Published for SISSA by Springer

Received: *December 12, 2012*
Revised: *February 25, 2013*
Accepted: *March 2, 2013*
Published: *March 22, 2013*

The toroidal block and the genus expansion

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Abstract: We study the correspondence between four-dimensional supersymmetric gauge theories and two-dimensional conformal field theories in the case of $\mathcal{N} = 2^*$ gauge theory. We emphasize the genus expansion on the gauge theory side, as obtained via geometric engineering from the topological string. This point of view uncovers modular properties of the one-point conformal block on a torus with complexified intermediate momenta: in the large intermediate weight limit, it is a power series whose coefficients are quasi-modular forms. The all-genus viewpoint that the conformal field theory approach lends to the topological string yields insight into the analytic structure of the topological string partition function in the field theory limit.

Keywords: Conformal and W Symmetry, Topological Strings, Conformal Field Models in String Theory

ArXiv ePrint: 1212.0722

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doi:10.1007/JHEP03(2013)133

Contents

1 Introduction

The two-dimensional / four-dimensional correspondence [1, 2] relates observables and structural properties of $\mathcal{N} = 2$ supersymmetric four-dimensional gauge theories to those of two-dimensional conformal field theory [3–7]. At the heart of the correspondence lies the observation that the ϵ-deformed $\mathcal{N} = 2$ instanton partition functions [8] map to conformal blocks of conformal field theory. The gauge group and field content of the gauge theory determine the worldsheet genus as well as the number and weight of the insertions of the respective conformal block. The partition sums on neither side of the correspondence are known as analytic functions of their parameters. The correspondence was established by comparing an expansion in the gauge coupling order by order in instanton number on the gauge theory side to an expansion in a complex structure parameter of the corresponding Riemann surface with punctures on the conformal field theory side [1, 9–11]. However, the $\mathcal{N} = 2$ gauge theories can be geometrically engineered within string theory [12] and this makes them amenable to worldsheet techniques which give rise to the holomorphic anomaly equations [13, 14]. The expressions one obtains for the instanton partition function from this vantage point are close in spirit to [15, 16], namely, they are exact in the gauge coupling, but obtained order by order in an ϵ expansion. The underlying theme of this paper is to contrast and exploit these two different expansions on the gauge theory and the conformal field theory side. Though many aspects of this paper will apply to a much larger class of theories, we will concentrate here on the example of $\mathcal{N} = 2^*$ ϵ-deformed four-dimensional SU(2) gauge theory (see also [17]). The corresponding quantity on the conformal field theory side is the one-point conformal block on the torus.

One of the hallmarks of both the Seiberg-Witten and the holomorphic anomaly vantage point is the emphasis on modular properties, with the effective coupling as modular parameter. It is by exploiting this symmetry that exact expressions in the effective coupling can be obtained [18–20]. Modularity is of course also a recurring theme on the conformal field theory side, but as a property of n-point functions rather than of their constituent conformal blocks. Imposing e.g. modular invariance of the torus one-point function or crossing symmetry of the four-point function on the sphere imposes constraints on sums over conformal blocks. But if the correspondence is to hold beyond the weak coupling regime, then each individual conformal block should have good modular properties. This suggests that we should be able to construct conformal blocks non-perturbatively with the appropriate complex structure parameter serving as modular variable. We indeed succeed at extracting such results in the semi-classical (infinite central charge) limit from null vector decoupling equations. That the solutions of these equations exhibit quasi-modular behavior is astounding. In studying the action of the modular group on the equations and its boundary conditions, we are led to complexify the exchanged momentum of the conformal block, corresponding to the vacuum expectation value a of the complex adjoint scalar field in the gauge theory. Enlarging the usual set of conformal blocks in this way is required to allow for a natural action of the modular group.

Topological string theory, just as physical string theory, was originally defined order by order in a genus expansion. An exciting aspect of the two-dimensional / four-dimensional correspondence is that it provides a non-perturbative definition of the topological string via the corresponding conformal field theory — albeit only on certain geometries and in the field theory limit [21–23]. In the following, we will freely use the terminology of the topological string. The action of the infinite dimensional chiral algebra on the conformal theory allows the derivation of recursion relations in the complex structure parameter $q = e^{2\pi i\tau}$ satisfied by conformal blocks [24–26] which are exact in the ϵ parameters, and hence in the string coupling from a topological string perspective. This allows us to derive all genus results for the topological string free energy at any given order in q. Furthermore, these recursion relations reveal a curious property regarding the structure of poles and zeros of the topological string partition function Z. When we factorize the partition function as $Z = Z_{pert}Z_{\text{inst}}$, with Z_{pert} encompassing contributions not involving base wrappings in the geometry un-

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derlying the engineering, the recursion relations reveal a surprising infinite number of poles of Z_{inst} in the fiber class variable. These are exactly canceled by zeros of Z_{pert}.

A characteristic of the Seiberg-Witten treatment of the problem, which extends to the treatment via the holomorphic anomaly, is the presence of two classes of variables: ultraviolet parameters such as the moduli space parameter u or the bare coupling, and infrared parameters, such as the vacuum expectation value a or the effective coupling. The two-dimensional / four-dimensional correspondence only translates a subset of these parameters into the conformal field theory context: the bare coupling maps to a certain parameterization of the complex structure of the punctured Riemann surface underlying the conformal block, whereas the parameter a maps to the exchanged momentum. A natural question is to identify the other parameters in the conformal field theory. In particular, the distinction between bare and effective coupling is noteworthy. The holomorphic anomaly equations yield results that are naturally modular in the effective coupling of the theory. In this paper, using conformal field theory methods, we will uncover modularity in $\mathcal{N} = 2^*$ amplitudes expressed in terms of the bare coupling (this is in the spirit of [10, 27]). Generally, the choice of bare coupling constant is ambiguous. Indeed, in the other SU(2) conformal theory, the theory with $N_f = 4$ fundamental hyper multiplets, two natural definitions of the bare coupling are possible. We will discuss further in a forthcoming paper how the choice with good modular properties is also distinguished from the conformal field theory point of view [28].

The structure of this paper is as follows. We will review relevant aspects of the torus one-point function and the topological string partition function in section 2, and introduce the correspondence between two-dimensional and four-dimensional observables. Section 3 is dedicated to the recursion relation satisfied by the torus one-point function, and lessons that can be drawn for both the conformal field theory and the topological string by combining it with modular results on the topological string side. In section 4, we analyze the constraint on the one-point function on the torus arising from null vector decoupling. We solve the resulting differential equation recursively in a semi-classical regime and check the result against topological string theory results, obtaining agreement. The modularity of the one-point function is discussed in detail. We draw conclusions in section 5.

2 The one-point conformal block in the correspondence

In this section, we exhibit the role of the one-point toroidal conformal block in conformal field theory, discuss the corresponding quantity in topological string theory, and then review how they are expected to match [1]. We will recall the engineering of gauge theory within topological string theory and freely use the language of the latter setup in the following.

Our formulas are based on the conformal algebra underlying any conformal field theory. The only parameter which enters is the central charge c. It will be useful however to also introduce the following parameterizations that have their origins in Liouville theory. For the central charge, we set

$$c = 1 + 6Q^2\,, \quad Q = b + \frac{1}{b}\,. \tag{2.1}$$

The semi-classical Liouville limit $c \to \infty$ has the incarnations $b \to 0$ or $b \to \infty$. In the semi-classical limit, we connect to a classical Liouville theory with action principle. To render an intuitive interpretation easier, we sometimes parameterize conformal weights h in terms of Liouville momenta α:

$$h = \alpha(Q - \alpha)\,. \tag{2.2}$$

2.1 The torus one-point function

We study a two-dimensional conformal field theory of central charge c on a torus $\Sigma = T^2$ with modular parameter τ. We will often write $q = e^{2\pi i\tau}$. The toroidal one-point function of a Virasoro primary field V_{h_m} with conformal dimension h_m is also the traced cylinder one-point function,

$$\langle V_{h_m}\rangle_\tau = \mathrm{Tr}\, V_{h_m} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}\,. \tag{2.3}$$

The trace here is over a basis of the Hilbert space. In a conformal field theory, this space is a direct sum of Verma modules associated to the primaries $|h\rangle$ in the spectrum of the theory of weight h. In principle, we need to indicate both the sum over left and right conformal dimensions. For simplicity only, we suppose the spectrum is diagonal. Since the basis of primaries and descendents is not orthogonal, we need to insert the overlaps of states when expressing the trace of an operator in terms of its matrix elements. The one-point function can thus be written as follows in terms of a basis of primaries and descendents:

$$\langle V_{h_m}\rangle_\tau = \sum_h \sum_{\mathbf{k},\bar{\mathbf{k}},\mathbf{k}',\bar{\mathbf{k}}'} (M^h)^{-1}_{(\mathbf{k}'\bar{\mathbf{k}}')(\mathbf{k}\bar{\mathbf{k}})} \langle h|L_{\mathbf{k}}\bar{L}_{\bar{\mathbf{k}}} V_{h_m} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} L_{-\mathbf{k}'}\bar{L}_{-\bar{\mathbf{k}}'}|h\rangle\,, \tag{2.4}$$

where we have introduced the non-trivial overlaps of the descendent states,

$$M^h_{(\mathbf{k}'\bar{\mathbf{k}}')(\mathbf{k}\bar{\mathbf{k}})} = \langle h|L_{\mathbf{k}}\bar{L}_{\bar{\mathbf{k}}} L_{-\mathbf{k}'}\bar{L}_{-\bar{\mathbf{k}}'}|h\rangle\,. \tag{2.5}$$

We have used the notation $L_{\mathbf{k}} = L_{k_1}\dots L_{k_m}$, with the sums running over vectors of increasing dimension, and the components of the vectors running over all positive integers. By invoking the commutator of the Virasoro generators with primary fields,[1]

$$[L_n, \mathcal{V}_h(\zeta)] = \zeta^{n+1}\partial\mathcal{V}_h(\zeta) + h(n+1)\zeta^n\mathcal{V}_h(\zeta)\,, \tag{2.6}$$

the matrix element in equation (2.4) can be expressed in terms of the three-point function $\langle h|\mathcal{V}_{h_m}(\zeta)|h\rangle$ and its derivatives. The latter can be evaluated explicitly, as the ζ dependence of the correlator $\langle h|\mathcal{V}_{h_m}(\zeta)|h\rangle$ is fixed by conformal invariance,[2] $\langle h|\mathcal{V}_{h_m}(z)|h\rangle \sim z^{-h_m}$. We hence obtain

$$\sum_{\mathbf{k},\bar{\mathbf{k}},\mathbf{k}',\bar{\mathbf{k}}'} (M^h)^{-1}_{(\mathbf{k}'\bar{\mathbf{k}}')(\mathbf{k}\bar{\mathbf{k}})} \langle h|L_{\mathbf{k}}\bar{L}_{\bar{\mathbf{k}}} V_{h_m} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}} L_{-\mathbf{k}'}\bar{L}_{-\bar{\mathbf{k}}'}|h\rangle = (q\bar{q})^{h-\frac{c}{24}}|\mathcal{F}^h_{h_m}(q)|^2\langle h|V_{h_m}|h\rangle\,. \tag{2.7}$$

[1] We distinguish notationally between operators on the cylinder, V_h, and on the plane, $\mathcal{V}_h$. With $\zeta = e^{-2\pi i z}$, they are related by $V_h(z) = (-2\pi i\,\zeta)^h\mathcal{V}_h(\zeta)$. When we wish to refer to the operator without specifying the coordinate system, we use the notation V to avoid introducing a third symbol.

[2] Note that the corresponding three-point function on the cylinder, $\langle h|V_{h_m}(z)|h\rangle$, is independent of the insertion point z, as it should be.

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The holomorphic quantity $\mathcal{F}^h_{h_m}(q)$ is referred to as the one-point conformal block on the torus. It encodes the contributions of the descendants of a given primary to the one-point function. Note that it is completely determined by the Virasoro algebra, hence depends only on the quantities explicitly indicated and the central charge of the conformal field theory. All of the dynamical information of the conformal field theory is encoded in the three-point functions,

$$C^h_{h_m,h} = \langle h|V_{h_m}|h\rangle\,. \tag{2.8}$$

Expressed in terms of these quantities, the one-point function on the torus thus finally takes the form

$$\langle V_{h_m}\rangle_\tau = \sum_h C^h_{h_m,h}(q\bar{q})^{h-\frac{c}{24}}|\mathcal{F}^h_{h_m}(q)|^2\,. \tag{2.9}$$

The formalism we have described is very general. In practice, the spectrum of conformal field theories can differ significantly from theory to theory. Rational conformal field theories will have a finite spectrum. Other theories can have discrete unitary spectra. Liouville theory has a continuous unitary spectrum. There also exist non-unitary conformal field theories with discrete or continuous spectra. All of these theories will differ in their three-point functions, and in the set of relevant conformal blocks.

2.2 The topological string theory

The two-dimensional / four-dimensional correspondence relates conformal field theory to gauge theory. The topological string enters our narrative as the relevant gauge theories can be geometrically engineered within string theory [12]. The tool we use to compute the gauge theory quantities is the holomorphic anomaly equations [14], whose natural habitat is the topological string. The holomorphic anomaly equations allow us to compute the gauge theory amplitudes in terms of modular forms. These can be expanded to yield the instanton contributions to arbitrarily high order in the instanton number.

The topological string partition function. The topological string partition function Z is traditionally assembled from the topological string amplitudes F^g which are generating functions for map counts from a Riemann surface of genus g to the target space X. The partition function weights the amplitudes with the string coupling g_s,

$$Z = \exp\sum_{g=0}^{\infty} F^g g_s^{2g-2}\,. \tag{2.10}$$

In the limit of large Kähler parameters, the amplitudes have an expansion

$$F^g = \sum_{\mathbf{k}} d^g_{\mathbf{k}}\, Q_1^{k_1}\cdots Q_n^{k_n}\,, \tag{2.11}$$

where the Kähler parameters $t_i = \int_{\Sigma_i} J$ are integrals of the Kähler form J on the space X, the parameters Q_i are the exponentials $Q_i = \exp(-t_i)$, and the surfaces Σ_i furnish a basis of homology two-cycles. The numbers $d^g_{\mathbf{k}}$ are rational, with denominators encoding multi-wrapping contributions. The sum is over curve classes labelled by the vector

$\mathbf{k} = (k_1, k_2, \ldots, k_n)$. The Gopakumar-Vafa form of the partition function re-expresses the partition function Z in terms of integer invariants $n^g_{\mathbf{k}}$,

$$Z = \exp \sum_g \sum_{n=0}^{\infty} \sum_{\mathbf{k}} \frac{n^g_{\mathbf{k}} \mathbf{Q}^{n\mathbf{k}}}{n(q_s^{n/2} - q_s^{-n/2})^{2-2g}}\,, \tag{2.12}$$

where the parameter $q_s = e^{g_s}$ is the exponential of the string coupling and we have written $\mathbf{Q}^{\mathbf{k}} = \prod_i Q_i^{k_i}$. Given a curve class $\mathbf{k}$, the invariants $n^g_{\mathbf{k}}$ are zero for large enough genus g. Therefore, knowing a finite number of invariants $n^g_{\mathbf{k}}$ yields the contribution of the curve class $\mathbf{k}$ to the partition function to all genus. The two expansions (2.10) and (2.12) are related by invoking the formula

$$-\frac{1}{\sinh^2 x} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} \qquad \text{for} \quad x^2 < \pi^2\,. \tag{2.13}$$

Geometric engineering. The path from type IIA string theory to four-dimensional $\mathcal{N} = 2$ SU(2) gauge theories proceeds via six dimensions. Each two-cycle Σ in the internal four-dimensional geometry gives rise to a perturbative state corresponding to a massless photon. D2-branes and anti-D2 branes wrapping the same cycle Σ yield non-perturbative massive states with mass m proportional to the Kähler parameter of the cycle divided by the string coupling, $m \sim \frac{t_\Sigma}{g_s}$. When the cycle Σ arises from the blowup of an A_1 singularity, the perturbative and non-perturbative states are elements of the same SU(2) multiplet. The symmetry is broken by the non-vanishing size of the cycle. In the blow-down limit $t_\Sigma \to 0$, the full SU(2) gauge symmetry is restored. To dimensionally reduce to four dimensions while preserving $\mathcal{N} = 2$ supersymmetry, the compactification manifold must be Calabi-Yau. Fibering the A_1 singularity appropriately over a $\mathbb{P}^1$ gives rise to such manifolds. In the following, we refer to the exceptional class resolving the A_1 singularity as the fiber class Σ_f and to the class of the $\mathbb{P}^1$ the A_1 singularity is fibered over as the base class Σ_b. The gauge coupling (squared) of the four-dimensional theory is inversely proportional to the volume of the compactification manifold, here the volume t_{Σ_b} of $\mathbb{P}^1$ (in the compactification from six dimensions to four). The weak coupling limit is therefore $t_{\Sigma_b} \to \infty$. Base wrapping number maps to instanton number in the gauge theory. To retain worldsheet instanton corrections wrapping the base (weighted by $e^{-t_{\Sigma_b}}$), a gauge theory argument [12] shows that one needs to simultaneously scale $t_{\Sigma_f} \to 0$. To maintain W-bosons at finite mass, this requires scaling the string coupling to zero as well. All in all, we can parameterize the field theory limit as follows:

$$e^{-t_{\Sigma_b}} = \left(\frac{\beta\Lambda}{M_{\text{string}}}\right)^4, \quad t_{\Sigma_f} = \frac{\beta a}{M_{\text{string}}}, \quad g_s = \beta \hat{g}_s\,,$$

where we take $\beta \to 0$. The scale Λ enters via dimensional transmutation. The power is determined by gauge theory considerations [12]. To add fundamental matter, one needs to blow up points on the base $\mathbb{P}^1$. Adjoint matter is obtained via partial compactification of the geometry [29].

The topological string partition function on this class of geometries can be computed via the topological vertex [30]. For the field theory limit to yield non-vanishing higher genus amplitudes, the zeros from powers of the parameter β in g_s^{2g-2} must be cancelled. This occurs via a resummation of contributions from fiber wrappings. Using vertex techniques, this resummation can be performed order by order in the base wrapping k. The calculation has been performed in the case of $N_f = 0$ for base wrapping number $k = 0, \dots, 4$ in [31], yielding the results

$$\sum_{n=1}^{\infty} \sum_{m} \frac{n^g_{(k,m)} {Q_b}^{nk} Q_f^{nm}}{n(q_s^{n/2} - q_s^{-n/2})^{2-2g}} = \sum_{n=1}^{\infty} \frac{P_k^g(Q_f^n)}{(1-Q_f^n)^{2g-2+4k}} \frac{{Q_b}^{nk}}{n(q_s^{n/2} - q_s^{-n/2})^{2-2g}} \tag{2.14}$$

$$\xrightarrow[\beta \to 0]{} \left(\frac{M_{\text{string}}}{a}\right)^{2g-2} P_k^g(1) \left(\frac{\Lambda}{a}\right)^{4k} \hat{g}_s^{2g-2}\,. \tag{2.15}$$

Note that multi-wrapping contributions $n > 1$ do not survive the field theory limit. Explicit expressions for the polynomials $P_k^g(x)$ can be found in [31].

It would be interesting to prove the pole structure in powers of the fiber parameter Q_f in the resummation formula (2.14) for all base wrapping numbers k and for more general gauge theories from the point of view of the vertex. In particular, the result for conformal theories, as determined via the holomorphic anomaly equations, differs from (2.15) in that Λ/a is replaced by an a independent factor q_{inst},

$$\left(\frac{\Lambda}{a}\right)^4 \to q_{\text{inst}}\,. \tag{2.16}$$

The a dependence of the amplitudes is hence independent of the instanton number k. This will be important for us in section 3.4.

The Ω deformation. For topological strings on non-compact Calabi-Yau target spaces, it is possible to introduce a second expansion parameter s in the partition function,

$$Z = \exp \sum_{n,g} F^{(n,g)} s^n g_s^{2g-2}\,. \tag{2.17}$$

The conventional partition function is obtained by setting $s = 0$. This generalization goes under the name of Ω deformation. It was first introduced in the field theory context in [8] in a localization calculation of integrals over instanton moduli space. The two equivariant parameters ϵ_1 and ϵ_2 related to spatial rotations in $\mathbb{R}^4$ considered there are related to g_s and s via

$$\begin{aligned} s &= (\epsilon_1 + \epsilon_2)^2\,, \\ g_s^2 &= \epsilon_1 \epsilon_2\,. \end{aligned} \tag{2.18}$$

The Ω deformation away from the field theory limit was studied in [32] and related to a motivic count in [33]. An interpretation of the free energies $F^{(n,g)}$ at $n \neq 0$ in terms of map counts has not been put forward.

Contributions with no fiber-wrapping. For the two- / four-dimensional correspondence as reviewed below, it will be important to factorize the topological string partition function into a $q_{\rm inst}$-independent perturbative factor Z_{pert} and a $q_{\rm inst}$-dependent factor $Z_{\rm inst}$. The nomenclature is inspired by the interpretation in the field theory limit. Contributions to Z_{pert} arise from maps that do not wrap the base Σ_b.

For the field theory limit of the topological string, an all genus (and all orders in the deformation parameter s) expression is known for Z_{pert}. For the $\mathcal{N} = 2^*$ theory, it is given by [29, 32, 34]

$$\log Z_{pert} = \frac{1}{g_s^2} \log \frac{\Gamma_2(2a + m + \frac{\epsilon_1+\epsilon_2}{2})\Gamma_2(-2a + m + \frac{\epsilon_1+\epsilon_2}{2})}{\Gamma_2(2a)\Gamma_2(-2a)} \,. \tag{2.19}$$

The function $\Gamma_2(x)$ is the Barnes' double Gamma function [35], defined as the exponential of the derivative of the Barnes' double zeta function ζ_2,

$$\Gamma_2(x|\epsilon_1, \epsilon_2) = \exp \frac{d}{ds}|_{s=0}\zeta_2(s, x|\epsilon_1, \epsilon_2) \,, \tag{2.20}$$

$$\zeta_2(s, x|\epsilon_1, \epsilon_2) = \sum_{m,n\geq 0} (m\epsilon_1 + n\epsilon_2 + x)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}e^{-tx}}{(1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})} \,. \tag{2.21}$$

The expression (2.19) possesses an asymptotic expansion in the arguments g_s and s whose coefficients match the results obtained via the holomorphic anomaly equations. The expansion is Borel summable, but not convergent. In section 3 below, we will demonstrate that the two-dimensional / four-dimensional correspondence allows us to derive such all genus results to any order in the parameter $q_{\rm inst}$.

The holomorphic anomaly equations. The topological string results we will use in this paper were obtained in [20] via application of the holomorphic anomaly equations [14]. These equations originate from studying the worldsheet definition of the topological string, and yield (together with appropriate boundary conditions) the topological string partition function in the genus expansion (2.10). A generalization of these equations was proposed in [19, 36] allowing to compute the refined amplitudes $F^{(n,g)}$ of equation (2.17). Holomorphic anomaly equations can be derived that yield the partition function directly in the field theory limit.

The partition function of $\mathcal{N} = 2^*$ SU(2) gauge theory. In this paper, we concentrate on the example of $\mathcal{N} = 2^*$ SU(2) gauge theory. The holomorphic anomaly equations for this theory, and the amplitudes $F^{(n,g)}$ they govern, depend on the gauge coupling constant, the coordinate on moduli space u, and the adjoint mass m. As the theory is conformal in the massless limit, the identification of the instanton expansion parameter $q_{\rm inst}$ is difficult. For an in depth discussion, we refer to [20].

The amplitudes simplify dramatically in the massless case. The instanton expansion parameter $q_{\rm inst}$ and the effective coupling can be identified in this case; they will simply be denoted as q in the following. Also, the square of the scalar vacuum expectation value a becomes a good global coordinate on moduli space in this limit, proportional to the global coordinate u. Furthermore, as the expectation value a becomes the only massive parameter in the theory aside from the deformation parameters $\epsilon_{1,2}$, it can serve as a genus counting parameter. The amplitudes thus take the form

$$F^{(n,g)} = \frac{1}{a^{2(g+n)-2}} \, p_{(n,g)} \left(E_2(q), E_4(q), E_6(q)\right) . \tag{2.22}$$

The polynomials $p_{(n,g)}$ are homogeneous polynomials in the Eisenstein series of weight $2(g+n)-2$. At order $g+n=2$, explicit expressions for the partition sums are [20][3]

$$F^{(2,0)} = \frac{E_2}{768\,a^2}, \qquad F^{(1,1)} = -\frac{E_2}{192\,a^2}, \qquad F^{(0,2)} = 0, \tag{2.23}$$

and at order $g+n=3$, one finds

$$\begin{aligned} F^{(3,0)} &= -\frac{1}{368640\,a^4}\left(5E_2^2+13E_4\right), \quad F^{(2,1)} = \frac{1}{184320\,a^4}\left(25E_2^2+29E_4\right), \\ F^{(1,2)} &= -\frac{1}{15360\,a^4}\left(5E_2^2+E_4\right), \quad F^{(0,3)} = 0\,. \end{aligned}$$

One can algorithmically generate the higher order terms.

2.3 The correspondence

The two-dimensional / four-dimensional correspondence in the case of $\mathcal{N}=2^*$ SU(2) theory identifies the torus one-point function with insertion of a primary field of dimension h_m with the generalized instanton partition function of the gauge theory with adjoint matter of mass m. Parameters are matched as follows:

$$\begin{aligned} b &= \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,, \\ h_m &= \frac{Q^2}{4} - \frac{m^2}{\epsilon_1\epsilon_2}\,, \\ h &= \frac{Q^2}{4} - \frac{a^2}{\epsilon_1\epsilon_2}\,. \end{aligned} \tag{2.24}$$

This dictionary results in the identifications

$$\begin{aligned} q^{h-\frac{c}{24}}\mathcal{F}^h_{h_m} &= Z_{\text{inst}}\,, \\ C^h_{h_m,h}(\text{Liouville}) &= Z_{pert}\,. \end{aligned} \tag{2.25}$$

The first equality, involving conformal blocks, does not depend on the two-dimensional field theory under consideration. The second equality invokes the three-point function of Liouville theory. One can think of the latter as the unitary theory with continuous spectrum fixed by demanding null vector decoupling in its correlation functions. As such, it is also governed by the Virasoro algebra.

[3] With regard to the reference [20], the normalization of the scalar vacuum expectation value is $a_{there} = 2a_{here}$.

The second equality would seem to restrict the regime of validity of the correspondence to the spectrum of Liouville theory. This would dictate strictly positive intermediate conformal dimensions h and purely imaginary vacuum expectation value $a \in i\,\mathbb{R}$. It will be interesting however to consider the first equality for general values of complex intermediate conformal dimensions h. This is natural from the gauge theory point of view, where the vacuum expectation value a is complex. Though this takes us outside the realm of unitary conformal field theories, we will see that it enables us to uncover interesting modular structure in more general representations of the Virasoro algebra.

3 Recursion, poles and modularity

The technical heart of this section is the recursion relation satisfied by the one-point conformal block on the torus. The conformal block exhibits poles at degenerate weights; the recursion relation is derived by determining the residues at these poles. In this section, after reviewing this relation, we will study its consequences in light of the two-dimensional / four-dimensional correspondence. Holomorphic anomaly results will allow us to relate sums over residues that occur in the recursion relation to modular expressions. In turn, we demonstrate that the recursion relation order by order in q provides all genus results for the topological string. Assuming modularity, this provides an alternative to the holomorphic anomaly equations for determining the topological string free energies. Finally, we point out a curious result regarding the zeros of Z_{pert} and the pole structure of Z_{inst} in the adjoint vacuum expectation value a.

3.1 The recursion relation for the one-point conformal block

In the semi-classical limit $h, c, h_m \to \infty$ and c/h and h_m/h small (namely, the semi-classical approximation to the integral over the momentum propagating in a given channel), the one-point toroidal conformal block simplifies. In the limit of large propagating conformal dimension h, the semi-classical result is corrected by an infinite power series in the modular parameter q with coefficients decreasing like negative powers of the exchanged conformal dimension h. The series is governed by a recursion relation derived in [24–26], and most pedagogically in [37]. The correction term to the leading asymptotics can be captured by a function $H^h_{h_m}(q)$,

$$\mathcal{F}^h_{h_m}(q) = \frac{q^{\frac{1}{24}}}{\eta(q)} H^h_{h_m}(q)\,, \tag{3.1}$$

which satisfies a recursion relation. We first define the coefficients of its q-expansion: $H^h_{h_m} = \sum_{n=0}^{\infty} H^{h,n}_{h_m} q^n$. In this expansion, the poles in the propagating conformal dimension h are manifest. The expansion coefficients satisfy the initialization condition and recursive formula

$$H^{h,0}_{h_m} = 1\,, \qquad H^{h,n>0}_{h_m} = \sum_{1 \le rs \le n} \frac{A_{rs} P_{rs}(m)}{h - h_{rs}} H^{h_{rs}+rs,n-rs}_{h_m}\,, \tag{3.2}$$

where

$$h_{rs}=\frac{Q^2}{4}-\frac{1}{4}(rb+sb^{-1})^2\,,\qquad A_{rs}=\frac{1}{2}\prod_{\substack{(p,q)=(1-r,1-s)\\(p,q)\neq(0,0),(r,s)}}^{(r,s)}\frac{1}{pb+qb^{-1}}\,,$$

$$P_{rs}(m)=\prod_{k=1}^{2r-1}\prod_{\substack{l=1\\(k,l)=(1,1)\bmod (2,2)}}^{2s-1}\left(\frac{m}{\sqrt{\epsilon_1\epsilon_2}}+\frac{kb+lb^{-1}}{2}\right)\left(\frac{m}{\sqrt{\epsilon_1\epsilon_2}}+\frac{kb-lb^{-1}}{2}\right)$$
$$\times\left(\frac{m}{\sqrt{\epsilon_1\epsilon_2}}-\frac{kb+lb^{-1}}{2}\right)\left(\frac{m}{\sqrt{\epsilon_1\epsilon_2}}-\frac{kb-lb^{-1}}{2}\right).$$

The recursion relation originates in the representation theory of the Virasoro algebra. The solution to the recursion relation can be obtained order by order in the parameter q (and to high order) using a symbolic manipulation program, and it can be successfully compared to the topological string partition function Z. The instanton partition function [8] provides a solution to the recursion relation in terms of sums over Young tableaux.

3.2 The correspondence between the conformal field theory and the topological string in the field theory limit

We now compare the conformal field theory and the topological string theory results on the partition function Z in the massless limit of $\mathcal{N}=2^*$ theory (i.e. $\mathcal{N}=4$ super Yang-Mills theory). The conformal field theory point of view yields results that are exact in the expectation value a order by order in the modular parameter q (see equation (3.2)), whereas the holomorphic anomaly approach yields exact results in the parameter q order by order in the expectation value a^{-1} (see equation (2.22)).

To identify quantities on both sides of the correspondence, note that an expansion of the recursion relation (3.2) in the large expectation value limit is of the form $H^h_{h_0}=1+\mathcal{O}\left(\frac{1}{a^2}\right)$. Given the dependence of the leading contribution to the one-point function on the expectation value a and the structure of the topological string partition function (2.22), we can make the identifications

$$\exp\sum_{n+g\leq 1}\left[F^{(n,g)}(a)\right]' s^n g_s^{2g-2}=q^{h-\frac{c}{24}}\frac{q^{\frac{1}{24}}}{\eta(q)} \tag{3.3}$$

and

$$H^h_{h_0}=\exp\sum_{n+g>1}\left[F^{(n,g)}(a)\right]' s^n g_s^{2g-2}\,. \tag{3.4}$$

The prime on the brackets $[\cdot]'$ indicates that terms constant in the modular parameter q have been dropped. In the topological string, these are the contributions from maps that do not wrap the base direction of the engineering geometry. They are captured by the Liouville three-point function (see equation (2.25)).

The $n+g \leq 1$ contribution. Using the formulas $c = 1 + 6\frac{s}{g_s^2}$ and $h = \frac{1}{4}\frac{s}{g_s^2} - \frac{a^2}{g_s^2}$, we find

$$q^{h-\frac{c}{24}} \frac{q^{\frac{1}{24}}}{\eta(q)} = \exp\left(-\frac{a^2}{g_s^2}\log q - \log\eta \right). \tag{3.5}$$

This allows us to determine the first terms in the topological string partition function

$$F^{(0,0)} = -a^2 \log q\,, \quad F^{(1,0)} = 0\,, \quad F^{(0,1)} = -\log\eta\,. \tag{3.6}$$

The part of the leading term which is physically significant in gauge theory (namely its a-dependence) matches the expected behavior of the gauge theory prepotential of $\mathcal{N}=4$ super Yang-Mills theory.[4]

3.3 A lesson for conformal field theory

The $n+g > 1$ contribution. Exploiting the exact results for the partition sums $F^{(n,g)}$ obtained via the holomorphic anomaly equations, we can use the identification (3.4) to obtain results to all orders in the modular parameter q for the semi-classical conformal block $H^h_{h_0}$. The simplest such relation is obtained by comparing the order a^{-2} terms on both sides of equation (3.4):

$$\begin{aligned} -g_s^2 \sum_{1\leq rs\leq n} q^{rs} A_{rs} P_{rs}(0) H_{h_{rs}+rs} &= \left[\frac{s^2}{g_s^2} F^{(2,0)} + sF^{(1,1)} + g_s^2 F^{(0,2)} \right]' \\ &= \left[\left(\frac{s^2}{g_s^2} - 4s \right) \frac{E_2(q)}{384} \right]'\,. \end{aligned}$$

In the second line, we have used the explicit results from [20].

Note that we have found a surprising constraint on the residues appearing in the conformal block recursion relation which is valid to all orders in the modular parameter q. It moreover implies that certain infinite sums over the residues have good modular properties. We will return to the interpretation of this modularity from the conformal field theory vantage point in section 4. Comparing higher order terms in a^{-2} on both sides of the identification (3.4) yields an infinite set of such constraints. Order by order in q, the validity of these constraints can be checked by invoking the recursion relation (3.2). The constraints are more powerful than these perturbative checks.

Finally, let us note that if we deform the $\mathcal{N}=4$ theory to $\mathcal{N}=2^*$ through a mass deformation, we can continue the above exploitation of the correspondence, order by order in m/a.

3.4 Lessons for the topological string in the field theory limit

Reconstructing the amplitudes from finitely many expansion coefficients in q. Using the relation to conformal field theory and expanding the block $\log H^h_{h_0}$ perturbatively

[4]The U(1) factor which enters the gauge theory side of the correspondence yields the semi-classical limit of the one-point toroidal conformal block. For a general gauge theory, it modifies $F^{(n,g)}$ for $n+g \leq 1$ by terms independent of the scalar vacuum expectation values.

in the modular parameter q, we can derive all genus results for coefficients of the partition function $\log Z$ at any order in the parameter q. E.g. to lowest order, we have

$$\log H^h_{h_0} = 1 - \frac{s(s-4g_s^2)}{8g_s^2(4a^2-s)} q + \mathcal{O}(q^2)\,, \tag{3.7}$$

which implies

$$\left[\sum_{n+g>1} F^{(n,g)} g_s^{2g-2} s^n\right]_q = -\frac{s(s-4g_s^2)}{8g_s^2(4a^2-s)}\,. \tag{3.8}$$

The notation $[\cdot]_q$ indicates the coefficient of q of the quantity enclosed in the square brackets. By the structure of the recursion relation (3.2) for $H^h_{h_0}$, it is evident that the coefficients of the monomial q^n (where $n > 0$) are rational functions in the parameters g_s and s, as explicitly exhibited here for the leading term. This is in contrast to the q independent contribution reviewed in section 2 above.

Note that together with the knowledge that the topological string amplitudes are quasi-modular, we can reconstruct the full amplitude at a given order in the string coupling g_s and the deformation parameter s by knowing a finite number of expansion coefficients in the modular parameter q. As the dimension of the vector space of quasi-modular forms of a given weight increases with weight, more coefficients are necessary to reconstruct the partition function $F^{(n,g)}$ at larger $n+g$. At $n+g=2$ for instance, the polynomial $p_{(n,g)}$ is of weight 2, and therefore proportional to E_2. The coefficient of $\frac{s^2}{g_s^2}$ and of s in equation (3.8) thus completely determine the partition functions $F^{(2,0)}$ and $F^{(1,1)}$. To determine the amplitudes at $n+g=3$, we require the coefficients of the forms E_2^2 and E_4, and we must thus expand the block $\log H^h_{h_0}$ to order q^2, etcetera.

Zeros and poles. It is manifest that the recursion relation (3.2) for the conformal blocks allows a resummation of the amplitudes (2.22). The increasing powers in a^{-2} yield a geometric series which is summed by expression (3.2). As can be seen explicitly in the example (3.8), this gives rise to poles in the a-plane at string coupling g_s and deformation parameter s dependent positions. The occurrences of these poles is unexpected from a physical point of view: singularities should arise only where particles become massless or, from a geometric point of view, when the target geometry is degenerating. In the case of the conformal field theory one-point function, the poles in the one-point conformal blocks are cancelled by the zeroes in the three-point function. The same mechanism is at work here, with the role of the three-point function played by the contributions to the partition function from curves not wrapping the base of the engineering geometry.

4 The one-point function via null vector decoupling

In this section, we derive a differential equation that will allow us to determine the one-point conformal block on the torus, in the semi-classical limit $c \to \infty$, in an ϵ_1 expansion.

Following [10, 38, 39], the strategy will be to insert an additional operator V_h into the one-point function correlator $\langle V_{h_m}\rangle_\tau$. By choosing V_h to be degenerate, the resulting two-point function will be constrained by a null vector decoupling differential equation. Choosing the operator V_h to simultaneously be light in the semi-classical limit will allow us to extract the one-point function from this two-point function. While the one-point conformal block on the torus is a universal conformal field theory quantity depending only on the central charge of the theory, the extraction of the one-point from the two-point function in the semi-classical limit is most readily argued for in the context of Liouville theory.

The two-point function with one degenerate insertion corresponds to a surface operator insertion in the gauge theory, or a brane insertion in topological string theory [5, 40]. In [41], following [42, 43], the open string topological partition function is computed using matrix model techniques. The closed topological string partition function is then extracted from the monodromies of the obtained result. Our approach to computing the closed topological string partition function relies on the fact that to leading order in ϵ_2, the one-point (closed) and the two-point (open) function coincide. The null vector decoupling equation thus permits the direct computation of the one-point function in the $\epsilon_2 \to 0$ limit.

4.1 Heavy and light insertions in Liouville theory

In this section, we follow the reasoning of [44], as reviewed and enhanced in [45], for the treatment of Liouville theory in the semi-classical limit. The limit $c \to \infty$ in any conformal field theory is referred to as semi-classical. In the parameterization introduced at the beginning of section 2, this corresponds to the limit $b \to 0$ (or $b \to \infty$). In this limit, primary operators in Liouville theory can be expressed in terms of the Liouville field ϕ via $V_\alpha = e^{2\alpha\phi}$. Deviating from the notation in the rest of this paper, we will label operators in this subsection by their Liouville momentum α rather than their conformal weight h.

The momenta of heavy operators in the semi-classical limit scale as $\frac{1}{b}$ in the $b \to 0$ limit. Their insertion changes the saddle point for the Liouville field. This effect can be incorporated into a modified semi-classical action S_L. Light operators have momentum scaling as b and do not modify the semi-classical saddle point. Their contribution to the correlation function is multiplicative. In terms of the rescaled field $\phi_{cl} = 2b\phi$ which is finite in the $b \to 0$ limit, one thus obtains [45]

$$\langle \mathcal{V}_{\frac{\eta_1}{b}}(\zeta_1, \bar{\zeta}_1) \cdots \mathcal{V}_{\frac{\eta_n}{b}}(\zeta_n, \bar{\zeta}_n) \mathcal{V}_{b\sigma_1}(\xi_1, \bar{\xi}_1) \cdots \mathcal{V}_{b\sigma_n}(\xi_n, \bar{\xi}_n) \rangle \approx e^{-\frac{1}{b^2} S_L[\phi_{cl}]} \prod_{i=1}^{m} e^{\sigma_i \phi_{cl}(\xi_i, \bar{\xi}_i)} \,. \tag{4.1}$$

Recall that we have parameterized $b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$, and we are interested in the one-point function $\langle \mathcal{V}_{\alpha_m} \rangle$ of momentum

$$\alpha_m = \frac{Q}{2} - \frac{m}{\sqrt{\epsilon_1 \epsilon_2}} \,. \tag{4.2}$$

To maintain the dependence on the mass m in the $b \to 0$ limit, we consider the limit $\epsilon_2 \to 0$ while keeping ϵ_1 fixed. The operator $\mathcal{V}_{\alpha_m}$ is heavy in this limit.

Degenerate operators $\mathcal{V}_{\alpha_{k,l}}$, parameterized by two positive integers $k, l \in \mathbb{N}_0$, have Liouville momentum

$$\alpha_{k,l} = \frac{Q}{2} - \frac{1}{2}\left(k\, b + \frac{l}{b}\right) \,. \tag{4.3}$$

These operators are light for $l = 1$. In the following, we will consider the lowest lying non-trivial such operator at $(k, l) = (2, 1)$. The semi-classical behavior of the corresponding two-point function is given by

$$\langle \mathcal{V}_{-\frac{b}{2}}(\zeta, \bar{\zeta}) \mathcal{V}_{\alpha_m}(0) \rangle \approx e^{-\frac{1}{2}\phi_{cl}(\zeta, \bar{\zeta})} \langle \mathcal{V}_{\alpha_m}(0) \rangle \,, \tag{4.4}$$

where $e^{-\frac{1}{2}\phi_{cl}}$ is constant whereas $\langle \mathcal{V}_{\alpha_m}(0) \rangle$ scales as $e^{-\frac{1}{b^2}} \sim e^{-\frac{\epsilon_1}{\epsilon_2}}$ in the $\epsilon_2 \to 0$ limit.

4.2 Isolating the contribution from a given channel

As reviewed in section 2.1 in the case of the one-point function, correlation functions on the torus can be defined as a trace over the spectrum of the conformal field theory. In the case of correlators with degenerate insertions, each summand will satisfy the null vector decoupling differential equation separately. To isolate particular summands, we study their monodromy behavior upon circling the A-cycle of the torus, i.e. under $z \to z + 1$, following [46]. Upon imposing this behavior as a boundary condition, the differential equation has a unique solution, which is essentially the conformal block we wish to compute.

To determine the monodromy under $z \to z + 1$, consider the correlator

$$\langle h | V_{h_m}(x) V_{(2,1)}(z) | h \rangle \sim \xi^{h_m} \zeta^{h_{(2,1)}} \langle h | \mathcal{V}_{h_m}(\xi) \mathcal{V}_{(2,1)}(\zeta) | h \rangle \,, \tag{4.5}$$

where Greek letters indicate coordinates on the plane, $\zeta = \exp[-2\pi i\, z]$ etcetera.[5] We denote the momenta corresponding to the weights $h_m, h_{(2,1)}, h$ as $\alpha_m, \alpha_{2,1}$, and α respectively. Considering only holomorphic dependence, we have

$$\begin{aligned} \mathcal{V}_{(2,1)}(\zeta) \mathcal{V}_h(0) = {} & C^{h_+}_{(2,1),h} \zeta^{h_+ - h_{(2,1)} - h} \left(\mathcal{V}_{h_+}(0) + \beta^{h_+}_{(2,1),h} \zeta \, (L_{-1} \mathcal{V}_{h_+})(0) + \dots \right) + \\ & + C^{h_-}_{(2,1),h} \zeta^{h_- - h_{(2,1)} - h} \left(\mathcal{V}_{h_-}(0) + \beta^{h_-}_{(2,1),h} \zeta \, (L_{-1} \mathcal{V}_{h_-})(0) + \dots \right) . \end{aligned} \tag{4.6}$$

We have here exploited the fact that fusion with degenerate vectors only involves a finite number of primaries [38]. For fusion with the vector $\mathcal{V}_{(2,1)}(z)$, the momenta of the two primaries appearing in the operator product expansion are $\alpha_\pm = \alpha \pm \frac{b}{2}$. Using the parameterization (2.24), we have

$$h_\pm - h = -\frac{b^2}{4} \pm \frac{ab}{\sqrt{\epsilon_1 \epsilon_2}} \,. \tag{4.7}$$

From this, we read off the monodromy under $z \to z + 1$ resulting from the two terms on the right hand side of (4.6) inserted into equation (4.5) to be

$$\zeta^{h_\pm - h} \to e^{\pi i \frac{b^2}{2} \mp 2\pi i \frac{ab}{\sqrt{\epsilon_1 \epsilon_2}}} \,. \tag{4.8}$$

Imposing this monodromy as a boundary condition on the differential equation will prove to be sufficient to isolate the sought after contribution to the torus one-point function.

[5] See footnote 1 for our conventions regarding the labeling of operators.

4.3 The null vector decoupling equation

We next turn to the derivation of the null vector decoupling equation on the torus [39, 47].

The conformal Ward identity on a torus. To determine the appropriate differential equation, we need to recall the conformal Ward identity on the torus [39]. It is determined by combining Ward identities for local reparameterization, local Lorentz and Weyl invariance, and involves an intricate cancellation of potentially non-holomorphic contributions. For the insertion of an energy-momentum tensor in the correlation function of a product of vertex operators V_i inserted at points z_i, $i = 1, 2, \ldots, n$, it can be written as [39]

$$\begin{aligned}&\langle T(z)\prod_{i=1}^{n} V_i(z_i)\rangle - \langle T\rangle\langle\prod_{i=1}^{n} V_i(z_i)\rangle = \\ &= \sum_{i=1}^{n}\Big(h_i(\wp(z-z_i)+2\eta_1)+(\zeta(z-z_i)+2\eta_1 z_i)\partial_{z_i}\Big)\langle\prod_{i=1}^{n} V_i(z_i)\rangle + 2\pi i\partial_\tau\langle\prod_{i=1}^{n} V_i(z_i)\rangle\,.\end{aligned} \tag{4.9}$$

For simplicity, we have assumed that the torus has periods $(1,\tau)$. We have written (4.9) in terms of the Weierstrass $\wp$-function and its primitive, the Weierstrass ζ-function,

$$\wp(z) = -\zeta'(z)\,, \quad \zeta(z) = \frac{\theta_1'(z)}{\theta_1(z)} + 2\eta_1 z\,, \tag{4.10}$$

and have introduced

$$\eta_1 = -\frac{1}{6}(\theta_1'''/\theta_1')|_{z=0}\,. \tag{4.11}$$

Imposing null vector decoupling. The primary field $V_{(2,1)}$ has a null vector at level two that decouples from the conformal field theory by assumption. This implies differential equations on correlators of the null vector with other vertex operators:

$$\langle((L_{-2}V_{(2,1)})(w) + \frac{1}{b^2}(L_{-1}^2 V_{(2,1)})(w))\prod_{i=1}^{n} V_i(z_i)\rangle = 0\,. \tag{4.12}$$

We use the conformal Ward identity to compute the relevant operator product expansions between the energy-momentum tensor and the primary. We can simplify the null vector equation further by using the relation between the partition function Z and the vacuum expectation value of the energy-momentum tensor on the torus, $2\pi i\partial_\tau \log Z = \langle T\rangle$. We then find

$$\begin{aligned}&\Big[\frac{1}{b^2}\partial_z^2 + 2\eta_1 z\partial_z + \sum_{i=1}^{n}(\zeta(z-z_i)+2\eta_1 z_i)\partial_{z_i} \\ &\quad +2\pi i\partial_\tau + 2h_{(2,1)}\eta_1 + \sum h_i(\wp(z-z_i)+2\eta_1)\Big] Z\langle V_{(2,1)}\prod_{i=1}^{n} V_i\rangle = 0\,.\end{aligned} \tag{4.13}$$

We now apply the null vector decoupling equation to the two-point function involving the degenerate field $V_{(2,1)}$ and one other insertion V_{h_m}. The result can be simplified through the ansatz [10, 48]

$$Z\langle V_{(2,1)}(z)V_{h_m}(0)\rangle_\tau = \theta_1(z|\tau)^{\frac{b^2}{2}}\eta(\tau)^{2(h_m-b^2-1)}\Psi(z|\tau)\,. \tag{4.14}$$

Plugging this ansatz into the null vector decoupling equation, we obtain the differential equation [48]

$$\left[-\frac{1}{b^2}\partial_z^2 - \left(\frac{1}{4b^2} - \frac{m^2}{\epsilon_1\epsilon_2}\right)\wp(z)\right]\Psi(z|\tau) = 2\pi i\partial_\tau\Psi(z|\tau)\,. \tag{4.15}$$

This form of the equation has the advantage of involving only operators and functions that behave simply under modular transformations.

Note that the factor $\theta_1(z|\tau)^{\frac{b^2}{2}}$ in the solution ansatz (4.14) has a monodromy that coincides with that of the channel independent contribution to the conformal block monodromy as determined in (4.8).

4.4 The semi-classical conformal block

In this section, we will provide an exponential ansatz for the two-point function, project it onto the channel h, and solve the differential equation (4.15) perturbatively in the semi-classical limit.

The exponential ansatz. Substituting $b = \sqrt{\epsilon_2/\epsilon_1}$ into the differential equation (4.15) for the rescaled one-point function Ψ, we obtain

$$\left[-\partial_z^2 - \left(\frac{1}{4} - \frac{1}{\epsilon_1^2}m^2\right)\wp(z)\right]\Psi(z|\tau) = \frac{\epsilon_2}{\epsilon_1}2\pi i\partial_\tau\Psi(z|\tau)\,. \tag{4.16}$$

This differential equation was derived in the strict $\epsilon_1 \to 0$ limit and solved in an m/a expansion in [10]. Its $\epsilon_1 \to 0$ limit is closely related to a renormalization group equation for instanton corrections to the $\mathcal{N} = 2^*$ gauge theory [49].

Recall that we wish to solve equation (4.16) in the semi-classical limit $\epsilon_2 \to 0$, as it is in this limit that we can disentangle the contribution of the auxiliary light insertion $V_{(2,1)}$ from that of the heavy insertion V_{h_m}. The factorization (4.4) of the two-point function in this limit suggests the ansatz

$$\Psi(z|\tau) = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau) + \frac{1}{\epsilon_1}\mathcal{W}(z|\tau) + O(\epsilon_2)\right]\,, \tag{4.17}$$

such that

$$\langle V_{h_m}\rangle \approx \exp\left[-\frac{1}{b^2}S_L[\phi_{cl}]\right] \approx \exp\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau)\,, \tag{4.18}$$

$$\exp\left[-\frac{1}{2}\phi_{cl}(z,\bar{z})\right] \approx \theta_1(z|\tau)^{\frac{b^2}{2}}\exp\frac{1}{\epsilon_1}\mathcal{W}(z|\tau)\,.$$

The powers of ϵ_2 in the ansatz (4.17) are determined by the b scaling behavior of (4.4). The leading ϵ_1 behavior of $\mathcal{W}(z|\tau)$ is motivated by the monodromy behavior (4.8). Finally, the same leading behavior of $\mathcal{F}(\tau)$ follows by requiring this monodromy to be compatible with the differential equation (4.16), as we will see below. Note that the factor $\eta(\tau)^{2(h_m-b^2-1)}$ in (4.14), with $\epsilon_{1,2}$ scaling behavior $\exp\mathcal{O}(1)$, cannot be unambiguously assigned to either $\langle V_{h_m}\rangle$ or $\exp\left[-\frac{1}{2}\phi_{cl}(z,\bar{z})\right]$.

Plugging the ansatz into the differential equation (4.16), we find

$$-\frac{1}{\epsilon_1}\mathcal{W}''(z|\tau) - \frac{1}{\epsilon_1^2}\mathcal{W}'(z|\tau)^2 + \left(\frac{1}{\epsilon_1^2}m^2 - \frac{1}{4}\right)\wp(z) = (2\pi i)^2\frac{1}{\epsilon_1^2}q\partial_q\mathcal{F}(\tau) + \frac{\epsilon_2}{\epsilon_1^2}2\pi i\partial_\tau\mathcal{W}(z|\tau)\,. \tag{4.19}$$

Boundary condition from monodromy. To project onto a channel with exchanged momentum a, we now impose the monodromy behavior determined in equation (4.8).

Since we already factored out $\theta_1^{b^2/2}$ in our ansatz (4.14), accounting for the monodromy $e^{\pi i b^2/2}$, we need to impose the monodromy $e^{\pm 2\pi i a b/\sqrt{\epsilon_1\epsilon_2}}$ on Ψ under the shift $z \to z+1$. This translates into

$$\mathcal{W}(z+1) - \mathcal{W}(z) = \pm 2\pi i a\,, \tag{4.20}$$

or equivalently,

$$\oint \mathcal{W}'(z) dz = \pm 2\pi i a\,. \tag{4.21}$$

Recursive definition of $\mathcal{W}_n$ and $\mathcal{F}_n$. In the semi-classical approximation, we can neglect the term proportional to ϵ_2 in equation (4.19), and obtain

$$-\frac{1}{\epsilon_1}\mathcal{W}''(z|\tau) - \frac{1}{\epsilon_1^2}\mathcal{W}'(z|\tau)^2 + \left(\frac{1}{\epsilon_1^2}m^2 - \frac{1}{4}\right)\wp(z) = (2\pi i)^2 \frac{1}{\epsilon_1^2} q\partial_q \mathcal{F}(\tau)\,. \tag{4.22}$$

This is an ordinary first order differential equation for the derivative $\mathcal{W}'(z)$ depending on an unknown function $\mathcal{F}(\tau)$. If we perform a formal expansion of $\mathcal{F}$ and $\mathcal{W}$ in the parameter ϵ_1,

$$\mathcal{F}(\tau) = \sum_{n=0}^{\infty} \mathcal{F}_n(\tau)\epsilon_1^n\,, \quad \mathcal{W}(z|\tau) = \sum_{n=0}^{\infty} \mathcal{W}_n(z|\tau)\epsilon_1^n\,, \tag{4.23}$$

we obtain a system of equations for the coefficients $\mathcal{F}_n$ and $\mathcal{W}_n$,

$$-\mathcal{W}_0'^2 + m^2\wp = (2\pi i)^2 q\partial_q \mathcal{F}_0\,, \tag{4.24}$$
$$-\mathcal{W}_0'' - 2\mathcal{W}_0'\mathcal{W}_1' = (2\pi i)^2 q\partial_q \mathcal{F}_1\,, \tag{4.25}$$
$$-\mathcal{W}_1'' - \mathcal{W}_1'^2 - 2\mathcal{W}_0'\mathcal{W}_2' - \frac{1}{4}\wp(z) = (2\pi i)^2 q\partial_q \mathcal{F}_2\,, \tag{4.26}$$
$$-\mathcal{W}_n'' - \sum_{i=0}^{n+1} \mathcal{W}_i'\mathcal{W}_{n+1-i}' = (2\pi i)^2 q\partial_q \mathcal{F}_{n+1} \quad \text{for} \quad n \geq 2\,. \tag{4.27}$$

The projection onto the h channel now reads

$$\oint \mathcal{W}_0' = \pm 2\pi i a\,, \quad \oint \mathcal{W}_i' = 0 \quad \text{for} \quad i > 0\,. \tag{4.28}$$

4.5 The structure of the perturbative solution

Consider the structure of the equations (4.27) at a given order n. Each increase in the order n introduces a function $\mathcal{W}_{n+1}'$ which has not occurred in previous equations, as well as a new function $q\partial_q\mathcal{F}_{n+1}$ on the right hand side of the equations. All other functions are known from lower order equations. The strategy is to eliminate $\mathcal{W}_{n+1}'$ from the equation by integrating along the $[0,1]$ cycle and invoking the boundary condition (4.28). We can thus determine $q\partial_q\mathcal{F}_n$ recursively for any n. By the two-dimensional / four-dimensional correspondence, these modular expressions should integrate to equal the topological string free energy $F^{(n,0)}$. We checked this equality to high degree in n.[6] Note that the fact that the q-dependence of the integral $\int d\tau\, q\partial_q \mathcal{F}_n$ for $n > 2$ is captured by a polynomial or power series in Eisenstein series (in the massless or massive case respectively) is non-trivial by itself.

For clarity of exposition, we will treat the massless case first, before turning to the case of arbitrary mass.

The massless case. In the massless case, the function $\mathcal{W}_0'$ does not depend on the position z of the insertion. Integrating both sides of the equation (4.27) along the $[0,1]$ cycle thus eliminates the unknown function $\mathcal{W}_{n+1}'$ from the equation by (4.28), allowing us to express $q\partial_q\mathcal{F}_{n+1}$ in terms of known quantities. This result in turn allows us to solve for the function $\mathcal{W}_{n+1}'$.

The first equation (4.24) has a slightly different structure compared to the rest, as only here the unknown, $\mathcal{W}_0'$, appears quadratically. We begin by solving the three equations (4.24)–(4.26) in the massless case. For simplicity, we will consider the positive sign on the right hand side of the boundary condition (4.28) for $\mathcal{W}_0$ in the following. The other sign can be easily accessed via the Weyl transformation $a \to -a$. In Liouville theory this transformation corresponds to reflection symmetry. We obtain

$$q\partial_q \mathcal{F}_0 = -a^2\,, \qquad \mathcal{W}_0' = 2\pi i a\,, \tag{4.29}$$

$$q\partial_q \mathcal{F}_1 = 0\,, \qquad \mathcal{W}_1' = 0\,, \tag{4.30}$$

and

$$(2\pi i)^2 q\partial_q \mathcal{F}_2 = -\frac{1}{4}\oint \wp(z)\,, \quad \mathcal{W}_2' = \frac{1}{4}\frac{\oint \wp(z) - \wp(z)}{4\pi i a}\,. \tag{4.31}$$

The result for $\mathcal{W}_0'$ follows from the boundary condition, while the differential equation guarantees that $\mathcal{W}_0'$ is z-independent.

The manipulations we have outlined above, as well as the results on monodromies of Weierstrass functions recorded in appendix A.2, will yield the functions $\mathcal{W}_n'$ as polynomials in the Weierstrass function $\wp(z)$ and its derivatives with coefficients in the ring of quasi-modular forms generated by the Eisenstein series E_2, E_4, E_6. We introduce a grading on this space of solutions as follows: we assign weight 2 to $\wp(z)$ and weight $2+n$ to $\wp^{(n)}(z)$, consistent with assigning weights $2,4,6$ to the Eisenstein series E_2, E_4, E_6. The integral $\oint$ is to commute with the grading. With this assignment, our claim is that

$$\mathcal{W}_{2n}' = \frac{p^e_{2n}(\wp)}{a^{2n-1}}\,, \quad \mathcal{W}_{2n+1}' = \frac{\wp'\, p^o_{2(n-1)}(\wp)}{a^{2n}}\,. \tag{4.32}$$

The polynomials p^e_n, p^o_n are homogeneous of weight n,

$$p^e_{2n}(\wp)\,, p^o_{2n}(\wp) \in \mathbb{C}[E_2, E_4, E_6][\wp]\,. \tag{4.33}$$

The proof follows easily by induction, upon invoking the equations

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3\,, \tag{4.34}$$

[6] $\mathcal{F}_n$ is determined up to a τ-independent integration constant.

$$\wp'' = 6\wp^2 - \frac{1}{2}g_2\,. \tag{4.35}$$

We note that the derivative $q\partial_q \mathcal{F}_{2n+1}$ is equal to the monodromy of the derivative of a periodic function, and therefore vanishes. We also have that

$$(2\pi i)^2 q\partial_q \mathcal{F}_{2n} = -\frac{1}{a^{2n-2}}\left(\sum_{i=0}^{n}\oint p^e_{2i}(\wp)p^e_{2(n-i)}(\wp) + \sum_{i=1}^{n-1}\oint p^o_{2(i-1)}(\wp)p^o_{2(n-i-2)}(\wp)\wp'^2\right), \tag{4.36}$$

which can be computed explicitly, recursively, and has weight $2n$. Note that the expansion coefficients $\mathcal{F}_n$ only depend on a^2.

The results up to $n=4$ are

$$\begin{aligned}
\partial_\tau \mathcal{F}_2 &= -i\frac{\pi}{24}E_2\,, & \mathcal{W}'_2 &= i\frac{\pi^2 E_2 + 3\wp}{48\pi a}\,,\\
\partial_\tau \mathcal{F}_3 &= 0\,, & \mathcal{W}'_3 &= -\frac{\wp'}{64\pi^2 a^2}\,,\\
\partial_\tau \mathcal{F}_4 &= i\frac{\pi(E_2^2 - E_4)}{4608a^2}\,, & \mathcal{W}'_4 &= -i\frac{(2E_2^2 - 25E_4)\pi^4 + 6\pi^2 E_2\wp + 225\wp^2}{9216\pi^3 a^3}\,,
\end{aligned} \tag{4.37}$$

and the next two non-vanishing orders of $\partial_\tau \mathcal{F}_n$ equal

$$\partial_\tau \mathcal{F}_6 = -i\frac{\pi(5E_2^3 + 21E_2E_4 - 26E_6)}{1105920a^4}\,, \quad \partial_\tau \mathcal{F}_8 = i\frac{\pi(35E_2^4 + 329E_2^2E_4 - 1402E_4^2 + 1038E_2E_6)}{297271296a^6}\,.$$

As promised, these integrate to elements of the polynomial ring generated by the Eisenstein series,

$$\mathcal{F}_4 = \frac{E_2}{768a^2}\,, \quad \mathcal{F}_6 = -\frac{5E_2^2 + 13E_4}{368640a^4}\,, \quad \mathcal{F}_8 = \frac{175E_2^4 + 1092E_2E_4 + 3323E_6}{743178240a^6}\,. \tag{4.38}$$

These results coincide, via the identification

$$\mathcal{F}_{2n} = F^{(n,0)} \tag{4.39}$$

with those obtained from the holomorphic anomaly equations [20] and quoted at the end of subsection 2.2. It is easy to compute the amplitudes to higher order.

The massive case. For non-vanishing mass, we expand all expansion coefficients $\mathcal{W}'_n$ and $q\partial_q \mathcal{F}_{2n}$ in powers of

$$v = \left(\frac{m}{2\pi a}\right)^2. \tag{4.40}$$

Solving for $\mathcal{W}'_{n+1}$ and integrating both sides of the resulting equations along the $[0,1]$ cycle, we can again eliminate the unknown function $\mathcal{W}'_{n+1}$ by invoking equation (4.28). Given $\mathcal{W}'_0$ to a certain order in the parameter v, expanding $1/\mathcal{W}'_0$ in the ratio v permits solving order by order for the coefficients of $q\partial_q \mathcal{F}_{2n}$ in a v expansion. This result in turn allows us to solve for the function $\mathcal{W}_{n+1}$. Assigning v the weight -2, the structural results (4.32) and (4.36) of the massless case apply here as well, with $p^e_{2n}(\wp)$ and $p^o_{2n}(\wp)$ now power series in the ratio v, with coefficients that are polynomials in $\wp$, with coefficients in turn in the ring of Eisenstein series,

$$p^e_{2n}(\wp)\,,p^o_{2n}(\wp)\in\mathbb{C}[E_2,E_4,E_6][\wp][[v]]\,. \tag{4.41}$$

The leading term in the power series in v is the massless result.

To compute the function $\mathcal{W}_0'$ (see also [10]), consider again equation (4.24), now for non-zero mass m. Together with the boundary conditions, it yields the relations

$$\oint_\alpha\sqrt{(2\pi)^2 q\partial_q\mathcal{F}_0+m^2\wp}=2\pi i a\,,\quad \mathcal{W}_0'=\sqrt{(2\pi)^2 q\partial_q\mathcal{F}_0+m^2\wp}\,. \tag{4.42}$$

We will solve these equations in a second perturbation series, in parallel to the ϵ_1 expansion, in the parameter v. Introducing the variable

$$\mathcal{G}=-\frac{1}{a^2}q\partial_q\mathcal{F}_0\,, \tag{4.43}$$

we can rewrite the monodromy condition as

$$\oint_\alpha\sqrt{\mathcal{G}-v\wp}=1. \tag{4.44}$$

For small m/a, we can expand the square root, and solve the equation order by order in v. We suppose that $\mathcal{G}$ also has a series expansion $\sum_{n=0}^\infty\mathcal{G}^n v^n$ in terms of the parameter v. Using the Taylor series of the square root and $\mathcal{G}^0=1$, we then find

$$\sum_{m=1}^\infty\frac{(-1)^m(2m)!}{(1-2m)(m!)^2 4^m}\oint\left[(\mathcal{G}^1-\wp)v+\sum_{n=2}^\infty\mathcal{G}^n v^n\right]^m=0. \tag{4.45}$$

At order p in the parameter v, we obtain a linear equation on the coefficient $\mathcal{G}^p$, which can be solved for in terms of the monodromies of powers of the Weierstrass function $\wp$. The latter can again be solved for recursively, as demonstrated in appendix A.2. We thus find that the coefficients $\mathcal{G}^p$ are polynomials in the Eisenstein series of total weight $2p$, in accord with our claim that $\mathcal{W}_0$ have overall weight 0.

The solutions up to fourth order in v are the following:

$$\begin{aligned}
\mathcal{G}^0&=1.\\
\mathcal{G}^1&=\oint_\alpha\wp=\frac{\pi^2}{3}(-E_2).\\
\mathcal{G}^2&=\oint_\alpha\frac{1}{4}(\mathcal{G}^1-\wp)^2=\frac{\pi^4}{36}(-E_2^2+E_4).\\
\mathcal{G}^3&=\oint_\alpha-\frac{1}{8}(\mathcal{G}^1-\wp)((\mathcal{G}^1)^2-4\mathcal{G}^2-2\mathcal{G}^1\wp+\wp^2)\\
&=\frac{\pi^6}{540}(-5E_2^3+3E_2E_4+2E_6)\\
\mathcal{G}^4&=\oint_\alpha\frac{1}{64}(5(\mathcal{G}^1)^4-24(\mathcal{G}^1)^2\mathcal{G}^2+16(\mathcal{G}^2)^2+32\mathcal{G}^1\mathcal{G}^3-20(\mathcal{G}^1)^3\wp+\\
&\qquad+48\mathcal{G}^1\mathcal{G}^2\wp-32\mathcal{G}^3\wp+30(\mathcal{G}^1)^2\wp^2-24\mathcal{G}^2\wp^2-20\mathcal{G}^1\wp^3+5\wp^4)\\
&=\frac{\pi^8}{9072}(-35E_2^4+7E_2^2E_4+10E_4^2+18E_2E_6).
\end{aligned} \tag{4.46}$$

These results in turn allow us to compute $\mathcal{W}_0'$ in a v expansion,

$$\mathcal{W}_0' = 2\pi i a\left(1 - \frac{v}{6}(\pi^2 E_2 + 3\wp) - \frac{v^2}{72}(2E_2^2\pi^4 - E_4\pi^4 + 6E_2\pi^2\wp + 9\wp^2) + O(v^3)\right). \quad (4.47)$$

With $\mathcal{W}_0'$ in hand, the above procedure then readily yields the higher $\mathcal{W}_n'$ and $\mathcal{F}_n$ recursively. We list here results for the first few orders in ϵ_1 and v,

$$\begin{aligned}
\mathcal{W}_1' &= \frac{v}{4}\wp' + \frac{v^2}{12}(3\wp\wp' + E_2\pi^2\wp') + O(v^3)\,, \\
\partial_\tau\mathcal{F}_2 &= -\pi i\frac{1}{24}E_2 - v\pi^3 i\frac{1}{144}(E_2^2 - E_4) + O(v^2)\,, \\
\mathcal{W}_2' &= \frac{i}{48a\pi}(E_2\pi^2 + 3\wp) + \frac{i}{288a\pi}(2E_2^2\pi^4 - 13E_4\pi^4 + 6E_2\pi^2\wp + 117\wp^2)v + O(v^2)\,, \\
\mathcal{W}_3' &= -\frac{1}{64a^2\pi^2}\wp' - \frac{1}{96a^2\pi^2}(E_2\pi^2 + 21\wp)\wp' v + O(v^2)\,, \\
\partial_\tau\mathcal{F}_4 &= \pi i\frac{1}{4608a^2}(E_2^2 - E_4) + \pi^3 i\frac{1}{23040a^2}(5E_2^3 + 13E_2E_4 - 18E_6)v + O(v^2)\,, \\
\mathcal{W}_4' &= -\frac{i}{9216a^3\pi^3}\Big((2E_2^2 - 25E_4)\pi^4 + 6E_2\pi^2\wp + 225\wp^2\Big) \\
&\quad -\frac{i}{276480a^3\pi^3}\Big((60E_2^3 + 657E_2E_4 - 4748E_6)\pi^6 + 45\wp(4E_2^2 - 705E_4)\pi^4 \\
&\quad \times 6885E_2\wp^2\pi^2 + 160245\wp^3\Big)v + O(v^2)\,.
\end{aligned} \quad (4.48)$$

Following [10], we can use formula (4.43) to integrate for the leading expansion coefficient $\mathcal{F}_0$. We find

$$\begin{aligned}
\mathcal{F}_0 &= -a^2\log q + 2m^2\log\eta + \frac{1}{48}\frac{m^4}{a^2}E_2 + \frac{1}{5760}\frac{m^6}{a^4}(5E_2^2 + E_4) \\
&\quad + \frac{1}{2903040}\frac{m^8}{a^6}(175E_2^3 + 84E_2E_4 + 11E_6) + O\left(\left(\frac{m}{a}\right)^{10}\right).
\end{aligned} \quad (4.49)$$

This coincides up to normalization with the prepotential determined in [27] via Seiberg-Witten techniques. We can go beyond this result by integrating the expansion coefficients $\mathcal{F}_{2n}$, obtaining expressions for the couplings $F^{(n,0)}$ in the dual gauge theory:

$$\mathcal{F}_2 = -\frac{\log\eta}{2} - \frac{E_2}{96}\frac{m^2}{a^2} + O\left(\left(\frac{m^2}{a^2}\right)^2\right), \quad (4.50)$$

$$\mathcal{F}_4 = \frac{E_2}{768a^2} + \frac{(5E_2^2 + 9E_4)}{7680a^2}\frac{m^2}{a^2} + O\left(\left(\frac{m^2}{a^2}\right)^2\right), \quad (4.51)$$

$$\mathcal{F}_6 = -\frac{5E_2^2 + 13E_4}{368640a^4} - \frac{(35E_2^3 + 168E_2E_4 + 355E_6)}{9289728a^4}\frac{m^2}{a^2} + O\left(\left(\frac{m^2}{a^2}\right)^2\right). \quad (4.52)$$

These solutions are perturbative in the parameter m/a, and non-perturbative in the modular parameter q. In the massive case, we generate an infinite series of primitives of polynomials in the Eisenstein series E_2, E_4, and E_6 for each expansion coefficient $\mathcal{F}_n$, $n > 2$. Note that the same amplitudes were obtained in [20] in closed form in terms of Eisenstein series in the effective coupling.

Changing the boundary condition. In this subsection, we want to analyze the role of the choice of cycle in the boundary conditions. It is clear from the modularity of the differential equation that modular transformations act on the space of solutions of the differential equation (see e.g. [50]). It is therefore natural to ask how the boundary condition satisfied by a given solution changes under modular transformations.

To answer this question, let us introduce a basis of cycles (α, β) of the torus, such that in the lattice representation, the path $[0,1]$ is a representative of α and $[0,\tau]$ a representative of β. We denote a solution of the differential equation (4.22) by

$$(w_{\gamma,a}, f_{\gamma,a}) \,, \tag{4.53}$$

where

$$\mathcal{W}(z|\tau) = w_{\gamma,a}(z|\tau) \,, \quad \mathcal{F}(\tau) = f_{\gamma,a}(\tau) \,, \tag{4.54}$$

and we impose the boundary condition on the cycle γ

$$\oint_\gamma \partial_z w_{\gamma,a}(z|\tau)\, dz = 2\pi i a(\tau) \,. \tag{4.55}$$

Note that by an extension of our above analysis to general boundary conditions, the function $\partial_z w_{\gamma,a}$ has no residue, and hence the integral with regard to a cycle is well-defined.

Given a solution $(w_{\alpha,a}, f_{\alpha,a})$ associated to the α cycle, define new functions:

$$w(z|\tau) = w_{\alpha,a}\left(\frac{z}{c\tau+d}\middle|\frac{a\tau+b}{c\tau+d}\right) \,, \quad f(\tau) = f_{\alpha,a}\left(\frac{a\tau+b}{c\tau+d}\right) \,. \tag{4.56}$$

In the following, we write

$$z' = \frac{z}{c\tau+d} \,, \quad \tau' = \frac{a\tau+b}{c\tau+d} = g(\tau) \,. \tag{4.57}$$

By

$$\partial_z w(z|\tau) = \frac{1}{c\tau+d}\partial_{z'} w_{\alpha,a}(z'|\tau') \,, \quad \partial_\tau f(\tau) = \frac{1}{(c\tau+d)^2}\partial_{\tau'} f_{\alpha,a}(\tau') \tag{4.58}$$

and taking into account the transformation properties of the Weierstrass function $\wp$, we see that the pair (w, f) is a solution to the differential equation (4.22) as well. To determine the boundary condition, we compute

$$\oint_{d\alpha+c\beta} dz\, \partial_z w(z|\tau) = \int_0^{c\tau+d} dz\, \partial_z w(z|\tau) = \int_0^1 dz'\, \partial_{z'} w_{\alpha,a}(z'|\tau') = 2\pi i a(\tau') \,.$$

Defining a cycle $\gamma = d\alpha + c\beta = g(\alpha)$, we can hence identify the solution (w, f) as

$$(w, f) = (w_{\gamma, a\circ\tau'}, f_{\gamma, a\circ\tau'}) \,. \tag{4.59}$$

We conclude that the action of $\mathrm{SL}(2,\mathbb{Z})$ on the space of solutions is given by

$$g \in \mathrm{SL}(2,\mathbb{Z}) : \quad (w_{\alpha,a}, f_{\alpha,a}) \mapsto (w_{g(\alpha),a\circ g}, f_{g(\alpha),a\circ g}). \tag{4.60}$$

When the function $a(\tau)$ does not depend on τ, it is always the same constant that appears on the right hand side of the boundary conditions on the various cycles γ.

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Colliding insertions. Finally, we want to address a subtle point. A glance at equation (4.22), keeping in mind the double pole of the Weierstrass $\wp$ function at $z = 0$, convinces us that the $z \to 0$ limit and the $\epsilon_1 \to 0$ limit are not independent. Our perturbation ansatz (4.23) tacitly assumed z fixed away from $z = 0$. Indeed, the perturbative solution we derive diverges as $z \to 0$.[7] In this limit, as the degenerate operator approaches V_{h_m}, the operator product expansion of the two operators suggests the behavior

$$\langle V_{(2,1)}(z) V_{h_m}(0)\rangle_\tau \underset{z\to 0}{\sim} z^{-h_m - h_{(2,1)} + h_\pm} \underset{b\to 0}{\longrightarrow} z^{\frac{1}{2}\pm\frac{m}{\epsilon_1}}\,, \tag{4.61}$$

where $h_\pm$ represents the conformal dimension of the two possible fusion products of the degenerate operator with the mass insertion. In this limit, it is more appropriate to directly analyze the $z \to 0$ limit of the differential equation (4.22) at fixed ϵ_1. This gives rise to the solutions $\mathcal{W} = \frac{1}{2}(\epsilon_1 \pm 2m)\log z$, indeed reproducing the expected behavior near $z = 0$.

5 Conclusions

In this paper, we have studied consequences of the correspondence [1] between two-dimensional conformal field theory and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions away from the weak coupling limit on the gauge theory side.

Combining recursion relations satisfied by the toroidal conformal blocks with modular results for $\mathcal{N} = 2^*$ and $\mathcal{N} = 4$ gauge theory [20], we have obtained new insights on both sides of the correspondence. On the one hand, we have demonstrated that the gauge theory results imply an infinite sequence of constraints, non-perturbative in $q = e^{2\pi i\tau}$, on the residua of the conformal blocks. On the other, we have seen how the recursion relation satisfied by the conformal block provides all genus results for the topological string order by order in q. Furthermore, we have identified the scaling of the topological string partition function $F^{(n,g)}$ in the scalar vacuum expectation value a in the field theory limit, $F^{(n,g)} \sim a^{2-2(g+n)}$, as the coefficients, order by order in q, of a geometric series in $\frac{1}{a^2}$. The resulting poles are exactly cancelled by zeros in the contribution to the partition function stemming from maps that do not wrap the base of the engineering geometry.

We have further demonstrated that the holomorphic anomaly results for $\mathcal{N} = 2^*$ in the semi-classical limit can be reproduced by a null vector decoupling equation of conformal field theory. The modularity properties, while manifest in the former approach, arise highly non-trivially in the solutions to this equation. The action of the modular group maps the intermediate conformal dimension off the real axis. We are thus led to consider one-point conformal blocks parameterized by complex intermediate conformal dimension (the image of the complex adjoint vacuum expectation value a on the gauge theory side). These form a larger class of conformal blocks than the class that underlies unitary (Liouville) conformal field theory. Unlike the traditional case, they exhibit modularity before being assembled into a physical one-point function correlator. This behavior is complementary to the intricate modular behavior of the subset of blocks that close into each other under the S-move in unitary Liouville theory [51, 52]. A detailed perturbative analysis of the null vector decoupling equation on the one-point toroidal conformal block

[7] We would like to thank Don Zagier for stressing this point.

allowed us to describe the solutions in terms of a recursion procedure, providing an explicit answer to all orders in q in the expansion parameters ϵ_1 and m/a, with m determining the conformal dimension of the insertion and a the propagating momentum.

Our analysis allows for many generalizations to other $\mathcal{N} = 2$ theories. A prime candidate is $\mathcal{N} = 2$ SU(2) gauge theory with $N_f = 4$ fundamental flavors to which we hope to return in the near future.

We believe that our analysis is one more indication that complexifying parameters in conformal field theories with continuous spectra is fruitful, as for instance indicated by the analysis of the structure of Verma modules [53], the appearance of interesting (analytically continued) operators for which there is no corresponding state in the theory [44], the analysis of analytically continued operator product expansions and correlation functions [45, 54] and the role of discrete states in the modularity of theories with continuous spectrum [55, 56].

Further avenues for exploration are to explain the intriguing analytical structure found for the topological string amplitudes directly in that framework. It would also be interesting to analyze the link between the modular properties and the wave-function interpretation of the conformal block with insertions in the analytic continuation of Chern-Simons theory further. Finally, the occurrence of quasi-modular forms in our analysis begs the questions whether their almost holomorphic brethren also play a role in this context, and whether the holomorphic anomaly equations can be derived purely within conformal field theory.

Acknowledgments

We would like to thank Jaume Gomis, Alessandro Tanzini and Don Zagier for interesting discussions and correspondence. Our work is supported in part by the grant ANR-09-BLAN-0157-02.

A Properties of modular and elliptic functions

In this appendix, we collect useful identities and modular and other properties of modular forms and elliptic functions.

A.1 Identities

The following identities hold:

$$\begin{aligned} \wp(z) &= (\theta_1'/\theta_1)^2 - \theta_1''/\theta_1 - 2\eta_1 \,, \\ \eta_1 &= -2\pi i \, \partial_\tau \log \eta(\tau) = \frac{\pi^2}{6} E_2(\tau) \,, \\ \partial_z^2 \theta_1 &= 4\pi i \partial_\tau \theta_1 \,. \end{aligned} \tag{A.1}$$

The η-function and θ-function have the following modular and elliptic properties:

$$\eta(\tau+1) = e^{\frac{i\pi}{12}} \eta(\tau) \,, \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau) \,, \tag{A.2}$$

$$\theta_1(z|\tau+1) = e^{\frac{\pi i}{4}} \theta_1(z|\tau) \,, \qquad \theta_1\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i\sqrt{-i\tau} e^{\frac{\pi i z^2}{\tau}} \theta_1(z|\tau) \,, \tag{A.3}$$

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau)\,, \qquad \theta_1(z+\tau|\tau) = -e^{-\pi i\tau}e^{-2\pi i z}\theta_1(z|\tau)\,. \tag{A.4}$$

The modular behavior of the second Eisenstein series is

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + c(c\tau+d)\frac{6}{\pi i}\,. \tag{A.5}$$

The derivatives of the Eisenstein series are

$$q\partial_q E_2 = \frac{E_2^2 - E_4}{12}\,, \qquad q\partial_q E_4 = \frac{E_2 E_4 - E_6}{3}\,, \qquad q\partial_q E_6 = \frac{E_2 E_6 - E_4^2}{2}\,. \tag{A.6}$$

A.2 The monodromies of powers of the Weierstrass function

In this subsection, we determine the monodromies of powers of the Weierstrass function $\wp$ along a cycle γ of an elliptic curve [57, 58]. The elliptic curve is given by the equation $Y^2 = 4X^3 - g_2 X - g_3$. The periods along γ of a basis of differentials of the second kind will be denoted by

$$\begin{aligned} \omega &= \int_\gamma dz = \frac{1}{2}\oint \frac{dX}{Y}\,, \\ \xi &= -\frac{1}{2}\oint \wp(z)dz = -\frac{1}{2}\oint \frac{XdX}{Y}\,. \end{aligned} \tag{A.7}$$

We can then write the monodromies of powers of the Weierstrass function $\wp$ along the cycle γ,

$$K_n = \oint_\gamma \wp^n(z)dz\,, \tag{A.8}$$

as a linear combination

$$K_n = A_n(g_2,g_3)\omega - B_n(g_2,g_3)\xi, \tag{A.9}$$

where A_n and B_n are polynomials of degree $2n$ and $2n-2$ respectively, where g_2 has degree 4 and g_3 has degree 6, and the coefficients of the polynomials are positive and rational. Moreover, they can be determined from the initialization

$$K_0 = 2\omega\,, \qquad K_1 = -2\xi\,, \qquad K_2 = \frac{1}{6}g_2\omega\,, \tag{A.10}$$

and the recursion relation

$$(8n-4)K_n = (2n-3)\,g_2\,K_{n-2} + (2n-4)\,g_3\,K_{n-3}\,. \tag{A.11}$$

The first few integrals of powers of the Weierstrass function are

$$\begin{aligned} K_3 &= \frac{1}{10}(2g_3\omega - 3g_2\xi)\,, \\ K_4 &= \frac{1}{168}(5g_2^2\omega - 48g_3\xi)\,, \\ K_5 &= \frac{1}{120}(8g_2g_3\omega - 7g_2^2\xi)\,, \\ K_6 &= \frac{1}{12320}((75g_2^3 + 448g_3^2)\omega - 1392g_2g_3\xi)\,, \\ K_7 &= \frac{1}{43680}(866g_2^2g_3\omega - (539g_2^3 + 2400g_3^2)\xi)\,. \end{aligned} \tag{A.12}$$

They can also be determined from the Halphen coefficients which determine the powers of the Weierstrass function $\wp$ in terms of its even derivatives. When $g_3 = 0$ or when $g_2 = 0$, the recursion relation can be solved explicitly.

When we use these integrals in the bulk of the paper, we work on a torus parameterized by a complex parameter z where $z \equiv z + 1 \equiv z + \tau$. Moreover, we restrict to determining the periods along the cycle α parameterized by $z \in [0, 1]$. Our initializations therefore read

$$\begin{aligned} K_0 &= \oint_\alpha dz = 1 = 2\omega\,, \\ K_1 &= \oint_\alpha \wp(z)dz = -2\eta_1 = -\frac{\pi^2}{3}E_2 = -2\xi\,, \\ K_2 &= \frac{\pi^4}{9}E_4\,. \end{aligned} \tag{A.13}$$

We moreover have the relations

$$\begin{aligned} g_2 &= \frac{4\pi^4}{3}E_4\,, \\ g_3 &= \frac{8\pi^6}{27}E_6\,. \end{aligned} \tag{A.14}$$

Under these circumstances, the first few period integrals of powers of the Weierstrass function are given by

$$\begin{aligned} K_0 &= 1\,, \\ K_1 &= \frac{\pi^2}{3}(-E_2)\,, \\ K_2 &= \frac{\pi^4}{9}E_4\,, \\ K_3 &= \frac{\pi^6}{135}(-9E_2E_4 + 4E_6)\,, \\ K_4 &= \frac{\pi^8}{567}(15E_4^2 - 8E_2E_6)\,, \\ K_5 &= \frac{\pi^{10}}{1215}(-21E_2E_4^2 + 16E_4E_6)\,, \\ K_6 &= \frac{\pi^{12}}{280665}(2025E_4^3 - 2088E_2E_4E_6 + 448E_6^2)\,, \\ K_7 &= \frac{\pi^{14}}{995085}(-4851E_2E_4^3 - 800E_2E_6^2 + 5196E_4^2E_6)\,. \end{aligned} \tag{A.15}$$

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