M. K. Sain, P. J. Antsaklis, B. F. Wyman, R. R. Gejji and J. L. Peczkowski, "The Total Synthesis Problem of Linear Multivariable Control-Part II: Unity Feedback and the Design Morphism" (Invited), P roc. o f t he 2 0th I EEE C onf. o n D ecision a nd C ontro l , pp. 875-884, San Diego, ÇA, Dec. 16-18, 1981.

THE TOTAL SYNTHESIS PROBLEM OF LINEAR MULTIVARIABLE CONTROL PART II: UNITY FEEDBACK AND THE DESIGN MORPHISM*

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Abstract

Under broad assumptions, it is known [1] that there is, in general, no "separation principle" to guarantee optimality of a division between control law design and filtering of plant uncertainty. It is possible, however, to develop parameterizations of nominal responses and to examine their capabilities in the feedback situation. In an earlier study, the authors presented a coordinate-free approach [34] to the Nominal Design Problem (NDP), which addresses control action and plant output syntheses, independently of the controller configuration. It was shown that NDP depended only upon plant structural matrices and a single design morphism. This paper studies the role of this design morphism in unity feedback synthesis (UFS). NDP, together with feedback synthesis, is understood as a Total Synthesis Problem.

Introduction

From an intuitive point of view, acceptable response is the hallmark of successful design in a linear multivariable control system. Even if we have perfect plant knowledge, and no system disturbances, the development of reasonable specifications and their achievement with available equipment is not a trivial matter. When, however, the plant is unstable, or uncertain, or acted upon by disturbances, the choice of feedback realization for the controller may place the response goals in competition with new goals such as internal stability, sensitivity suppression, and disturbance rejection.

Under broad assumptions, Zames [1] has observed that there is, in general, no "separation principle" to guarantee optimality of a division between control law design and filtering of plant uncertainty. It thus makes sense to develop parameterizations of response and to examine their capabilities in the feedback situation. Because feedback features, such as internal stability, are certainly dependent upon controller configuration, it is desirable that a response parameterization not depend upon controller configuration. In that way, it would permit comparisons and contrasts among specific control arrangements. This suggests that response be studied as a problem in synthesis.

Control synthesis has, indeed, a rich history. As early as 1951, Guillemin proposed that synthesis of feedback control systems should involve a determination of the closed loop transfer function from specifications, followed by construction of appropriate compensation networks [2]. In due course, Truxal [3] discussed the Guillemin method, as it related to basic feedback issues of plant pole cancellation, imperfect cancellation, and controller complexity. Not surprisingly, some of the same ideas then appeared in texts on sampled-data control [4] and are now re-occurring in modern works on digital control [5].

As one solves classical transfer function equations for the compensation required by a given closed loop specification, one of course inverts the plant, thereby obtaining the equivalent series compensator. By 1957, at least for stable plants, authors [6] began to discuss equivalent series compensation as a parameter of feedback synthesis.

The literature of many inputs and many outputs began to follow the trend [7,8,9,10]. As part of the general state space development in the area of control, the problem of synthesizing a given closed loop command/output-response map became known as model matching, and was solved in that context by Morse [11]. Subsequently, model matching has been studied from an input/output view [12,13,14,15], where focus was placed upon the matrix equation

$$[Z_1(s)][Z(s)] = [Z_2(s)], \qquad (1)$$

with $[Z_i(s)]$, $i = 1,2$, being given rational matrices and with $[Z(s)]$ to be determined. A principal issue was the fact that $[Z(s)]$ might be required to have certain properties, such as being stable or proper, while $[Z_i(s)]$, $i = 1,2$, might not be so restricted. Investigators then began the use of transfer function rings and subrings [16,17], after which began a gradual coalescence [18] with methods based upon function spaces and operator algebras, as well as generalizations of the rings [19]. In 1977, working outside the ring context, in a mixed state space/transfer function format, Bengtsson [20] solved a broad class of problems concern-

*This research was supported in part by the Office of Naval Research, Contracts N00014-79-C-0475, P00001 and P00002, and by the National Aeronautics and Space Administration, under Grants NAG-2-34 and 2-35.

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ing feedback realization in a model matching context. Pernebo [21] extended this work on model matching, with the aid of matrices of rings.

In 1977, as part of an application study dealing with gas turbine control, Peczkowski and Sain [22] solved a model matching problem using transfer functions. Building upon this experience, Peczkowski, Sain, and Leake [23] proposed in 1979 the Total Synthesis Problem (TSP), wherein both the command/output-response and command/control-response are to be synthesized, subject to the plant constraint. From an algebraic viewpoint, TSP carries with it the idea of tradeoff between control response and output response, one of the important features of approaches based upon optimal control. From the outset, TSP was rooted in application studies, which have continued [24,25, 26,27,28,29,30,31,32].

A useful feature of the TSP idea is that it may be subdivided immediately into a Nominal Design Problem (NDP), which is not dependent upon specific controller structures, and a Feedback Synthesis Problem (FSP), which is. Gejji [33] made the first study of this separation, in a semi-coordinate-free context. Gejji found that NDP was characterized in terms of the plant structural matrices and a single, "good" transfer function matrix. In 1981, this NDP work was extended [34] to a coordinate-free treatment, in which the role of the plant zero module [35] was explicity addressed.

In this paper, we begin a study of FSP---for the unity feedback case. For ease of reading, the terminology of [34] is summarized. The treatment is coordinate-free. Added matrix formulae, and discussion, may be found in [36].

Notation and Preliminaries

Denote an arbitrary field by k. The principal ideal domain of polynomials in one indeterminate s, with coefficients in k, is $k[s]$. The quotient field of $k[s]$ is $k(s)$.

For $i = 1,2$, let V_i be a k-vector space. Then

$$V_1 \otimes_k V_2 \tag{2}$$

is a k-vector space, the tensor product of V_1 and V_2, regarded as vector spaces over k. Since $k[s]$ is a k-vector space, one can write

$$k[s] \otimes_k V \tag{3}$$

for V any k-vector space of finite dimension. Denote this k-vector space by $V[s]$. It can be shown that $V[s]$ is also a $k[s]$-module. Similarly,

$$k(s) \otimes_k V \tag{4}$$

admits the structure of a $k(s)$-vector space, which we denote by $V(s)$. One has an insertion $i : V[s] \to V(s)$, which is a morphism of $k[s]$-modules. For W another k-vector space of finite dimension, develop $W[s]$ and $W(s)$, as above. One has the projection $p : W(s) \to W(s)/W[s]$, a morphism of $k[s]$-modules, onto the quotient module $W(s)/W[s]$.

Let $L(s) : V(s) \to W(s)$ be a morphism of $k(s)$-vector spaces. We call $L(s)$ a transfer function.

Remark: $L(s)$ is not a matrix. However, if we choose bases in V and in W, these bases induce bases in $V(s)$ and in $W(s)$ and define a matrix, denoted $[L(s)]$, for the morphism $L(s)$.

Because $V(s)$ and $W(s)$ are $k[s]$-modules as well, the transfer function is also a morphism of $k[s]$-modules, so that the Kalman input/output map

$$L^{\#}(s) = p \circ L(s) \circ i, \tag{5}$$

a morphism of $k[s]$-modules, is defined. It can be shown that

$$V[s]/\ker L^{\#}(s) \tag{6}$$

is a torsion $k[s]$-module, which we term the pole module of $L(s)$ and denote by $X(L)$.

Let $S_g \subset k[s]$ be closed under ring multiplication and multiplicative unit; and let S_g exclude the additive unit. Then $p(s) \in k[s]$ is a good polynomial if $p(s) \in S_g$. Now denote by $m(L)$ the minimal polynomial of the pole module $X(L)$. We say that $L(s)$ is a good transfer function if $m(L)$ is a good polynomial.

The Nominal Design Problem

Let R, U, and Y be q, m, and p-dimensional k-vector spaces. On these spaces, we can develop $k[s]$-modules $R[s]$, $U[s]$, and $Y[s]$, as well as $k(s)$-vector spaces $R(s)$, $U(s)$, and $Y(s)$. We shall understand $R(s)$ as a space of exogenous, reference or command inputs; $U(s)$ as a space of control inputs to a plant with transfer function $P(s) : U(s) \to Y(s)$; and $Y(s)$ as a space of plant outputs.

The Nominal Design Problem (NDP) is to find a general solution for all pairs $(M(s), T(s))$ of good transfer functions which satisfy the commutative diagram of Figure 1. Intuitively, NDP deals

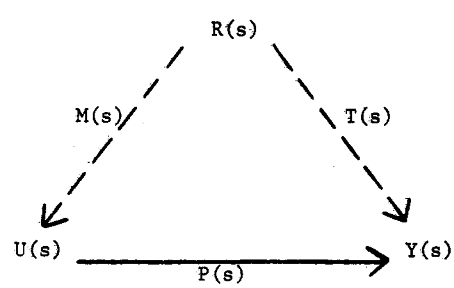

Figure 1. The Nominal Design Problem.

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with the problem of designing, simultaneously, the complete set of controlled outputs from a plant and a corresponding complete set of control inputs, subject to the plant constraint. $M(s)$ and $T(s)$, respectively, characterize these sets, which would be the entries in $[M(s)]$ and in $[T(s)]$, for specific bases.

Remark: $P(s)$ is not required to be a good transfer function. In case it were, $[P(s)]$, $[M(s)]$, and $[T(s)]$ have entries in the same ring, so that the theory of free modules could be applied directly.

The solution to NDP can be given in terms of the kernel of an abstract morphism. We sketch briefly here the concepts involved in this characterization.

Let $S_g^{-1}k[s]$ be the localization of $k[s]$ induced by S_g. It satisfies

$$k[s] \subset S_g^{-1}k[s] \subset k(s). \quad (7)$$

This localization of $k[s]$ by S_g can be used to induce a localization of a $k[s]$-module $V[s]$, according to the construction

$$S_g^{-1}V[s] = (S_g^{-1}k[s]) \otimes_{k[s]} V[s]. \quad (8)$$

Morphisms, also, can then be localized. Indeed, if $J(s) : V[s] \rightarrow W[s]$ is a morphism of $k[s]$-modules, there is a morphism

$$S_g^{-1}J(s) : S_g^{-1}V[s] \rightarrow S_g^{-1}W[s] \quad (9)$$

of $S_g^{-1}k[s]$-modules with action given by

$$S_g^{-1}J(s)\ \{\sum_{i=1}^{n} (p_i(s),q_i(s)) \otimes_{k[s]} v_i(s)\} = \sum_{i=1}^{n} (p_i(s),q_i(s)) \otimes_{k[s]} J(s)v_i(s). \quad (10)$$

Return now to the solution of NDP. Write $H(R,U)$ for the k-vector space $Hom_k(R,U)$ of morphisms $R \rightarrow U$ of k-vector spaces. Next consider $Hom_{k[s]}(R[s],U[s])$, which is naturally isomorphic to

$$H(R,U)[s] = k[s] \otimes_k H(R,U), \quad (11)$$

and identify the two. Similarly, identify $Hom_{k(s)}(R(s),U(s))$ with

$$H(R,U)(s) = k(s) \otimes_k H(R,U). \quad (12)$$

Denote localization in the manner

$$H(R,U)(s)_g = S_g^{-1}H(R,U)[s], \quad (13)$$

so that

$$H(R,U)[s] \subset H(R,U)(s)_g \subset H(R,U)(s). \quad (14)$$

Similar developments follow on $H(R,Y)$ and on $H(U,Y)$. On the $k(s)$-vector space level, define a morphism

$$F : H(R,U)(s) \oplus H(R,Y)(s) \rightarrow H(R,Y)(s) \quad (15)$$

with action

$$F(M(s),T(s)) = P(s) \circ M(s) - T(s). \quad (16)$$

Pairs $(M(s),T(s))$, on the $k(s)$-vector space level, satisfy Figure 1 if and only if they lie in ker F, which is an mq-dimensional $k(s)$-vector space, denoted by $K(s)$. Restricted to $H(R,U)(s)_g \oplus H(R,Y)(s)_g$ and $H(R,U)[s] \oplus H(R,Y)[s]$, respectively, F has kernels $K(s)_g$ and $K[s]$, respectively, which are free $S_g^{-1}k[s]$-and free $k[s]$-modules of rank mq, respectively, satisfying

$$K[s] \subset K(s)_g \subset K(s). \quad (17)$$

Remark: A $k[s]$-basis for $K[s]$ gives a $S_g^{-1}K[s]$-basis for $K(s)_g$. This means that NDP computations may be carried out in $k[s]$. Thus, although $K[s]$ is only a $k[s]$-submodule of $K(s)_g$, it can be used to generate $K(s)_g$. This generation process may be intuitively viewed as pole assignment, together with gain and zero adjustment.

$K(s)_g$ characterizes the solutions of NDP. It turns out, however, that the pairs of good transfer functions in $K(s)_g$ can be more explicitly described in terms of a fixed pair of $k[s]$-module morphisms and a single good transfer function.

The Design Morphism

Given $P(s)$, there exist morphisms $D(s) : U[s] \rightarrow U[s]$ and $N(s) : U[s] \rightarrow Y[s]$ of $k[s]$-modules, with $D(s)$ monic, such that (1) the diagram of Figure 2 commutes and (2) there exist morphisms

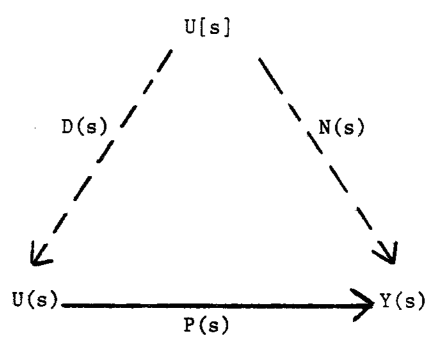

Figure 2. Right-Prime Factorization.

$A(s) : U[s] \rightarrow U[s]$ and $B(s) : Y[s] \rightarrow U[s]$ of $k[s]$-modules with the property

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$$A(s) \circ D(s) + B(s) \circ N(s) = 1_{U[s]}. \qquad (18)$$

The pair $(N(s),D(s))$ is known as a right-prime factorization of $P(s)$.

According to our foregoing discussion, Figure 2 can be localized. Write $D_g(s)$ for $S_g^{-1}D(s)$, $N_g(s)$ for $S_g^{-1}N(s)$, and $U(s)_g$ for $S_g^{-1}U[s]$. Also observe that $S_g^{-1}U(s)$, with $U(s)$ understood as a k[s]-module, is equal to $U(s)$. The localized diagram is that of Figure 3.

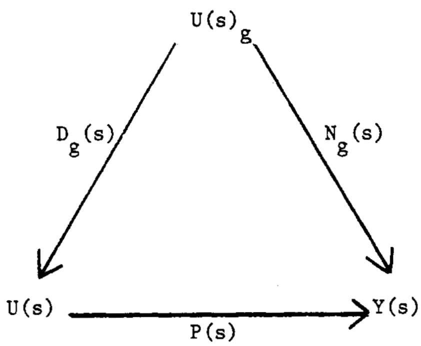

Figure 3. Localization of Figure 2.

Remark: Figure 3 is a natural way of viewing what might be defined as a good right-prime factorization of $P(s)$. Again we see that the classical right-prime factorization is adequate to generate good right-prime factorizations which may be of interest.

Next consider a good transfer function $X_g(s) : R(s)_g \to U(s)_g$. It is clear that one may construct solutions to NDP according to the scheme indicated in Figure 4. Indeed,

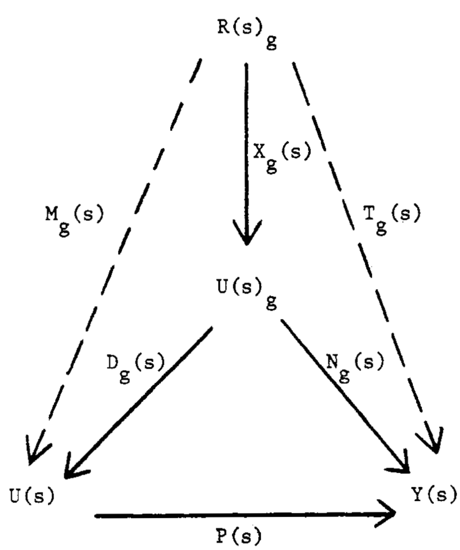

Figure 4. $X_g(s)$ => NDP Solution $(M_g(s),T_g(s))$.

$(M_g(s),T_g(s))$ generate a good transfer function pair $(M(s),T(s))$ satisfying Figure 1 by a process of localization up to $k(s)$. It can also be shown [34] that this is the only way that NDP solution pairs $(M(s),T(s))$ are generated. Thus, in Figure 5, if $(M(s),T(s))$ is a solution to NDP, there must exist a good transfer function $X(s)$ such that the diagram commutes.

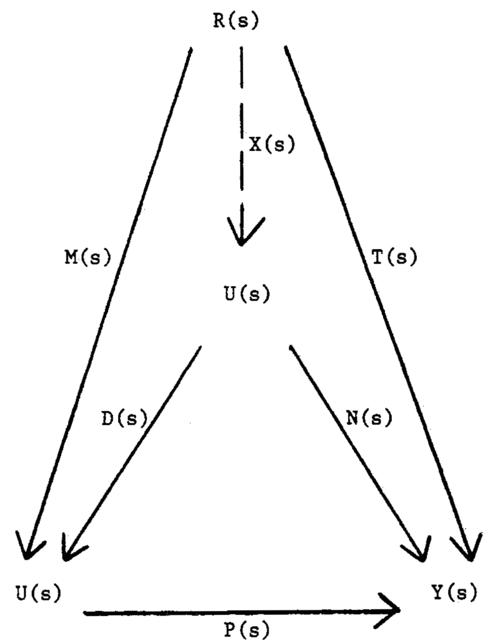

Figure 5. NDP Solution $(M(s),T(s))$=> Good $X(s)$.

Remark: We use the same symbol for $D(s) : U[s] \to U[s]$ and its localization $D(s) : U(s) \to U(s)$, to avoid proliferation of symbols; and we follow the same convention for $N(s)$. The diagrams make the situation clear.

The right-prime pair $(N(s),D(s))$, and the good transfer function $X(s)$, give a specific characterization of solutions to NDP. In a genuine sense, $X(s)$ contains all the design freedom available. We call $X(s)$ the design morphism.

Output Feedback Synthesis (OFS)

By an output feedback, we shall understand a morphism $C(s) : R(s) \oplus Y(s) \to U(s)$ of k(s)-vector spaces. Let i_R and i_Y be the insertions of $R(s)$ and $Y(s)$ into the biproduct. Then each output feedback determines morphisms $R(s) \to U(s)$ and $Y(s) \to U(s)$ of k(s)-vector spaces by

$$C_R(s) = C(s) \circ i_R \ , \quad C_Y(s) = C(s) \circ i_Y. \qquad (19)$$

Moreover, each pair $(C_R(s),C_Y(s))$ of such morphisms induces a unique output feedback. In view of this, output feedback generates the step

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$$y(s) = P(s)u(s) \tag{20}$$

$$= P(s)\,\{C_R(s)r(s) + C_Y(s)y(s)\}, \tag{21}$$

which may be written as

$$\{1_Y(s) - P(s) \circ C_Y(s)\}y(s) = P(s) \circ C_R(s)r(s). \tag{22}$$

We shall say that $C(s)$ is an <u>output feedback synthesis</u> (OFS) <u>of</u> $M(s)$ if $\{1_{Y(s)} - P(s) \circ C_Y(s)\}$ is an isomorphism and if

$$M(s) = (1_{U(s)} - C_Y(s) \circ P(s))^{-1} \circ C_R(s). \tag{23}$$

By itself, output feedback synthesis of $M(s)$ poses no problem, as we may choose $C_R(s)$ to be $M(s)$ and $C_Y(s)$ to be the zero morphism.

In order to attack a more interesting problem, both in application and in theory, we choose to constrain $C(s)$. In particular, we specify that

$$C(s)(r(s),y(s)) = G(s)\,\{r(s) - y(s)\} \tag{24}$$

for $G(s) : R(s) \to U(s)$ a morphism of $k(s)$-vector spaces. Here $(C_R(s),C_Y(s))$ is given by $(G(s), - G(s))$. We can refer to this well known case as <u>unity feedback synthesis</u> (UFS). For UFS, $C(s) = 0 \iff G(s) = 0$; thus trivialities do not occur. $G(s)$ determines a UFS for $M(s)$ if $(1_{Y(s)} + P(s) \circ G(s))$ is an isomorphism and if

$$M(s) = (1_{U(s)} + G(s) \circ P(s))^{-1} \circ G(s). \tag{25}$$

<u>Remark</u>: A design morphism $X(s)$ may fail to meet the conditions for UFS. Notice also that UFS requires R and Y to be identified, or, if preferred, to be regarded as isomorphic copies of one another.

In what follows, it is useful to have in mind the following Lemma.

<u>Lemma 1</u>. Let $\mathcal{R}$ be a ring, and let $L_1 : M_1 \to M_2$ and $L_2 : M_2 \to M_1$ be morphisms of $\mathcal{R}$-modules. Then

$$(1_{M_2} + L_1 \circ L_2) : M_2 \to M_2 \tag{26}$$

has an inverse morphism if and only if

$$(1_{M_1} + L_2 \circ L_1) : M_1 \to M_1 \tag{27}$$

has an inverse morphism.

Proof: For necessity suppose that $(1_{M_2} + L_1 \circ L_2)^{-1} : M_2 \to M_2$ exists and is a morphism of $\mathcal{R}$-modules. Then

$$\{1_{M_1} - L_2 \circ (1_{M_2} + L_1 \circ L_2)^{-1} \circ L_1\} : M_1 \quad M_1 \tag{28}$$

is a morphism of $\mathcal{R}$-modules. Futhermore,

$$(1_{M_1} + L_2 \circ L_1) \circ \{1_{M_1} - L_2 \circ (1_{M_2} + L_1 \circ L_2)^{-1} \circ L_1\}$$
$$= 1_{M_1} - L_2 \circ (1_{M_2} + L_1 \circ L_2)^{-1} \circ L_1 + L_2 \circ L_1$$
$$- L_2 \circ L_1 \circ L_2 \circ (1_{M_2} + L_1 \circ L_2)^{-1} \circ L_1$$
$$= 1_{M_1} + L_2 \circ \{- (1_{M_2} + L_1 \circ L_2)^{-1} + 1_{M_2}$$
$$- L_1 \circ L_2\,(1_{M_2} + L_1 \circ L_2)^{-1}\} \circ L_1$$
$$= 1_{M_1} + L_2 \circ \{1_{M_2} - (1_{M_2} + L_1 \circ L_2)$$
$$\circ\,(1_{M_2} + L_1 \circ L_2)^{-1}\} \circ L_1$$
$$= 1_{M_1}, \tag{29}$$

and a like calculation can be made from the right. Thus $(1_{M_1} + L_2 \circ L_1)$ has an inverse morphism as well. A similar sequence establishes sufficiency. In the process, we have established the explicit expression relating the inverses.

<u>Corollary 2</u>. In Lemma 1, when the inverse morphisms exist,

$$(1_{M_1} + L_2 \circ L_1)^{-1} = 1_{M_1} - L_2 \circ (1_{M_2} + L_1 \circ L_2)^{-1} \circ L_1. \tag{30}$$

<u>Corollary 3</u>. $(1_{U(s)} + G(s) \circ P(s)) : U(s) \to U(s)$ is an isomorphism when and only when $(1_{R(s)} + P(s) \circ G(s)) : R(s) \to R(s)$ is an isomorphism.

Conditions on the design morphism $X(s)$, for a UFS in NDP, are mild, as shown in the next Proposition.

<u>Proposition 4</u>. A solution $(M(s),T(s))$ to the Nominal Design Problem, characterized by the right-prime pair $(N(s),D(s))$ and the design morphism $X(s)$, admits unity feedback synthesis if and only if $(1_{R(s)} - N(s) \circ X(s)) : R(s) \to R(s)$ is an isomorphism of $k(s)$-vector spaces.

Proof: In view of Figure 5, $N(s) \circ X(s)$ is $T(s)$. For sufficiency, choose

$$G(s) = M(s) \circ \{1_{R(s)} - T(s)\}^{-1}, \tag{31}$$

which achieves USF. For necessity, observe that

$$1_{R(s)} - T(s) = 1_{R(s)} - P(s) \circ M(s)$$
$$= 1_{R(s)} - P(s) \circ \{1_{U(s)} + G(s) \circ P(s)\}^{-1} \circ G(s)$$
$$= \{1_{R(s)} + P(s) \circ G(s)\}^{-1}. \tag{32}$$

<u>Remark</u>: The condition of this proposition would

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typically be met by adding "rolloff" to [X(s)].

Because of Proposition 4, we can picture NDP and UFS together in one commutative diagram, Figure 6. In this diagram, we have denoted the composition $P(s) \circ G(s)$ by $Q(s) : R(s) \to Y(s)$.

UFS in NDP does not yet fulfill application needs. What is needed is a concept of good UFS. Of course, a special case in point is the familiar idea of unity feedback synthesis with internal stability.

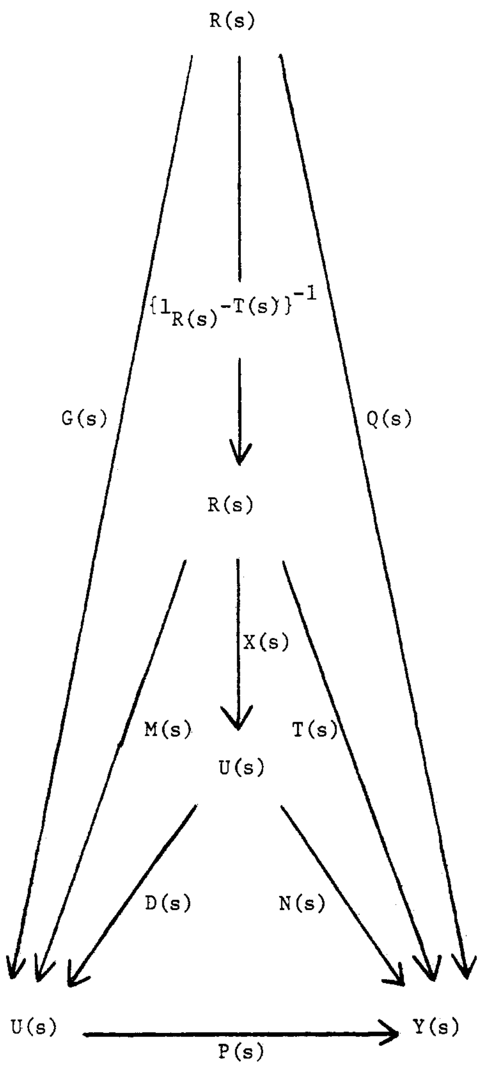

Figure 6. UFS for NDP.

Good Unity Feedback Synthesis (GUFS)

Given $G(s)$, there exist morphisms $\tilde{D}_G(s) : U[s] \to U[s]$ and $\tilde{N}_G(s) : R[s] \to U[s]$ of k[s]-modules, with $\tilde{D}_G(s)$ monic, such that (1) their localizations up to $k(s)$ make the diagram of Figure 7 commute and (2) there exist morphisms

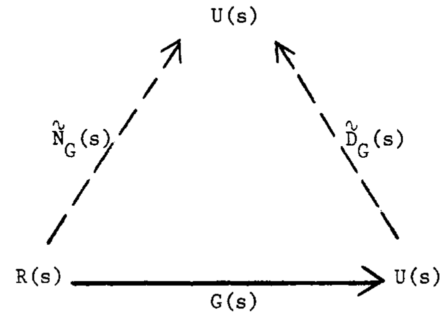

Figure 7. Left-Prime Factorization.

$\tilde{A}(s) : U[s] \to U[s]$ and $\tilde{B}(s) : U[s] \to R[s]$ of k[s]-modules with the property

$$\tilde{D}_G(s) \circ \tilde{A}(s) + \tilde{N}_G(s) \circ \tilde{B}(s) = 1_{U[s]}. \quad (33)$$

The pair $(\tilde{D}_G(s), \tilde{N}_G(s))$ is known as a left-prime factorization of $G(s)$. Using this factorization, we can give an alternate statement of the condition for UFS.

Proposition 5. Suppose that $(N(s),D(s))$ is a right-prime factorization of $P(s)$, and that $(\tilde{D}_G(s), \tilde{N}_G(s))$ is a left-prime factorization of $G(s)$. If $G(s)$ generates a unity feedback synthesis for $M(s)$, then $D_i(s)$, given by

$$\tilde{D}_G(s) \circ D(s) + \tilde{N}_G(s) \circ N(s) : U[s] \to U[s], \quad (34)$$

is a monomorphism of k[s]-modules.

Proof: Notice that the result is true when and only when localization of the morphism up to $k(s)$ produces an isomorphism $U(s) \to U(s)$ of $k(s)$-vector spaces. For UFS, $(1_{U(s)} + G(s) \circ P(s))$ must be an isomorphism. But

$$1_{U(s)} + G(s) \circ P(s) = \tilde{D}_G^{-1}(s) \circ$$

$$\{D_G(s) \circ D(s) + N_G(s) \circ N(s)\} \circ D^{-1}(s); \quad (35)$$

and the result follows.

The manner in which $X(s)$ is constrained by a requirement to achieve UFS can be expressed in terms of $D_i(s)$ and of $\tilde{N}_G(s)$.

Theorem 6. Suppose that the right-prime factorization $(N(s),D(s))$ and the design morphism $X(s)$ characterize a solution $(M(s),T(s))$ to NDP. If $G(s)$, described by its left-prime factorization $(\tilde{D}_G(s), \tilde{N}_G(s))$, generates a UFS for $M(s)$, then

$$X(s) = D_i^{-1}(s) \circ \tilde{N}_G(s). \quad (36)$$

Proof: The result follows from the calculation

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M. K. Sain, P. J. Antsaklis, B. F. Wyman, R. R. Gejji and J. L. Peczkowski, "The Total Synthesis Problem of Linear Multivariable Control-Part II: Unity Feedback and the Design Morphism" (Invited), Proc. of the 20th IEEE Conf. on Decision and Control, pp. 875-884, San Diego, CA, Dec. 16-18, 1981.

$$
\begin{aligned}
X(s) &= D^{-1}(s) \circ M(s) \\
&= D^{-1}(s) \circ \{1_{U(s)} + G(s) \circ P(s)\}^{-1} \circ G(s) \\
&= D^{-1}(s) \circ \{1_{U(s)} + \tilde{D}_G^{-1}(s) \circ \tilde{N}_G(s) \circ N(s) \circ \\
&\quad D^{-1}(s)\}^{-1} \circ \tilde{D}_G^{-1}(s) \circ \tilde{N}_G(s) \\
&= D_i^{-1}(s) \circ \tilde{N}_G(s) . \qquad (37)
\end{aligned}
$$

Remark: The result does not say that $(D_i(s), \tilde{N}_G(s))$ is a left prime factorization of $X(s)$.

The idea of the theorem can be reversed, in a sense.

Theorem 7. Let $A(s) : U[s] \to U[s]$ and $B(s) : R[s] \to U[s]$ be morphisms of k[s]-modules, with $A(s)$ monic. Further, let $(A(s), B(s))$ be a left-prime factorization of the transfer function $A^{-1}(s) \circ B(s) : R(s) \to U(s)$ constructed after localization up to $k(s)$. If

$$A(s) \circ D(s) + B(s) \circ N(s) : U[s] \to U[s], \quad (38)$$

denoted by $D_{AB}(s)$, is monic, and if the inverse of its localization up to $k(s)$, when composed with $B(s)$, is a good transfer function, then the design morphism

$$X(s) = D_{AB}^{-1}(s) \circ B(s) \quad (39)$$

admits UFS with

$$G(s) = A^{-1}(s) \circ B(s). \quad (40)$$

Proof: From Figure 6, we must have

$$
\begin{aligned}
G(s) &= M(s) \circ (1_{R(s)} - N(s) \circ X(s))^{-1} \\
&= D(s) \circ X(s) \circ (1_{R(s)} - N(s) \circ X(s))^{-1} \\
&= D(s) \circ D_{AB}^{-1}(s) \circ B(s) \circ \{1_{R(s)} - N(s) \circ D_{AB}^{-1}(s) \\
&\quad \circ B(s)\}^{-1} \\
&= D(s) \circ D_{AB}^{-1}(s) \circ \{1_{U(s)} - B(s) \circ N(s) \circ \\
&\quad D_{AB}^{-1}(s)\}^{-1} \circ B(s) \\
&= D(s) \circ \{D_{AB}(s) - B(s) \circ N(s)\}^{-1} \circ B(s) \\
&= D(s) \circ \{A(s) \circ D(s)\}^{-1} \circ B(s) \\
&= A^{-1}(s) \circ B(s), \qquad (41)
\end{aligned}
$$

as desired.

Remark: It is possible to relax the left-prime and right-prime assumptions in such results, provided corresponding technical assumptions are made upon the systems represented by $P(s)$ and $G(s)$.

Remark: This theorem assumes only that the composition $D_{AB}^{-1}(s) \circ B(s)$ is a good transfer function. In practice, this permits hidden internal behavior which may not be acceptable.

We shall then say that a UFS is good, with acronym GUFS, if $D_i^{-1}(s) : U(s) \to U(s)$ is a good transfer function. Theorem 7 then has an immediate Corollary.

Corollary 8. If $D_{AB}^{-1}(s) : U(s) \to U(s)$ is a good transfer function, then the construction of Theorem 7 gives a GUFS.

In the terminology of Theorem 7, it is now clear that the family of design morphisms $X(s)$ admitting GUFS can be parameterized in the manner

$$X(s) = \{A(s) \circ D(s) + B(s) \circ N(s)\}^{-1} \circ B(s). \quad (42)$$

Notice that

$$
\begin{aligned}
1_{U(s)} - X(s) \circ N(s) &= 1_{U(s)} - \{A(s) \circ D(s) + B(s) \\
&\quad \circ N(s)\}^{-1} \circ B(s) \circ N(s) \\
&= 1_{U(s)} - \{1_{U(s)} + D^{-1}(s) \circ A^{-1}(s) \circ B(s) \circ N(s)\}^{-1} \\
&\quad \circ D^{-1}(s) \circ A^{-1}(s) \circ B(s) \circ N(s) \\
&= \{1_{U(s)} + D^{-1}(s) \circ A^{-1}(s) \circ B(s) \circ N(s)\}^{-1} \\
&= \{A(s) \circ D(s) + B(s) \circ N(s)\}^{-1} \circ A(s) \circ D(s), \quad (43)
\end{aligned}
$$

so that

$$\{1_{U(s)} - X(s) \circ N(s)\} \circ D^{-1}(s) : U(s) \to U(s) \quad (44)$$

is a good transfer function. We have established necessity in the following representation theorem.

Theorem 9. Suppose that the right-prime factorization $(N(s), D(s))$ and the design morphism $X(s)$ describe a solution $(M(s), T(s))$ to NDP. Suppose further that $M(s)$ admits UFS. Then $M(s)$ admits GUFS if and only if $Z(s) : U(s) \to U(s)$, defined by

$$Z(s) = \{1_{U(s)} - X(s) \circ N(s)\} \circ D^{-1}(s), \quad (45)$$

is a good transfer function.

Proof: We need only show sufficiency. Consider the diagram of Figure 8, where p_R and p_U are the biproduct projections.

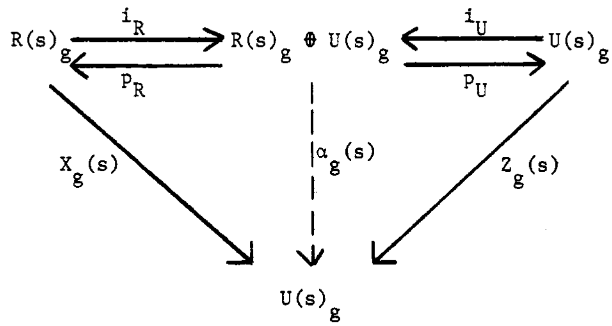

Figure 8. Module Biproduct Property.

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There exists a unique morphism $\alpha_g(s)$ of $S_g^{-1}k[s]$-modules which makes this diagram commute. Localize $\alpha_g(s)$ up to $k(s)$, and denote it by $\alpha(s) : R(s) \oplus U(s) \to U(s)$, where it is a good transfer function. Perform a left-prime factorization $(\tilde{D}_\alpha(s), \tilde{N}_\alpha(s))$ for $\alpha(s)$, as in Figure 9. Then the diagrams of these two figures

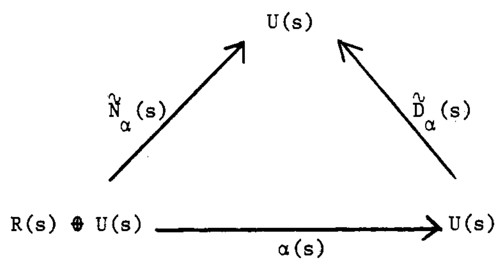

Figure 9. Left-Prime Factorization of $\alpha(s)$.

imply that

$$\tilde{N}_\alpha(s) = \tilde{D}_\alpha(s) \circ X(s) \circ p_R \qquad (46)$$

$$\tilde{N}_\alpha(s) = \tilde{D}_\alpha(s) \circ Z(s) \circ p_U, \qquad (47)$$

so that we may restrict $\tilde{D}_\alpha(s) \circ X(s)$ and $\tilde{D}_\alpha(s) \circ Z(s)$ in the manner

$$\tilde{D}_\alpha(s) \circ Z(s) = A(s) : U[s] \to U[s] \qquad (48)$$

$$\tilde{D}_\alpha(s) \circ X(s) = B(s) : R[s] \to U[s] \qquad (49)$$

with the former monic because $M(s)$ admits UFS. It is easy to see that

$$G(s) = A^{-1}(s) \circ B(s) = Z^{-1}(s) \circ X(s) \qquad (50)$$

is the $G(s)$ <u>required</u> from the assumption that $M(s)$ admits UFS. Indeed,

$$\begin{aligned} G(s) &= M(s) \circ \{1_{R(s)} - N(s) \circ X(s)\}^{-1} \\ &= D(s) \circ X(s) \circ \{1_{R(s)} - N(s) \circ X(s)\}^{-1} \\ &= D(s) \circ \{1_{U(s)} - X(s) \circ N(s)\}^{-1} \circ X(s) \\ &= Z^{-1}(s) \circ X(s) . \end{aligned} \qquad (51)$$

Thus it remains to calculate

$$\begin{aligned} A(s) \circ D(s) + B(s) \circ N(s) &= \tilde{D}_\alpha(s) \circ Z(s) \circ D(s) + \tilde{D}_\alpha(s) \circ X(s) \circ N(s) \\ &= \tilde{D}_\alpha(s) \circ \{Z(s) \circ D(s) + X(s) \circ N(s)\} \\ &= \tilde{D}_\alpha(s) \end{aligned} \qquad (52)$$

from which $D_{AB}(s)$ is monic and the inverse of its localization up to $k(s)$ is a good transfer function.

Corollary 10. If $P(s) : U(s) \to Y(s)$ is a good transfer function, then $M(s)$ admits UFS if and only if $M(s)$ admits GUFS.

Discussion

In earlier papers, the authors have introduced the idea of a Total Synthesis Problem (TSP) for linear multivariable control. Broadly speaking, TSP decomposes into the Nominal Design Problem (NDP), which addresses control action and plant output syntheses, independently of the controller configuration, and the Feedback Synthesis Problem (FSP), which incorporates the advantages of feedback realization. NDP can be characterized [33,34] in terms of a right-prime plant factorization and a good design morphism. In this paper, we have examined the effect of requiring feedback synthesis to take the form of good, unity feedback.

Various feedback parameterizations may be found in the literature. For example, Porter [37] discusses a parameter for adaptation, parameter estimation, and sensitivity reduction, and attributes such structures to earlier literature [38]. Moreover, parameterizations of internal stability [39], and their generalizations [19] have been made. Though the design morphism [33,34] of TSP does not seem to have received prior study, it is to be expected that it should relate to feedback parameters already studied for unity feedback. For example, Desoer and Chen [18] have used the stable parameter of Zames [1] for UFS in the model matching case. When the plant is a good transfer function, and for the case of unity feedback, there exists a good isomorphism relating the Zames parameter to the design morphism. Otherwise, there is only a $k(s)$-relationship.

More importantly, however, model matching does not determine the design morphism associated with TSP, as may be seen easily in Figure 5. Thus TSP, characterized uniquely by the design morphism, materializes as a problem with fundamental distinctions from model matching.

In the nonlinear case [29,31,32] application studies have been made. Moreover, Liu and Sung [40] have reported nonlinear model matching results for unity feedback with a parameter in the same spirit as the design morphism. Of course, [40] can be reduced to the linear situation. However, TSP and model matching remain distinct problems, as described above.

Dedication

The first author wishes to dedicate this paper to his father, Charles G. Sain, who passed away while it was being written.

$$0 \to CGS \to MKS \to MKS/CGS \to 0$$

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Addendum

In the preparation of the introduction to this paper, the authors inadvertently omitted reference [41]. It should be inserted between references [16] and [17] in the Introduction.

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