stability group is C.C.R. and every orbit is closed. The more difficult part of the proof involves generalizing a result of Effros and Hahn [9, Theorem 5.11] which requires that all the stability groups be abelian. Our methods are considerably different from either Schochetman's or Effros and Hahn's and they are very algebraic in nature. Our methods are modeled after and depend heavily upon those of Rieffel and Green [19], [23]. In particular, we make free use of the theory of strong Morita equivalence and imprimitivity algebras (cf. [23], [24], and [25]).

In the course of proving the above "C.C.R. Result," we obtain some results which allow us to calculate explicitly the topology on the primitive ideal space of a large class of transformation group C^* -algebras. Namely, we work with transformation group C^* -algebras which have the property that every primitive ideal is

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induced in an appropriate sense from a stability group; such algebras are called *EH*-regular. In the second countable case, it is known [17], [19] that $C^*(G, \Omega)$ is *EH*-regular if G is amenable or if the orbits are locally closed. We are able to calculate $\text{Prim } C^*(G, \Omega)$ up to homeomorphism when $C^*(G, \Omega)$ is *EH*-regular and either the action is essentially free or the stability groups are contained in a fixed abelian subgroup.

In §2 we make some preliminary definitions and state some of the basic results from the literature concerning transformation group C^* -algebras. Although none of the results or proofs in §2 are new, we have included them for the sake of completeness and for the reader's convenience.

In §3 we state our C.C.R. result and prove the easier half of the theorem. Then we introduce the various notions of induced representations of $C^*(G, \Omega)$ which will be needed in completing the proof of the C.C.R. theorem. The reader will undoubtedly find some familiarity with Marc Rieffel's induced representations of C^* -algebras helpful [23].

In §4 we finish the proof of the C.C.R. theorem and prove some related results.

In §5 we obtain the results which allow us to calculate $\text{Prim } C^*(G, \Omega)$ in the cases mentioned above.

The reader is assumed to be familiar with the basic theory of C^* -algebras as presented in the first five chapters of [6]. In addition some knowledge of the multiplier algebra of a C^* -algebra [4] will be assumed. If A is a C^* -algebra, we will always denote the multiplier algebra by $M(A)$. Unless stated otherwise, all ideals are assumed to be closed and two-sided.

This paper is based on part of the author's doctoral dissertation at the University of California at Berkeley. The author would like to take this opportunity to thank Professor Marc Rieffel for his supervision and helpful suggestions during the writing of this dissertation.

2. Preliminaries and notation. Let Ω be a locally compact Hausdorff space. We will denote the space of continuous complex valued functions vanishing at infinity on Ω by $C_0(\Omega)$. We let $C_c(\Omega)$ be the subspace of functions in $C_0(\Omega)$ which have compact support. If \mathcal{A} is a normed algebra, we denote the analogous spaces of \mathcal{A} -valued functions by $C_0(\Omega, \mathcal{A})$ and $C_c(\Omega, \mathcal{A})$.

In addition, we suppose G is a locally compact group with a jointly continuous action on Ω ; more precisely, we have a continuous map $G \times \Omega \rightarrow \Omega$, so that $r \cdot (s \cdot x) = (rs) \cdot x$ where $s \cdot x$ denotes the image of (s, x) . Thus (G, Ω) is a locally compact transformation group and we may form the associated transformation group C^* -algebra, $C^*(G, \Omega)$. For a precise treatment of the construction of $C^*(G, \Omega)$ and the basic facts concerning transformation group C^* -algebras, the reader should consult [5], [7], [9], [19], [29]. For the reader's convenience, we give some of the essential details. $C^*(G, \Omega)$ is the enveloping C^* -algebra of the Banach $*$ -algebra $L^1(G, C_0(\Omega))$ of all Bochner integrable $C_0(\Omega)$ -valued measurable functions on G with respect to a fixed Haar measure. Multiplication and involution are

defined by

$$f * g(s, x) = \int_G f(r, x)g(r^{-1}s, r^{-1} \cdot x) dr,$$

$$f * (s, x) = \Delta(s^{-1})f(s^{-1}, s^{-1} \cdot x)^{\bar{}}$$

where $f, g \in L^1(G, C_0(\Omega))$, $s \in G, x \in \Omega$, Δ is the modular function on G , and $\bar{}$ denotes complex conjugation.

Notice that if Ω consists of a single point, then the above construction is nothing more than the group C^* -algebra of G . We also remark that $C^*(G, \Omega)$ is the same as the covariance algebra $C^*(G, C_0(\Omega))$ where the strongly continuous action of G on $C_0(\Omega)$ is given by ${}^s\phi(x) = \phi(s^{-1} \cdot x)$ for $\phi \in C_0(\Omega)$ and $s \in G$. We will often find it convenient to work with the subalgebra $C_c(G \times \Omega)$ rather than $L^1(G, C_0(\Omega))$ or $C^*(G, \Omega)$ itself. To indicate that we are viewing $C_c(G \times \Omega)$ as a subalgebra of $C^*(G, \Omega)$ as opposed to a subalgebra of $C_0(G \times \Omega)$ will denote the subalgebra of $C^*(G, \Omega)$ by $C_c(G, \Omega)$.

DEFINITION 2.1. A covariant representation L of (G, Ω) on a Banach space \mathfrak{B}_L consists of a uniformly bounded strongly continuous representation, V_L , of G on \mathfrak{B}_L and a norm-decreasing nondegenerate representation, M , of $C_0(\Omega)$ on \mathfrak{B}_L such that $V_L(s)M_L(\phi)V_L(s^{-1}) = M_L({}^s\phi)$ for every $s \in G$ and $\phi \in C_0(\Omega)$. If \mathfrak{B}_L is also a Hilbert space we require that V_L be unitary and M_L be $*$ -preserving.

For convenience, henceforth all representations will be assumed to be nondegenerate. The next lemma is an immediate consequence of the definitions.

LEMMA 2.2. A covariant representation, $L = (V_L, M_L)$, gives a representation of $L^1(G, C_0(\Omega))$, also called L , defined by

$$L(f) = \int_G M_L(f(s, \cdot))V_L(s) ds$$

for $f \in L^1(G, C_0(\Omega))$.

For example, the natural representation of $L^1(G, C_0(\Omega))$ on itself by left multiplication is the integrated form of (V, M) where

$$(V(s)f)(r, x) = f(s^{-1}r, s^{-1} \cdot x), \quad (M(\phi)f)(r, x) = \phi(x)f(r, x)$$

for $r, s \in G, x \in \Omega, f \in L^1(G, C_0(\Omega))$, and $\phi \in C_0(\Omega)$. These actions induce homomorphisms R_G and R_Ω of G and $C_0(\Omega)$ into the multiplier algebra [4] $M(C^*(G, \Omega))$. Phil Green (cf. [19, p. 195]) has combined this observation with the basic work in [7] to show the following proposition.

PROPOSITION 2.3. If H is a Hilbert space then there is a one to one correspondence between $*$ -representations of $C^*(G, \Omega)$ on H and covariant representations of (G, Ω) on H . The correspondence is given in one direction by Lemma 2.2 (recall that $*$ -representations of $L^1(G, C_0(\Omega))$ are in one to one correspondence with $*$ -representations of its enveloping C^* -algebra, $C^*(G, \Omega)$). In the other direction, let L be a $*$ -representation of $C^*(G, \Omega)$. Then L has a unique extension to $M(C^*(G, \Omega))$, which we also denote by L , and

$$V_L(s) = L(R_G(s)), \quad M_L(\phi) = L(R_\Omega(\phi)).$$

Again motivated by Green's work, we choose to discuss the continuity properties of inducing and of restricting representations in terms of maps between spaces of ideals. If \mathcal{Q} is a C^* -algebra, we give the space $\mathcal{I}(\mathcal{Q})$ of closed two-sided ideals of \mathcal{Q} the topology having as a subbase for its open sets the family $\{\mathcal{O}_J\}_{J \in \mathcal{I}(\mathcal{Q})}$, where $\mathcal{O}_J = \{I \in \mathcal{I}(\mathcal{Q}) : I \not\perp J\}$. Alternately, we may identify an ideal in \mathcal{Q} with a closed subset of $\hat{\mathcal{Q}}$ in the usual way. The topology on $\mathcal{I}(\mathcal{Q})$ can then be described in terms of a subbase for the topology on $\mathcal{X}(\hat{\mathcal{Q}})$, the closed subsets of $\hat{\mathcal{Q}}$. This subbase is the family of $U(\mathcal{O}) = \{F \in \mathcal{X}(\hat{\mathcal{Q}}) : F \cap \mathcal{O} \neq \emptyset\}$ where \mathcal{O} runs over all open subsets of $\hat{\mathcal{Q}}$. The reader can now see our topology is essentially Fell's "inner hull-kernel" topology (cf. [11]). The following observation will be of great importance in §5.

LEMMA 2.4. *Let $\{I_\alpha\}_{\alpha \in \Lambda}$ be a net of ideals in $\mathcal{I}(\mathcal{Q})$ converging to I . Suppose also that I_α corresponds to $F_\alpha \in \mathcal{X}(\hat{\mathcal{Q}})$. Then, given any $P \in F$, there is a subnet, $\{I_\beta\}_{\beta \in \Lambda'}$, such that there are $P_\beta \in F_\beta$ with $\{P_\beta\}_{\beta \in \Lambda'}$ converging to P in $\text{Prim } \mathcal{Q}$.*

PROOF. By the above remarks, we may assume the F_α converge to F in $\mathcal{X}(\hat{\mathcal{Q}})$. Let U be any neighborhood of P . Then $\{C \in \mathcal{X}(\hat{\mathcal{Q}}) : C \cap U \neq \emptyset\}$ is an open neighborhood of F . Therefore, given any $\alpha_0 \in \Lambda$ there is an $\alpha \geq \alpha_0$ such that $F_\alpha \cap U \neq \emptyset$. Now let $\Lambda' = \{(U, \alpha) : U \text{ is a neighborhood of } F, \alpha \in \Lambda, \text{ and } F_\alpha \cap U \neq \emptyset\}$, and pick $P_{(U, \alpha)} \in F_\alpha \cap U$; this will suffice. Q.E.D.

DEFINITION 2.5. Let D' and F be C^* -algebras and D an ideal of D' . Suppose P is a $*$ -homomorphism of F into D' . Define

(a) $P_* : \mathcal{I}(F) \rightarrow \mathcal{I}(D)$ by

$$P_*(J) = \text{ideal generated by } \{P(f)d : f \in J, d \in D\}.$$

(b) $P^* : \mathcal{I}(D) \rightarrow \mathcal{I}(F)$ by

$$P^*(I) = \{f \in F : P(f) \cdot D \subseteq I\}.$$

Note that $P^*(I)$ is an ideal of F because I is an ideal of D' . The essential properties we need are given by the next lemma.

LEMMA 2.6 (GREEN). (i) P^* is continuous from $\mathcal{I}(D) \rightarrow \mathcal{I}(F)$.

(ii) P^* preserves arbitrary intersections while P_* preserves arbitrary unions (the union of a family of ideals being the ideal they generate).

PROOF. This is part of [19, Proposition 9(i)]. Q.E.D.

Let H be a closed subgroup of G .

DEFINITION 2.7. If L is a $*$ -representation of $C^*(G, \Omega)$, let $\text{Res}_H^G(L)$ denote the $*$ -representation of $C^*(H, \Omega)$ corresponding to $(V_{L|_H}, M_L)$.

Let $R_H = R_{G|_H}$ and note that $R = (R_H, R_\Omega)$ is a covariant representation of (H, Ω) on $C^*(G, \Omega)$. The integrated form of R gives a $*$ -homomorphism of $L^1(H, C_0(\Omega))$ into $M(C^*(G, \Omega))$. Since the latter is a C^* -algebra, the homomorphism "lifts" to a $*$ -homomorphism of $C^*(H, \Omega)$ into $M(C^*(G, \Omega))$. Call this homomorphism \mathcal{L} .

DEFINITION 2.8. Define $\text{Res}_H^G : \mathcal{I}(C^*(G, \Omega)) \rightarrow \mathcal{I}(C^*(H, \Omega))$ by $\text{Res}_H^G = \mathcal{L}^*$. When $H = e$ we let $\text{Res} = \text{Res}_e^G : \mathcal{I}(C^*(G, \Omega)) \rightarrow \mathcal{I}(C_0(\Omega))$.

Our ambiguous notation is justified by the next lemma.

LEMMA 2.9 (GREEN). *If L is a representation of $C^*(G, \Omega)$, then $\ker \text{Res}_H^G(L) = \text{Res}_H^G(\ker L)$.*

PROOF. This is [19, Proposition 9(ii)]. Q.E.D.

Let Γ be the integrated version of the homomorphism R_G of G into $M(C^*(G, \Omega))$.

LEMMA 2.10. *If $L = (V_L, M_L)$ is a representation of $C^*(G, \Omega)$, then $\Gamma^*(\ker L) = \ker V_L$. In particular, if $N = (V_N, M_N)$ is another representation and $\ker L = \ker N$, then $\ker V_L = \ker V_N$.*

PROOF. The second statement follows from the first. Moreover, $\Gamma^*(\ker L) = \{f \in C^*(G, \Omega) : \Gamma(f)C^*(G, \Omega) \subseteq \ker L\}$. Since $L(\Gamma(f)g) = V_L(f)L(g)$ and L is non-degenerate, the desired result follows. Q.E.D.

By the orbit space Ω/G we mean the quotient topological space obtained from Ω by identifying all the points in the same orbit. In many cases Ω/G will not even be T_0 . When it is necessary to work with T_0 spaces, we follow [9] and make the following definition:

DEFINITION 2.11. If X is a topological space denote the quotient topological space obtained from X by identifying points with identical closures by X^\sim . X^\sim is called the T_0 -ization of X . Notice that X^\sim is always T_0 and that $X^\sim = X$ if X is already T_0 .

Recall that $S_x = \{r \in G : r \cdot x = x\}$ is called the stability group at x . Let Σ denote the space of closed subgroups of G endowed with the compact Hausdorff topology introduced by Fell [11]. Also, suppose that f_0 is a nonnegative, real-valued function in $C_c(G)$ which does not vanish at the identity. For the remainder of this paper, let α_H be the left Haar measure on H defined by

$$\int_H f_0(t) d\alpha_H(t) = 1.$$

Such a choice is called a continuous choice of Haar measures and has the property that $H \rightarrow \int_H f d\alpha_H$ is continuous for each $f \in C_c(G)$ [14, p. 908].

Also, Let Δ_H be the modular function on H and let $Y = \{(H, t) \in \Sigma \times G : t \in H\}$. Notice that Y is closed in $\Sigma \times G$. The proof of the next lemma is routine, so we omit it.

LEMMA 2.12. (i) *Suppose $\{f_\gamma\} \subseteq C_c(G)$ converges to f in the inductive limit topology and $H_\gamma \rightarrow H$ in Σ . Then $\int_{H_\gamma} f_\gamma d\alpha_{H_\gamma}$ converges to $\int_H f d\alpha_H$.*

(ii) $\Delta_S(t)$ is continuous on $Y \subseteq \Sigma \times G$.

(iii) Let $F \in C_c(Y)$, then

$$H \mapsto \int_H F(H, t) d\alpha_H(t)$$

is a continuous function on Σ .

The reader should be aware that the natural map of Ω to Σ defined by $x \mapsto S_x$ is not continuous in general. It is precisely this difficulty that we will need to overcome in many of our results. This leads us to make the following definition.

DEFINITION 2.13. By a point of discontinuity of Ω we mean a point $y \in \Omega$ where the map $x \mapsto S_x$ fails to be continuous.

We pause to give two results of Glimm which are related to the above. When G is a Lie group, Ω is second countable, and the action is smooth, then there is an open dense subset of Ω on which the map $x \mapsto S_x$ is continuous [13, Theorem 3]. In the same paper Glimm also gives an example where $x \mapsto S_x$ is continuous on no open set.

Suppose that H is a closed subgroup of G . In the following sections, it will be convenient to make use of the various methods of inducing representations from $C^*(H, \Omega)$ to $C^*(G, \Omega)$. For future reference, we outline the basic definitions here.

In [19] Phil Green described a method for inducing representations of $C^*(H, \Omega)$ to $C^*(G, \Omega)$ using the techniques developed by Marc Rieffel in [23]. First, Green defines $C^*(G, \Omega)$ in terms of a symmetric Haar measure, $\Delta_G(s)^{-1/2}d\alpha_G(s)$. To avoid confusion, we remark that his definitions are mapped onto ours by $f \mapsto \Delta_G^{-1/2}f$. Thus, the $C_c(H, \Omega)$ -valued inner product on the imprimitivity algebra, $C_c(G, \Omega)$, using our definitions is

$$\langle f, g \rangle_{B_0}(t, x) = \gamma_H(t) \int_G f(s, s \cdot x)^- g(st, s \cdot x) d\alpha_G(s), \tag{1}$$

where $\gamma_H(t) = \Delta_G(t)^{1/2} \Delta_H(t)^{-1/2}$ [19, p. 200]. Recall that if $L = (\pi, \rho)$ is a representation of $C^*(H, \Omega)$ on V_L , then the induced representation is denoted by $\text{Ind}_H^G(L)$ or $\text{Ind}_H^G(\pi, \rho)$, and acts on the completion of $C_c(G, \Omega) \otimes_{B_0} V_L$ with respect to the inner product defined by

$$\langle f \otimes \xi, g \otimes \eta \rangle = \langle L(\langle g, f \rangle_{B_0})\xi, \eta \rangle_L. \tag{2}$$

The action of $h \in C^*(G, \Omega)$ on the class of $f \otimes \xi$ is given by $(h * f) \otimes \xi$ ([19, p. 204] and [23, Theorem 5.2]).

Notice that if Ω is a single point and if π is a unitary representation of H , then the above construction yields the representation of G induced from π on H , $\text{Ind}_H^G(\pi)$, as defined in §4 of [23]. In this case, the $C_c(H)$ valued inner product on $C_c(G)$ is denoted by simply $\langle \cdot, \cdot \rangle_H$, and $\langle f, g \rangle_H(t) = \gamma_H(t)g^* * f(t)$.

Of course, using techniques developed by Mackey and extended to the nonseparable case by Blattner [2], it is possible to realize the space of the induced representation as a set of V_L -valued functions on G . When H is normal in G , this set has a particularly easy description which it will be convenient to use in §5.

Let μ be a Haar measure on G/H such that

$$\int_G f(s) d\alpha_G(s) = \int_{G/H} \int_H f(st) d\alpha_H(t) d\mu_s.$$

Consider the subspace, \mathfrak{F} , of the continuous V_L -valued functions f on G , which have compact support modulo H (i.e. the support of F is contained in CH for C compact in G) and which satisfy

$$f(st) = \pi(t^{-1})f(s).$$

Note that $\|f(s)\|$ may be viewed as a continuous function with compact support on G/H . More generally, given f and g in \mathfrak{F} , the function $s \rightarrow \langle f(s), g(s) \rangle_{V_L}$ is an

element of $C_c(G/H)$ and we can define an inner product by

$$\langle f, g \rangle = \int_{G/H} \langle f(s), g(s) \rangle_{V_L} d\mu_s. \tag{3}$$

Let \mathfrak{H} denote the completion of \mathfrak{F} with respect to the above inner product. Notice that \mathfrak{F} may be viewed as a subset of the Hilbert space, V^L , defined in [2] as the space of the representation of G induced by the unitary part of L (since $\Delta_H(t) = \Delta_G(t)$ for H normal and a moment's reflection shows that the norms agree). But, [2, Lemma 2] implies \mathfrak{F} is total in V^L ; thus \mathfrak{H} and V^L coincide.

LEMMA 2.14. *Let $L = (\pi, \rho)$ be a representation of $C^*(H, \Omega)$. Then the map U , defined on elementary tensors by*

$$U(f \otimes v)(r) = \int_H \rho(f(rt, r \cdot)) \pi(t)v d\alpha_H(t),$$

defines a unitary map from the completion of $C_c(G, \Omega) \otimes V_L$ onto \mathfrak{H} . In particular, the unitary part of $\text{Ind}_H^G(\pi, \rho)$ is equivalent to $\text{Ind}_H^G(\pi)$.

PROOF. The map U is essentially the map defined in Theorem 5.12 of [23], and straightforward computations show that it has the required properties. The last statement may be verified using Proposition 2.3 and [23, Theorem 5.12]. Q.E.D.

3. Induced representations of $C^*(G, \Omega)$. In this section we state the first of our main results. We also prove the easier half of this result and introduce the constructions needed to prove the second half.

In [19], Phil Green defines a covariance algebra to be quasi-regular if every primitive ideal lives on a quasi-orbit (cf. [19, p. 221]). In the transformation group C^* -algebra case, quasi-regularity means that, for every $P \in \text{Prim } C^*(G, \Omega)$, $\text{hull}(\text{Res}_e^G(P)) = \overline{G \cdot x}$ for some $x \in \Omega$. In [9], Effros and Hahn show that if G and Ω are second countable then $C^*(G, \Omega)$ is quasi-regular. More generally, Green shows $C^*(G, \Omega)$ is quasi-regular whenever $(\Omega/G)^\sim$ is second countable or almost Hausdorff [19, Corollary 19]. It is not known if every covariance algebra is quasi-regular. The question is closely related to the question of whether every prime ideal in a C^* -algebra is primitive (cf. remark on p. 223 of [19]).

It will be for the class of quasi-regular algebras that we can prove many of our results. However, our results concerning C.C.R. transformation group C^* -algebras can be stated much more succinctly in the second countable case. The results in the nonseparable case are stated in Proposition 3.2 and Proposition 4.17.

THEOREM 3.1. *Let G and Ω be second countable and suppose the stability group at every point of discontinuity of Ω is amenable. Then $C^*(G, \Omega)$ is C.C.R. if and only if Ω/G is T_1 and every stability group is C.C.R.*

The first part of Theorem 3.1 follows from the next proposition.

PROPOSITION 3.2. *Suppose that Ω/G is T_1 . If G and Ω are not second countable we also assume that $C^*(G, \Omega)$ is quasi-regular and that the natural map of G/S_x onto $G \cdot x$ is a homeomorphism for each $x \in \Omega$. Then $C^*(G, \Omega)$ is C.C.R. if and only if every stability group is C.C.R.*

PROOF. Notice that in the second countable case, all the additional hypotheses are automatically satisfied. We have already pointed out that $C^*(G, \Omega)$ is quasi-regular in this case and it follows from a result of Glimm's [13], that $\Omega/G \cong T_0$ implies that the natural map of G/S_x onto $G \cdot x$ is a homeomorphism in the second countable case.

Thus, if $L = (V, M)$ is an irreducible representation of $C^*(G, \Omega)$, then $\text{Res}(\ker L) = I_{G \cdot x}$ for some x . Here $I_{G \cdot x}$ means the ideal of functions in $C_0(\Omega)$ vanishing on $G \cdot x$. But by Lemma 2.9, $\text{Res}(\ker L) = \ker \text{Res } L = \ker M$. By [17, Lemma 1] we may view $C^*(G, G \cdot x)$ as a factor algebra of $C^*(G, \Omega)$ and L as a representation of the former algebra. It follows that $C^*(G, \Omega)$ is C.C.R. if and only if each $C^*(G, G \cdot x)$ is C.C.R. However, since the homeomorphism $G/S_x \rightarrow G \cdot x$ is obviously G -equivariant, it is easy to see that $C^*(G, G \cdot x)$ is isomorphic to $C^*(G, G/S_x)$. Moreover, $C^*(G, G/S_x)$ is (strongly) Morita equivalent to $C^*(S_x)$ (cf. [24]). In particular, $C^*(G, G/S_x)$ is C.C.R. if and only if $C^*(S_x)$ is C.C.R. (see for example [23, Corollary 6.24]). Q.E.D.

We remark that the hypothesis that $G/S_x \rightarrow G \cdot x$ be a homeomorphism is essential in the nonseparable case. Let \mathbf{R}_d be the real numbers with the discrete topology and let \mathbf{R}_d act on \mathbf{R} , with the usual topology, by left translation. Form $C^*(\mathbf{R}_d, \mathbf{R})$ and the representation $\pi = (\lambda, \rho)$ on $L^2(\mathbf{R})$ where

$$\lambda(s)f(r) = f(r - s), \quad \rho(\phi)f(r) = \phi(r)f(r).$$

Note that π is irreducible. This follows because the analogous representation of $C^*(\mathbf{R}, \mathbf{R})$ is irreducible (cf. [26]) and because any operator which commutes with $\lambda(s)$ for all $s \in \mathbf{R}_d$ commutes with $\lambda(g)$ for any $g \in C^*(\mathbf{R})$. However, the range of π is not just the compact operators. In fact, let ϕ be any nonzero function in $C_0(\mathbf{R})$. Define $f \in C_c(\mathbf{R}_d, \Omega)$ by

$$f(r, x) = \begin{cases} 0, & r \neq 0, \\ \phi(x), & r = 0. \end{cases}$$

Then a simple computation shows that $\pi(f) = \rho(\phi)$. Since $\rho(\phi)$ does not have discrete spectrum it cannot be compact.

The method of attack in proving the other direction of Theorem 3.1 is as follows. Under suitable hypotheses, we will produce a continuous map $\pi: \text{Prim } C^*(G, \Omega) \rightarrow \Omega/G$ and a cross section. Effros and Hahn [9] have shown this cross section is continuous when all the stability groups are abelian. Thus, Ω/G is identified with a subset of $\text{Prim } C^*(G, \Omega)$ and must be T_1 when $C^*(G, \Omega)$ is C.C.R. The remainder of the proof of Theorem 3.1 involves generalizing the above to require only that the stability groups be amenable at points of discontinuity of Ω . Since the definition of the cross section involves the notion of the representation of $C^*(G, \Omega)$ induced from a stability group, it will be necessary to make that notion precise and to investigate the continuity properties. It is to that task that we now turn while postponing the rest of the proof of 3.1 to §4.

Suppose that H is a closed subgroup of G , and that $H \subseteq S_x$ for some $x \in \Omega$. Then, if ω is a unitary representation of H on V_ω , there is a natural covariant

representation of $C^*(H, \Omega)$, (ω, ρ_x) , where ρ_x is defined by $\rho_x(\phi)v = \phi(x)v$. Thus, each representation of H is associated with an induced representation of $C^*(G, \Omega)$, $\text{Ind}_H^G(\omega, \rho_x)$.

It will be convenient to realize the above representation on the space of $\text{Ind}_H^G(\omega)$, that is, on the completion of $C_c(G) \otimes_H V$ with respect to the inner product described in equation (1) of §2. Towards this end, if $f \otimes v \in C_c(G, \Omega) \otimes V$, then define $U(f \otimes v) = \psi(f) \otimes v$, where $\psi(f)(s) = f(s, s \cdot x)$. The next proposition follows from routine calculations.

PROPOSITION 3.3. *Suppose that $H \in \Sigma$, and that $H \subseteq S_x$ for some $x \in \Omega$. Then the map U defined above extends to a unitary map from the space of $\text{Ind}_H^G(\omega, \rho_x)$ to the space of $\text{Ind}_H^G(\omega)$. Moreover, U intertwines $\text{Ind}_H^G(\omega, \rho_x)$ with the covariant representation $(\text{Ind}_H^G(\omega), \rho)$ where*

$$\rho(\phi)(f \otimes \xi) = \phi_x \cdot f \otimes \xi \quad \text{and} \quad \phi_x \cdot f(s) = \phi(s \cdot x)f(s).$$

DEFINITION 3.4. Let $\text{Ind}_{(x,H)}^G(\omega)$ denote the representation $(\text{Ind}_H^G(\omega), \rho)$ defined above. When $H = S_x$, we call $\text{Ind}_{(x,H)}^G(\omega)$ the representation of $C^*(G, \Omega)$ induced from ω on the stability group at x .

REMARK. Adopting the point of view of [23], one may view $\omega \mapsto \text{Ind}_H^G(\omega, \rho_x)$ and $\omega \mapsto \text{Ind}_{(x,H)}^G(\omega)$ as functors from the category of Hermitian $C^*(H)$ -modules to Hermitian $C^*(G, \Omega)$ -modules. These functors are naturally unitarily equivalent.

LEMMA 3.5 (GREEN). *There is a continuous map, Ind_H^G , from $\mathcal{G}(C^*(H, \Omega))$ to $\mathcal{G}(C^*(G, \Omega))$ such that, if L is a representation of $C^*(H, \Omega)$, then $\text{Ind}_H^G(\ker L) = \ker(\text{Ind}_H^G(L))$.*

PROOF. The proof is due to Green [19, Proposition 9]. We include it for completeness. Let \mathfrak{E} denote the transformation group C^* -algebra

$$C^*(G, G/H \times \Omega),$$

where the G -action is the diagonal one; $s \cdot (rH, x) = (srH, s \cdot x)$. Thus, \mathfrak{E} is Morita equivalent to $C^*(H, \Omega)$ [19, Proposition 3]. Recall from [25, Theorem 3.2] that Morita equivalent C^* -algebras have isomorphic lattices of ideals, and let $I^\mathfrak{E}$ denote the ideal of \mathfrak{E} corresponding to $I \in \mathcal{G}(C^*(H, \Omega))$.

Let P denote the natural homomorphism of $C^*(G, \Omega)$ into $M(\mathfrak{E})$ (i.e. the integrated form of the homomorphisms of G and $C_0(\Omega)$ into $M(\mathfrak{E})$). It is not difficult to see that $\text{Ind}_H^G(L)$ is the composition of P with the canonical extension to $M(\mathfrak{E})$ of the representation of \mathfrak{E} induced from L via the imprimitivity bimodule $C_c(G, \Omega)$ (cf. [19, Proposition 3 and following remarks]). Since the kernel of the representation of \mathfrak{E} induced from L is $(\ker L)^\mathfrak{E}$ [25, Proposition 3.7], it follows that $\ker(\text{Ind}_H^G(L)) = \{f \in C^*(G, \Omega) : P(f) \cdot \mathfrak{E} \subseteq (\ker L)^\mathfrak{E}\}$.

Thus, by Lemma 2.6, it will suffice to define $\text{Ind}_H^G(I) = P^*(I^\mathfrak{E})$. Q.E.D.

4. C.C.R. transformation group C^* -algebras. It is well known that the representation induced from a stability group is irreducible. The original proof in the second countable case goes back to Mackey ([21, §6]; see also [14, pp. 900–901]). The proof of this fact given in Proposition 4.2 is for general transformation groups and

is based on an idea of Marc Rieffel's and some helpful suggestions from William Arveson. However, first we need to establish some notation and prove a preliminary lemma.

For the moment fix $x \in \Omega$ and consider the transformation group $(G, G/S_x)$. Recall from §1 that we have homomorphisms of $C_0(G/S_x)$ and G into $M(C^*(G, G/S_x))$, which we denote by R_x and R_G . Let the extension of R_x to $BC(G/S_x)$, the bounded continuous functions on G/S_x , also be denoted by R_x . Since the natural map of G/S_x into Ω defined by $rS_x \rightarrow r \cdot x$ is continuous, we obtain a homomorphism of $C_0(\Omega)$ into $BC(G/S_x)$, and hence, into $M(C^*(G, G/S_x))$. Denote this map by R_1 . It is easy to see that (R_G, R_1) is a covariant representation of (G, Ω) on $C^*(G, G/S_x)$. The integrated version of (R_G, R_1) gives a *-homomorphism of $C^*(G, \Omega)$ into $M(C^*(G, G/S_x))$, which we will denote by R^x .

LEMMA 4.1. *Suppose $x \in \Omega$ and ω is a unitary representation of S_x . Then if $L = \text{Ind}_{(x, S_x)}^G(\omega)$ we have the commutative diagram:*

$$\begin{array}{ccc} C^*(G, \Omega) & \xrightarrow{R^x} & M(C^*(G, G/S_x)) \\ L \searrow & & \swarrow U^\omega \\ & B(V) & \end{array}$$

Here, U^ω is the representation of $C^*(G, G/S_x)$ induced from ω , V is the space of U^ω defined in §2, and $B(V)$ is the algebra of bounded operators on V .

PROOF. Recall that U acts on the completion of $C_c(G) \otimes_H V_\omega$ with respect to the inner product defined in equation (1); thus our diagram at least makes sense. Moreover, U^ω is the integrated version of (π, M) , where

$$\begin{aligned} \pi(s)(f \otimes \xi) &= \lambda(s)(f) \otimes \xi, & s \in G, \\ M(\phi)(f \otimes \xi) &= M_1(\phi)(f) \otimes \xi, & \phi \in C_0(G/S_x), \end{aligned}$$

and $\lambda(s)f(r) = f(s^{-1}r)$ while $M_1(\phi)f(r) = \phi(rS_x)f(r)$. Let $L_1 = U^\omega \circ R^x$ be the representation of $C^*(G, \Omega)$ and R_G^1 and R_Ω the homomorphisms of G and $C_0(\Omega)$ into $M(C^*(G, \Omega))$. Then $V_{L_1}(s) = L_1(R_G^1(s)) = U^\omega(R_G(s)) = \pi(s) = V_L(s)$, while $M_{L_1}(\phi) = L_1(R_\Omega(\phi)) = U^\omega(R_1(\phi)) = M(\tilde{\phi})$, where $\tilde{\phi}$ is the function in $BC(G/S_x)$ defined by $\tilde{\phi}(rS_x) = \phi(r \cdot x)$. Thus, $M_{L_1}(\phi) = M_L(\phi)$. Q.E.D.

PROPOSITION 4.2. *If ω is an irreducible representation of S_x , then $L = \text{Ind}_{(x, S_x)}^G(\omega)$ is also an irreducible representation of $C^*(G, \Omega)$.*

PROOF. Let $U^\omega = (\pi, M)$ be as in Lemma 4.1. Since $C^*(S_x)$ and $C^*(G, G/S_x)$ are Morita equivalent, U^ω is irreducible. Thus, the only operators in $B(V)$ commuting with $\{\pi(s), M(\phi) : s \in G, \phi \in C_0(G/S_x)\}$ are the scalars. Since $\pi(s) = V_L(s)$ for each $s \in G$, it suffices by Lemma 4.1 to show that for each $\phi \in C_0(G/S_x)$, $M(\phi)$ can be weakly approximated by operators of the form $U^\omega(R^x(\psi))$ for $\psi \in C_0(X)$. We may also assume that ϕ is real valued.

Let C be a compact subset of G/S_x . Notice that the map of $G/S_x \rightarrow \Omega$ restricts to a homeomorphism of C onto its compact image. Thus, if we fix $\phi \in C_0(G/S_x)$, we may construct via the Tietze extension theorem, a function ψ_C in $C_c(X)$ whose

image in $BC(G/S_x)$ agrees with ϕ on C and such that $\|\psi_C\|_\infty < \|\phi\|_\infty$. Thus, the $\{U^\omega(R^x(\psi_C))\}$ form a net in $B(V)$, indexed by increasing C , which we claim converges weakly to $M(\phi)$. Since our net is bounded, it suffices to show the convergence on a dense subset, and in particular, on elements of the form $f \otimes \xi$ in $C_c(G) \otimes_H V_\omega$.

Let $f \otimes \xi$ and $g \otimes \eta$ be in $C_c(G) \otimes_H V$ with the support of $g \subseteq K$. Then, if $C \supseteq K$,

$$\begin{aligned} &\langle U^\omega(R^x(\psi_C))(f \otimes \xi), g \otimes \eta \rangle_\omega - \langle M(\phi)(f \otimes \xi), g \otimes \eta \rangle_\omega \\ &= \int_H \gamma_H(t) \int_G g^*(s)(\psi_C(s^{-1} \cdot x) - \phi(s^{-1}S_x))f(s^{-1}t) d\alpha_G(s)(\omega(t)\xi, \eta)_{V_\omega} d\alpha_H(t) \\ &= \int_H \gamma_H(t) \int_G \overline{g(s)} (\psi_C(s \cdot x) - \phi(sS_x))f(st) d\alpha_G(s)(\omega(t)\chi, \eta)_{V_\omega} d\alpha_H(t) \\ &= 0, \end{aligned}$$

since $\psi_C(s \cdot x) = \phi(sS_x)$ on the support of g . Q.E.D.

For quasi-regular algebras, we may extend slightly a definition of [9, p. 62] and make the following definition.

DEFINITION 4.3. If $C^*(G, \Omega)$ is quasi-regular, then we define π from $\text{Prim } C^*(G, \Omega)$, to $(\Omega/G)^\sim$ by $\pi(P) = \text{hull}(\text{Res}(P))$.

Note that we identify orbit closures with the corresponding quasi-orbit (i.e. equivalence class) in $(\Omega/G)^\sim$. The fact that this identification is appropriate is demonstrated by the following lemma of Green's [19, p. 221].

LEMMA 4.4. For $x \in \Omega$, let $[x]$ denote the class of x in $(\Omega/G)^\sim$. The map from $(\Omega/G)^\sim$ to $\mathcal{X}(\Omega)$ defined by $[x] \rightarrow \overline{G \cdot x}$ is a homeomorphism onto its image.

Notation. If χ is a character of G and $H \in \Sigma$, then χ_H will denote the restriction of χ to H .

LEMMA 4.5. Suppose $C^*(G, \Omega)$ is quasi-regular:

- (1) π is continuous and surjective.
- (2) If ω is a representation of S_x , then

$$\text{hull}(\text{Res}(\ker(\text{Ind}_{(x,S_x)}^G(\omega)))) = \overline{G \cdot x}.$$

- (3) If χ is a character of G , $\pi(\ker(\text{Ind}_{(x,S_x)}^G(\chi_{S_x}))) = \overline{G \cdot x}$.

PROOF. In the second countable case, (1) follows from Lemma 5.5 and Corollary 5.10 of [9], and (2) is a variant of Mackey's induction-restriction theorem (see [12, Theorem 3.3 and §4] for the group case and [10, Proposition 2.1] for covariance algebra case).

In general, continuity follows from the continuity of Res (Lemma 2.6) and the previous lemma. Surjectivity will follow from (3), and (3) will follow from (2) (recall, $\ker(\text{Ind}_{(x,S_x)}^G(\chi_{S_x}))$ is primitive by Proposition 4.2).

For the proof of (2), let $L = \text{Ind}_{(x,S_x)}^G(\omega)$. By Lemma 4.1, we may write $L = U^\omega \circ R^x$. We identify $C^*(G, \Omega \setminus \overline{G \cdot x})$ with an ideal of $C^*(G, \Omega)$ via [17, Lemma 1]. Then $R^x(C^*(G, \Omega \setminus \overline{G \cdot x})) = 0$. In particular, $\ker(L)$ contains $C^*(G, \Omega \setminus \overline{G \cdot x})$.

Then, by Lemma 2.9, we have $\text{Res}(\ker L) = \ker M_L \supseteq \text{Res}(C^*(G, \Omega \setminus \overline{G \cdot x})) \supseteq \{\phi \in C_0(\Omega) : \phi \text{ vanishes on } \overline{G \cdot x}\}$.

Let $U^\omega = (\pi, M)$ and $\xi = s \cdot x$ for some $s \in G$. As we showed in Lemma 4.1, if $\phi \in C_0(\Omega)$ then $M_L(\phi) = M(\tilde{\phi})$ where $\tilde{\phi}$ is the appropriate function in $BC(G/S_x)$. It is clear that $M(\psi) \neq 0$ if $\psi \neq 0$ in $BC(G/S_x)$. Thus, if $\phi(\xi) \neq 0$ then $M_L(\phi) \neq 0$. In particular, $\ker M_L$ is contained in $\{\phi \in C_0(\Omega) : \phi(g \cdot x) = 0 \ \forall g \in G\}$. Combining this with the above, we see that $\ker(M_L)$ is the ideal of functions in $C_0(\Omega)$ vanishing on $\overline{G \cdot x}$. Q.E.D.

The following lemma will be a fundamental tool in the rest of this paper.

LEMMA 4.6. *Let $x \in \Omega$ and $C \subseteq H \subset S_x$. Suppose that H is amenable and that π is an irreducible unitary representation of H . Then $\text{Ind}_{(x,C)}^G(\pi|_C)$ weakly contains $\text{Ind}_{(x,H)}^G(\pi)$.*

PROOF. By Proposition 3.3

$$\text{Ind}_{(x,C)}^G(\pi|_C) \cong \text{Ind}_C^G(\pi|_C, \rho_x),$$

which, by [19, Proposition 8], is equivalent to

$$\text{Ind}_H^G(\text{Ind}_C^H(\pi|_C, \rho_x)).$$

On the other hand, $\text{Ind}_C^H(\pi|_C, \rho_x)$ is equivalent to $\text{Ind}_{(x,C)}^H(\pi|_C)$. It follows from Proposition 3.3 that the latter representation is the integrated form of $(\text{Ind}_C^H(\pi|_C), \rho_x)$. Thus, if $\eta = \text{Ind}_C^H(\pi|_C)$, it suffices, by Proposition 3.3, to show that $\text{Ind}_{(x,H)}^G(\eta)$ weakly contains $\text{Ind}_{(x,H)}^G(\pi)$. We claim it suffices to show that η weakly contains π . If so, then U^η weakly contains U^π , where U^η and U^π are the corresponding representations of $C^*(G, G/S_x)$. By Lemma 4.1., $\text{Ind}_{(x,H)}^G(\eta) = U^\eta \circ R^x$ while $\text{Ind}_{(x,H)}^G(\omega) = U^\pi \circ R^x$; the sufficiency of our claim follows provided the canonical extension of U^η to $M(C^*(G, G/S_x))$ weakly contains the extension of U^π . But, if $g \in M(C^*(G, G/S_x))$, then $U^\eta(g) = 0$ if and only if $U^\eta(gf) = 0$ for every $f \in C^*(G, G/S_x)$. Thus, $U^\pi(gf) = 0$ for every $f \in C^*(G, G/S_x)$, and $U^\pi(g) = 0$.

When H is amenable, the fact that η weakly contains π is the content of Theorem 5.1 of [20]. Q.E.D.

Let H be a normal subgroup of G . Let $s \in G$ and K be a subgroup of H . Also, set $L = s \cdot Ks^{-1} \subseteq H$. By the uniqueness of Haar measure, there is a continuous homomorphism $\lambda = \lambda_{G,H} : G \rightarrow \mathbf{R}^+$ such that

$$\int_H f(srs^{-1}) d\alpha_H(r) = \lambda(s) \int_H f(r) d\alpha_H \quad \text{for } f \in C_c(H).$$

Similar considerations allow us to define Haar measures on K and L , say μ and ν , such that

$$\int_L f(\xi) d\nu_{(\xi)} = \int_K f(sts^{-1}) d\mu_{(t)} \quad \text{for } f \in C_c(L)$$

and

$$\int_K g(t) d\mu_{(t)} = \int_L g(s^{-1}\xi s) d\nu_{(\xi)} \quad \text{for } g \in C_c(K).$$

Now, let (ω, ρ) be a covariant representation of (K, Ω) . Let $(\omega, \rho)^s$ denote the covariant representation of (L, Ω) given by

$$\omega^s(r) = \omega(s^{-1}rs), \quad \rho^s(\phi) = \rho(s^{-1}\phi).$$

Recall that $s^{-1}\phi(x) = \phi(s \cdot x)$.

We remark that by [19, p. 198], $(G, C^*(H, \Omega))$ forms a covariant system when H is normal in G . The strongly continuous G -action is given on $F \in C_c(H, \Omega)$ by

$${}^s f(h, x) = \lambda_{G,H}(s)f(s^{-1}hs, s^{-1} \cdot x).$$

The following lemma is certainly well known. In the second countable case, a proof appears in [14, Theorem 2.1].

LEMMA 4.7. *If $\pi = \text{Ind}_K^H(\omega, \rho)$ and $\sigma = \text{Ind}_L^H((\omega, \rho)^s)$, then σ is unitarily equivalent to π^s , where $\pi^s(f) = \pi(s^{-1}f)$ (the G -action is described above). Moreover, the class of π^s depends only on the class of s in G/H .*

PROOF. The space of π, V_π , is the completion of $C_c(H, \Omega) \otimes_{B_K} V_\omega$ with respect to the inner product given by

$$\langle f \otimes \xi, g \otimes \eta \rangle_\pi = \int_K (\rho(\langle g, f \rangle_{B_K}(t, \cdot))\omega(t)\xi, \eta)_\omega d\mu_{(t)},$$

where $\langle \cdot, \cdot \rangle_{B_K}$ is the $C^*(K, \Omega)$ -valued inner product defined in [18] (cf. equation (1) of this paper). The space of σ, V_σ , is the completion of $C_c(H, \Omega) \otimes_{B_L} V_\omega$ with respect to the inner product given by

$$\langle f \otimes \xi, g \otimes \eta \rangle_\sigma = \int_L (\rho(\langle g, f \rangle_{B_L}(\zeta, s \cdot))\omega(s^{-1}\zeta s)\xi, \eta)_\omega dv_\zeta,$$

where $\langle \cdot, \cdot \rangle_{B_L}$ is the $C^*(L, \Omega)$ -valued inner product.

Define U from $C_c(H, \Omega) \otimes V_\omega$ to $C_c(H, \Omega) \otimes V_\omega$ by $U(f \otimes \xi) = \tilde{f} \otimes \xi$ with $\tilde{f}(h, x) = \lambda(s)^{-1/2}f(s^{-1}hs, s^{-1} \cdot x)$. Using the definitions of μ and ν , it is not hard to see that U extends to a unitary map of V_π onto V_σ which intertwines the two representations. Another routine computation verifies that for all $r \in G$ and $t \in H$

$$\pi^{rs}(f) = V(t^{-1})\pi^r(f)V(t),$$

where V is the unitary part of π . Q.E.D.

COROLLARY 4.8. *If χ is a character of $G, K \subseteq S_x$, and $s \in G$, then $\text{Ind}_{(x,K)}^G(\chi_K)$ is unitarily equivalent to $\text{Ind}_{(s \cdot x, L)}^G(\chi_L)$ where $L = sKs^{-1}$.*

PROOF. Recall that the stability group at $s \cdot x$ is $sS_x s^{-1}$. Thus, by Proposition 3.3, it suffices to apply the last lemma with $H = G$. Q.E.D.

Let \tilde{G} denote the set of characters of G endowed with the topology of uniform convergence on compact sets. If $\omega \in \tilde{G}$ and $x \in \Omega$ let $\tau(x, \omega)$ denote the representation of $C^*(G, \Omega)$ induced from ω on the stability group at x (i.e. $\text{Ind}_{(x, S_x)}^G(\omega_{S_x})$). We define ϕ from $\Omega \times \tilde{G}$ to $\text{Prim}(C^*(G, \Omega))$ by

$$\phi(x, \omega) = \ker(\tau(x, \omega)).$$

LEMMA 4.9. *If the stability groups are amenable at each point of discontinuity of Ω , then ϕ is continuous.*

PROOF. Let $F = \{I \in \text{Prim}(C(G, \Omega)): I \supseteq J\}$ be an arbitrary closed subset of $\text{Prim}(C^*(G, \Omega))$. Moreover, suppose that $(x_\alpha, \omega_\alpha)$ converges to (x, ω) in $\Omega \times \tilde{G}$ and that $\phi(x_\alpha, \omega_\alpha) \supseteq J$ for each α . We need to show $\phi(x, \omega) \supseteq J$. For convenience we will denote S_{x_α} by S_α and $\tau(x_\alpha, \omega_\alpha)$ by simply τ_α .

Now since Σ is compact, we may assume that $S_\alpha \rightarrow C$. As is shown in [9], $C \subseteq S_x$.

Let $\pi = \text{Ind}_{(x,C)}^G(\omega_C)$. The space of τ_α can be identified with the completion of $C_c(G)$ with respect to the inner product defined in equation (2); namely, for $g, h \in C_c(G)$

$$\begin{aligned} \langle g, h \rangle_\alpha &= \int_{S_\alpha} \langle h, g \rangle_{S_\alpha}(t) \omega_\alpha(t) d\alpha_{S_\alpha}(t) \\ &= \int_{S_\alpha} \gamma_{S_\alpha}(t) h^* * g(t) \omega_\alpha(t) d\alpha_{S_\alpha}(t). \end{aligned}$$

Using Lemma 2.12(ii) we see that the integrand may be viewed as a continuous function on $Y = \{(t, H) \in G \otimes \Sigma: t \in H\}$. Therefore, by part (iii) of that lemma, $\langle g, h \rangle_\alpha$ converges to

$$\int_C \gamma_C(t) h^* * g(t) \omega(t) d\alpha_C(t) = \langle g, h \rangle_\pi.$$

We claim it suffices to show that, for all $F \in C^*(G, \Omega)$ and $f, g \in C_c(G)$,

$$\langle \tau_\alpha(F)(g), f \rangle_\alpha \text{ converges to } \langle \pi(F)(g), h \rangle_\pi. \tag{4}$$

Let $F \in J$. As $\tau_\alpha(F) = 0$ for every α , we have $\pi(F) = 0$ since $C_c(G)$ is dense in the completion. Thus, $J \subseteq \ker \pi$. If x is a point of continuity of Ω , then $C = S_x$. Otherwise the sufficiency of the claim follows from Lemma 4.6.

Also, it suffices to show the above only for F in a dense subset. Since elements of the form $F = f \cdot \phi$, where $f \in C_c(G)$ and $\phi \in C_c(\Omega)$, span a dense subset, we need only show (4) for elements of this form. However, if $\tau_\alpha = (V_\alpha, M_\alpha)$ and $\pi = (V, M)$, then

$$\langle \tau_\alpha(f \cdot \phi)h, g \rangle_\alpha = \langle M_\alpha(\phi)V_\alpha(f)(h), g \rangle_\alpha = \langle M_\alpha(\phi)f * h, g \rangle_\alpha.$$

It follows that it will be enough to show that $\langle M_\alpha(\phi)g, h \rangle_\alpha$ converges to $\langle M(\phi)g, h \rangle_\pi$ for all $g, h \in C_c(G)$. As above, it suffices, by Lemma 2.12, to show that $h^* * M_\alpha(\phi)(g)(t)\gamma_{S_\alpha}(t)\omega_\alpha(t)$ converges to $h^* * M(\phi)(g)(t)\gamma_{S_x}(t)\omega(t)$ in the inductive limit topology. Clearly, we need only show $h^* * M_\alpha(\phi)(g)$ converges to $h^* * M(\phi)(g)$, or more simply, that $M_\alpha(\phi)(g)$ converges to $M(\phi)(g)$ in the inductive limit topology. Since $M_\alpha(\phi)(g)(t) = \phi(t \cdot x_\alpha)g(t)$, the supports are all contained in the support of g . If $M_\alpha(\phi)(g)$ did not converge uniformly to $M(\phi)(g)$, then there would exist t in the support of g such that

$$|\phi(t_\alpha \cdot x_\alpha)g(t_\alpha) - \phi(t_\alpha \cdot x)g(t_\alpha)| \geq \epsilon > 0$$

for every α . Since we may assume the $t_\alpha \rightarrow t$, we get a contradiction. Q.E.D.

Using the above lemma, we obtain a generalization of [9, Theorem 5.11] and [9, Corollary 5.13]. Note that we continue to identify orbit closures and quasi-orbits.

LEMMA 4.10. *Suppose that $C^*(G, \Omega)$ is quasi-regular and that the stability groups are amenable at each point of discontinuity of Ω . If χ is a character of G , then π (see Definition 4.3) admits a continuous cross section, s_χ , with the property that $s_\chi(\overline{G \cdot x}) = \ker(\text{Ind}_{(x, S_x)}^G(\chi_{S_x}))$.*

PROOF. As in [9], we define a map ψ from Ω to $\text{Prim } C^*(G, X)$ by $\psi(x) = \ker(\text{Ind}_{(x, S_x)}^G(\chi_{S_x}))$. ψ is continuous by the previous lemma.

To complete the proof, we need to show that ψ factors through $(\Omega/G)^\sim$. Suppose that $\overline{G \cdot x} = \overline{G \cdot y}$. Then, in particular, there is a net $s_\alpha \cdot x \rightarrow y$ in Ω . By Corollary 4.8, $\psi(s_\alpha \cdot x) = \psi(x)$. Since ψ is continuous, we must have $\psi(y) \subseteq \{\psi(x)\}^-$. We obtain the desired result by symmetry. Q.E.D.

THEOREM 4.11. *If $C^*(G, \Omega)$ is quasi-regular and if the stability group at every point of discontinuity of Ω is amenable, then π admits a continuous cross section.*

PROOF. There is always at least one character of G , namely the trivial one. Now apply the previous lemma. Q.E.D.

In [9] Effros and Hahn conjectured that, when G is amenable, every primitive ideal of a transformation group C^* -algebra is induced from a stability group. To be more precise, we borrow the following definition from Phil Green [19, p. 223].

DEFINITION 4.12. We say that $C^*(G, \Omega)$ is *EH-regular* if

- (a) $C^*(G, \Omega)$ is quasi-regular,
- (b) for every $P \in \text{Prim } C^*(G, \Omega)$, there is a $x \in \Omega$ and an irreducible representation ω of S_x such that $P = \ker \text{Ind}_{S_x}^G(\omega, \rho_x)$.

A number of special cases of the conjecture have been worked out by Effros and Hahn [9], Gootman [16], Green [19], and Sauvageot [27]. Recently, Jon Rosenberg and Elliot Gootman [17] were able to prove the conjecture in the second countable case, for general amenable G . In fact, their theorem holds for covariance algebras. In [19, Proposition 20], Green was also able to show that if each $G \cdot x$ is locally closed and the natural map of G/S_x to $G \cdot x$ is a homeomorphism for each $x \in \Omega$, then $C^*(G, \Omega)$ is *EH-regular* whenever it is quasi-regular. For example, in the second countable case, if $C^*(G, \Omega)$ is type I, then it is *EH-regular* (cf. [8] and [15]).

THEOREM 4.13. *Suppose G is abelian. If G and Ω are not second countable, we assume that $C^*(G, \Omega)$ is *EH-regular*. Then, if $P \in \text{Prim } C^*(G, \Omega)$, there is a continuous cross section for π with P in its range.*

PROOF. By [17] we may assume that $C^*(G, \Omega)$ is *EH-regular* and thus $P = \ker(\text{Ind}_{S_x}^G(\chi, \rho_x))$ where χ is a character of S_x . Let $\tilde{\chi}$ be an extension of χ to G . We complete the proof by applying Lemma 4.10. Q.E.D.

Notice that if $C^*(G, \Omega)$ is C.C.R. and G and Ω are second countable then Ω/G is T_0 by [15]. Thus, $(\Omega/G)^\sim = \Omega/G$ and Theorem 4.11 and Proposition 3.2 imply Theorem 3.1 (see the remarks following Proposition 3.2). If we replace the assumption that (G, Ω) be second countable by assuming that the stability groups are

amenable at points of discontinuity of Ω , then we can get the same result. To prove this, two lemmas are needed.

Let $x \in \Omega$. There is a continuous real-valued function, ρ in G such that $\rho(s) > 0$ for all $s \in G$, and

$$\rho(st) = \Delta_{S_x}(t)\Delta_G(t)^{-1}\rho(s) \quad \text{for all } s \in G, t \in S_x.$$

Then there is a unique quasi-invariant regular measure, μ , on G/S_x such that, for all $f \in C_c(G)$,

$$\int_G f(s)\rho(s)d\alpha_G(s) = \int_{G/S_x} \int_{S_x} f(st)d\alpha_{S_x}(t)d\mu(s). \tag{5}$$

The existence of ρ and μ follows from [3, Théorème 2]. The uniqueness follows from [3, Théorème 1 and Théorème 2]. The following lemma is certainly well known, but we were unable to find a reference in the literature.

LEMMA 4.14. *If μ is a measure on G/S_x such that equation (5) holds and χ is a character of G , then $\pi = \text{Ind}_{(x,S_x)}^G(\chi_{S_x})$ is unitarily equivalent to the representation, $L = (V_x, M_x)$, of $C^*(G, \Omega)$ on $L^2(G/S_x, \mu)$ where*

$$\begin{aligned} V_x(s)f(\dot{r}) &= \sqrt{\frac{\rho(s^{-1}r)}{\rho(r)}} \chi(s)f(s^{-1} \cdot \dot{r}), & s \in G, \\ M_x(\phi)f(\dot{r}) &= \phi(r \cdot x)f(\dot{r}), & \phi \in C_0(\Omega). \end{aligned}$$

PROOF. We may identify the space of π with the completion of $C_c(G)$ with respect to the inner product

$$\langle f, g \rangle_\pi = \chi(\langle g, f \rangle_H).$$

We then define U from $C_c(G)$ into $V_L = L^2(G/S_x, \mu)$ by

$$U(f)(\dot{r}) = \int_{S_x} f(rt)\rho(rt)^{-1/2}\chi(rt)d\alpha_{S_x}(t).$$

Once we show that U maps $C_c(G)$ onto a dense subset of $L^2(G/S_x)$, it is an easy matter to check that U extends to an unitary map of the space of π onto $L^2(G/S_x)$ which provides the desired equivalence. The fact that U is onto follows from the existence of Bruhat approximate cross sections. That is, there is a real-valued, positive, continuous function, b , on G such that

$$\int_{S_x} b(st)d\alpha_{S_x}(t) = 1 \quad \text{for all } s \in G,$$

and the support of b has compact intersection with the saturant of every compact set $C \subseteq G$ (i.e. CS_x). The existence of such functions is shown in [3, Proposition 8], for example. Q.E.D.

The next lemma will provide the essential step in circumventing the separability problems in Proposition 4.16. The lemma and its proof were shown to me by Marc Rieffel. Since the measure theory in arbitrary locally compact spaces can be tricky, we will use a set-up exactly as in [21, §§11 and 12]. We let Y and Z be arbitrary locally compact spaces together with nonnegative linear functionals I and J on

$C_c(Y)$ and $C_c(Z)$ respectively. We let μ and ν be the regular borel measures obtained from the Riesz representation theorem [21, 11.37 and 11.34]. For example, let $Y = G/S_x$ and I the functional defined as follows: for $f \in C_c(G/S_x)$ let $\tilde{f} \in C_c(G)$ be such that $f(\dot{s}) = \int_{S_x} \tilde{f}(st) d\alpha_{S_x}(t)$ and set $I(f) = \int_G \tilde{f}(s) \rho(s) d\alpha_G(s)$. Then, the corresponding measure is the quasi-invariant measure μ defined above.

LEMMA 4.15. *Let X be a locally compact space with $Y, Z, \mu,$ and ν as above. Let i and j be continuous injections of Y and Z into X and let M_1 and M_2 denote the representations of $C_0(X)$ on $L^2(Y, \mu)$ and $L^2(Z, \nu)$ coming from i and j . Then, if $i(Y) \cap j(Z) = \emptyset, M_1$ and M_2 have no equivalent subrepresentations.*

PROOF. We first show that the image of $C_0(X)$ generates $L^\infty(Y, \mu)$ and $L^\infty(Z, \nu)$. To do this, it will suffice to show that we can weakly approximate any element of $C_0(Y) \subseteq L^\infty(Y)$ by elements of the form $M_1(\phi)$ for $\phi \in C_0(x)$ (the proof for $L^\infty(Z, \nu)$ being exactly the same). However, this follows in much the same way as in the proof of Proposition 4.2.

Now, we show that nonzero invariant subspaces, $H,$ of $M_1,$ and hence also of $L^\infty(Y),$ are of the form $H = L^2(\mathcal{E}, \mu)$ for \mathcal{E} measurable (as in Definition 11.28 of [21]). Moreover, \mathcal{E} must contain a nonnull measurable subset of finite measure, and by regularity of $\mu,$ a compact subset of nonzero measure. (For a nonnull measurable subset of a locally compact group without the latter property, see [21, 11.33].)

Let $C_H = \{F \subseteq Y: F \text{ is measurable and } \chi_F \in H\}$. Also, let P be the orthogonal projection on $H.$ If $g \in H,$ we set $\mathcal{E}_n = \{x \in Y: |g(x)| \geq 1/n\}.$ Since $\chi_{\mathcal{E}_n}/g(x) \in L^\infty(Y),$ we have $\chi_{\mathcal{E}_n} \in H.$ Moreover, the Lebesgue dominated convergence theorem implies $g_n = \chi_{\mathcal{E}_n} \cdot g$ converges to g in $L^2(Y).$ Thus, if $F \supseteq \mathcal{E}_n$ then $\|\chi_F \cdot g - g\| \leq \|g_n - g\|.$ Therefore, $\{\chi_F\}_{F \in C_H}$ converges strongly to the identity on H as F increases. On the other hand, if $p \in H^\perp$ then $\chi_F \cdot p = 0$ for all $F \in C_H.$ In fact, if $p \geq 0$ and $\chi_F(x)p(x) \neq 0$ a.e., then $\int_Y \chi_F(x)p(x) d\mu(x) \neq 0$ and $\langle \chi_F, p \rangle \neq 0.$

Thus, if $f \in L^2(Y)$ and we let $f_0 \in H$ and $f_\perp \in H^\perp$ be such that $f = f_0 + f_\perp,$ then

$$\chi_F(f) = \chi_F(f_0 + f_\perp) = \chi_F(f_0) \rightarrow f_0.$$

In particular, $\{\chi_F\}$ converges strongly to $P.$ This implies $P \in L^\infty(Y, \mu),$ and since P is a projection, $P = \chi_{\mathcal{E}}$ for some measurable set $\mathcal{E}.$ Our assertions now follow. Of course, similar results hold for invariant subspaces on $L^2(Z, \nu).$

Finally, we suppose some subrepresentation, say corresponding to $\mathcal{E} \subseteq Y$ is equivalent to a subrepresentation on $L^2(Z, \nu).$ By the remarks above, there is a compact set, $C \subseteq \mathcal{E},$ such that $\mu(C) > 0.$ Thus, the representation on $L^2(C)$ is equivalent to a subrepresentation on $L^2(F)$ for some $F \subseteq Z.$ Let $K \subseteq F$ be compact with $\nu(F) > 0.$ Of course, the representation on $L^2(K)$ is equivalent to some subrepresentation on $L^2(C).$

Since $i(C)$ and $j(K)$ are disjoint compact sets, there is a $f \in C_0(X)$ with f identically one on $i(C)$ and identically zero on $j(K).$ In particular f is the zero

operator on $L^2(K)$ and the identity on $L^2(C)$. This contradiction finishes the proof. Q.E.D.

The next result extends part of a result of Elliot Gootman’s [15, Theorem 3.3] to nonseparable algebras when the stability groups are amenable at points of discontinuity.

PROPOSITION 4.16. *Suppose that $C^*(G, \Omega)$ is G.C.R. and that the stability groups are amenable at points of discontinuity of Ω , then Ω/G is T_0 .*

PROOF. If Ω/G is not T_0 , then there are x and y in Ω such that $G \cdot x \cap G \cdot y = \emptyset$ and $\overline{G \cdot x} = \overline{G \cdot y}$. By Lemma 4.10,

$$\text{Ker}(\text{Ind}_{(x,S_x)}^G(1)) = \text{Ker}(\text{Ind}_{(y,S_y)}^G(1)),$$

where 1 denotes the trivial representation. If the above representations were equivalent, then using the notation of Lemma 4.14, M_x and M_y would be equivalent representations of $C_0(\Omega)$. However, this is impossible by the previous lemma. Therefore, $C^*(G, \Omega)$ is not G.C.R. Q.E.D.

The following proposition now follows immediately from the above and the remarks following Proposition 3.2 and Theorem 4.13.

PROPOSITION 4.17. *Suppose that $C^*(G, \Omega)$ is C.C.R. and that the stability groups are amenable at points of discontinuity of Ω , then the orbits are closed.*

After having worked out a proof of Proposition 4.17, the author learned that Elliot Gootman had worked out such a proof in the separable case several years ago, but had never published it. In fact, the proof given in this paper is simpler than the author’s original proof and was influenced by suggestions of both Gootman and Rieffel.

The reader should notice that the trivial character played no special role in the proof of Propositions 4.16 and 4.17. In fact, if χ is a character of G and either $y \mapsto S_y$ is continuous at x or S_x is amenable, then in order for $\text{Ind}_{(x,S_x)}^G(\chi_{S_x})$ to be a C.C.R. representation, $G \cdot x$ must be closed. Similarly, “local” versions of Theorem 3.1 and Proposition 3.2 follow immediately.

Unfortunately, we do not know whether the hypothesis on the stability groups is a necessary one in either Proposition 4.16 or the above. In fact we are unable to decide the necessity of this hypothesis in Theorem 4.11. However, Lemma 4.6, the essential lemma on which Lemma 4.9 depends, and hence on which all the above results depend, is false in general without an amenability assumption as can be seen in the case where Ω is a single point.

5. The topology of $\text{Prim } C^*(G, \Omega)$. In this section, we will study the structure of the primitive ideal space of a variety of classes of transformation group C^* -algebras. To obtain detailed information we will eventually have to assume that we know all of the primitive ideals in terms of the stability groups and the orbit structure. This is, we will have to assume our algebras are *EH*-regular (cf. Definition 4.12). For the moment, we assume only that $C^*(G, \Omega)$ is quasi-regular.

We start our investigation by assuming that G is abelian. The following lemma must be well known, and since the proof is straightforward, we omit it.

LEMMA 5.1. *Suppose that G is abelian and that $\omega \in \hat{G}$. Then, if we identify $C^*(G)$ with $C_0(\hat{G})$, the kernel of $\pi = \text{Ind}_{\omega_H}^G$ is $\{f \in C_0(\hat{G}): f \text{ vanishes on } \omega H^\perp\}$, where $H^\perp = \{\sigma \in \hat{G}: \sigma(h) = 1 \text{ for } h \in H\}$.*

Now we define a map ψ from $\Omega \times \hat{G}$ to $\text{Prim } C^*(G, \Omega)$ by

$$\psi(x, \omega) = s_\omega(\overline{G \cdot x}),$$

where s_ω is defined in Lemma 4.10.

Suppose $\psi(x, \omega) = \psi(y, \sigma)$. Since $\pi(\psi(x, \omega)) = \pi(\psi(y, \sigma))$, we have $\overline{G \cdot x} = \overline{G \cdot y}$ by Lemma 4.5. Recall from the proof of Lemma 4.10 that $\overline{G \cdot x} = \overline{G \cdot y}$ implies that $\text{Ind}_{(x, S_x)}^G(\sigma_{S_x})$ has the same kernel as $\text{Ind}_{(y, S_y)}^G(\sigma_{S_y})$. Thus, $\psi(x, \omega) = \psi(y, \sigma)$ implies $\psi(x, \omega) = \psi(x, \sigma)$. If $L^\sigma = \text{Ind}_{(x, S_x)}^G(\sigma_{S_x})$, then we have $\ker L^\omega = \ker L^\sigma$. Then if $L^\sigma = (V_\sigma, M_\sigma)$, it follows from Lemma 2.10 that $\ker V_\sigma = \ker V_\omega$. However, by definition, $V_\omega = \text{Ind}_{S_x}^G(\omega_{S_x})$. By Lemma 5.1, we have $\omega S_x^\perp = \sigma S_x^\perp$.

This leads us to make the following definition.

DEFINITION 5.2. Let Λ be the quotient topological space obtained from the product $\Omega \times \hat{G}$ where (x, ω) is identified with (y, σ) if and only if $\overline{G \cdot x} = \overline{G \cdot y}$ and $\omega S_x^\perp = \sigma S_y^\perp$.

The identification makes sense since $\overline{G \cdot x} = \overline{G \cdot y}$ implies $S_x = S_y$ for abelian groups. Notice that Λ may also be viewed as a quotient of $(\Omega/G) \sim \times \hat{G}$.

THEOREM 5.3. *If $C^*(G, \Omega)$ is EH -regular, then ψ factors through Λ and defines a homeomorphism of Λ onto $\text{Prim } C^*(G, \Omega)$.*

PROOF. From the above discussion, ψ factors through Λ and defines a one to one map of Λ into $\text{Prim } C^*(G, \Omega)$. Since $\tilde{G} = \hat{G}$, ψ is continuous by Lemma 4.9.

Notice that ψ is onto by Theorem 4.13. Finally, let F be closed in $\Omega \times \hat{G}$ and saturated with respect to the equivalence relation (Definition 5.2). Now, we need only show that $\psi(F)$ is closed in $\text{Prim } C^*(G, \Omega)$.

Suppose that $\{P_\alpha\} \subseteq \psi(F)$ and let $L^\alpha = \text{Ind}_{(x_\alpha, S_{x_\alpha})}^G(\omega_{\alpha S_{x_\alpha}})$, where $(s_\alpha, \omega_\alpha) \in F$ and $\ker L^\alpha = P_\alpha$. Suppose also that $P_\alpha \rightarrow P$. By virtue of the fact that $C^*(G, \Omega)$ is EH -regular, we may assume that $P = \ker \text{Ind}_{(x, S_x)}^G(\omega_{S_x})$. By Lemma 4.5, $\pi(P_\alpha) \rightarrow \pi(P)$, and therefore, $\overline{G \cdot x_\alpha} \rightarrow \overline{G \cdot x}$ in $(\Omega/G) \sim$. Since the natural map of Ω onto $(\Omega/G) \sim$ is open [19, p. 221], we may assume that there are $y_\alpha \in \Omega$ such that $\overline{G \cdot y_\alpha} = \overline{G \cdot x_\alpha}$ and the y_α converge to x .

On the other hand, Lemmas 2.6 and 2.10 imply that $\Gamma^*(P_\alpha) \rightarrow \Gamma^*(P)$, and hence, that $\ker V_\alpha \rightarrow \ker V$, where V_α is the unitary part of L^α . By Lemma 5.1, $\omega_\alpha S_{x_\alpha}^\perp \rightarrow \omega S_x^\perp$ in $\chi(\hat{G})$. In particular by Lemma 2.4, there is a net $\omega_\beta \sigma_\beta \rightarrow \omega$ in \hat{G} , where $\sigma_\beta \in S_{x_\beta}^\perp$. Thus, $(y_\beta, \omega_\beta \sigma_\beta)$ converges to (x, ω) in $\Omega \times \hat{G}$. Since F is saturated and closed, we have (x, ω) in F . Q.E.D.

We remark that the above proof shows that the natural map of $\Omega \times \hat{G}$ onto Λ is open (since convergent nets in Λ can be "lifted" to $\Omega \times \hat{G}$). In particular, the topology on Λ , and hence the topology on $\text{Prim } C^*(G, \Omega)$, can be easily computed. In fact, a base for the topology on Λ can always be obtained as the forward image of a base for the topology on $\Omega \times \hat{G}$.

For the remainder of this section, we drop the assumption that G be abelian and assume that H is a normal subgroup of G . As was pointed out in the discussion

preceding Lemma 4.7, $(G, C^*(H, \Omega))$ forms a covariant system. Also, if $L = (\pi, M)$ is a representation of $C^*(H, \Omega)$ on V then, for $s \in G$, we defined L^s to be the representation of $C^*(H, \Omega)$ such that $L^s(f) = L^{(s^{-1}f)}$.

Let \mathcal{F} be the space of V -valued functions on G described in §2 and \mathcal{K} the completion with respect to the inner product defined in equation (3). For $f \in C^*(H, \Omega)$ and $F \in \mathcal{F}$, define $R(f)(F)(s) = L^s(f)(F(s))$. Recall from the proof of Lemma 4.7 that $L^{st}(f) = \pi(t^{-1})L^s(f)\pi(t)$. Thus, $R(f)(F)(st) = \pi(t^{-1})R(f)(F)(s)$. In particular, $R(f)(F) \in \mathcal{F}$.

LEMMA 5.4. *R extends to a representation of $C^*(H, \Omega)$ on \mathcal{K} with kernel equal to $\bigcap_{s \in G} \ker(L^s)$.*

PROOF. Since G acts by *-automorphisms on $C^*(H, \Omega)$ and L is a representation of $C^*(H, \Omega)$, to show that R extends to a representation, it suffices to show that R is norm decreasing. But $\|R(f)F\|^2$ is equal to

$$\int_{G/H} \langle L^{(s^{-1}f)}F(s), L^{(s^{-1}f)}F(s) \rangle_V d\mu_s \leq \|f\|^2 \|F\|^2.$$

On the other hand, $R(F) = 0$ if and only if $R(f)(F) = 0$ for every $F \in \mathcal{F}$. Since $s \rightarrow \|R(f)F(s)\|$ is continuous and positive, we see that $R(f) = 0$ if and only if $\|L^s(f)F(s)\| = 0$ for every $s \in G$ and $F \in \mathcal{F}$. That is, $R(f) = 0$ if and only if $L^s(f) = 0$ for every $s \in G$. Q.E.D.

When G and Ω are second countable, the next proposition is a special case of Takesaki’s generalization of Mackey’s subgroup theorem to covariance algebras [29, Theorem 7.1] and a theorem of Fell’s [12, Theorem 3.2]. Since the unitary defined in Lemma 2.14 intertwines $\text{Res}_H^G \text{Ind}_H^G(L)$ and the representation, R , defined above, our proposition follows from the previous lemma.

PROPOSITION 5.5. *Let H be a normal subgroup of G and L a representation of $C^*(H, \Omega)$. Then*

$$\ker(\text{Res}_H^G \text{Ind}_H^G(L)) = \bigcap_{s \in G} L^s.$$

For the remainder of this paper we will assume that both $C^*(H, \Omega)$ and $C^*(G, \Omega)$ are EH -regular. In addition, we assume that all the stability groups of (G, Ω) are contained in H . As a consequence of these assumptions, we may now make the following definition.

DEFINITION 5.6. Let $\Gamma = \text{Prim } C^*(H, \Omega)$ and define ψ from Γ into $\text{Prim } C^*(G, \Omega)$ by $\Psi(P) = \text{Ind}_H^G(P)$.

This definition requires some comment. First, $\text{Ind}_H^G(P)$ is defined in Lemma 3.5. In fact, by that lemma, if L is a representation of $C^*(H, \Omega)$ with kernel P , we have $\text{Ind}_H^G(P) = \ker \text{Ind}_H^G(L)$. Now, since we have assumed that $C^*(H, \Omega)$ is EH -regular, $P = \ker \text{Ind}_{S_x}^H(\omega, \rho_x)$ for some $\omega \in \hat{S}_x$. Thus, $\Psi(P) = \text{Ind}_H^G(\ker \text{Ind}_H^G(\omega, \rho_x))$, which by Lemma 3.5 is equal to $\ker(\text{Ind}_H^G \text{Ind}_{S_x}^H(\omega, \rho_x))$. Thus, by Proposition 8 of [19], $\Psi(P) = \ker \text{Ind}_{S_x}^G(\omega, \rho_x)$, which is primitive by Proposition 4.2.

By [14, Lemma 1.3], (G, Γ) is a topological transformation group via the G -action, ${}^sP = \{{}^s f : f \in P\}$.

DEFINITION 5.7. Let Υ denote $(\Gamma/G)^\sim$, the T_0 -ization of Γ/G (cf. Definition 2.1).

THEOREM 5.8. *Let (G, Ω) be a locally compact transformation group. Suppose that H is normal in G , that all the stability groups are contained in H , and that both $C^*(H, \Omega)$ and $C^*(G, \Omega)$ are EH -regular. Then ψ factors through Υ , and defines a homeomorphism of Υ onto $\text{Prim } C^*(G, \Omega)$.*

PROOF. Ind_H^G is continuous from Γ into $\text{Prim } C^*(G, \Omega)$ by Lemma 2.6. By the above discussion ψ maps Γ into $\text{Prim } C^*(G, \Omega)$, and since $C^*(G, \Omega)$ is EH -regular, ψ is surjective.

Fix $P \in \Gamma$ and let L be a representation of $C^*(H, \Omega)$ with kernel P . By Lemma 4.7, $\text{Ind}_H^G(L^s)$ is equivalent to ${}^s\text{Ind}_H^G(L)$. Again by Lemma 4.7, ${}^s\text{Ind}_H^G(L)$ is equivalent to $\text{Ind}_H^G(L)$. In particular, $\ker(\text{Ind}_H^G(L^s)) = \ker(\text{Ind}_H^G(L))$, or by Lemma 3.5, $\psi(P) = \psi({}^sP)$.

Suppose $\overline{G \cdot J} = \overline{G \cdot P}$. Then there is a net ${}^sJ \rightarrow P$. Since ψ is continuous and constant on G -orbits, $\psi(P) \supseteq \psi(J)$. By symmetry, $\psi(P) = \psi(J)$. Thus, ψ factors through Υ .

Now consider $\text{Res}_H^G \psi(P) = \text{Res}_H^G(\ker \text{Ind}_H^G(L))$. The latter is equal to $\ker(\text{Res}_H^G \text{Ind}_H^G(L))$, which by Proposition 5.5 equals

$$\bigcap_{s \in G} \ker L^s = \bigcap_{s \in G} {}^sP.$$

It follows that the hull($\text{Res}_H^G(\psi(P))$) = $\overline{G \cdot P}$. In particular, $\psi(P) = \psi(J)$ if and only if $\overline{G \cdot P} = \overline{G \cdot J}$. In other words, the map of Υ onto $\text{Prim } C^*(G, \Omega)$ is a continuous bijection.

To complete the proof, we only need to show that $\psi(F)$ is closed in Υ provided that F is a closed G -invariant subset of Γ .

Let $J_\alpha = \psi(P_\alpha)$ for $\{P_\alpha\} \subseteq F$ and suppose that $J_\alpha \rightarrow J$. We may assume that $J = \psi(P)$ for some $P \in \Gamma$. Since Res_H^G is continuous (Lemma 2.6), we have $\text{Res}_H^G(\psi(P_\alpha))$ converging to $\text{Res}_H^G(\psi(P))$ in $\mathcal{G}(C^*(H, \Omega))$. By the remarks preceding Lemma 2.4, we have $\overline{GP_\alpha}$ converging to $\overline{G \cdot P}$ in $\chi(\Gamma)$. By Lemma 2.4, we have subnet $\{\overline{GP_\beta}\}$ and $I_\beta \in \overline{GP_\beta}$ with I_β converging to P . Let U be any neighborhood of P . We eventually have an I_{β_U} in U . Hence, there is a $s_{\beta_U} \in G$ such that ${}^{s_{\beta_U}}P_{\beta_U}$ is in U . In short, ${}^{s_{\beta_U}}P_{\beta_U}$ is a subnet which converges to P ; thus, $P \in F$.

COROLLARY 5.9. *The map ψ from Γ to Υ is open. In fact, if a net $\{\overline{G \cdot P_\alpha}\}$ converges to $\overline{G \cdot P}$ then there is a subnet $\{\overline{G \cdot P_\beta}\}$ and $s_\beta \in G$ such that ${}^{s_\beta}P_\beta \rightarrow P$.*

PROOF. The first assertion follows from the second. The second follows from the fact that Υ is homeomorphic to $\text{Prim } C^*(G, \Omega)$ and the proof of the last theorem. Q.E.D.

We remark that the first assertion is a special case of a result by Phil Green which shows that the natural map of $\text{Prim } A$ onto the quasi-orbit space is open for any covariant system (G, A) (p. 221 of [19]).

Suppose now that all the stability groups are contained in a fixed, but not necessarily normal, subgroup K . Then there is a normal subgroup, H , of G contained in K which contains all the stability groups. Thus, if all the stability

groups are contained in a fixed abelian subgroup, we can assume H is abelian. By Theorem 5.3, Γ is homeomorphic to Λ_H , the quotient of $\Omega \times \hat{H}$ as in Definition 5.2. Recall that the homomorphism is implemented by the map ϕ from $\Omega \times \hat{H}$ to Λ , where ϕ is defined by $\phi(x, \omega) = \ker(\text{Ind}_{S_x}^H(\omega, \rho_x))$. We have $\tau\phi(x, \omega) = \ker(\tau(\text{Ind}_{S_x}^H(\omega, \rho_x)))$ which, by Lemma 4.7 is equal to $\ker(\text{Ind}_{S_{r \cdot x}}^H(\omega', \rho_{r \cdot x})) = \phi(r \cdot x, \omega')$. Thus the G -action on Γ is carried over to the following action on Γ : for $r \in G$, $\tau[x, \omega] = [r \cdot x, \omega']$, where $[x, \omega]$ denotes the class of (x, ω) in Λ_H .

We summarize the above in the following corollary to Theorem 5.8.

COROLLARY 5.10. *Suppose all the stability groups of (G, Ω) are contained in a normal abelian subgroup, H . If $C^*(G, \Omega)$ and $C^*(H, \Omega)$ are EH -regular then $\text{Prim } C^*(G, \Omega)$ is homeomorphic to $(\Lambda_H/G)^\sim$.*

The reader may find the following alternate description of $\text{Prim } C^*(G, \Omega)$ more descriptive. Let Δ be the quotient topological space obtained from $\Omega \times \hat{H}$ by identifying (x, ω) and (y, σ) if and only if $x = y$ and $\omega S_x^\perp = \sigma S_x^\perp$. Notice that we have a well defined G -action on Δ where the s -action on the class of (x, ω) gives the class of $(s \cdot x, \omega^s)$.

COROLLARY 5.11. *In the situation of Corollary 5.10, $\text{Prim } C^*(G, \Omega)$ is homeomorphic to $(\Delta/G)^\sim$.*

PROOF. It will suffice to show that $(\Delta/G)^\sim$ is homeomorphic to $(\Lambda_H/G)^\sim$. Let α be the natural map of $\Omega \times \hat{H}$ into $(\Lambda_H/G)^\sim$. We remark that α is continuous and, since α is the composition of the natural maps of $\Omega \times \hat{H}$ onto Λ_H and the natural map of $\Lambda_H \approx \Gamma$ into $(\Gamma/G)^\sim \approx \Upsilon$, α is open by the discussion following Theorem 5.3 and Corollary 5.9. It will suffice to show α factors through $(\Delta/G)^\sim$ and is injective on $(\Delta/G)^\sim$.

It is clear that α factors through Δ . Since α is obviously G -equivariant, it factors through Δ/G . Since $(\Lambda_H/G)^\sim$ is T_0 , an argument similar to those in Lemma 4.10 and Theorem 5.8 shows that α factors through $(\Delta/G)^\sim$.

Suppose that (x, ω) and (y, δ) belong to the same class in $(\Lambda_H/G)^\sim$. Then, there are r_α in G such that $[r_\alpha \cdot x, \omega^{r_\alpha}]$ converges to $[y, \delta]$ in Λ_H . Since the natural map of $\Omega \times \hat{H}$ to $(\Lambda_H/G)^\sim$ is open, there are $y_\beta \in \Omega$ and $\sigma_\beta \in \hat{H}$ with $\overline{H \cdot y_\beta} = \overline{H r_\alpha \cdot x}$ and $\sigma_\beta \in S_{y_\beta}^\perp$ such that $(y_\beta, \sigma_\beta \omega^{r_\beta})$ converges to (y, δ) in $\Omega \times \hat{H}$.

Now let U be a neighborhood of y in Ω . Since y_β is eventually in U and $y_\beta \in \overline{H r_\beta \cdot x}$, there is a $t_{\beta\nu} \in H$ such that $t_{\beta\nu} r_{\beta\nu} \cdot x$ is in U . In particular, there is a net $(t_\gamma r_\gamma \cdot x, \sigma_\gamma \omega^{r_\gamma})$ converging to (y, δ) in $\Omega \times \hat{H}$. Moreover, since H is abelian, $S_{t_\gamma r_\gamma \cdot x} = S_{r_\gamma \cdot x}$ and $\omega^{r_\gamma} = \omega^{t_\gamma r_\gamma}$. Therefore, $(y, \delta) \in \overline{G \cdot (x, \omega)}$ in Δ . By symmetry,

$$\overline{G \cdot (y, \delta)} = \overline{G \cdot (x, \omega)}. \quad \text{Q.E.D.}$$

As a final example of Theorem 5.8, we consider essentially free actions. That is, we suppose that all of the stability groups of (G, Ω) are equal to the same subgroup, say H . Notice that H must be normal and that $C^*(H, \Omega)$ is isomorphic to $C^*(H) \otimes C_0(\Omega)$ or $C_0(\Omega, C^*(H))$. Thus, $\text{Prim } C^*(H, \Omega) = \Gamma$ is homeomorphic to $\Omega \times \text{Prim}(H)$. The G -action of Γ transforms to the following action on $\Omega \times \text{Prim}(H)$: ${}^s(x, P) = (s \cdot x, {}^sP)$, where sP denotes the natural action of G on

$\text{Prim}(H)$ coming from the strongly continuous action of G on $C^*(H)$. We have

COROLLARY 5.12. *If (G, Ω) is an essentially free transformation group with stability group H and if $C^*(G, \Omega)$ is EH -regular then $\text{Prim } C^*(G, \Omega)$ is homeomorphic to $(\Omega \times \text{Prim}(H)/G)^\sim$.*

We conclude this section with several examples of the use of Theorem 5.3. We first consider the transformation group of the multiplicative positive reals acting on \mathbf{R}^2 by $(a, b) = (a/t, b/t)$. The orbits are rays originating from the origin together with the origin which is a fixed point. Note that the action is free everywhere but at the origin. The orbit space may be identified with the unit circle, \mathbf{T} , union the origin, O . The primitive ideal space of $C^*(\mathbf{R}^+, \mathbf{R}^2)$ is, by Theorem 5.3, easily identified with \mathbf{T} union \mathbf{R}^+ . The topology may be computed easily since the map from $\mathbf{R}^2 \times \mathbf{R}^+$ onto $\mathbf{T} \cup \mathbf{R}^+$ is open; the open sets are $\{U \subseteq \mathbf{T}: U \text{ is open in } \mathbf{T}\} \cup \{\mathbf{T} \cup V: V \text{ is open in } \mathbf{R}^+\}$.

Notice that the picture is changed dramatically if we alter the action. Now, let \mathbf{R}^+ act on \mathbf{R}^2 by $(a, b) = (a/t, tb)$. The orbits now consist of the origin, the four remaining branches of the coordinate axes, and the set of hyperbolas given by $\{(x, y): xy = b\}$ for $b \in \mathbf{R} - \{0\}$. Note that each $b \in \mathbf{R} - \{0\}$ corresponds to two orbits. Thus, the orbit space may be identified as a set with the union of the two lines $y = x$ and $y = -x$ (denoted A) together with the four points: $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (-1, 0)$, $a_4 = (0, -1)$. We let Q_i be the right angle which is formed by A and which contains a_i in its interior. That is, $Q_1 = \{(x, y) \in A: x \geq 0\}$, $Q_2 = \{(x, y): y \geq 0\}$, $Q_3 = \{(x, y): x \leq 0\}$, and $Q_4 = \{(x, y): y \leq 0\}$. A , Q_1 , Q_2 , Q_3 , and Q_4 have the relative topology coming from \mathbf{R}^2 . Using the openness of the natural map of \mathbf{R}^2 into the orbit space, we see that every neighborhood of $(0, 0)$ must contain a_1, a_2, a_3 , and a_4 . Also, every neighborhood of a_i must contain a neighborhood of $(0, 0)$ in Q_i (possibly with $(0, 0)$ deleted). We will denote the orbit space by F . The open sets of F are as follows:

$$\{U \subseteq A: (0, 0) \notin U \text{ and } U \text{ is open in } A\}$$

together with

$$\{U \subseteq F: U \cap A \text{ is open in } A, (0, 0) \in U, \text{ and } a_i \in U \text{ for } i = 1, 2, 3, 4\}$$

and

$$\{V \subseteq Q_j \cup a_j: a_j \in V \text{ and } V \cap Q_j \text{ is a neighborhood of } (0, 0) \text{ with } (0, 0) \text{ deleted for } j = 1, 2, 3, 4\}.$$

Since the action is free everywhere but at the origin, we may identify $\text{Prim } C^*(\mathbf{R}^+, \mathbf{R}^2)$ with F where the origin is replaced by \mathbf{R} . The open sets are $\{U: U \subseteq F \setminus (0, 0), U \text{ open in } F\} \cup \{U \cup V: U \text{ is an open neighborhood of } (0, 0) \text{ in } F \text{ with } (0, 0) \text{ deleted and } V \text{ is open in } \mathbf{R}\}$.

We can also use Theorem 5.3 to compute the topology of the primitive ideal space of the group C^* -algebra of groups which are the semi-direct product of two second countable abelian groups. Suppose N and K are abelian subgroups of G such that $G = KN$, $K \cap N = \{e\}$, and N is normal in G . Since (K, \hat{N}) is a

topological transformation group with respect to the K -action given by ${}^s\chi(a) = \chi(sas^{-1})$, we may form $C^*(K, \hat{N})$. Using the fact that a left Haar measure on C is given by

$$\int_G f = \int_N \int_K f(sa) ds da,$$

one may show without difficulty that

$$\Phi(f)(s, \chi) = \int_N f(sa)^{s^{-1}} \chi(a) da$$

defines a *-isomorphism between $C^*(G)$ and $C^*(K, \hat{N})$. Thus, Theorem 5.3 applies. In particular, one can use the above remarks to work out the topology on the dual spaces of such classical examples as the “ $ax + b$ ” group or the Heisenberg group; although when the action of K on \hat{N} is smooth (as in the two examples mentioned), Larry Baggett has already done this (cf. [1, Theorem 3.3]).

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