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The trace formula in invariant form

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Introduction

The trace formula for GL_2 has yielded a number of deep results on automorphic forms. The same results ought to hold for general groups, but so far, little progress has been made. One of the reasons has been the lack of a suitable trace formula.

In [1(d)] and [1(e)] we presented a formula

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi\in\mathfrak{X}} J_{\chi}^T(f), \tag{1}$$

or, as we wrote it in [1(e), §5],

$$\operatorname{tr} R_{\text{cusp}}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^T(f) - \sum_{\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)} J_{\chi}^T(f). \tag{1*}$$

G is a reductive group defined over $\mathbf{Q}$, and f is any function in $C_c^\infty(G(\mathbf{A})^1)$. The left hand side of (1*) is the trace of the convolution operator of f on the space of cusp forms on $G(\mathbf{Q})\setminus G(\mathbf{A})^1$. It is a distribution which is of great importance in the study of automorphic representations. One would hope to study it through the distributions $\{J_{\mathfrak{o}}^T\colon\ \mathfrak{o}\in\mathcal{O}\}$ and $\{J_{\chi}^T\colon\ \chi\in\mathfrak{X}\setminus\mathfrak{X}(G)\}$. Unfortunately, these distributions depend on a number of unpleasant things. There is the parameter T, as well as a choice of maximal compact subgroup of $G(\mathbf{A})^1$ and a choice of minimal parabolic subgroup. What is worse, they are not invariant; their values change when f is replaced by a conjugate of itself. In any generalization of the applications of the trace formula for GL_2, we would not be handed the function f. We could only expect to be given a function such as

$$\phi(f)\colon \pi \to \operatorname{tr}\pi(f),$$

whose values are invariant in f. Here π ranges over the irreducible tempered representations of $G(\mathbf{A})^1$. The decomposition of $\operatorname{tr} R_{\text{cusp}}(f)$ into the right hand side of (1*) would then be of uncertain value, for the individual terms actually depend on f and not just $\phi(f)$.

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The purpose of this paper is to modify the terms in (1) so that they are invariant. Under certain assumptions on the local groups $G(\mathbf{Q}_v)$, we will obtain a formula

$$\sum_{\mathfrak{o}\in\mathcal{O}} I_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} I_{\chi}(f) \tag{2}$$

in which the individual terms are invariant distributions. The definitions will be such that $I_\chi = J_\chi^T$ if χ belongs to $\mathfrak{X}(G)$. We will therefore also have

$$\operatorname{tr}(R_{\mathrm{cusp}}(f)) = \sum_{\mathfrak{o}\in\mathcal{O}} I_{\mathfrak{o}}(f) - \sum_{\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)} I_\chi(f), \tag{2*}$$

the analogue of (1*).

The main assumptions on the local groups $G(\mathbf{Q}_v)$ are set forth in Section 5. One expects them to hold for all groups, but they are a little beyond the present state of harmonic analysis. They are, essentially, that any invariant distribution, I, on $G(\mathbf{A})^1$ can be identified with a distribution, $\hat{I}$, on the space

$$\mathcal{I}_c(G(\mathbf{A})^1) = \{\phi = \phi(f): f \in C_c^\infty(G(\mathbf{A})^1)\}.$$

This will apply in particular to the invariant distributions $I_{\mathfrak{o}}$ and I_χ. In Section 13 we shall show that $\hat{I}_{\mathfrak{o}}$ and $\hat{I}_\chi$ are natural objects on G. They are independent of any choice of maximal compact subgroup, maximal split torus, or even Haar measure. In this sense they are similar to the terms in the trace formula for compact quotient.

A formula akin to (2*) is proved for $G = \mathrm{GL}_2$ in [10(b), §8]. (See also [9].) The main step is the Poisson summation formula on the group of idèles. At the right moment a sum over the multiplicative group of the field is replaced by a sum over Grössencharakters. Likewise, our main step is to apply the trace formula to the Levi components, M, of proper parabolic subgroups of G. To do this, we need to derive a function in $C_c^\infty(M(\mathbf{A})^1)$ from f. Therein lies the difficulty. We can always assume inductively that (2) is valid on M, and use it instead of (1). Then we need only produce a function in $\mathcal{I}_c(M(\mathbf{A})^1)$. However, this is difficult enough. It will not be done completely until Section 12. The main step is a splitting formula for some tempered distributions, $I_{M,\gamma}$, proved in Section 11.

In the applications of the trace formula for GL_2 it is important to show that on certain functions, ϕ, many of the distributions vanish. In Section 14 we shall study this phenomenon on a group of higher rank. The group will be GL_n. ϕ will be the function in $\mathcal{I}_c(G(\mathbf{A})^1)$ one expects to associate to a function on the general linear group of a division algebra. Our main tool is again the splitting formula of Section 11. The reduction to this formula is an elementary exercise.

The formula (2) actually follows rather formally from the existence of some auxiliary data. We present the formal manipulations in Section 4, along with an attempt to motivate our definitions. It is the proof of Theorem 4.2 that contains the crucial application of (2) to Levi subgroups. The best way to first read this paper is to look at (2.5) and the statement of Theorem 3.2, and then go directly to Sections 4 and 5. Section 4 is in fact intended as a second, more technical introduction to the paper. After Section 5 the reader might return to the earlier sections. Section 6 contains some lemmas which are used frequently throughout the rest of the paper. They are best motivated by the calculations of Sections 2 and 3.

We shall conclude this introduction by illustrating how our methods apply to the trace formula of $G = \mathrm{GL}_2$. Let I be the group of idèles (on $\mathbf{Q}$) and let I^1 be the subgroup of idèles of norm 1. We have subgroups

$$M(\mathbf{A}) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in I \right\},$$

$$P(\mathbf{A}) = \left\{ \begin{pmatrix} \alpha & u \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in I,\, u \in \mathbf{A} \right\}$$

and

$$K = O_2(\mathbf{R}) \times \prod_p \mathrm{GL}_2(\mathbf{Z}_p) = \prod_v K_v$$

of $G(\mathbf{A})$. Suppose that f is a smooth function of compact support on

$$G(\mathbf{A})^1 = \{ x \in \mathrm{GL}_2(\mathbf{A}) : |\det x| = 1 \}.$$

Let $J_{\mathfrak{o}}(f)$ and $J_\chi(f)$ be the values of $J_{\mathfrak{o}}^T(f)$ and $J_\chi^T(f)$, respectively, at $T = 0$. Then the trace formula for GL_2 is

$$\mathrm{tr}(R_{\mathrm{cusp}}(f)) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f) - \sum_{\chi \in \mathfrak{X} \setminus \mathfrak{X}(G)} J_\chi(f).$$

Since we are in the special case of $G = \mathrm{GL}_2$, the distributions on the right can be evaluated explicitly (see [7], [4]). We shall copy them from [7], with minor modifications to fit our setting.

Associated to each $\mathfrak{o} \in \mathcal{O}$ is a semisimple conjugacy class $\{\gamma\}$ in $G(\mathbf{Q})$. Let $\mathcal{O}(G)$ be the set of $\mathfrak{o}$ for which the eigenvalues of γ are not rational. For $\mathfrak{o} \in \mathcal{O}(G)$, $J_{\mathfrak{o}}(f)$ equals

(a) $$\mathrm{vol}\left(G(\mathbf{Q},\gamma) \setminus G(\mathbf{A})^1 \cap G(\mathbf{A},\gamma)\right) \int_{G(\mathbf{A})^1 \cap G(\mathbf{A},\gamma) \setminus G(\mathbf{A})^1} f(x^{-1}\gamma x)\, dx,$$

where $G(\mathbf{A}, \gamma)$ denotes the centralizer of γ in $G(\mathbf{A})$. If $\mathfrak{o}$ belongs to $\mathcal{O} \setminus \mathcal{O}(G)$ we can take $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Then we can and do identify $\mathfrak{o}$ with the set

$\{(\alpha,\beta),(\beta,\alpha)\}$ (consisting of one or two elements) of ordered pairs of nonzero rational numbers. If $\mathfrak{o}$ contains two elements, $J_{\mathfrak{o}}(f)$ equals

$$\text{(b)} \qquad \operatorname{vol}(\mathbf{Q}^{\times}\backslash I^{1})^{2} \sum_{(\alpha,\beta)\in\mathfrak{o}} \int_{G(\mathbf{A})^{1}\cap M(\mathbf{A})\backslash G(\mathbf{A})^{1}} f\left(x^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}x\right)v(x)\,dx,$$

where if $x = a\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}k$, for $a \in M(\mathbf{A})$, $k \in K$, and $u = \prod_v u_v$ in $\mathbf{A}$, then

$$v(x) = \log\sqrt{1+|u_{\mathbf{R}}|^{2}} + \sum_{p}\log\left(\operatorname{Max}\{1,|u_p|\}\right).$$

If $\mathfrak{o}$ consists of one element, (α,α), $J_{\mathfrak{o}}(f)$ equals the sum of

$$\text{(c)} \quad \operatorname{vol}(\mathbf{Q}^{\times}\backslash I^{1})^{2} \lim_{u\to 1}\left\{|u-u^{-1}|\int_{G(\mathbf{Q}_S)^{1}\cap M(\mathbf{Q}_S)\backslash G(\mathbf{Q}_S)^{1}} f\left(x^{-1}\begin{pmatrix}\alpha u & 0\\ 0 & \alpha u^{-1}\end{pmatrix}x\right)\right.$$
$$\left.\cdot\left(v(x)+r_S(u)\right)dx\right\}$$

and

$$\text{(d)} \qquad \operatorname{vol}\left(G(\mathbf{Q})\backslash G(\mathbf{A})^{1}\right) f\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}.$$

Here S is a sufficiently large finite set of valuations,

$$\mathbf{Q}_S = \prod_{v\in S}\mathbf{Q}_v,$$

and $r_S(u)$ is a certain smooth function on $\{u = \prod_{v\in S}u_v\colon u_v \in \mathbf{Q}_v^{\times} - \{1\}\}$ with the property that the expression in the brackets in (c) extends to a continuous function of $u \in \prod_{v\in S}\mathbf{Q}_v^{\times}$.

An irreducible unitary representation of $M(\mathbf{A})$ consists of an ordered pair

$$(\mu,\nu) = \left(\bigotimes_v \mu_v, \bigotimes_v \nu_v\right)$$

of characters on I. There corresponds a representation

$$I(\mu,\nu) = \bigotimes_v I(\mu_v,\nu_v),$$

induced from $P(\mathbf{A})$ to $G(\mathbf{A})$. If $s \in \mathbf{C}$, write

$$\mu_s(u) = \mu(u)|u|^{s}, \qquad u \in I,$$

and

$$I(\mu,\nu,s,f) = \int_{G(\mathbf{A})^{1}} f(x)I(\mu_{s/2},\nu_{-s/2},x)\,dx.$$

Note that the restriction, $\bar{\mu}$, of μ to I^{1} can be identified with the orbit $\{\mu_s\colon s \in i\mathbf{R}\}$. Let

$$R(\mu,\nu,s) = \bigotimes_v R(\mu_v,\nu_v,s)$$

be the normalized intertwining operator between $I(\mu, \nu, s)$ and $I(\nu, \mu, -s)$ defined as in [7, p. 521]. (Here, as in the rest of the paper, we shall agree that $I(\mu, \nu, s)$ acts on a space of functions on K. The space is independent of s, so the derivative $R'(\mu, \nu, s)$ makes sense.) If

$$m(\mu, \nu, s) = \frac{L(1-s, \mu\nu^{-1})}{L(1+s, \mu\nu^{-1})},$$

then

$$M(\mu, \nu, s) = m(\mu, \nu, s)R(\mu, \nu, s)$$

is the operator which arises in the functional equation of Eisenstein series. Suppose that $\chi \in \mathfrak{X} \setminus \mathfrak{X}(G)$. It corresponds to a Weyl group orbit of automorphic representations of $M(\mathbf{A})^1$, or as we prefer, a set $\{(\bar{\mu}, \bar{\nu}), (\bar{\nu}, \bar{\mu})\}$ (containing one or two elements) of ordered pairs of characters on $\mathbf{Q}^x \setminus I^1$. Then $J_\chi(f)$ equals the sum of

(e) $$-\tfrac{1}{4}\varepsilon_\chi \operatorname{tr}(M(\mu, \mu, 0)I(\mu, \mu, 0, f)),$$

(f) $$\frac{1}{4\pi} \sum_{(\bar{\mu}, \bar{\nu}) \in \mathcal{O}} \int_{-i\infty}^{i\infty} m(\mu, \nu, s)^{-1} m'(\mu, \nu, s) \operatorname{tr}(I(\mu, \nu, s, f))\, d|s|,$$

and

(g) $$\frac{1}{4\pi} \sum_{(\bar{\mu}, \bar{\nu}) \in \mathcal{O}} \int_{-i\infty}^{i\infty} \operatorname{tr}\left(R(\mu, \nu, s)^{-1} R'(\mu, \nu, s) I(\mu, \nu, s, f)\right) d|s|,$$

where $\varepsilon_\chi = 1$ if $\bar{\mu} = \bar{\nu}$ and is 0 if they are distinct.

If we sum over $\mathfrak{o} \in \mathcal{O}$ and $\chi \in \mathfrak{X} \setminus \mathfrak{X}(G)$, the contributions to the trace formula from the expressions (a), (b), (c), (d), (e), (f), and (g) are the respective analogues of terms (ii), (iv), (v), (i), (vi), (vii) and (viii) in [7]. As distributions in f, the expressions (a), (d), (e) and (f) are invariant. They remain unchanged if f is replaced by the function

$$f^y \colon x \to f(yxy^{-1}), \qquad y \in G(\mathbf{A})^1.$$

This is not true of the other terms. If $\mathfrak{o}$ belongs to $\mathcal{O} \setminus \mathcal{O}(G)$, $J_{\mathfrak{o}}(f^y - f)$ does not vanish, and can in fact be calculated explicitly. It equals

(b*) $$\operatorname{vol}(\mathbf{Q}^x \setminus I^1)^2 \sum_{(\alpha, \beta) \in \mathfrak{o}} \left(f_{P,y}(\alpha, \beta) + f_{\bar{P},y}(\alpha, \beta)\right),$$

where

$$f_{P,y}(\alpha, \beta) = |\alpha\beta^{-1}|^{1/2} \int_K \int_{\mathbf{A}} f\left(k^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} k\right) u'_P(k, y)\, dv\, dk,$$

$$f_{\bar{P},y}(\alpha, \beta) = f_{P,y}(\beta, \alpha), \qquad \alpha, \beta \in I^1,$$

and

$$u'_P(k, y) = -\tfrac{1}{2}\log|\alpha_1\beta_1^{-1}|,$$

if

$$ky = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}\begin{pmatrix} 1 & v_1 \\ 0 & 1 \end{pmatrix}k_1, \qquad \alpha_1, \beta_1 \in I^1, v_1 \in \mathbf{A}, k_1 \in K.$$

Both $f_{P,y}$ and $f_{\bar{P},y}$ are smooth functions on $I^1 \times I^1$. Similarly, one can calculate the result of replacing f by $f^y - f$ in (g). It equals

$$\text{(g*)} \qquad \sum_{(\bar{\mu}, \bar{\nu}) \in \chi} \left(\hat{f}_P(\bar{\mu}, \bar{\nu}) + \hat{f}_{\bar{P}}(\bar{\mu}, \bar{\nu})\right).$$

Suppose that $\tilde{f}$ is any function in $C_c^\infty(G(\mathbf{A}))$ whose restriction to $G(\mathbf{A})^1$ is f. If μ and ν are characters on I set

$$\phi_M(\tilde{f}, \mu, \nu) = \operatorname{tr}\left(R(\mu, \nu, 0)^{-1}R'(\mu, \nu, 0)I(\mu, \nu, \tilde{f})\right).$$

Then

$$\phi_M(\tilde{f}) \colon (\mu, \nu) \to \phi_M(\tilde{f}, \mu, \nu)$$

is a Schwartz function on $\hat{I} \times \hat{I}$. It is the Fourier transform of a Schwartz function on $I \times I$. Let $\phi_M^\vee(f)$ denote the restriction of this latter function to $I^1 \times I^1$. It depends only on f, and not $\tilde{f}$. The expression (g) equals

$$\frac{1}{16\pi^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \phi_M(\tilde{f}, \mu_s, \nu_t)\, d|s|\, d|t|.$$

This is just the sum over $(\bar{\mu}, \bar{\nu}) \in \chi$ of the values of the Fourier transform of $\phi_M^\vee(f)$ at $(\bar{\mu}, \bar{\nu})$. We would like to apply the Poisson summation formula on $I^1 \times I^1$ to $\phi_M^\vee(f)$. In this case it is enough to know that $\phi_M^\vee(f)$ is a Schwartz function; however if there is to be any hope for the general case it will be essential to show that $\phi_M^\vee(f)$ is *compactly supported*. The proof of this fact for general G and M, as we have already noted, is the main goal of this paper.

The proof for GL_2 is actually rather trivial. At first glance, it might appear hopeless, for the function

$$(s, t) \to \phi_M(\tilde{f}, \mu_s, \nu_t), \qquad s, t \in \mathbf{C},$$

certainly has poles. Remember, however, that $\phi_M^\vee(f)$ is not a function on $I \times I$, but the restriction of a function to $I^1 \times I^1$. This is what saves the day. We can assume that

$$\tilde{f} = \bigotimes_v f_v, \qquad f_v \in C_c^\infty(G(\mathbf{Q}_v)).$$

Since $R(\mu, \nu, 0)$ is a restricted tensor product $\otimes_v R_v(\mu_v, \nu_v, 0)$ of local intertwining operators, $\phi_M(f,\mu,\nu)$ equals

$$\sum_v \operatorname{tr}\big(R_v(\mu_v,\nu_v,0)^{-1}R'_v(\mu_v,\nu_v,0)I(\mu_v,\nu_v,f_v)\big)\prod_{\{w:\,w\neq v\}}\operatorname{tr}I(\mu_w,\nu_w,f_w).$$

Almost all the terms in this sum over v equal 0. For any w,

$$(\mu_w,\nu_w)\to \operatorname{tr}I(\mu_w,\nu_w,f_w))$$

is the Fourier transform on $\mathbf{Q}_w^x\times\mathbf{Q}_w^x$ of the function

$$f_{w,P}(\alpha,\beta)=|\alpha\beta^{-1}|^{1/2}\int_{K_w}\int_{\mathbf{Q}_w} f_w\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}xk\right)dx\,dk,\qquad \alpha,\beta\in\mathbf{Q}_w^x.$$

It is clear that $f_{w,P}$ is a smooth, compactly supported function on $\mathbf{Q}_w^x\times\mathbf{Q}_w^x$. For any v,

$$(\mu_v,\nu_v)\to\operatorname{tr}\big(R_v(\mu_v,\nu_v,0)^{-1}R'_v(\mu_v,\nu_v,0)I(\mu_v,\nu_v,f_v)\big)$$

is the Fourier transform of a Schwartz function on $\mathbf{Q}_v^x\times\mathbf{Q}_v^x$. It need not have compact support. However, if C^w is any compact subset of $\prod_{\{w:\,w\neq v\}}\mathbf{Q}_w^x$, and $C=\mathbf{Q}_v^x\times C^w$, the intersection of $C\times C$ with $I^1\times I^1$ is compact. The support of $\phi_M^\vee(f)$ is certainly contained in a finite union of such sets and is therefore compact. It depends only on the support of f.

Now apply the Poisson summation formula on $I^1\times I^1$. The sum over $\chi \bullet \mathfrak{X}\setminus\mathfrak{X}(G)$ of (g) equals

$$\operatorname{vol}(\mathbf{Q}^x\setminus I^1)^2\sum_{\alpha,\beta\in\mathbf{Q}^x}\phi_M^\vee(f,\alpha,\beta)=\sum_{\mathfrak{o}\in\mathcal{O}\setminus\mathcal{O}(G)}\operatorname{vol}(\mathbf{Q}^x\setminus I^1)^2\sum_{(\alpha,\beta)\in\mathfrak{o}}\phi_M^\vee(f,\alpha,\beta).$$

If $\mathfrak{o}$ belongs to $\mathcal{O}\setminus\mathcal{O}(G)$, define

$$I_{\mathfrak{o}}(f)=J_{\mathfrak{o}}(f)-\operatorname{vol}(\mathbf{Q}^x\setminus I^1)^2\sum_{(\alpha,\beta)\in\mathfrak{o}}\phi_M^\vee(f,\alpha,\beta).$$

It follows from (b*) and (g*) that

$$I_{\mathfrak{o}}(f^y-f)=0,$$

so that $I_{\mathfrak{o}}$ is an invariant distribution. If $\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)$, define $I_\chi(f)$ to be the sum of the expressions (e) and (f). Then I_χ is clearly an invariant distribution. Finally, set $I_{\mathfrak{o}}=J_{\mathfrak{o}}$ if $\mathfrak{o} \bullet \mathcal{O}(G)$. This of course is also an invariant distribution. Then we have

$$\operatorname{tr}R_{\text{cusp}}(f)=\sum_{\mathfrak{o}\in\mathcal{O}}I_{\mathfrak{o}}(f)-\sum_{\chi\in\mathfrak{X}\setminus\mathfrak{X}(G)}I_\chi(f).$$

This is our trace formula in invariant form for GL_2. The reader familiar with the invariant trace formula for GL_2 in [10(b)] will observe that it is different from the formula we have just given. For in [10(b)], Poisson summation was applied to the contribution from terms (b) and (c) above, whereas we have applied it to the contribution from (g). While being more immediately suited for the applications, the formula in [10(b)] is harder to prove for GL_2, and may be impossible to establish directly for arbitrary G.

Contents

1. A review of the trace formula

Suppose that G is a reductive algebraic group defined over a field F of characteristic 0. More than anything else this paper concerns Levi components of parabolic subgroups of G defined over F, or as we shall call them, *Levi subgroups* of G. If M is a Levi subgroup, let $A = A_M$ be the split component of the center of M; set

$$\mathfrak{a} = \mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbf{R}),$$

where $X(M)_F$ is the group of rational characters of M defined over F. Now $\mathfrak{a}$ is a real vector space whose dimension equals that of the split torus A. Suppose that L is a Levi subgroup of G which contains M. Then L is also a reductive group defined over F, and M is a Levi subgroup of L. We shall denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Let us also write $\mathcal{F}^L(M)$ for the set of parabolic subgroups of L, defined over F, which contain M, and let $\mathcal{P}^L(M)$ denote the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. Each of these three sets is finite. If $L = G$, we shall usually denote the sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$. (In general, if a superscript L is used to denote the dependence of some object in this paper on a Levi subgroup, we shall often omit the superscript when $L = G$.)

We shall try to reserve the letters L and M for Levi subgroups of G, and to use the letters P, Q and R for parabolic subgroups. If $M_0 \subset L$ are Levi subgroups of G, and $P \in \mathcal{F}^L(M_0)$, there is a unique Levi component M_P of P which contains M_0. It is defined over F. The unipotent radical, N_P, of P is also defined over F. We shall write A_P and $\mathfrak{a}_P$ for A_{M_P} and $\mathfrak{a}_{M_P}$. Suppose that $M \subset M_1 \subset L$ are Levi subgroups of G. If $Q \in \mathcal{P}^L(M_1)$ and $R \in \mathcal{P}^{M_1}(M)$, there is a unique group, $Q(R)$, in $\mathcal{P}^L(M)$ which is contained in Q and whose intersection with M_1 is R. Notice that there is a natural map from $\mathfrak{a}_M$ to $\mathfrak{a}_L$. We shall denote its kernel by $\mathfrak{a}_M^L$.

Suppose for the moment that F is a local field and that M_0 is a Levi subgroup of G. We will want to work with particular maximal compact subgroups of $G(F)$, which we will call *admissible relative to* M_0. If F is Archimedean we will take this to mean that the Lie algebras of A_{M_0} and K are orthogonal with respect to the Killing form of G. If F is non-Archimedean the vertex of K in the Bruhat-Tits building of G must be special and must belong to the apartment associated to a maximal split torus of M_0. Any K which is admissible relative to M_0 has the following properties.

(i) $G = P(F)K$ for any $P \in \mathcal{P}(M_0)$.

(ii) Any coset in $G(F)/M_0(F)$ which normalizes $M_0(F)$ has a representative in K.

(iii) $K = (N_P(F) \cap K)\cdot(M_P(F) \cap K)$ for any $P \in \mathcal{F}(M_0)$. If L is a group in $\mathcal{L}(M_0)$, $K \cap L(F)$ is a maximal compact subgroup of $L(F)$ which is admissible relative to M_0.

For the rest of this paper G will be a fixed reductive group defined over the field $\mathbf{Q}$ of rational numbers. We fix a minimal Levi subgroup M_0 of G. Then $A_0 = A_{M_0}$ is a maximal $\mathbf{Q}$-split torus of G. We shall say that a maximal compact subgroup

$$K = \prod_v K_v$$

of $G(\mathbf{A})$ is *admissible relative to* M_0 if for each valuation v on $\mathbf{Q}$, K_v is a maximal compact subgroup of $G(\mathbf{Q}_v)$ which is admissible relative to M_0, and if for any embedding of G into GL_n, defined over $\mathbf{Q}$,

$$K_v = \mathrm{GL}_n(\mathfrak{o}_v) \cap G(\mathbf{Q}_v)$$

for almost all v. Fix such a K. Then K satisfies the conditions of [1(d)]. For any $P \in \mathcal{F}(M_0)$ we can define the function

$$H_P(nmk) = H_{M_P}(m), \qquad n \in N_P(\mathbf{A}), m \in M_P(\mathbf{A}), k \in K,$$

from $G(\mathbf{A})$ to $\mathfrak{a}_P$ as in [1(d), §1]. Let Ω be the Weyl group of (G, A_0). For any $s \in \Omega$ let w_s be a fixed representative of s in $G(\mathbf{Q})$. w_s is determined only modulo $M_0(\mathbf{Q})$, but for any $P \in \mathcal{F}(M_0)$, $H_P(w_s^{-1})$ is uniquely determined. In [1(c)], thinking of the standard maximal compact subgroup of $GL_n(\mathbf{A})$, we mistakenly stated that w_s could also be chosen in K. However, we can chose an element $\tilde{w}_s \in K$ such that

$$\tilde{w}_s a \tilde{w}_s^{-1} = w_s a w_s^{-1}$$

for all $a \in A_0(\mathbf{A})$. It follows that w_s belongs to $KM_0(\mathbf{A})$ for every $s \in \Omega$.

LEMMA 1.1. *There is a vector $T_0 \in \mathfrak{a}_0$, uniquely determined modulo $\mathfrak{a}_G$, such that*

$$H_{P_0}(w_s^{-1}) = T_0 - s^{-1}T_0$$

for any $P_0 \in \mathcal{P}(M_0)$ and $s \in \Omega$.

Proof. The uniqueness follows from the fact that $\mathfrak{a}_G$ is the set of fixed points of the group Ω acting on $\mathfrak{a}_0$. Since w_s lies in $KM_0(\mathbf{A})$, $H_{P_0}(w_s^{-1})$ is independent of $P_0 \in \mathcal{P}(M_0)$. Fix P_0. We have the set, Δ_{P_0}, of simple roots of (P_0, A_0), the set $\{\alpha^\vee : \alpha \in \Delta_{P_0}\}$ of co-roots, and the basis $\{\tilde{\varpi}_\alpha : \alpha \in \Delta_{P_0}\}$ of $(\mathfrak{a}_{M_0}^G)^*$ which is dual to Δ_{P_0}. For each simple reflection s_α, $\alpha \in \Delta_{P_0}$, there is a real number h_α such that $H_{P_0}(w_{s_\alpha}^{-1}) = h_\alpha \alpha^\vee$. Define

$$T_0 = \sum_{\alpha \in \Delta_{P_0}} h_\alpha \tilde{\varpi}_\alpha .$$

The lemma will be proved inductively on the length of s. Suppose that

$$s_1 = s_\alpha s, \qquad \alpha \in \Delta_{P_0},$$

and that the length of s_1 is greater by one than the length of s. If we write $w_s^{-1} = m_s \tilde{w}_s^{-1}$, for $m_s \in M_0(\mathbf{A})$, we see that

$$H_{P_0}(w_{s_1}^{-1}) = H_{P_0}(m_s) + s^{-1}H_{P_0}(w_{s_\alpha}^{-1}) = H_{P_0}(w_s^{-1}) + s^{-1}H_{P_0}(w_{s_\alpha}^{-1}).$$

It follows by induction that $H_{P_0}(w_{s_1}^{-1}) - (T_0 - s_1^{-1}T_0)$ equals

$$H_{P_0}(w_{s_1}^{-1}) - H_{P_0}(w_s^{-1}) - (T_0 - s_1^{-1}T_0) + (T_0 - s^{-1}T_0)$$
$$= s^{-1}H_{P_0}(w_{s_\alpha}^{-1}) - s^{-1}(T_0 - s_\alpha T_0).$$

By the definition of T_0, this equals 0. The lemma follows. □

We will eventually end up with objects which are independent of any choice of Haar measures, as well as our choices of K and M_0. In the meantime, however, we had best fix some measures. Suppose that v is a valuation on $\mathbf{Q}$. If M_v is any

Levi subgroup of G defined over $\mathbf{Q}_v$, for which K_v is admissible, we assign $K_v \cap M(\mathbf{Q}_v)$ the Haar measure for which the total volume is one. Suppose that $L_v \in \mathcal{L}(M_v)$ and $P \in \mathcal{F}^{L_v}(M_v)$. If v is discrete, we take Haar measures on $N_P(\mathbf{Q}_v)$ and $M_P(\mathbf{Q}_v)$ such that the intersection of each group with K_v has volume one. Then if $f \in C_c(L(\mathbf{Q}_v))$,

$$(1.1)\quad \int_{L_v(\mathbf{Q}_v)} f(x)\,dx = \int_{N_P(\mathbf{Q}_v)}\int_{M_P(\mathbf{Q}_v)}\int_{K_v \cap L_v(\mathbf{Q}_v)} f(nmk)\delta_P(m)^{-1}\,dk\,dm\,dn.$$

Here, δ_P is the modular function of the group $P(\mathbf{Q}_v)$. If v is Archimedean, simply fix Haar measures on all groups $\{N_P(\mathbf{Q}_v), M_P(\mathbf{Q}_v)\}$ given as above, so that (1.1) holds, and so that groups which are conjugate under K_v have compatible Haar measures. Now suppose that S is a set, possibly infinite, of valuations on $\mathbf{Q}$. Suppose that $L \in \mathcal{L}(M_0)$ and $P \in \mathcal{F}^L(M_0)$. We take the restricted product measures on all the groups $\prod_{v \in S} M_P(\mathbf{Q}_v)$ and $\prod_{v\in S} N_P(\mathbf{Q}_v)$. Then the analogue of (1.1) holds for functions f on $\prod_{v \in S} L(\mathbf{Q}_v)$. In this way we obtain Haar measures on the groups $N_P(\mathbf{A})$ and $M_P(\mathbf{A})$. By further restricting our choice of measure on $N_P(\mathbf{Q}_v)$, v Archimedean, we can assume that for each P, the volume of $N_P(\mathbf{Q}) \backslash N_P(\mathbf{A})$ is one. Then our measures on adèle groups satisfy the conditions of [1(d)]. We take the Haar measure on $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$ associated to some Euclidean metric which is invariant under the Weyl group Ω. The metric also gives us a measure on any subspace of $\mathfrak{a}_0$. If $P \in \mathcal{P}(M_0)$, $i\mathfrak{a}_P^*$ is isomorphic to the group of unitary characters on $\mathfrak{a}_P$. We take the Haar measure on $i\mathfrak{a}_P^*$ which is dual to that on $\mathfrak{a}_P$. The measures on $M(\mathbf{A})$ and $\mathfrak{a}_M$ yield a measure on $M(\mathbf{A})^1$, the kernel of the map

$$H_M\colon M(\mathbf{A}) \to \mathfrak{a}_M$$

defined in [1(d)]. $M(\mathbf{A})$ is the direct product of $M(\mathbf{A})^1$ and $A(\mathbf{R})^0$, so we also obtain a Haar measure on $A(\mathbf{R})^0$, the identity component of $A(\mathbf{R})$.

In the first three sections of the paper we shall examine the trace formula presented in [1(d), (e)]. In these sections we will try to use the notation of [1(d), (e)], so any undefined symbols will have the meaning assigned there. In particular, if P and Q are groups in $\mathcal{F}(M_0)$, with $P \subset Q$, $\mathfrak{a}_P^Q$ is the subspace $\mathfrak{a}_{M_P}^{M_Q}$ of $\mathfrak{a}_P$. To the set, Δ_P^Q, of simple roots of $(P \cap M_Q, A_P)$ there was associated a basis $\{\alpha^\vee\colon \alpha \in \Delta_P^Q\}$ of $\mathfrak{a}_P^Q$; $\hat{\Delta}_P^Q$ was defined to be the corresponding dual basis of $(\mathfrak{a}_P^Q)^*$. Then Δ_P^Q and $\hat{\Delta}_P^Q$ are naturally embedded subsets of $\mathfrak{a}_0^*$. Remember also that τ_P^Q and $\hat{\tau}_P^Q$ denote the characteristic functions of $\{H \in \mathfrak{a}_0\colon \alpha(H) > 0,\ \alpha \in \Delta_P^Q\}$ and $\{H \in \mathfrak{a}_0\colon \tilde{\omega}(H) > 0,\ \tilde{\omega} \in \hat{\Delta}_P^Q\}$. When there is an obvious meaning, we shall allow notation established for parabolic subgroups of G to carry over to parabolic subgroups of a Levi subgroup of G. For example, if $R \supset Q$, and Q_1 and P_1 are the intersections of Q and P with M_R, $\Delta_{P_1}^{Q_1} = \Delta_P^Q$, $\tau_{P_1}^{Q_1} = \tau_P^Q$, $\mathfrak{a}_{Q_1} = \mathfrak{a}_Q$, etc.

As it is given in [1(d), (e)], the trace formula depends on a fixed minimal parabolic subgroup $P_0 \in \mathcal{P}(M_0)$. Until we remove this dependence at the end of Section 2, P_0 will be fixed, and the letters P, Q or R will denote groups in $\mathcal{P}(M_0)$ which contain P_0. The terms in the formula are indexed by sets $\mathcal{O}$ and $\mathfrak{X}$. $\mathcal{O}$ can be defined as the set of semisimple conjugacy classes in $G(\mathbf{Q})$. The elements in $\mathfrak{X}$ are Weyl orbits of irreducible cuspidal automorphic representations of Levi subgroups of G (Ω clearly acts on the set of pairs $\{(M, \rho)\}$, M a group in $\mathcal{L}(M_0)$ and ρ a cuspidal representation of $M(\mathbf{A})^1$). The trace formula is an identity

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^T(f) = \sum_{\chi\in\mathfrak{X}} J_\chi^T(f),$$

associated to functions f in $C_c^\infty(G(\mathbf{A})^1)$. T is any suitably regular point (depending on the support of f) in $\mathfrak{a}_0^+$, the positive chamber in $\mathfrak{a}_0$ defined by P_0. $J_{\mathfrak{o}}^T$ and J_χ^T are distributions whose definitions we will recall in the next section.

Many of our arguments will be inductive, so we will need to keep track of distributions on Levi components $L = M_P$ of parabolic subgroups P, $P_0 \subset P$. We can certainly define the sets of equivalence classes $\mathcal{O}^L$ and $\mathfrak{X}^L$ associated to L. If $\mathfrak{o}$ is a class in $\mathcal{O}$, $\mathfrak{o} \cap L(\mathbf{Q})$ is a union, possibly empty, of classes $\mathfrak{o}_1, \ldots, \mathfrak{o}_n$ in $\mathcal{O}^L$. $P_0 \cap L$ is a fixed minimal parabolic subgroup of L and T remains a point in the associated positive chamber of $\mathfrak{a}_0$. We therefore have the distributions $J_{\mathfrak{o}_i}^{L,T}$ on $C_c^\infty(L(\mathbf{A})^1)$. Define

$$J_{\mathfrak{o}}^{L,T} = \sum_{i=1}^n J_{\mathfrak{o}_i}^{L,T}.$$

Similarly, suppose that χ belongs to $\mathfrak{X}$. Then χ is a G-Weyl orbit of irreducible cuspidal automorphic representations on Levi subgroups. This decomposes into a finite union, again possibly empty, of L-Weyl orbits $\chi_1, \ldots, \chi_n$ in $\mathfrak{X}^L$. Again define

$$J_\chi^{L,T} = \sum_{i=1}^n J_{\chi_i}^{L,T}.$$

The trace formula for L then implies that

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{L,T}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi^{L,T}(f), \tag{1.2}$$

for all $f \in C_c^\infty(L(\mathbf{A})^1)$.

2. The distributions $J_{\mathfrak{o}}$ and J_χ

In this section we shall show that $J_{\mathfrak{o}}^T(f)$ and $J_\chi^T(f)$ are polynomials in T; that is, as functions of T they belong to the symmetric algebra on $\mathfrak{a}_{0,\mathbf{C}}^*$. We will also take the opportunity to recall the definitions of the distributions. Fix $f \in$

$C_c^\infty(G(\mathbf{A})^1)$ and $\mathfrak{o} \in \mathfrak{O}$. Then $J_{\mathfrak{o}}^T(f)$ is the integral over x in $G(\mathbf{Q}) \setminus G(\mathbf{A})^1$ of

$$\sum_{\{P:\, P \supset P_0\}} (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q}) \setminus G(\mathbf{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x) \hat{\tau}_P(H_P(\delta x) - T),$$

where $Z = A_G$, and

$$K_{P,\mathfrak{o}}(x, y) = \sum_{\gamma \in M_P(\mathbf{Q}) \cap \mathfrak{o}} \int_{N_P(\mathbf{A})} f(x^{-1}\gamma n y)\, dn.$$

Suppose that T_1 is a fixed suitably regular point in $\mathfrak{a}_0^+$. We shall let T vary freely in $T_1 + \mathfrak{a}_0^+$, and try to relate $J_{\mathfrak{o}}^T$ with the distributions $J_{\mathfrak{o}}^{M,T_1}$. It evidently will be a question of expressing $\hat{\tau}_P(H_P(\delta x) - T)$ in terms of the functions $\hat{\tau}_P^Q(H_P(\delta x) - T_1)$, Q ranging over parabolic subgroups that contain P. This suggests making the following inductive definition:

If X is a point in $\mathfrak{a}_0$, define functions

$$\Gamma_Q'(H, X), \qquad H \in \mathfrak{a}_0,$$

on $\mathfrak{a}_0$, indexed by parabolic subgroups $Q \supset P_0$, by demanding that for all $Q \supset P_0$,

$$\hat{\tau}_Q(H - X) = \sum_{\{R:\, R \supset Q\}} (-1)^{\dim(A_R/Z)} \hat{\tau}_Q^R(H) \Gamma_R'(H, X).$$

The definition is indeed inductive; if $\Gamma_R'(H, X)$ has been defined for all $R \supsetneqq Q$, then $\Gamma_Q'(H, X)$ is specified uniquely by the formula. $\Gamma_Q'(H, X)$ depends only on the projections of H and X onto $\mathfrak{a}_Q$ and it is invariant under $\mathfrak{a}_G$. To express it in another way, consider the sum

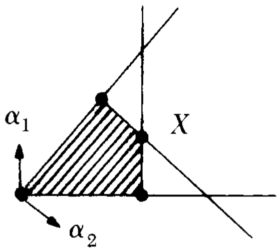
$$\sum_{\{R:\, R \supset Q\}} (-1)^{\dim(A_R/Z)} \tau_Q^R(H) \hat{\tau}_R(H - X). \tag{2.1}$$

This equals

$$\sum_{\{R_1:\, R_1 \supset Q\}} \Gamma_{R_1}'(H, X) \sum_{\{R:\, Q \subset R \subset R_1\}} (-1)^{\dim(A_R/A_{R_1})} \tau_Q^R(H) \hat{\tau}_R^{R_1}(H).$$

It is easy to verify that if $Q \subsetneqq R_1$, the sum over R vanishes (see the remark following [1(d), Corollary 6.2]). It follows that $\Gamma_Q'(H, X)$ equals (2.1).

If $G = \mathrm{GL}_3$, $\mathfrak{a}_{P_0}/\mathfrak{a}_G$ is two dimensional. If X is in $\mathfrak{a}_0^+$, $\Gamma_{P_0}(\cdot, X)$ is the characteristic function of the shaded region.



It is the algebraic sum of the characteristic functions of the chambers at each of the four vertices. In general we have

LEMMA 2.1. *For each X in a fixed compact subset of $\mathfrak{a}_P/\mathfrak{a}_G$, the support of the function*

$$H \to \Gamma_P'(H, X), \qquad H \in \mathfrak{a}_P/\mathfrak{a}_G,$$

is contained in a fixed compact set, which is independent of X.

Proof. If $Q \supset P$, set $\hat{\tau}^{Q/P}(H)$ equal to the characteristic function of

$$\{H: \tilde{\omega}(H) > 0,\ \tilde{\omega} \in \hat{\Delta}_P \backslash \hat{\Delta}_Q\}.$$

Mimicking the construction of $\Gamma_Q'(H, X)$, we define functions $\tilde{\Gamma}_Q(H, X)$ inductively by demanding that for all $Q \supset P_0$,

$$\hat{\tau}_Q(H - X) = \sum_{\{R:\, R \supset Q\}} (-1)^{\dim(A_R/Z)} \hat{\tau}^{R/Q}(H) \tilde{\Gamma}_R(H, X).$$

Then

$$\tilde{\Gamma}_Q(H, X) = \sum_{\{R:\, R \supset Q\}} (-1)^{\dim(A_R/Z)} \hat{\tau}^{R/Q}(H) \hat{\tau}_R(H - X).$$

The values of these functions are easily seen from inspection. Modulo sign, $\tilde{\Gamma}_Q(\cdot, X)$ is just the characteristic function in $\mathfrak{a}_Q/\mathfrak{a}_G$ of a parallelepiped with opposite vertices 0 and X. In particular, $\tilde{\Gamma}_Q(H, X)$ is compactly supported as a function of $H \in \mathfrak{a}_Q/\mathfrak{a}_G$.

The lemma will be proved by induction on $\dim G$. We have

$$\begin{aligned} \Gamma_P'(H, X) &= \sum_{\{Q:\, Q \supset P\}} (-1)^{\dim(A_Q/Z)} \tau_P^Q(H) \hat{\tau}_Q(H - X) \\ &= \sum_{\{R:\, R \supset P\}} \tilde{\Gamma}_R(H, X) \sum_{\{Q:\, P \subset Q \subset R\}} (-1)^{\dim(A_Q/A_R)} \tau_P^Q(H) \hat{\tau}^{R/Q}(H). \end{aligned}$$

We have already noted that if $P \neq G$,

$$\sum_{\{Q:\, P \subset Q \subset G\}} (-1)^{\dim(A_Q/Z)} \tau_P^Q(H) \hat{\tau}_Q(H) = 0.$$

Therefore the outer sum may be taken over only those R not equal to G. For a given $R \neq G$, and $H \in \mathfrak{a}_P^G$, put

$$H = H^* + H_*, \qquad H^* \in \mathfrak{a}_P^R, \qquad H_* \in \mathfrak{a}_R^G.$$

Then $\tilde{\Gamma}_R(H, X) = \tilde{\Gamma}_R(H_*, X)$. Moreover,

$$\hat{\tau}^{R/Q}(H) = \hat{\tau}_Q^R(H^* - L(H_*)),$$

where

$$H_* \to L(H_*)$$

is a linear map from $\mathfrak{a}_R^G$ to $\mathfrak{a}_P^R$ which is independent of Q. If the summand corresponding to R does not vanish, H_* will lie in a fixed compact set. So, therefore, will $L(H_*)$. Applying the induction assumption to the group M_R, we see that H^* must lie in a fixed compact subset of $\mathfrak{a}_P^R$. It follows that H is contained in a fixed compact subset of $\mathfrak{a}_P^G$. $\square$

The Fourier transform of $\Gamma_P'(\cdot, X)$ will be an entire function on $\mathfrak{a}_{P,\mathbf{C}}^*$. It is easy to calculate. Let λ be a point in $\mathfrak{a}_{P,\mathbf{C}}^*$ whose real part belongs to $-(\mathfrak{a}_P^*)^+$. Then

$$\int_{\mathfrak{a}_P^G} \Gamma_P'(H,X)e^{\lambda(H)}\,dH = \sum_{\{Q:\, Q\supset P\}} (-1)^{\dim(A_Q/Z)} \int_{\mathfrak{a}_P^G} \tau_P^Q(H)\hat{\tau}_Q(H-X)e^{\lambda(H)}\,dH.$$

In this integral set

$$H = \sum_{\tilde{\omega}\in\hat{\Delta}_P^Q} t_{\tilde{\omega}}\tilde{\omega}^\vee + \sum_{\alpha\in\Delta_Q} t_\alpha\alpha^\vee.$$

With this change of variables we must multiply by the volume of $\mathfrak{a}_P^G$ modulo $\hat{L}_P^Q \times L_Q$, where $\hat{L}_P^Q$ and L_Q are the lattices generated by $\{\tilde{\omega}^\vee\colon \tilde{\omega} \in \hat{\Delta}_P^Q\}$ and $\{\alpha^\vee\colon \alpha \in \Delta_Q\}$ respectively. The result is the sum over $Q \supset P$ of the product of

$$(-1)^{\dim(A_Q/Z)}(-1)^{\dim(A_P/Z)}\mathrm{vol}\big(\mathfrak{a}_P^G/\hat{L}_P^Q \times L_Q\big)$$

with

$$\prod_{\tilde{\omega}\in\hat{\Delta}_P^Q} \lambda(\tilde{\omega}^\vee)^{-1}\cdot \prod_{\alpha\in\Delta_Q} e^{\lambda(\alpha^\vee)\cdot\tilde{\omega}_\alpha(X)}\lambda(\alpha^\vee)^{-1}.$$

Here, $\tilde{\omega}_\alpha$ is the element in $\hat{\Delta}_Q$ dual to $\alpha^\vee$. Let λ_Q denote the projection of λ onto $\mathfrak{a}_{Q,\mathbf{C}}^*$. Then

$$\prod_{\alpha\in\Delta_Q} e^{\lambda(\alpha^\vee)\tilde{\omega}_\alpha(X)} = e^{\lambda_Q(X)}.$$

Define

$$\hat{\theta}_P^Q(\lambda) = \mathrm{vol}\big(\mathfrak{a}_P^Q/\hat{L}_P^Q\big)^{-1}\prod_{\tilde{\omega}\in\hat{\Delta}_P^Q}\lambda(\tilde{\omega}^\vee),$$

and

$$\theta_Q^R(\lambda) = \mathrm{vol}\big(\mathfrak{a}_Q^R/L_Q^R\big)^{-1}\prod_{\alpha\in\Delta_Q^R}\lambda(\alpha^\vee),$$

if $R \supset Q$. (As suggested in §1, we sometimes write θ_Q^R as θ_Q if $R = G$.) We have proved

LEMMA 2.2. *The Fourier transform of the function*

$$H \to \Gamma_P'(H,X), \qquad H \in \mathfrak{a}_P/\mathfrak{a}_G$$

is

$$\sum_{\{Q:\, Q\supset P\}} (-1)^{\dim(A_P/A_Q)} e^{\lambda_Q(X)} \hat{\theta}_P^Q(\lambda)^{-1}\theta_Q(\lambda)^{-1}, \qquad \lambda\in\mathfrak{a}_{P,\mathbf{C}}^*.$$

In particular, this latter function of λ *is regular.* □

To evaluate the integral over $\mathfrak{a}_P/\mathfrak{a}_G$ of $\Gamma_P'(\cdot, X)$, replace λ by $t\lambda$, $t>0$, in the formula and let t approach 0. The resulting limit must exist and be independent of λ. Since $\hat{\theta}_P^Q(\lambda)^{-1}\theta_Q(\lambda)^{-1}$ is homogeneous of degree $q = \dim(A_P/Z)$, the result is

$$\frac{1}{q!}\sum_{\{Q:\, Q\supset P\}} (-1)^{\dim(A_P/A_Q)} \lambda_Q(X)^q \hat{\theta}_P^Q(\lambda)^{-1}\theta_Q(\lambda)^{-1}.$$

It is a polynomial in X which is homogeneous of degree q.

Now we can return to our discussion of $J_\circ^T(f)$. In the expression for $J_\circ^T(f)$, make the substitution

$$\hat{\tau}_P(H_P(\delta x) - T) = \sum_{\{Q:\, Q\supset P\}} (-1)^{\dim(A_Q/Z)} \hat{\tau}_P^Q(H_P(\delta x) - T_1) \times \Gamma_Q'(H_Q(\delta x) - T_1, T - T_1).$$

Take the sum over Q outside the sum over P, and write the integral over (x,δ) in $(G(\mathbf{Q})\backslash G(\mathbf{A})^1)\times(P(\mathbf{Q})\backslash G(\mathbf{Q}))$ as an integral over $(Q(\mathbf{Q})\backslash G(\mathbf{A})^1)\times(P(\mathbf{Q})\backslash Q(\mathbf{Q}))$. Then $J_\circ^T(f)$ is the sum over $\{Q\colon Q\supset P_0\}$, and the integral over x in $Q(\mathbf{Q})\backslash G(\mathbf{A})^1$, of

$$\sum_{\{P:\, P_0\subset P\subset Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in P(\mathbf{Q})\backslash Q(\mathbf{Q})} K_{P,\circ}(\delta x, \delta x)\hat{\tau}_P^Q(H_P(\delta x) - T_1) \cdot \Gamma_Q'(H_Q(x) - T_1, T - T_1).$$

Decompose the integral over x into an integral over n in $N_Q(\mathbf{Q})\backslash N_Q(\mathbf{A})$, $m\in M_Q(\mathbf{Q})\backslash M_Q(\mathbf{A})^1$, $a\in A_Q(\mathbf{R})^0\cap G(\mathbf{A})^1$ and $k\in K$. Since

$$K_{P,\circ}(\delta namk, \delta namk) = \delta_Q(a)K_{P,\circ}(\delta mk, \delta mk),$$

for $\delta\in M_Q(\mathbf{Q})$, $J_\circ^T(f)$ equals the sum over $Q\supset P_0$ of the product of

$$\int_{\mathfrak{a}_Q/\mathfrak{a}_G} \Gamma_Q'(H, T-T_1)\,dH$$

with

$$(2.2)\quad \int_K\int_{M_Q(\mathbf{Q})\backslash M_Q(\mathbf{A})^1} \sum_{\{P:\, P_0\subset P\subset Q\}} (-1)^{\dim(A_P/A_Q)} \cdot\sum_{\delta\in P(\mathbf{Q})\cap M_Q(\mathbf{Q})\backslash M_Q(\mathbf{Q})} K_{P,\circ}(\delta mk, \delta mk)\cdot\hat{\tau}_P^Q(H_P(\delta mk) - T_1).$$

If u_1, u_2 belong to $M_Q(\mathbf{A})^1$,

$$\int_K K_{P,\mathfrak{o}}(u_1k, u_2k)\,dk$$

equals

$$\sum_{\gamma\in M_P(\mathbf{Q})\cap\mathfrak{o}} \int_{N_P(\mathbf{A})\cap M_Q(\mathbf{A})} f_Q(u_1^{-1}\gamma n u_2)\,dn,$$

where

$$f_Q(m) = \delta_Q(m)^{1/2}\int_K\int_{N_Q(\mathbf{A})} f(k^{-1}mnk)\,dn\,dk,$$

a smooth compactly supported function on $M_Q(\mathbf{A})^1$. The sum over P can be regarded as a sum over standard parabolic subgroups of M_Q. It follows that (2.2) equals $J_{\mathfrak{o}}^{M_Q,T_1}(f_Q)$. We therefore have

$$J_{\mathfrak{o}}^T(f) = \sum_{\{Q:\,Q\supset P_0\}} J_{\mathfrak{o}}^{M_Q,T_1}(f_Q)\cdot\int_{\mathfrak{a}_Q/\mathfrak{a}_G} \Gamma_Q'(H, T-T_1)\,dH. \tag{2.3}$$

In particular, $J_{\mathfrak{o}}^T(f)$ is a polynomial in T.

Next, take $\chi\in\mathfrak{X}$. Then $J_\chi^T(f)$ is the integral over x in $G(\mathbf{Q})\setminus G(\mathbf{A})^1$ of

$$\sum_{\{P:\,P\supset P_0\}} (-1)^{\dim(A_P/Z)} \sum_{\delta\in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\chi}(\delta x,\delta x)\hat\tau_P(H_P(\delta x)-T).$$

$K_{P,\chi}(x,y)$ is the kernel of the restriction of the operator $R_P(f)$ to the invariant subspace $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\setminus G(\mathbf{A})^1)_\chi$ of $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\setminus G(\mathbf{A})^1)$. (See [1(d), §3, §4].) It can be obtained by projecting

$$K_P(x,y) = \sum_{\gamma\in M_P(\mathbf{Q})}\int_{N_P(\mathbf{A})} f(x^{-1}\gamma n y)\,dn,$$

regarded either as a function of x or y, onto $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\setminus G(\mathbf{A})^1)_\chi$. Then $K_{P,\chi}(x,y)$, regarded as a function of either x or y, is smooth. The analogue of (2.3) is established as above. The argument for $J_\chi^T(f)$ follows that for $J_{\mathfrak{o}}^T(f)$ identically until we come to the integral

$$\int_K K_{P,\chi}(u_1k, u_2k)\,dk.$$

However, if we allow u_1, u_2 to belong to $G(\mathbf{A})^1$, this integral is just the kernel of the restriction of the operator

$$\int_K R_P(k)R_P(f)R_P(k)^{-1}\,dk$$

to the invariant subspace $L^2(N_P(\mathbf{A})M_P(\mathbf{Q})\setminus G(\mathbf{A}))_\chi$. Suppose that $Q\supset P$. There is a representation $R^{M_Q}_{P\cap M_Q}$ of $M_Q(\mathbf{A})^1$ on the Hilbert space $L^2(N_P(\mathbf{A})M_P(\mathbf{Q}) \cap M_Q(\mathbf{A})^1 \backslash M_Q(\mathbf{A})^1)$. Associated to χ, we have a subspace $L^2(N_P(\mathbf{A})M_P(\mathbf{Q}) \cap M_Q(\mathbf{A})^1 \backslash M_Q(\mathbf{A})^1)_\chi$ which is invariant under the operator $R^{M_Q}_{P\cap M_Q}(f_Q)$. If we take u_1 and u_2 to be elements in $M_Q(\mathbf{A})^1$, we obtain the kernel of the restriction of $R^{M_Q}_{P\cap M_Q}(f_Q)$ to this subspace. We therefore have

$$J^T_\chi(f) = \sum_{\{Q:\, Q\supset P_0\}} J^{M_Q,T_1}_\chi(f_Q)\cdot \int_{\mathfrak{a}_Q/\mathfrak{a}_G} \Gamma'_Q(H, T - T_1)\, dH. \tag{2.4}$$

Analogues of (2.3) and (2.4) certainly hold for the distributions $J^{L,T}_{\mathfrak{o}}$ and $J^{L,T}_\chi$. In particular, we have

PROPOSITION 2.3. *Suppose $L = L_Q$ for $Q \supset P_0$, and that $f \in C^\infty_c(L(\mathbf{A})^1)$, $\mathfrak{o} \in \mathfrak{O}$, and $\chi \in \mathfrak{X}$. Then $J^{L,T}_{\mathfrak{o}}(f)$ and $J^{L,T}_\chi(f)$ are polynomial functions of T.* □

These polynomials can be defined for all T. We shall denote the values assumed at T_0, the vector defined by Lemma 1.1, by $J^L_{\mathfrak{o}}(f)$ and $J^L_\chi(f)$ respectively. It follows from [1(d), Theorem 7.1] and [1(e), Theorem 2.1] that the series $\Sigma_{\mathfrak{o}}|J^L_{\mathfrak{o}}(f)|$ and $\Sigma_\chi|J^L_\chi(f)|$ converge. We obtain the identity

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J^L_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J^L_\chi(f) \tag{2.5}$$

from (1.2).

Suppose that P_0' is another minimal parabolic subgroup in $\mathcal{P}(M_0)$. There is a unique element s in Ω such that $P_0' = w_s^{-1}P_0 w_s$. If P is a parabolic subgroup that contains P_0, $P' = w_s^{-1}Pw_s$ is a parabolic subgroup containing P_0'. Suppose that

$$y = nmk, \quad n \in N_P(\mathbf{A}), \quad m \in M_P(\mathbf{A}), \quad k \in K,$$

is an arbitrary element in $G(\mathbf{A})$. Writing

$$w_s^{-1}y = w_s^{-1}nw_s\cdot w_s^{-1}mw_s\cdot w_s^{-1}k,$$

we see that

$$\begin{aligned} H_{P'}(w_s^{-1}y) &= H_{P'}(w_s^{-1}mw_s) + H_{P'}(w_s^{-1}) \\ &= s^{-1}H_P(y) + H_P(w_s^{-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \hat\tau_P(H_P(y) - T) &= \hat\tau_P(sH_{P'}(w_s^{-1}y) - sH_P(w_s^{-1}) - T) \\ &= \hat\tau_{P'}(H_{P'}(w_s^{-1}y) - H_{P_0}(w_s^{-1}) - s^{-1}T). \end{aligned}$$

It follows that $J_{\mathfrak{o}}^T(f)$ is the integral over $G(\mathbf{Q}) \setminus G(\mathbf{A})^1$ of

$$\sum_{\{P: P \supset P_0\}} (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q}) \setminus G(\mathbf{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x)$$

$$\cdot \hat{\tau}_{P'}\big(H_{P'}(w_s^{-1}\delta x) - H_{P_0}(w_s^{-1}) - s^{-1}T\big)$$

$$= \sum_{\{P': P' \supset P_0'\}} (-1)^{\dim(A_{P'}/Z)} \sum_{\delta \in P'(\mathbf{Q}) \setminus G(\mathbf{Q})} K_{P,\mathfrak{o}}(w_s\delta x, w_s\delta x)$$

$$\cdot \hat{\tau}_{P'}\big(H_{P'}(\delta x) - H_{P_0}(w_s^{-1}) - s^{-1}T\big).$$

Now

$$K_{P,\mathfrak{o}}(w_s y, w_s y) = \sum_{\gamma \in M_P(\mathbf{Q}) \cap \mathfrak{o}} \int_{N_P(\mathbf{A})} f\big(y^{-1} \cdot w_s^{-1}\gamma w_s \cdot w_s^{-1} n w_s \cdot y\big)\, dn$$

$$= \sum_{\gamma \in M_{P'}(\mathbf{Q}) \cap \mathfrak{o}} \int_{N_{P'}(\mathbf{A})} f\big(y^{-1}\gamma n y\big)\, dn$$

$$= K_{P',\mathfrak{o}}(y, y).$$

We have shown that if P_0 and T are replaced by P_0' and $H_{P_0}(w_s^{-1}) + s^{-1}T$ in the definition of $J_{\mathfrak{o}}^T(f)$, the result is the same. By Lemma 1.1,

$$H_{P_0}(w_s^{-1}) + s^{-1}T_0 = T_0.$$

It follows that $J_{\mathfrak{o}}^{T_0}(f) = J_{\mathfrak{o}}(f)$ is independent of P_0. The same argument applies to $J_\chi^T(f)$ and also to the corresponding distributions on Levi subgroups. Thus, each of the distributions $J_{\mathfrak{o}}^L$ and J_χ^L depends on M_0 and K, but not on a minimal parabolic subgroup of L. They are defined for any L in $\mathcal{L}(M_0)$.

Suppose that L' and L belong to $\mathcal{L}(M_0)$, and that $L' = w_s^{-1} L w_s$ for $s \in \Omega$. Suppose that $f \in C_c^\infty(L(\mathbf{A})^1)$, and

$$f'(m') = f\big(\tilde{w}_s m' \tilde{w}_s^{-1}\big), \qquad m' \in L'(\mathbf{A})^1.$$

Then from the argument above we see that

$$J_{\mathfrak{o}}^{L'}(f') = J_{\mathfrak{o}}^L(f)$$

and

$$J_\chi^{L'}(f') = J_\chi^L(f)$$

for all $\mathfrak{o}$ and χ.

3. Noninvariance

If f is a function on $G(\mathbf{A})^1$ and $y \in G(\mathbf{A})^1$, define

$$f^y(x) = f\big(yxy^{-1}\big).$$

A distribution J on $G(\mathbf{A})^1$ is said to be *invariant* if $J(f^y) = J(f)$ for all f and y. Our distributions $J_{\mathfrak{o}}$ and J_χ are definitely not invariant. In this section we shall evaluate them on functions of the form $f^y - f$.

The calculation is similar to that of the last section. Fix a minimal parabolic subgroup P_0 in $\mathcal{P}(M_0)$. Given f and y, fix a suitably regular point T in $\mathfrak{a}_0^+$. In the formula for $J_{\mathfrak{o}}^T(f^y)$, $K_{P,\mathfrak{o}}(\delta x, \delta x)$ will be replaced by

$$\sum_{\gamma \in M_P(Q) \cap \mathfrak{o}} \int_{N_P(\mathbf{A})} f^y(x^{-1}\delta^{-1}\gamma n \delta x)\, dn = K_{P,\mathfrak{o}}(\delta x y^{-1}, \delta x y^{-1}).$$

Thus, $J_{\mathfrak{o}}^T(f^y)$ equals

$$\int_{G(\mathbf{Q})\backslash G(\mathbf{A})^1} \sum_{\{P:\, P \supset P_0\}} (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\mathfrak{o}}(\delta x y^{-1}, \delta x y^{-1}) \cdot \hat{\tau}_P(H_P(\delta x) - T)\, dx,$$

which, after a change of variables, may be written as the integral over x in $G(\mathbf{Q}) \backslash G(\mathbf{A})^1$ of

$$\sum_{\{P:\, P \supset P_0\}} (-1)^{\dim(A_P/Z)} \sum_{\delta \in P(\mathbf{Q})\backslash G(\mathbf{Q})} K_{P,\mathfrak{o}}(\delta x, \delta x) \hat{\tau}_P(H_P(\delta x y) - T).$$

Let $K_P(\delta x)$ be any element in K such that $\delta x \cdot K_P(\delta x)^{-1}$ belongs to $P(\mathbf{A})$. Then

$$\begin{aligned} \hat{\tau}_P(H_P(\delta x y) - T) &= \hat{\tau}_P(H_P(\delta x) - T + H_P(K_P(\delta x) y)) \\ &= \sum_{\{Q:\, Q \supset P\}} (-1)^{\dim(A_Q/Z)} \hat{\tau}_P^Q(H_P(\delta x) - T) \\ &\qquad \cdot \Gamma_Q'(H_P(\delta x) - T, -H_P(K_P(\delta x) y)). \end{aligned}$$

The argument can now proceed as in Section 2. As a function of x,

$$\Gamma_Q'(H_P(x) - T, -H_P(K_P(x) y)) = \Gamma_Q'(H_Q(x) - T, -H_Q(K_Q(x) y))$$

is left $Q(\mathbf{Q})$-invariant. Set

$$\begin{aligned} u_Q'(x, y) &= \int_{A_Q(\mathbf{R})^0 \cap G(\mathbf{A})^1} \Gamma_Q'(H_Q(ax) - T, -H_Q(K_Q(ax) y))\, da \\ &= \int_{\mathfrak{a}_Q/\mathfrak{a}_G} \Gamma_Q'(H, -H_Q(K_Q(x) y))\, dH, \end{aligned}$$

and

$$f_{Q,y}(m) = \delta_Q(m)^{1/2} \int_K \int_{N_Q(\mathbf{A})} f(k^{-1} m n k) u_Q'(k, y)\, dn\, dk,$$

for $m \in M_Q(\mathbf{A})^1$. Then $u_Q'(k, y)$ is a smooth function of $k \in K$, and $f_{Q,y}$ is a smooth compactly supported function on $M_Q(\mathbf{A})^1$. The net result of the calculation is the formula

$$J_{\mathfrak{o}}^T(f^y) = \sum_{\{Q:\, Q \supset P_0\}} J_{\mathfrak{o}}^{M_Q, T}(f_{Q,y}). \tag{3.1}$$

Next, take $\chi \in \mathfrak{X}$. The analogous formula holds for $J_\chi^T(f^y)$. It is proved the same way. The only additional point is that

$$R_P(f^y) = R_P(y)^{-1} R_P(f) R_P(y).$$

Therefore the kernel of the restriction of $R_P(f^y)$ to $L^2(N_P(\mathbf{A}) M_P(\mathbf{Q}) \backslash G(\mathbf{A})^1)_\chi$ is

$$K_{P,\chi}(u_1 y^{-1}, u_2 y^{-1}), \qquad u_1, u_2 \in G(\mathbf{A})^1.$$

Modifying the discussion above (in the way we obtained (2.4) in the last section), we come to the formula

$$J_\chi^T(f^y) = \sum_{\{Q:\, Q \supset P_0\}} J_\chi^{M_Q, T}(f_{Q,y}). \tag{3.2}$$

Both sides of (3.1) and (3.2) are polynomials in T. If we take the values at $T = T_0$ of each side, the resulting distributions are all independent of P_0. However, the sums are still only over those Q which contain P_0. Suppose that Q' is any parabolic subgroup in $\mathfrak{F}(M_0)$. There is a unique Q containing P_0 such that $Q' = w_s^{-1} Q w_s$ for some s in Ω. Then $u'_{Q'}(\tilde{w}_s^{-1} k, y)$ equals

$$\int_{\mathfrak{a}_{Q'}/\mathfrak{a}_G} \Big(\sum_{\{R':\, Q' \subset R'\}} (-1)^{\dim(A_{R'}/Z)} \tau_{Q'}^{R'}(H) \cdot \hat{\tau}_{R'}\big(H + H_{Q'}(\tilde{w}_s^{-1} k y)\big) \Big)\, dH$$

$$= \int_{\mathfrak{a}_{Q'}/\mathfrak{a}_G} \sum_{R'} (-1)^{\dim(A_{R'}/Z)} \tau_{Q'}^{R'}(H) \hat{\tau}_{R'}\big(H + s^{-1} H_Q(ky)\big)\, dH$$

$$= \int_{\mathfrak{a}_Q/\mathfrak{a}_G} \sum_{\{R:\, Q \subset R\}} (-1)^{\dim(A_R/Z)} \tau_Q^R(H) \hat{\tau}_R(H + H_Q(ky))\, dH$$

$$= u'_Q(k, y).$$

If m belongs to $M_Q(\mathbf{A})^1$, $f_{Q',y}(\tilde{w}_s^{-1} m \tilde{w}_s)$ equals

$$\delta_{Q'}(\tilde{w}_s^{-1} m \tilde{w}_s)^{1/2} \int_K \int_{N_{Q'}(\mathbf{A})} f(k^{-1} \tilde{w}_s^{-1} m \tilde{w}_s n k) u'_{Q'}(k, y)\, dn\, dk$$

$$= \delta_Q(m)^{1/2} \int_K \int_{N_Q(\mathbf{A})} f(k^{-1} m n k) u'_{Q'}(\tilde{w}_s^{-1} k, y)\, dn\, dk$$

$$= f_{Q,y}(m).$$

It follows that $J_{\mathfrak{o}}^{M_{Q'}}(f_{Q',y}) = J_{\mathfrak{o}}^{M_Q}(f_{Q,y})$ and $J_\chi^{M_{Q'}}(f_{Q',y}) = J_\chi^{M_Q}(f_{Q,y})$ for all $\mathfrak{o}$ and

χ. Thus, the sums from (2.1) and (2.2) may be taken over all $Q \in \mathfrak{F}(M_0)$. The number of Q which are conjugate to a given parabolic subgroup containing P_0 equals the order of Ω divided by the order of the Weyl group of the given Levi component. The corresponding summands must then be multiplied by $|\Omega^{M_Q}|/|\Omega|$.

We summarize the results we have just established as a theorem. We want to leave room for future induction arguments, so we shall state the results for distributions on $L(\mathbf{A})^1$ rather than on $G(\mathbf{A})^1$.

THEOREM 3.2. *Suppose that* $L \in \mathfrak{L}(M_0)$, *that* $f \in C_c^\infty(L(\mathbf{A})^1)$, *and that* $y \in L(\mathbf{A})^1$. *Then*

$$J_{\mathfrak{o}}^L(f^y) = \sum_{Q \in \mathfrak{F}^L(M_0)} |\Omega^{M_Q}||\Omega^L|^{-1} J_{\mathfrak{o}}^{M_Q}(f_{Q,y})$$

and

$$J_{\chi}^L(f^y) = \sum_{Q \in \mathfrak{F}^L(M_0)} |\Omega^{M_Q}||\Omega^L|^{-1} J_{\chi}^{M_Q}(f_{Q,y}),$$

for all $\mathfrak{o} \in \mathfrak{O}$ *and* $\chi \in \mathfrak{X}$. □

Here $f_{Q,y}$ is defined in the obvious way. That is, $f_{Q,y}(m)$ equals

$$\delta_Q^L(m)^{1/2} \cdot \int_{K \cap L(\mathbf{A})} \int_{N_Q(\mathbf{A})} f(k^{-1}mnk) \cdot u_Q'(k,y)\, dn\, dk, \tag{3.3}$$

where

$$u_Q'(k,y) = \int_{\mathfrak{a}_Q/\mathfrak{a}_L} \sum_{\{R \in \mathfrak{F}^L(M_0):\, Q \subset R\}} \tau_Q^R(H)\hat{\tau}_R(H + H_Q(ky))\, dH.$$

This theorem is the basis of all that follows. We shall explain its role in the next section.

4. The main problem: Discussion and motivation

The object of this paper is to derive a trace formula whose terms are invariant. We want an identity which is of the form (2.5) but such that the distributions indexed by $\mathfrak{O}$ and $\mathfrak{X}$ are invariant. The computation of the last section gives a measure of the failure of the distributions $J_{\mathfrak{o}}$ and J_χ to be invariant. We will later use this information to construct the required invariant distributions.

Suppose that S is a fixed finite set of places in $\mathbf{Q}$. Write $\mathbf{Q}_S = \prod_{v \in S} \mathbf{Q}_v$. Then if H is any subgroup of G defined over $\mathbf{Q}$,

$$H(\mathbf{Q}_S) = \prod_{v \in S} H(\mathbf{Q}_v).$$

Suppose that for every $L \in \mathcal{L}(M_0)$, $U(L)$ is some vector space of functions with common domain a subset of $L(\mathbf{Q}_S)$. We assume that $U(L)$ is complete with respect to some topology. We assume in addition that for any $y \in L(\mathbf{Q}_S)^1 = L(\mathbf{Q}_S) \cap L(\mathbf{A})^1$, the map

$$f \to f^y, \qquad f \in U(L),$$

is a continuous endomorphism of $U(L)$; and that for any $Q \in \mathcal{F}^L(M_0)$, the map

$$f \to f_{Q,y},$$

given by the formula (3.3), sends $U(L)$ continuously to $U(M_Q)$. By a distribution on $U(L)$ we mean an element in $U(L)'$, the dual topological vector space of $U(L)$. We suppose, finally, that we have been given a family of distributions $\{J^L \in U(L)'\}$ and a family of nonzero complex numbers $\{c(L)\}$, each indexed by $\mathcal{L}(M_0)$, such that for any L, $y \in L(\mathbf{Q}_S)$, and $f \in U(L)$,

$$J^L(f^y) = \sum_{Q \in \mathcal{F}^L(M_0)} c(M_Q)c(L)^{-1} J^{M_Q}(f_{Q,y}). \tag{4.1}$$

Our primary example of such a scheme comes from letting $U(L) = C_c^\infty(L(\mathbf{Q}_S)^1)$ and $c(L) = |\Omega^L|$. If S contains the Archimedean place, we can take $\{J^L\}$ to be one of the families $\{J_{\mathfrak{o}}^L\}$ or $\{J_\chi^L\}$. For if f is a function in $C_c^\infty(L(\mathbf{Q}_S)^1)$, the product of f with the characteristic function of

$$\prod_{v \notin S} (K_v \cap L(Q_v))$$

is a function in $C_c^\infty(L(\mathbf{A})^1)$. Conversely, any function in $C_c^\infty(L(\mathbf{A})^1)$ can be obtained in this way, for a large enough set S. It is in this sense that $J_{\mathfrak{o}}^L$ and J_χ^L are regarded as distributions on $C_c^\infty(L(\mathbf{Q}_S)^1)$.

We would like to be able to associate a natural family of invariant distributions to each family $\{J^L\}$. This will be possible if we are given some additional data. Suppose that for every $M \in \mathcal{L}(M_0)$, $V(M)$ is a second complete topological vector space. Suppose that for every pair $M \subset L$, we are given a continuous map

$$\phi_M^L \colon U(L) \to V(M)$$

such that for every $y \in L(\mathbf{Q}_S)^1$,

$$\phi_M^L(f^y) = \sum_{Q \in \mathcal{F}^L(M)} \phi_M^{M_Q}(f_{Q,y}). \tag{4.2}$$

We shall sometimes write ϕ for ϕ_M^M. In this case, (4.2) says that $\phi(f^y) = \phi(f)$ for each y and f. It follows that for every i in $V(M)'$, the distribution

$$\phi'(i) \colon f \to i(\phi(f)), \qquad f \in U(M),$$

is invariant. We make the further assumption:

(4.3) For every $M \in \mathfrak{L}(M_0)$, ϕ maps $U(M)$ onto $V(M)$; the image of the transpose, ϕ', is the space of all invariant distributions on $U(M)$.

The first statement of (4.3) implies that ϕ' is injective; the second states that any invariant distribution on $U(M)$ is of the form $(\phi)'(i)$. If I is any invariant distribution on $U(M)$ we shall let $\hat{I}$ be the unique element i in $V(M)'$ such that $\phi'(i) = I$.

Proposition 4.1. *Suppose that* $\{\phi_M^L\}$ *satisfies* (4.2) *and* (4.3). *Then for every family* $\{J^L\}$ *of distributions satisfying* (4.1) *there is a unique family* $\{I^L \in U(L)'\}$ *of invariant distributions such that for every* $f \in U(L)$,

$$J^L(f) = \sum_{M \in \mathfrak{L}^L(M_0)} c(M)c(L)^{-1}\hat{I}^M(\phi_M^L(f)).$$

Proof. Fix $\{J^L\}$. Assume inductively that I^M has been defined for all groups $M \in \mathfrak{L}(M_0)$ such that $\dim M < \dim L$. Define

$$I^L(f) = J^L(f) - \sum_{\substack{M \in \mathfrak{L}^L(M_0) \\ M \neq L}} c(M)c(L)^{-1}\hat{I}^M(\phi_M^L(f)),$$

for $f \in U(L)$. We want to evaluate $I^L(f^y - f)$, for $y \in L(\mathbf{Q}_S)$. The function f equals $f_{L,y}$. Therefore $J^L(f^y - f)$ equals the sum on the right hand side of (4.1), but taken only over those $Q \neq L$. The same observation gives a formula for $\phi_M^L(f^y - f)$. It follows that $I^L(f^y - f)$ equals the difference between

$$\sum_{\substack{Q \in \mathfrak{F}^L(M_0) \\ Q \neq L}} c(M_Q)c(L)^{-1}J^{M_Q}(f_{Q,y})$$

and

$$\sum_{M \in \mathfrak{L}^L(M_0)} \sum_{\substack{Q \in \mathfrak{F}^L(M) \\ Q \neq L}} c(M)c(L)^{-1}\hat{I}^M(\phi_M^{M_Q}(f_{Q,y})).$$

Now $\mathfrak{F}^L(M)$ is a subset of $\mathfrak{F}^L(M_0)$. A group $Q \in \mathfrak{F}^L(M_0)$ belongs to $\mathfrak{F}^L(M)$ if and only if $M \subset M_Q$. Therefore $I^L(f^y - f)$ is the sum over all $Q \in \mathfrak{F}^L(M_0)$, $Q \neq L$, of the product of $c(M_Q)c(L)^{-1}$ with

$$J^{M_Q}(f_{Q,y}) - \sum_{\{M:\, M \subset M_Q\}} c(M)c(M_Q)^{-1}\hat{I}^M(\phi_M^{M_Q}(f_{Q,y})).$$

This last expression is 0 by our induction assumption. Thus, I^L is an invariant distribution, as required. □

Suppose that $U(L) = C_c^\infty(L(\mathbf{Q}_S)^1)$ and that maps ϕ_M^L, satisfying the hypotheses of the proposition, have been defined. If S contains the Archimedean valuation, we can regard $J_\mathfrak{o}^L$ and J_χ^L as distributions on $C_c^\infty(L(\mathbf{Q}_S)^1)$, as we noted above. Then by Theorem 3.2 we obtain two families $\{I_\mathfrak{o}^L\colon \mathfrak{o} \in \mathcal{O}\}$ and $\{I_\chi^L\colon \chi \in \mathfrak{X}\}$ of invariant distributions on $C_c^\infty(L(\mathbf{Q}_S)^1)$. Our invariant trace formula is a formal consequence of the definitions.

THEOREM 4.2. *Suppose that $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$. Then the series $\Sigma_\mathfrak{o} I_\mathfrak{o}^L(f)$ and $\Sigma_\chi I_\chi^L(f)$ converge absolutely, and*

$$\sum_\mathfrak{o} I_\mathfrak{o}^L(f) = \sum_\chi I_\chi^L(f).$$

Proof. Assume inductively that the theorem holds if L is replaced by any group $M \in \mathcal{L}(M_0)$ with $\dim M < \dim L$. The series

$$\sum_\mathfrak{o} |I_\mathfrak{o}^L(f)|$$

equals

$$\sum_\mathfrak{o} \left| J_\mathfrak{o}^L(f) - \sum_{\substack{M \in \mathcal{L}^L(M_0) \\ M \neq L}} c(M)c(L)^{-1} \hat{I}_\mathfrak{o}^M\big(\phi_M^L(f)\big) \right|.$$

It is bounded by the sum of

$$\sum_\mathfrak{o} |J_\mathfrak{o}^L(f)|$$

and

$$\sum_{\{M:\, M \subsetneqq L\}} c(M)c(L)^{-1} \sum_{\mathfrak{o} \in \mathcal{O}} \left|\hat{I}_\mathfrak{o}^M\big(\phi_M^L(f)\big)\right|.$$

The first term is finite (see the remark following Proposition 2.3). By the assumption (4.3) $\phi_M^L(f)$ can be regarded as the image under ϕ_M^M of a function in $C_c^\infty(M(\mathbf{Q}_S)^1)$. The second term is then finite by our induction assumption. Thus, $\Sigma_\mathfrak{o} |I_\mathfrak{o}^L(f)|$ is finite. The same argument shows that

$$\sum_\mathfrak{o} I_\mathfrak{o}^L(f) = \sum_\mathfrak{o} J_\mathfrak{o}^L(f) - \sum_{\substack{M \in \mathcal{L}^L(M_0) \\ M \neq L}} c(M)c(L)^{-1} \sum_\mathfrak{o} \hat{I}_\mathfrak{o}^M\big(\phi_M^L(f)\big).$$

Similarly, $\Sigma_\chi |I_\chi^L(f)|$ is finite and

$$\sum_\chi I_\chi^L(f) = \sum_\chi J_\chi^L(f) - \sum_{\substack{M \in \mathcal{L}^L(M_0) \\ M \neq L}} c(M)c(L)^{-1} \sum_\chi \hat{I}_\chi^M\big(\phi_M^L(f)\big).$$

The required identity now follows from (2.5) and our induction assumption. □

We have tried to motivate *why* we must define maps

$$\phi_M^L\colon U(L) \to V(M),$$

with $U(M) = C_c^\infty(L(\mathbf{Q}_S)^1)$. It is a task which will consume the rest of the paper. We conclude the section with an attempt to motivate *how* we will make the definition.

The first step is obviously to define the spaces $V(M)$ and the maps $\phi = \phi_M^M$. There are two apparent possibilities. We could try taking $\phi(f)$ to be what is sometimes called the Harish-Chandra transform of f, obtained by taking orbital integrals of f. Then $V(M)$ would be a space of functions on the regular semisimple conjugacy classes of $M(\mathbf{Q}_S)^1$. This was the approach taken in [5(a)]. Alternatively, we might take $V(M)$ to be a space of complex valued functions on the irreducible tempered characters of $M(\mathbf{Q}_S)^1$. Then $\phi(f)$ would be defined by the character values of f. It is this second alternative that we will choose. The solution that it eventually yields seems quite natural. We will discuss our candidates $V(M)$ and ϕ in Section 5, and the extent to which the hypothesis (4.3) is known.

Once $V(M)$ has been chosen we will define the maps ϕ_M^L. This amounts to associating distributions

$$J_{M,\pi}^L, \qquad L \in \mathcal{P}(M_0),$$

on $C_c^\infty(L(\mathbf{Q}_S)^1)$ to irreducible tempered representations π of M, which vanish if L does not contain M, and for which (4.2) holds. The only distributions that we know at present which satisfy (4.2) are the families $\{J_\chi^L\}$ and $\{J_{\mathfrak{o}}^L\}$. But χ can be represented by a cuspidal automorphic representation on a Levi subgroup of G (which is a Levi subgroup of L if J_χ^L does not vanish). For certain χ (those we called unramified in [1(e)]) $J_\chi(f)$ can be expressed explicitly in terms of these corresponding cuspidal representations [1(e), §4]. We shall simply define $J_{M,\pi}^L$ by the appropriate analogue of this formula. At the same time, we define distributions

$$J_{M,\gamma}^L, \qquad L \in \mathcal{P}(M_0),$$

for regular elements γ in $L(\mathbf{Q}_S)$. They are obtained from an appropriate analogue of a formula [1(d), (8.7)] proved for $J_{\mathfrak{o}}(f)$ for unramified classes $\mathfrak{o}$. In Section 8 we shall show that these distributions satisfy (4.2).

We can then define the value of ϕ_M^L at π to be $J_{M,\pi}^L(f)$. It is still necessary to show that $\phi_M^L(f)$ belongs to $V(M)$. This can be regarded as the main problem of our paper. We will first solve the analogous problem for $\mathcal{C}(L(\mathbf{Q}_S))$, the Schwartz space on $L(\mathbf{Q}_S)$. For if $U(L)$ is taken to be $\mathcal{C}(L(\mathbf{Q}_S))$, the spaces $V(M)$ and maps ϕ_M^L can also be defined by the prescription outlined above. In Section 9 we shall show that ϕ_M^L maps $\mathcal{C}(L(\mathbf{Q}_S))$ continuously to $V(M)$. The distributions $\{J_{M,\gamma}^L\}$ will all be tempered and satisfy (4.2), so Proposition 4.1 will provide us with a family $\{I_{M,\gamma}^L\}$ of invariant distributions on $\mathcal{C}(L(\mathbf{Q}_S))$. In Section 11 we show how each $I_{M,\gamma}^L$ decomposes into distributions on the local groups $L(\mathbf{Q}_v)$. This allows us in Section 12 to prove finally that ϕ_M^L maps $C_c^\infty(L(\mathbf{Q}_S)^1)$ continuously into the associated space $V(M)$.

5. Invariant harmonic analysis

We shall now discuss candidates for the spaces $U(M)$ and $V(M)$, and also for the map

$$\phi = \phi_M^M\colon U(M) \to V(M).$$

Condition (4.3) becomes a question in local harmonic analysis, which has not yet been answered in complete generality. It is possible that an affirmative answer is not too far distant. At any rate, we shall simply assume what is needed.

If H is any locally compact group, let $\Pi(H)$ denote the set of equivalence classes of irreducible (continuous) unitary representations of H. If the notion of a *tempered* representation is defined for H, we will let $\Pi_{\mathrm{temp}}(H)$ stand for those classes that are tempered. Suppose that v is a valuation on $\mathbf{Q}$. If v is discrete there corresponds a rational prime p_v. If v is real, set $p_v = e$. Suppose that M_v is a Levi subgroup of G defined over $\mathbf{Q}_v$. Harish-Chandra defines the map H_{M_v} from $M_v(\mathbf{Q}_v)$ to $\mathfrak{a}_v = \mathrm{Hom}(X(M_v)_{\mathbf{Q}_v}, \mathbf{R})$ by setting

$$p_v^{\langle H_{M_v}(m_v), \chi\rangle} = |\chi_v(m_v)|_v, \qquad \chi \in X(M_v)_{\mathbf{Q}_v}, m_v \in M_v(\mathbf{Q}_v).$$

Suppose that $\pi_v \in \Pi(M_v(\mathbf{Q}_v))$. If ζ_v is a vector in $\mathfrak{a}_{v,\mathbf{C}}^*$, we set

$$\pi_{v,\zeta_v}(m_v) = \pi_v(m_v)\cdot p_v^{\zeta_v(H_{M_v}(m_v))}.$$

Suppose that M_v' is another Levi subgroup of G defined over $\mathbf{Q}_v$, $M_v' \subset M_v$, and that $\sigma_v \in \Pi(M_v'(\mathbf{Q}_v))$. If $P_v \in \mathcal{P}^{M_v}(M_v')$, we can lift σ_v to the parabolic subgroup $P_v(\mathbf{Q}_v)$, and then induce up to $M_v(\mathbf{Q}_v)$. The class of the resulting representation of $M_v(\mathbf{Q}_v)$ is independent of P_v. We denote it by $\sigma_v^{M_v}$. It is convenient to define, in a noncanonical way, a "norm" function on $\Pi(M_v(\mathbf{Q}_v))$. If v is discrete and $\pi_v \in \Pi(M_v(\mathbf{Q}_v))$, set $\|\pi_v\| = 0$. However, if v is Archimedean, let Δ be a fixed left invariant differential operator on $M_v(\mathbf{Q}_v)$ of order two. We assume that Δ is positive definite and that it commutes with right translations on $M_v(\mathbf{Q}_v)$ by $K_v \cap M_v(\mathbf{Q}_v)$. For any $\pi_v \in \Pi(M_v(\mathbf{Q}_v))$ we obtain an operator $\pi_v(\Delta)$ on the space on which π_v acts. Let $\|\pi_v\|$ be its smallest eigenvalue. It is a positive number. We can follow the same prescription to define $\|W\|$ for any W in $\Pi(K_{\mathbf{R}})$.

Now suppose that S is a finite set of valuations on $\mathbf{Q}$. Suppose that for each $v \in S$, M_v is a Levi subgroup of G defined over $\mathbf{Q}_v$. We shall refer to $\mathfrak{M} = \prod_{v \in S} M_v$ as a *Levi S-subgroup* of G, and we write $\mathfrak{M}_S = \prod_{v \in S} M_v(\mathbf{Q}_v)$. Any $\pi \in \Pi(\mathfrak{M}_S)$ is a unique tensor product $\otimes_{v \in S} \pi_v$ of irreducible representations of the groups $M_v(\mathbf{Q}_v)$ [3(a)]. If $\zeta = \oplus_{v \in S} \zeta_v$ is a vector in $\oplus_v \mathfrak{a}^*_{v,\mathbf{C}}$ we shall put

$$\pi_\zeta = \bigotimes_{v \in S} \pi_{v,\zeta_v}.$$

We shall also put

$$\|\pi\| = \sup_{v \in S} \|\pi_v\|.$$

If $\mathfrak{M}' = \prod_{v \in S} M'_v$ is contained in $\mathfrak{M}$, and $\sigma = \otimes_{v \in S} \sigma_v$ belongs to $\Pi(\mathfrak{M}'_S)$, set

$$\sigma^{\mathfrak{M}} = \bigotimes_{v \in S} \sigma_v^{M_v}.$$

Most of the time we will take each M_v equal to a fixed $M \in \mathfrak{L}(M_0)$. Then we shall write σ^M for $\sigma^{\mathfrak{M}}$. In this situation, we shall sometimes want to embed a vector $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$ diagonally into $\oplus_{v \in S} \mathfrak{a}^*_{v,\mathbf{C}}$ by

$$\lambda \to \zeta = \bigoplus_{v \in S} (\log p_v)^{-1} \lambda.$$

We shall then write π_λ for π_ζ, so that we have

$$\pi_\lambda(m) = \pi(m) e^{\lambda(H_M(m))}, \qquad m \in M(\mathbf{Q}_S).$$

Our first candidate for $U(M)$ will be the Schwartz space on $M(\mathbf{Q}_S)$. Actually, only the Schwartz spaces $\mathcal{C}(M(\mathbf{Q}_v))$, $v \in S$, appear in the literature. However, Harish Chandra's definition ([5(a)], [5(b)]) extends easily to $M(\mathbf{Q}_S)$. Indeed, if $v \in S$ let Ξ_v^M and σ_v^M be the functions on $M(\mathbf{Q}_v)$ used in [5(a)] and [5(b)] in the definition of $\mathcal{C}(M(\mathbf{Q}_v))$. Given $m = \prod_{v \in S} m_v$ in $M(\mathbf{Q}_S)$, set

$$\Xi^M(m) = \prod_{v \in S} \Xi_v^M(m_v)$$

and

$$\sigma^M(m) = \prod_{v \in S} \sigma_v^M(m_v).$$

If the Archimedean valuation v belongs to S, let X_L and X_R be operators on $C^\infty(M(\mathbf{Q}_S))$ which act, through $C^\infty(M(\mathbf{Q}_v))$, as left and right invariant differential operators. If the Archimedean valuation does not belong to S, set $X_L = X_R = 1$. For any $n \geq 0$ and $f \in C^\infty(M(\mathbf{Q}_S))$, put

$$\|f\|_{X_L, X_R, n} = \sup_{m \in M(\mathbf{Q}_S)} \left\{ |(X_L X_R f)(m)| \Xi^M(m) \big(1 + \sigma^M(m)\big)^{-n} \right\}.$$

Now, for any open compact subgroup K_0 of

$$K_S = \prod_{\{v \in S:\, v \text{ discrete}\}} K_v$$

let $\mathcal{C}_{K_0}(M(\mathbf{Q}_S))$ be the space of smooth, $K_0 \cap M(\mathbf{Q}_S)$ bi-invariant functions f on $M(\mathbf{Q}_S)$ such that

$$\| f \|_{X_L, X_R, n} < \infty$$

for all X_L, X_R and n. The seminorms $\| \cdot \|_{X_L, X_R, n}$ induce a topology on $\mathcal{C}_{K_0}(M(\mathbf{Q}_S))$. The Schwartz space $\mathcal{C}(M(\mathbf{Q}_S))$ can then be defined as the topological direct limit, over all K_0, of the spaces $\mathcal{C}_{K_0}(M(\mathbf{Q}_S))$.

When $U(M) = \mathcal{C}(M(\mathbf{Q}_S))$, we will take $V(M)$ to be a space of functions on $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$. If V is any real vector space, let $\text{Diff}(V)$ denote the space of differential operators with constant coefficients on V. Now, suppose that ϕ is a complex valued function on $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$. If π is a finite sum of representations $\{\pi^i\}$ in $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$, put $\phi(\pi) = \Sigma_i \phi(\pi^i)$. Suppose that $\mathfrak{M} = \Pi_{v \in S} M_v$ is a Levi S-subgroup of M and that $D \in \text{Diff}(\oplus_{v \in S} i\mathfrak{a}_v^*)$. If $\sigma \in \Pi_{\text{temp}}(\mathfrak{M}_S)$ and $\zeta \in \oplus_{v \in S} i\mathfrak{a}_v^*$, $(\sigma_\zeta)^M$ is a finite sum of classes in $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$. Then $\phi((\sigma_\zeta)^M)$ is defined. If it is a smooth function of ζ, we shall denote its derivative with respect to D at $\zeta = 0$ by $D_\sigma \phi(\sigma^M)$. Otherwise, put $D_\sigma \phi(\sigma^M) = \infty$. Suppose that K_0 is an open compact subgroup of K_S. Let $\mathcal{I}_{K_0}(M(\mathbf{Q}_S))$ be the space of complex valued functions ϕ on $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$ such that

(i) $\phi(\pi) = 0$ unless π has a $(K_0 \cap M(\mathbf{Q}_S))$-fixed vector.

(ii) For any $\mathfrak{M}$, D the $n \geq 0$, the seminorm

$$\| \phi \|_{D, n} = \sup_{\sigma \in \Pi_{\text{temp}}(\mathfrak{M}_S)} (1 + \| \sigma \|)^n | D_\sigma \phi(\sigma^M) |$$

is finite.

We topologize $\mathcal{I}_{K_0}(M(\mathbf{Q}_S))$ with the seminorms $\| \cdot \|_{D, n}$. Define $\mathcal{I}(M(\mathbf{Q}_S))$ to be the union over all K_0 of the spaces $\mathcal{I}_{K_0}(M(\mathbf{Q}_S))$, equipped with the direct limit topology. $\mathcal{I}(M(\mathbf{Q}_S))$ is our first candidate for the space $V(M)$.

Given M and $f \in \mathcal{C}(M(\mathbf{Q}_S))$, let $\phi(f)$ be the function on $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$ whose value, $\phi(f, \pi)$, at π is the trace of the operator

$$\pi(f) = \int_{M(\mathbf{Q}_S)} f(x) \pi(x)\, dx.$$

(For the existence of the integral and of trace class see [5(a)] and [12].) It follows fairly readily from the definition of $\mathcal{C}(M(\mathbf{Q}_S))$ that

$$\phi_M^M = \phi : f \to \phi(f), \qquad f \in \mathcal{C}(M(\mathbf{Q}_S)),$$

maps $\mathcal{C}(M(\mathbf{Q}_S))$ continuously into $\mathcal{I}(M(\mathbf{Q}_S))$. Then (4.3) is the following assumption, which we take for granted from now on.

ASSUMPTION 5.1. *For every $M \in \mathcal{L}(M_0)$, ϕ maps $\mathcal{C}(M(\mathbf{Q}_S))$ onto $\mathcal{I}(M(\mathbf{Q}_S))$. The image of the transpose, ϕ', is the space of all tempered invariant distributions on $M(\mathbf{Q}_S)$.*

The assumption will hold for S if it holds for each v in S. If v is Archimedean, it can be established from the results of [1(a)] and [8]. If v is discrete, the first statement of the assumption can probably be proved with the results of [5(c)], but the second statement is not known. However in the case that $G = \mathrm{GL}_n$, the induced representations σ^M are all irreducible (see [2] and [6]), and the second statement of the assumption can presumably be proved from this fact.

Important examples of invariant tempered distributions on M are the orbital integrals. Let $M(\mathbf{Q}_S)_{\mathrm{reg}}$ be the set of regular semisimple elements in $M(\mathbf{Q}_S)$. An element γ belongs to $M(\mathbf{Q}_S)_{\mathrm{reg}}$ if and only if $M(\mathbf{Q}_S)_\gamma$, the centralizer of γ in $M(\mathbf{Q}_S)$, is of the form $\mathcal{T}_S = \prod_{v \in S} T_v(\mathbf{Q}_v)$, where each T_v is a maximal torus M defined over $\mathbf{Q}_v$. Let

$$D^M(m), \qquad m \in M(\mathbf{Q}_S),$$

be the coefficient of degree equal to the rank of M in the characteristic polynomial of $1 - \mathrm{Ad}(m)$. If $m = \prod_{v \in S} m_v$ belongs to $M(\mathbf{Q}_S)$, set

$$|D^M(m)| = \prod_{v \in S} |D^M(m_v)|_v.$$

The orbital integral of $f \in \mathcal{C}(M(\mathbf{Q}_S))$ at $\gamma \in M(\mathbf{Q}_S)_{\mathrm{reg}}$ is then defined as

$$I_\gamma^M(f) = |D^M(\gamma)|^{1/2} \int_{M(\mathbf{Q}_S)_\gamma \backslash M(\mathbf{Q}_S)} f(x^{-1}\gamma x)\, dx.$$

I_γ^M is an invariant tempered distribution. In view of our assumption, we can identify I_γ^M with a uniquely determined linear function of $\hat{I}_\gamma^M$ on $\mathcal{I}(M(\mathbf{Q}_S))$.

Suppose that $L \in \mathcal{L}(M)$ and that $P \in \mathcal{P}^L(M)$. If $f \in \mathcal{C}(L(\mathbf{Q}_S))$,

$$f_P(m) = \delta_P(m)^{1/2} \int_{K \cap L(\mathbf{Q}_S)} \int_{N_P(\mathbf{Q}_S)} f(k^{-1}mnk)\, dn\, dk$$

is a Schwartz function on $M(\mathbf{Q}_S)$. For $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$, it is a simple exercise to show that $\phi(f, \pi^L)$, the character of the induced representation π^L evaluated at f, equals $\phi(f_P, \pi)$. In particular, the element $\phi(f_P)$ in $\mathcal{I}(M(\mathbf{Q}_S))$ depends only on M and not on P. We denote it by f_M. As a function of f, $f_M(\pi)$ is an invariant distribution. Assumption 5.1 then implies that the map $f \to f_M$ factors through a map $\phi \to \phi_M$ from $\mathcal{I}(L(\mathbf{Q}_S))$ to $\mathcal{I}(M(\mathbf{Q}_S))$. It satisfies the formula

$$(5.1) \qquad \phi_M(\pi) = \phi(\pi^L),\ \phi \in \mathcal{I}(L(\mathbf{Q}_S)),\ \pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S)).$$

The map behaves well with respect to orbital integrals. If γ belongs to $L(\mathbf{Q}_S)_{\text{reg}} \cap M$, it is easy to show that

$$\hat{I}^L_\gamma(\phi) = \hat{I}^M_\gamma(\phi_M), \tag{5.2}$$

for all ϕ.

As we suggested in Section 4, our second and main candidate for $U(M)$ is the space $C_c^\infty(M(\mathbf{Q}_S)^1)$. It could be defined as the space of *compactly supported* functions on $M(\mathbf{Q}_S)^1$ which are restrictions to $M(\mathbf{Q}_S)^1$ of functions in $\mathcal{C}(M(\mathbf{Q}_S))$. The orbital integrals of a compactly supported function should differ from those of an arbitrary Schwartz function only by being of bounded support in the variables γ. This suggests a definition for our corresponding candidate for $V(M)$. It will be a space of functions on $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$. Now $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$ is the set of orbits of $i\mathfrak{a}^*$ in $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$ under the action

$$(\pi, \Lambda) \to \pi_\Lambda, \ \pi \in \Pi_{\text{temp}}(M(\mathbf{Q}_S)), \qquad \Lambda \in i\mathfrak{a}^*.$$

Let $\text{Lat}(S) = \text{Lat}(M, S)$ be the stabilizer in $i\mathfrak{a}^*$ of any π. If $\mathcal{T}$ is as above, put $\mathcal{T}_S^1 = \mathcal{T}_S \cap M(\mathbf{Q}_S)^1$ and $\mathcal{T}_{S,\text{reg}}^1 = \mathcal{T}_S^1 \cap M(\mathbf{Q}_S)_{\text{reg}}$. We can project any $\phi \in \mathcal{I}(M(\mathbf{Q}_S))$ onto a function

$$\pi \to \int_{i\mathfrak{a}^*/\text{Lat}(S)} \phi(\pi_\Lambda)\, d\Lambda, \ \pi \in \Pi_{\text{temp}}(M(\mathbf{Q}_S)), \tag{5.3}$$

on $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$. If γ belongs to $\mathcal{T}_S^1$, $I^M_\gamma(\phi)$ depends only on the function (5.3).

We can define the notion of the *support* of a function in $\mathcal{I}(M(\mathbf{Q}_S))$, or as we prefer, of the function (5.3) on $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$. If ϕ^1 is the function (5.3), let $\text{supp}(\phi^1)$ be the set of pairs $(\mathcal{T}, \text{supp}_{\mathcal{T}}(\phi^1))$, where $\mathcal{T}$ is as above and $\text{supp}_{\mathcal{T}}(\phi^1)$ is the closure in $\mathcal{T}_S^1$ of the support of the function

$$\gamma \to \hat{I}^M_\gamma(\phi), \qquad \gamma \in \mathcal{T}_{S,\text{reg}}^1.$$

Suppose that $\mathcal{S} = (\mathcal{T}, S(\mathcal{T}))$ is any collection of pairs such that $S(\mathcal{T})$ is a compact subset of $\mathcal{T}_S^1$ for each $\mathcal{T}$. Let $\mathcal{I}_{\mathcal{S}}(M(\mathbf{Q}_S)^1)$ be the set of functions ϕ^1 of the form (5.3) such that $\text{supp}(\phi^1) \subset \mathcal{S}$; that is, such that $\text{supp}_{\mathcal{T}}(\phi^1)$ is contained in $S(\mathcal{T})$ for every $\mathcal{T}$. The inverse image of $\mathcal{I}_{\mathcal{S}}(M(\mathbf{Q}_S)^1)$ under the map (5.3) is a closed subset of $\mathcal{I}(M(\mathbf{Q}_S))$. We give $\mathcal{I}_{\mathcal{S}}(M(\mathbf{Q}_S)^1)$ the topology induced by this map. We then define $\mathcal{I}_c(M(\mathbf{Q}_S)^1)$ to be the union over all such collections $\mathcal{S}$ of the spaces $\mathcal{I}_{\mathcal{S}}(M(\mathbf{Q}_S)^1)$, equipped with the direct limit topology. Now if $\phi = \phi(f)$, for $f \in \mathcal{C}(M(\mathbf{Q}_S))$, the function (5.3) depends only on the restriction of f to $M(\mathbf{Q}_S)^1$. In other words, for every $f \in C_c^\infty(M(\mathbf{Q}_S)^1)$ we obtain a function on $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$, which we continue to denote by $\phi(f)$. It follows from standard properties of orbital integrals that

$$\phi^M_M = \phi: f \to \phi(f), \qquad f \in C_c^\infty(M(\mathbf{Q}_S)^1),$$

is a continuous map from $C_c^\infty(M(\mathbf{Q}_S)^1)$ to $\mathcal{I}_c(M(\mathbf{Q}_S)^1)$. Then with $U(M) = C_c^\infty(M(\mathbf{Q}_S)^1)$, $V(M) = \mathcal{I}_c(M(\mathbf{Q}_s)^1)$ and ϕ equal to this map, (4.3) becomes the following assumption, which we also take for granted.

ASSUMPTION 5.2. *For every* $M \in \mathcal{L}(M_0)$, ϕ *maps* $C_c^\infty(M(\mathbf{Q}_S)^1)$ *onto* $\mathcal{I}_c(M(\mathbf{Q}_S)^1)$. *The image of the transpose,* ϕ', *is the space of all invariant distributions on* $M(\mathbf{Q}_S)^1$.

This assumption, too, will hold for S if it holds for each v in S. For archimedean v, it is essentially the characterization of orbital integrals of smooth functions of compact support, a well known problem. It has been solved for $G = GL_2$ in [10(b)] but there has apparently been nothing established for other groups. However, it will certainly be needed in any of the applications of the trace formula, so there seems no harm in assuming it at this point. If v is discrete the assumption does not amount to anything new. The orbital integrals of compactly supported functions can be characterized in terms of their Shalika germs. Moreover, Harish-Chandra has shown that the linear span of the orbital integrals is dense in the space of all invariant distributions on a p-adic group. His unpublished argument also uses Shalika germs.

In summary, Assumptions 5.1 and 5.2 each contain two assertions; the statements in each case apply separately to real and p-adic groups, so there are eight assertions in all. The two assertions of Assumption 5.1 are known for real groups and unknown for p-adic groups, although probably within reach of present methods. The two assertions of Assumption 5.2 are known for p-adic groups and unknown for real groups.

6. Convex sets and some related functions

Throughout this section, M will be a fixed Levi subgroup in $\mathcal{L}(M_0)$. We shall establish some elementary properties for smooth functions on the real vector space $i\mathfrak{a}_M^*$. Suppose that $P \in \mathcal{P}(M)$. We saw in Lemma 2.2 that for fixed $X \in \mathfrak{a}_P = \mathfrak{a}_M$, the function

$$\sum_{\{Q:\,Q\supset P\}} (-1)^{\dim(A_P/A_Q)} e^{\lambda_Q(X)} \hat{\theta}_P^Q(\lambda)^{-1}\theta_Q(\lambda)^{-1},$$

could be extended to a smooth function on $i\mathfrak{a}_M^*$. We proved this geometrically, by exhibiting the function as the Fourier transform of a compactly supported function. We could have proved the result directly by transcribing the proof of Lemma 2.1. We shall in fact do this. We will obtain a more general statement, in which $e^{\lambda(X)}$ is replaced by an arbitrary function of λ.

In the discussion of the functions $\Gamma_Q'(\cdot, X)$ we used the fact that if $P \subsetneq R$,

$$\sum_{\{Q:\,P\subset Q\subset R\}} (-1)^{\dim(A_Q/A_R)} \tau_P^Q(H)\hat{\tau}_Q^R(H) = 0.$$

If the real part of λ belongs to $-\mathfrak{a}_P^+$, we can integrate each summand against $e^{\lambda(H)}$ (see the proof of Lemma 2.2). We obtain the formula

$$(6.1) \qquad \sum_{\{Q:\, P\subset Q\subset R\}} (-1)^{\dim(A_P/A_Q)}\hat{\theta}_P^Q(\lambda)^{-1}\theta_Q^R(\lambda)^{-1} = 0.$$

By analytic continuation, it is valid for all $\lambda \in \mathfrak{a}_{P,\mathbf{C}}^*$.

Now, suppose that $c_P(\lambda)$ is any smooth function on $i\mathfrak{a}_M^* = i\mathfrak{a}_P^*$. If $Q \supset P$, define

$$c_Q(\lambda) = c_P(\lambda_Q),$$

where, as before, λ_Q is the projection of λ onto $i\mathfrak{a}_Q^*$. Copying our construction in Section 2, we define functions

$$c'_Q(\lambda), \qquad Q \supset P,$$

inductively by demanding that for all $Q \supset P$,

$$(6.2) \qquad c_Q(\lambda)\theta_Q(\lambda)^{-1} = \sum_{\{R:\, R\supset Q\}} c'_R(\lambda)\theta_Q^R(\lambda)^{-1}.$$

They are each defined on the complement of a finite set of hyperplanes in $i\mathfrak{a}_P^*$. It follows from (6.1) and (6.2) that if $Q \supset P$,

$$(6.3) \qquad c'_Q(\lambda) = \sum_{\{R:\, R\supset Q\}} (-1)^{\dim(A_Q/A_R)}\hat{\theta}_Q^R(\lambda)^{-1}c_R(\lambda)\theta_R(\lambda)^{-1}.$$

If $c_P(\lambda) = e^{\lambda(X)}$, $c'_P(\lambda)$ is just the function mentioned above.

LEMMA 6.1. *If $c_P(\lambda)$ is a smooth function on $i\mathfrak{a}_P^*$, $c'_P(\lambda)$ extends to a smooth function on $i\mathfrak{a}_P^*$.*

Proof. Any element λ in $i\mathfrak{a}_P^*$ can be written

$$\sum_{\tilde{\omega}\in\hat{\Delta}_P} c_{\tilde{\omega}}\tilde{\omega} + \lambda_Z$$

where each $c_{\tilde{\omega}}$ is a complex number and λ_Z belongs to $\mathfrak{a}_{G,\mathbf{C}}^*$. If $Q \supset P$, $\hat{\Delta}_Q$ is a subset of $\hat{\Delta}_P$. Let $\lambda_{Q/P}$ denote the element

$$\sum_{\tilde{\omega}\in\hat{\Delta}_Q} c_{\tilde{\omega}}\tilde{\omega} + \lambda_Z$$

in $\mathfrak{a}_{Q,\mathbf{C}}^*$. If $\phi_Q(\lambda)$ is a function on $i\mathfrak{a}_P^*$ which depends only on λ_Q, set

$$\phi_{Q/P}(\lambda) = \phi_Q(\lambda_{Q/P}).$$

We then have the function $c_{Q/P}$ and also the function $\theta_{Q/P}^R$, for $R \supset Q$. Suppose that α is a root in Δ_P, which does not belong to Δ_P^Q. If $\alpha_Q^\vee$ denotes the projection of $\alpha^\vee$ onto $\mathfrak{a}_Q$,

$$\lambda_{Q/P}(\alpha_Q^\vee) = \lambda_{Q/P}(\alpha^\vee) = \lambda(\alpha^\vee).$$

Since

$$\mathrm{vol}(\mathfrak{a}_Q^G/L_Q^G) = \mathrm{vol}(\mathfrak{a}_Q^R/L_Q^R)\mathrm{vol}(\mathfrak{a}_R^G/L_R^Q),$$

it follows that

$$\theta_{Q/P}(\lambda) = \theta_{Q/P}^R(\lambda)\theta_{R/P}(\lambda),$$

whenever $R \supset Q \supset P$. Similarly, $\theta_{Q/P}^{R_1}(\lambda) = \theta_{Q/P}^R(\lambda)\theta_{R/P}^{R_1}(\lambda)$ if $R_1 \supset R \supset Q \supset P$.

Mimicking the construction of $c_Q'(\lambda)$, we define functions $\tilde{c}_{Q/P}(\lambda)$ on E inductively by demanding that for all $Q \supset P$,

$$c_{Q/P}(\lambda)\theta_{Q/P}(\lambda)^{-1} = \sum_{\{R:\, R \supset Q\}} \tilde{c}_{R/P}(\lambda)\theta_{Q/P}^R(\lambda)^{-1}.$$

Then

$$\sum_{\{R:\, R\supset Q\}} (-1)^{\dim(A_Q/A_R)}\theta_{Q/P}^R(\lambda)^{-1}c_{R/P}(\lambda)\theta_{R/P}(\lambda)^{-1}$$

$$= \sum_{\{R,\, R_1:\, R_1\supset R\supset Q\}} (-1)^{\dim(A_Q/A_R)}\theta_{Q/P}^R(\lambda)^{-1}\tilde{c}_{R_1/P}(\lambda)\theta_{R/P}^{R_1}(\lambda)^{-1}$$

$$= \sum_{\{R_1:\, R_1\supset Q\}} \theta_{Q/P}^{R_1}(\lambda)^{-1}\tilde{c}_{R_1/P}(\lambda) \sum_{\{R:\, R_1\supset R\supset Q\}} (-1)^{\dim(A_Q/A_R)}$$

$$= \tilde{c}_{Q/P}(\lambda).$$

Thus $\tilde{c}_{Q/P}(\lambda)$ equals the product of $\theta_{Q/P}(\lambda)^{-1}$ with

$$\sum_{\{R:\, R\supset Q\}} (-1)^{\dim(A_Q/A_R)}c_{R/P}(\lambda). \tag{6.4}$$

Now $\theta_{Q/P}(\lambda)^{-1}$ is a product of linear forms defined by those roots α in $\Delta_P\backslash\Delta_P^Q$. Fix such an α. Then the parabolic subgroups R, with $R \supset Q$, occur in pairs (R, R'); if R is such that α does not vanish on $\mathfrak{a}_R$, we define R' by setting

$$\mathfrak{a}_{R'} = \{H \in \mathfrak{a}_R: \alpha(H) = 0\}.$$

If $\lambda(\alpha^\vee) = 0$, it is clear from the definitions that

$$c_{R/P}(\lambda) = c_{R'/P}(\lambda).$$

Since

$$\dim(A_Q/A_{R'}) = \dim(A_Q/A_R) + 1,$$

(6.4) vanishes whenever $\lambda(\alpha^\vee) = 0$. It follows from Taylor's theorem that (6.4) is divisible as a smooth function by the linear form $\lambda(\alpha^\vee)$. Therefore, $\tilde{c}_{Q/P}(\lambda)$ is a smooth function of λ.

The lemma will now be proved by induction on $\dim G$. Notice that if $R \supset Q \supset P$,

$$c_Q(\lambda)\theta_Q(\lambda)^{-1} = c_Q(\lambda_Q)\theta_Q(\lambda_Q)^{-1} = c_{Q/P}(\lambda_Q)\theta_{Q/P}(\lambda_Q)^{-1},$$

and

$$\theta^R_{Q/P}(\lambda_Q) = \theta^R_Q(\lambda).$$

Therefore,

$$\begin{aligned} c'_P(\lambda) &= \sum_{\{Q:\, Q\supset P\}} (-1)^{\dim(A_P/A_Q)} \hat{\theta}^Q_P(\lambda)^{-1} \cdot c_Q(\lambda)\theta_Q(\lambda)^{-1} \\ &= \sum_{\{R,Q:\, R\supset Q\supset P\}} (-1)^{\dim(A_P/A_Q)} \hat{\theta}^Q_P(\lambda)^{-1} \tilde{c}_{R/P}(\lambda_Q)\theta^R_Q(\lambda)^{-1}. \end{aligned}$$

Suppose that $R = G$. Then $\tilde{c}_{R/P}(\lambda_Q)$ equals $c_{G/P}(\lambda)$, and is independent of Q. In view of (6.1), the sum over Q will vanish. It follows that $c'_P(\lambda)$ equals the sum over $\{R\colon P \subset R \subsetneq G\}$ of $(\tilde{c}_{R/P})'_{P\cap M_R}(\lambda)$, the function defined by (6.3), but with $(G, c_P(\lambda), Q)$ replaced by $(M_R, \tilde{c}_{R/P}(\lambda), P\cap M_R)$. Since $\tilde{c}_{R/P}(\lambda)$ is a smooth function on F, our lemma follows from the induction assumption. □

To motivate the next lemmas we turn again to the example

$$c_P(\lambda) = e^{\lambda(X)},$$

where X is any point in $\mathfrak{a}_P^+$. Then $c'_P(\lambda)$ is the Fourier transform of the function $\Gamma'_P(\cdot, X)$. Consider first the function $\Gamma'_P(\cdot, -X)$. If there is no $Q\supset P$ with $\tau^Q_P(H)\hat{\tau}_Q(H+X) = 1$, then $\Gamma'_P(H, -X)$ equals 0. Otherwise, let R be the largest group with this property. (R is defined by letting Δ^R_P be the union over all such Q of the sets Δ^Q_P.) As X is in the positive chamber, $\tau^R_P(H+X)\hat{\tau}_R(H+X)$ equals 1. This implies $\hat{\tau}^R_P(H+X)\hat{\tau}_R(H+X) = 1$, from which one can verify that $\hat{\tau}_P(H+X) = 1$ (see [1(b), Lemma 2.2 and the ensuing discussion]). Thus, $\tilde{\omega}(H+X) > 0$ for all $\tilde{\omega} \bullet \hat{\Delta}_P$. It follows that

$$\tau^Q_P(H)\hat{\tau}_Q(H+X) = 1$$

for all Q with $P\subset Q\subset R$. Therefore $\Gamma'_P(H, -X)$ equals

$$\sum_{\{Q:\, P\subset Q\subset R\}} (-1)^{\dim(A_Q/Z)},$$

which is $(-1)^{\dim(A_P/Z)}$ if $R = P$, and 0 otherwise. Thus, $\Gamma'_P(\cdot, -X)$ is the product of $(-1)^{\dim(A_P/Z)}$ and the characteristic function of

$$\{H\in\mathfrak{a}_P\colon \alpha(H)\le 0,\ \alpha\in\Delta_P;\ \tilde{\omega}(H+X)>0,\ \tilde{\omega}\in\hat{\Delta}_P\}.$$

It follows from Lemma 2.2 that $(-1)^{\dim(A_P/Z)}\Gamma'_P(-H, -X)$ and $\Gamma'_P(H, X)$ have the same Fourier transforms. Therefore, modulo a set of measure 0, $\Gamma'_P(\cdot, X)$ is the characteristic function of

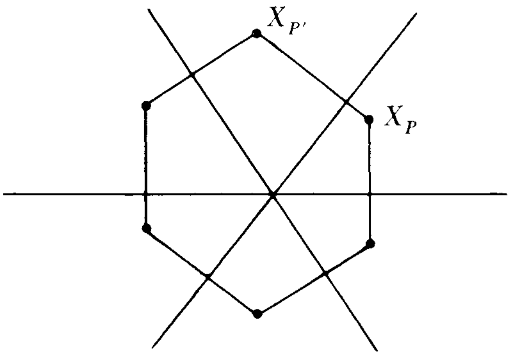
$$\{H\in\mathfrak{a}_P\colon \alpha(H)>0,\ \alpha\in\Delta_P;\ \tilde{\omega}(H-X)<0,\ \tilde{\omega}\in\hat{\Delta}_P\}.$$

In other words, the figure drawn for GL_3 in Section 2 is valid in general.

In [1(b)] we studied families of points $\{X_P\colon P \in \mathcal{P}(M)\}$ which we called A_M-orthogonal. This means that for every pair (P, P') of adjacent groups in $\mathcal{P}(M)$, $X_P - X_{P'}$ is a multiple of the co-root associated to the unique root in $\Delta_P \cap (-\Delta_{P'})$. Suppose that this multiple is always positive, and that, in addition, each point X_P lies in $(\mathfrak{a}_P^G)^+$. It follows from [1(b), Lemma 3.2] and what we have just shown, that the characteristic function of the convex hull of $\{X_P\colon P \in \mathcal{P}(M)\}$ equals the function

$$\sum_{P \in \mathcal{P}(M)} \Gamma'_P(H, X), \qquad H \in \mathfrak{a}_M^G,$$

almost everywhere. For $G = \mathrm{GL}_3$, the convex hull is the region



In [1(b)] we calculated the Fourier transform of the characteristic function of the convex hull. It equals

$$\sum_{P \in \mathcal{P}(M)} e^{\lambda(X_P)}\theta_P(\lambda)^{-1}.$$

In particular, this function of λ extends to a smooth function on $i\mathfrak{a}_M^*$.

Suppose that for each $P \in \mathcal{P}(M)$, $c_P(\lambda)$ is a smooth function on $i\mathfrak{a}_M^*$. We shall call the collection

$$\{c_P(\lambda)\colon P \in \mathcal{P}(M)\}$$

a (G, M)*-family* if the following condition holds: suppose that P and P' are adjacent groups in $\mathcal{P}(M)$, and that λ belongs to the hyperplane spanned by the common wall of the chambers of P and P'. Then

$$c_P(\lambda) = c_{P'}(\lambda).$$

This condition is equivalent to the property that whenever P and P' are elements in $\mathcal{P}(M)$ which are contained in a given parabolic subgroup Q, and λ belongs to $i\mathfrak{a}_Q^*$, then $c_P(\lambda) = c_{P'}(\lambda)$. In particular, there is a well defined function, $c_Q(\lambda)$, on $i\mathfrak{a}_Q^*$. The collection $\{e^{\lambda(X_P)}\}$ is a (G, M) family if and only if $\{X_P\}$ is A_M-orthogonal, in the sense defined above.

LEMMA 6.2. *If* $\{c_P(\lambda)\colon P \in \mathcal{P}(M)\}$ *is a* (G, M) *family,*

$$c_M(\lambda) = \sum_{P \in \mathcal{P}(M)} c_P(\lambda)\theta_P(\lambda)^{-1}$$

can be extended to a smooth function on $i\mathfrak{a}_M^*$.

Proof. The only possible singularities are along hyperplanes $\lambda(\alpha^\vee) = 0$ where α is a reduced root of (G, A_M). Such a singularity occurs only in the terms corresponding to those P for which either α or $-\alpha$ is a simple root. But such groups in $\mathcal{P}(M)$ occur in pairs (P, P') where P and P' are adjacent, and have α and $-\alpha$ respectively as a simple root. If we multiply the corresponding pair of terms by $\lambda(\alpha^\vee)$, and then take λ to be a point in general position on the hyperplane $\lambda(\alpha^\vee) = 0$, the result is 0, since $c_P(\lambda) = c_{P'}(\lambda)$. It follows from Taylor's theorem that c_M does not have a singularity on the hyperplane. ∎

Fix a (G, M) family $\{c_P(\lambda)\}$. We shall often denote the value of $c_M(\lambda)$ at $\lambda = 0$ simply by c_M. To calculate it, set

$$\lambda = t\Lambda, \qquad t \in \mathbf{R},\ \Lambda \in F,$$

and let t approach 0. If $p = \dim(A_M/A_G)$ we obtain

$$c_M = \frac{1}{p!} \sum_{P \in \mathcal{P}(M)} \left(\lim_{t \to 0} \left(\frac{d}{dt} \right)^p c_P(t\Lambda) \right) \theta_P(\Lambda)^{-1}. \tag{6.5}$$

In particular, this expression is independent of Λ. Likewise, if Q contains some group in $\mathcal{P}(M)$, we shall write c_Q' for $c_Q'(0)$. It equals

$$\frac{1}{q!} \sum_{\{R:\, R \supset Q\}} (-1)^{\dim(A_Q/A_R)} \hat{\theta}_Q^R(\Lambda)^{-1} \left(\lim_{t \to 0} \left(\frac{d}{dt} \right)^q c_R(t\Lambda) \right) \theta_R(\Lambda)^{-1},$$

where $q = \dim(A_Q/A_G)$. Now, fix a group L in $\mathcal{L}(M)$. If $Q \in \mathcal{P}(L)$, $P \in \mathcal{P}(M)$, and $P \subset Q$, the function

$$\lambda \to c_P(\lambda), \qquad \lambda \in i\mathfrak{a}_Q^*,$$

depends only on Q and not on P. We have agreed to denote it by $c_Q(\lambda)$. Then

$$\{c_Q(\lambda)\colon Q \in \mathcal{P}(L)\}$$

is a (G, L) family. Suppose that $Q \bullet \mathcal{P}(L)$ is fixed. If $R \bullet \mathcal{P}^L(M)$, $Q(R)$ is the unique group in $\mathcal{P}(M)$ such that $Q(R) \subset Q$ and $Q(R) \cap L = R$. Define a function c_R^Q on $i\mathfrak{a}_M^*$ by

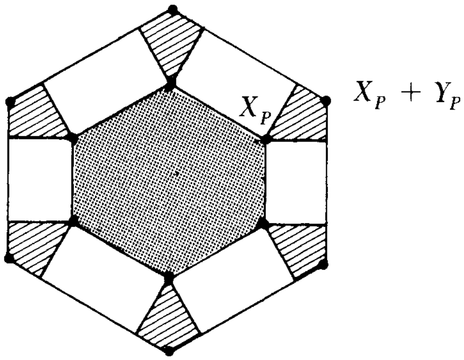
$$c_R^Q(\lambda) = c_{Q(R)}(\lambda).$$

Then $\{c_R^Q(\lambda): R \in \mathcal{P}^L(M)\}$ is an (L, M) family. In particular, we have functions $c_M^Q(\lambda)$ and $(c_P^Q)'(\lambda)$, $P \in \mathcal{P}^L(M)$, and their values c_M^Q and $(c_P^Q)'$ at $\lambda = 0$. In general, c_M^Q depends on Q, and not just on L. If it is independent of Q, we shall sometimes denote it by c_M^L. If each of the functions

$$c_R^Q(\lambda), \qquad R \in \mathcal{P}^L(M),$$

depends only on L, and not on Q, we shall denote it by $c_R^L(\lambda)$, or even $c_R(\lambda)$, since R determines L uniquely.

Suppose that $\{d_P(\lambda)\}$ is a second (G, M) family. Then $\{(cd)_P(\lambda) = c_P(\lambda)d_P(\lambda)\}$ is also a (G, M) family. There is a very simple formula for $(cd)_M(\lambda)$. For geometric intuition, consider the case that $G = \mathrm{GL}_3$, $M = M_0$, and $c_P(\lambda)d_P(\lambda) = e^{\lambda(X_P)}e^{\lambda(Y_P)}$.



The volume of the shaded region equals c_M. The volumes of the six hatched regions add up to d_M. Each of the other six regions has volume equal to $c_M^Q d_Q'$, for a maximal parabolic subgroup Q.

LEMMA 6.3. *We have, in general,*

$$(cd)_M(\lambda) = \sum_{Q \in \mathcal{F}(M)} c_M^Q(\lambda) d_Q'(\lambda).$$

Proof. $(cd)_M(\lambda)$ equals

$$\sum_{P \in \mathcal{P}(M)} c_P(\lambda) d_P(\lambda) \theta_P(\lambda)^{-1} = \sum_{P \in \mathcal{P}(M)} \sum_{\{Q:\, Q \supset P\}} c_P(\lambda) \theta_P^Q(\lambda)^{-1} d_Q'(\lambda).$$

Since $\{d_P(\lambda)\}$ is a (G, M) family, each function $d_Q'(\lambda)$ is well defined, in that it depends only on Q and not on the group $P \subset Q$. Interchanging the order of summation yields the sum over Q of the product of $d'_Q(\lambda)$ with

$$\sum_{\{P \in \mathcal{P}(M):\, P \subset Q\}} c_P(\lambda)\theta_P^Q(\lambda)^{-1}.$$

This last expression is just $c_M^Q(\lambda)$. □

COROLLARY 6.4 $d_M(\lambda) = \sum_{Q \in \mathcal{P}(M)} d'_Q(\lambda)$.

Proof. Set

$$c_P(\lambda) = 1 = \lim_{X_P \to 0} e^{\lambda(X_P)}$$

for each $P \in \mathcal{P}(M)$. In view of our earlier remarks on convex hulls, $c_M^Q(\lambda)$ will vanish if Q belongs to the complement of $\mathcal{P}(M)$ in $\mathcal{F}(M)$; if Q belongs to $\mathcal{P}(M)$, $c_M^Q(\lambda)$ trivially equals 1. The corollary follows from the lemma. □

COROLLARY 6.5. *Suppose that if $L \in \mathcal{L}(M)$,*

$$c_M^L = c_M^Q, \qquad Q \in \mathcal{P}(L)$$

is independent of Q. Then

$$(cd)_M = \sum_{L \in \mathcal{L}(M)} c_M^L d_L.$$

Proof. This follows from Lemma 6.3 and Corollary 6.4. ■

We remark that all of the results of this section are valid if the functions $c_P(\lambda)$ take values in a complete topological vector space, instead of just **C**. Of course for Lemma 6.3 we would need to assume the space was also an algebra.

7. Some examples

Examples of (G, M) families occur naturally in harmonic analysis. One elementary example is obtained from the Weyl group translates of a point. Take $M = M_0$, fix $P_0 \in \mathcal{P}(M_0)$ and let T be a point in $\mathfrak{a}_0$. Any $P \in \mathcal{P}(M_0)$ equals $w_s^{-1}P_0 w_s$ for a unique element $s \in \Omega$. Define $X_P = s^{-1}T$. Suppose that $P' = (w_{s'})^{-1}P_0 w_{s'}$ is adjacent to P. Then s' equals $s_\alpha s$, s_α the simple reflection corresponding to $\alpha \in \Delta_{P_0}$. The point $X_P - X_{P'}$ equals $s^{-1}(T - s_\alpha^{-1}T)$, which is a multiple of $s^{-1}\alpha^\vee$. But $s^{-1}\alpha^\vee = \beta^\vee$, where β is the unique root in $\Delta_P \cap (\Delta_{P'})$. Thus

$$c_P(\lambda) = e^{\lambda(X_P)}, \qquad P \in \mathcal{P}(M_0),$$

is a (G, M_0) family.

There is another elementary example, which we will use later, in Section 11. Fix $M \in \mathcal{L}(M_0)$. If β is any reduced root of (G, A), we can form the co-root $\beta^\vee$. For any $P \in \mathcal{P}(M)$, set $X_P^\beta = \beta^\vee$ if β is a root of (P, A), and let $X_P^\beta = 0$ otherwise. Let $\{r_\beta\}$ be a set of real numbers, and define X_P to be the sum over all reduced roots β of $r_\beta X_P^\beta$. Suppose that P and P' are adjacent. If β is a root of both (P, A) and (P', A), X_P^β equals $X_{P'}^\beta$. If it is a root of neither, both vectors are zero. The only reduced root of (P, A) which is not a root of (P', A) is the unique root β in $\Delta_P \cap (-\Delta_{P'})$. It follows that $X_P - X_{P'}$ is always a multiple of $\beta^\vee$. Thus,

$$c_P(\lambda) = e^{\lambda(X_P)}, \qquad P \in \mathcal{P}(M),$$

is a (G, M) family. Suppose that $L \in \mathcal{L}(M)$, and that $Q \in \mathcal{P}(L)$. We have the (L, M) family

$$\left\{c_R^Q(\lambda) = e^{\lambda(X_{Q(R)})}: R \in \mathcal{P}^L(M)\right\}.$$

Write

$$X_{Q(R)} = X_R + X_Q$$

where X_Q is the sum of $r_\beta X_{Q(R)}^\beta$ over those reduced roots β which do not vanish on $\mathfrak{a}_L$, and X_R is the sum over the remaining β, namely, the reduced roots of (L, A_M). It is clear from the definitions that X_R is independent of Q and that X_Q is independent of R. Therefore,

$$\begin{aligned} c_M^Q &= \lim_{\lambda \to 0} e^{\lambda(X_Q)} \sum_{R \in \mathcal{P}^L(M)} e^{\lambda(X_R)} \theta_R^L(\lambda)^{-1} \\ &= \lim_{\lambda \to 0} \sum_R e^{\lambda(X_R)} \theta_R^L(\lambda)^{-1}. \end{aligned}$$

It is independent of Q. If $L_1 \in \mathcal{L}(L)$ and $Q \in \mathcal{P}(L_1)$, the same is true of the (L_1, L) family

$$\left\{c_R^Q(\lambda): R \in \mathcal{P}^{L_1}(L)\right\}.$$

That is, c_L^Q depends only on L_1 and L, and not on Q. We denote it by $c_L^{L_1}$.

For our next example, fix a finite set S of valuations on $\mathbf{Q}$, and fix $M \in \mathcal{L}(M_0)$. Now $G(\mathbf{Q}_S)$ is a subgroup of $G(\mathbf{A})$, so for any $x \in G(\mathbf{Q}_S)$ and $P \in \mathcal{P}(M)$, we have the vector $H_P(x)$ in $\mathfrak{a}$. Suppose that P and P' are adjacent and that α is the unique root in $\Delta_P \cap (-\Delta_{P'})$. Then $-H_P(x) + H_{P'}(x)$ is a multiple of $\alpha^\vee$. For the case that $S = \{\mathbf{R}\}$ this is [1(b), Lemma 3.6]. The proof for general S is identical. Therefore,

$$v_P(\lambda, x) = e^{-\lambda(H_P(x))}, \qquad P \in \mathcal{P}(M),$$

is a (G, M) family. Suppose that $L \in \mathcal{L}(M)$. For $\lambda \in i\mathfrak{a}_L^*$ there are two possible ways to define

$$v_Q(\lambda, x), \qquad Q \in \mathcal{P}(L).$$

There is the (G, L) family $\{e^{-\lambda(H_Q(x))}\}$ or there is the (G, L) family derived as in Section 6 from the (G, M) family $\{v_P(\lambda, x)\colon P \in \mathcal{P}(M)\}$. The two are obviously the same. Notice also that if $Q \in \mathcal{P}(L)$ and x belongs to $L(\mathbf{Q}_S)$, the functions

$$v_R^Q(\lambda, x) = v_{Q(R)}(\lambda, x), \qquad R \in \mathcal{P}^L(M),$$

depend only on R and not on Q. We denote them by $v_R^L(\lambda, x)$ or $v_R(\lambda, x)$. Sometimes, however, we will take x to be a general point in $G(\mathbf{Q}_S)$ and use the function

$$v_M^Q(x) = \lim_{\lambda \to 0} \sum_{\{P \in \mathcal{P}(M)\colon P \subset Q\}} v_P(\lambda, x)\theta_P^Q(\lambda)^{-1}.$$

As a function on $G(\mathbf{Q}_S)$ it is left $M(\mathbf{Q}_S)$ invariant. Indeed

$$v_M^Q(\lambda, mx) = e^{-\lambda(H_M(m))}v_M^Q(\lambda, x), \qquad m \in M(\mathbf{Q}_S),$$

since $H_P(m) = H_M(m)$ is independent of P. Setting $\lambda = 0$ we see that $v_M^Q(mx) = v_M^Q(x)$.

Our final three examples, which are all basically the same, are derived from the intertwining operators between induced representations. Let v be a valuation on $\mathbf{Q}$, and suppose that M_v is a Levi subgroup of G defined over $\mathbf{Q}_v$. (From now on, unless we state otherwise, we will only consider groups M_v for which K_v is admissible.) Take a representation $\pi_v \in \Pi(M_v(\mathbf{Q}_v))$ and a vector $\zeta_v \in \mathfrak{a}^*_{v,\mathbf{C}}$. If $P \in \mathcal{P}(M_v)$, we can lift the representation π_{v,ζ_v} to $P(\mathbf{Q}_v)$, and then induce to $G(\mathbf{Q}_v)$. This gives a representation $I_P(\pi_{v,\zeta_v})$ of $G(\mathbf{Q}_v)$ on a Hilbert space $\mathcal{H}_P(\pi_v)$. We take $\mathcal{H}_P(\pi_v)$ to be the space of square integrable functions ϕ from K_v to the space on which π_v acts such that

$$\phi(k_1 k) = \pi_v(k_1)\phi(k), \qquad k_1 \in K_v \cap P(\mathbf{Q}_v).$$

Then $\mathcal{H}_P(\pi_v)$ depends only on π_v and not on ζ_v.

Suppose that v is real. Then there are canonically normalized intertwining operators

$$R_{P'|P}(\pi_v)\colon \mathcal{H}_P(\pi_v) \to \mathcal{H}_{P'}(\pi_v), \qquad P, P' \in \mathcal{P}(M_v),$$

such that for all $x \in G(\mathbf{Q}_v)$,

$$(7.1) \qquad R_{P'|P}(\pi_v)I_P(\pi_v, x) = I_{P'}(\pi_v, x)R_{P'|P}(\pi_v).$$

The operators are unitary, and

$$(7.2) \qquad R_{P''|P}(\pi_v) = R_{P''|P'}(\pi_v)R_{P'|P}(\pi_v),$$

whenever P, P' and P'' belong to $\mathcal{P}(M_v)$. Suppose that ψ_0 is a function in $\mathcal{H}_P(\pi_v)$ such that $\psi_0(k) = \psi_0(1)$ for all $k \in K_v$. (In particular, the representation π_v is of class 1.) Then the normalizations have the property that

$$R_{P'|P}(\pi_v)\psi_0 = \psi_0 \tag{7.3}$$

for all P'. Suppose that $L_v \in \mathcal{L}(M_v)$ and $R \in \mathcal{P}^{L_v}(M_v)$. Consider the induced representation

$$\pi_v' = I_R^{L_v}(\pi_v) = I_R(\pi_v)$$

of $L_v(\mathbf{Q}_v)$. Then $\mathcal{H}_Q(\pi_v') = \mathcal{H}_{Q(R)}(\pi_v)$ and $I_Q(\pi_v') = I_{Q(R)}(\pi_v)$ for any Q in $\mathcal{P}(L_v)$. The intertwining operators are related by

$$R_{Q'|Q}(\pi_v') = R_{Q'(R)|Q(R)}(\pi_v), \qquad Q, Q' \in \mathcal{P}(L_v). \tag{7.4}$$

If ψ is a continuous function in $\mathcal{H}_{Q(R)}(\pi_v)$ and $k \in K_v$, the function

$$\psi_k \colon k_1 \to \psi(k_1 k), \qquad k_1 \in K_v \cap L_v(\mathbf{Q}_v),$$

belongs to $\mathcal{H}_R^{L_v}(\pi_v)$. If R' is another group in $\mathcal{P}^{L_v}(M_v)$,

$$\left(R_{Q(R')|Q(R)}(\pi_v)\psi\right)_k = R_{R'|R}(\pi_v)\psi_k. \tag{7.5}$$

Finally, suppose that $W \in \Pi(K_v)$. The space, $\mathcal{H}_P(\pi_v)_W$, of vectors in $\mathcal{H}_P(\pi_v)$ which transform under K_v according to W, is finite dimensional. Let $R_{P'|P}(\pi_v)_W$ be the restriction of $R_{P'|P}(\pi_v)$ to this space. Then for any $D \in \mathrm{Diff}(i\mathfrak{a}_v^*)$ there are constants C and N such that

$$\| D_{\pi_v} R_{P'|P}(\pi_v)_W \| \leq C(1 + \|\pi_v\|)^N (1 + \|W\|)^N, \tag{7.6}$$

for all $\pi_v \in \Pi(M_v(\mathbf{Q}_v))$, and $W \in \Pi(K_v)$. These properties were established in [1(f)].

Now take v to be a discrete valuation. There should also be a canonical way to normalize the intertwining operators between induced representations (see [10(a), p. 282]). However to check the required properties, or even make the definitions in full generality would require a better understanding of harmonic analysis than is now available. Shahidi [11] has introduced a normalization which applies to a large number of cases and is presumably the one of [10(a)]. At any rate, given Harish Chandra's work on the unnormalized intertwining operators (see [12]), it should be possible to define ad hoc normalizations with the right properties. From now on, we shall just assume the existence of operators

$$R_{P'|P}(\pi_v) \colon \mathcal{H}_P(\pi_v) \to \mathcal{H}_{P'}(\pi_v), \qquad P, P' \in \mathcal{P}(M_v),$$

for which properties (7.1)–(7.6) hold.

Suppose that S is a finite set of valuations. Let M be a group in $\mathcal{L}(M_0)$ and let $\pi = \otimes_{v \in S} \pi_v$ be a representation in $\Pi(\mathfrak{M}_S)$. For $P \in \mathcal{P}(M)$ we define the induced representation

$$I_P(\pi) = \bigotimes_{v \in S} I_P(\pi_v)$$

which acts on the Hilbert space

$$\mathcal{H}_P(\pi) = \bigotimes_{v \in S} \mathcal{H}_P(\pi_v).$$

We have intertwining operators

$$R_{P'|P}(\pi) = \bigotimes_{v \in S} R_{P'|P}(\pi_v)$$

which satisfy the analogues of (7.1)–(7.6).

Suppose that $P_0 \in \mathcal{P}(M)$ is fixed. Then, with π_λ as defined in Section 5,

$$R_P(\lambda, \pi, P_0) = R_{P|P_0}(\pi)^{-1} R_{P|P_0}(\pi_\lambda), \qquad P \in \mathcal{P}(M),$$

is a family of operator valued functions of $\lambda \in i\mathfrak{a}^*$. We can interpret each operator as a direct sum of operators on finite dimensional spaces. In fact, let K_0 be an open compact subgroup of K_S, and let W be an irreducible unitary representation of $K_{\mathbf{R}}$. Let $\mathcal{H}_P(\pi)_{K_0, W}$ be the space of vectors in $\mathcal{H}_P(\pi)$ which are invariant under K_0 and which transform under $K_{\mathbf{R}}$ according to W. Then $\mathcal{H}_{P_0}(\pi)_{K_0, W}$ is finite dimensional, and is an invariant subspace of each operator $R_P(\lambda, \pi, P_0)$. Suppose that $Q \in \mathcal{F}(M)$, and that groups $P, P' \in \mathcal{P}(M)$ are both contained in Q. Then $P = Q(R)$ and $P' = Q(R')$ for uniquely determined groups $R, R' \bullet \mathcal{P}^{M_Q}(M)$. It follows from (7.5) that if $\lambda \bullet i\mathfrak{a}_Q^*$,

$$R_{P'|P}(\pi_\lambda) = R_{P'|P}(\pi).$$

Consequently for any such λ,

$$\begin{aligned} R_{P'}(\lambda, \pi, P_0) &= R_{P|P_0}(\pi)^{-1} R_{P'|P}(\pi)^{-1} R_{P'|P}(\pi_\lambda) R_{P|P_0}(\pi_\lambda) \\ &= R_P(\lambda, \pi, P_0). \end{aligned}$$

Therefore $\{R_P(\lambda, \pi, P_0) \colon P \in \mathcal{P}(M)\}$ is a (G, M) family. More generally, if $L \in \mathcal{L}(M)$ and $P_0 \in \mathcal{P}^L(M)$ we have the (L, M) family

$$R_P(\lambda, \pi, P_0) = R_P^L(\lambda, \pi, P_0), \qquad P \in \mathcal{P}^L(M),$$

composed of the intertwining operators acting on $\mathcal{H}_{P_0}^L(\pi)$. We can form the operators $R_M^L(\pi, P_0)$ and $R_Q'(\pi, P_0)$, $Q \bullet \mathcal{F}^L(M)$, (the values of $\lambda = 0$ of $R_M^L(\lambda, \pi, P_0)$ and $R_Q'(\lambda, \pi, P_0)$.) In general, each of these new operators is unbounded, but it can still be regarded as a direct sum of operators on finite dimensional spaces. It follows from the definition that if P_1 is a second fixed group in $\mathcal{P}^L(M)$,

$$R_M^L(\pi, P_1) = R_{P_0|P_1}(\pi)^{-1} R_M^L(\pi, P_0) R_{P_0|P_1}(\pi). \tag{7.7}$$

For any $L \in \mathcal{L}(M)$, we can define a scalar valued (L, M)-family. Fix $f \in \mathcal{C}(L(\mathbf{Q}_S))$, $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$ and $P_0 \in \mathcal{P}^L(M)$. Define

$$\phi_P(\lambda, f, \pi, P_0) = \Phi_P^L(\lambda, f, \pi, P_0) = \mathrm{tr}\big(I_{P_0}(\pi, f)R_P(\lambda, \pi, P_0)\big), \qquad P \in \mathcal{P}^L(M).$$

This is certainly an (L, M)-family. That each function is smooth in λ can be obtained from (7.6) and the differentiation variant of the dominated convergence theorem. In particular, any differentiation with respect to λ can be interchanged with the trace operation. It follows that for any $Q \in \mathcal{F}^L(M)$ the number $\phi'_Q(f, \pi, P_0)$, obtained a priori from the family $\{\phi_P(\lambda, f, \pi, P_0): P \in \mathcal{P}^L(M)\}$, also equals $\mathrm{tr}(I_{P_0}(\pi, f)R'_Q(\pi, P_0))$. Similarly,

$$\phi_M^L(f, \pi, P_0) = \mathrm{tr}\big(I_{P_0}(\pi, f)R_M^L(\pi, P_0)\big).$$

Combining this last formula with (7.7), we find that $\phi_M^L(f, \pi, P_0)$ is independent of P_0. We shall denote it simply by $\phi_M^L(f, \pi)$. The similarity to our notation of Section 4 is of course intentional. Incidentally, the same reasoning establishes more generally that for any $Q \in \mathcal{F}^L(M)$, the number $\phi_M^Q(f, \pi, P_0)$ is independent of the group $P_0 \in \mathcal{P}^L(M)$.

It is obvious that each function $\phi_P^L(\lambda, f, \pi, P_0)$ depends only on the unitary equivalence class of π. Suppose that $M_1 \in \mathcal{L}^L(M)$. Fix $Q_0 \in \mathcal{P}^L(M_1)$ and $P_0 \in \mathcal{P}^L(M)$, with $P_0 \subset Q_0$. Then $P_0 = Q_0(R_0)$ for $R_0 \in \mathcal{P}^{M_1}(M)$. Given $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$, let π_1 be the induced representation $I_{R_0}^{M_1}(\pi)$ of $M_1(\mathbf{Q}_S)$. If Q is any other group in $\mathcal{P}^L(M_1)$, and $\lambda \in i\mathfrak{a}_{M_1}^*$,

$$\begin{aligned} R_Q^L(\lambda, \pi_1, Q_0) &= R_{Q|Q_0}(\pi_1)^{-1}R_{Q|Q_0}(\pi_{1,\lambda}) \\ &= R_{Q(R_0)|Q_0(R_0)}(\pi)^{-1}R_{Q(R_0)|Q_0(R_0)}(\pi_\lambda) \\ &= R_Q^L(\lambda, \pi, P_0), \end{aligned}$$

by (7.4). It follows that

$$\phi_Q^L(\lambda, f, \pi, P_0) = \phi_Q^L(\lambda, f, \pi^{M_1}, Q_0), \tag{7.8}$$

for $f \in \mathcal{C}(L(\mathbf{Q}_S))$. In other words, the (L, M_1) family associated to π^{M_1} and Q_0 is the same as the one derived from the (L, M) family $\{\phi_P^L(\lambda, f, \pi, P_0): P \in \mathcal{P}^L(M)\}$. The (M_1, M) families derived from (L, M) families are also related to intrinsically defined (M_1, M) families:

Lemma 7.1. *Suppose that $Q \in \mathcal{P}^L(M_1)$, and $P_0 \in \mathcal{P}^L(M)$, with $P_0 \subset Q$. Then for $f \in \mathcal{C}(L(\mathbf{Q}_S))$ and $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$,*

$$\phi_R^Q(\lambda, f, \pi, P_0) = \phi_R^{M_1}(\lambda, f_Q, \pi, P_0 \cap M_1), \qquad R \in \mathcal{P}^{M_1}(M).$$

Proof. Let $R_0 = P_0 \cap M_1$. Then $P_0 = Q(R_0)$. We need to evaluate the trace of an operator on $\mathcal{H}_{P_0}^L(\pi)$. Now $\mathcal{H}_{P_0}^L(\pi)$ can be regarded as the space of square integrable, $K \cap Q(\mathbf{Q}_S)$ equivariant functions from $K \cap L(\mathbf{Q}_S)$ to the Hilbert space $\mathcal{H}_{R_0}^{M_1}(\pi)$. Then $I_{P_0}^L(f)$ is an integral operator with kernel $K(k_1, k_2)$ equal to

$$\int_{M_1(\mathbf{Q}_S)} I_{R_0}^{M_1}(\pi, m) \cdot \delta_{R_0}^{M_1}(m)^{1/2} \int_{N_Q(\mathbf{Q}_S)} f(k_1^{-1} mnk_2)\, dn\, dm.$$

By (7.4) the operator

$$R_{Q(R)}^L(\lambda, \pi, P_0), \qquad R \in \mathcal{P}^{M_1}(M),$$

on $\mathcal{H}_{P_0}^L(\pi)$ is just fiber multiplication by the operator

$$R_R^{M_1}(\lambda, \pi, R_0)$$

on $\mathcal{H}_{R_0}^{M_1}(\pi)$. Therefore, the trace of

$$I_{P_0}^L(f) R_{Q(R)}^L(\lambda, \pi, P_0)$$

equals

$$\int_{K \cap M_1(\mathbf{Q}_S) \backslash K \cap L(\mathbf{Q}_S)} \operatorname{tr}\big(K(k, k) R_R^{M_1}(\lambda, \pi, R_0)\big)\, dk.$$

This is just

$$\operatorname{tr}\big(I_{R_0}^{M_1}(\pi, f_Q) R_R^{M_1}(\lambda, \pi, R_0)\big) = \phi_R^{M_1}(\lambda, f_Q, \pi, R_0).$$

This establishes the lemma. □

Finally, suppose that f belongs to $C_c^\infty(L(\mathbf{Q}_S))$ and that $\pi \in \Pi(M(\mathbf{Q}_S))$. Fix $P_0 \in \mathcal{P}^L(M)$ and consider the functions

$$\operatorname{tr}\big(I_{P_0}(\pi, f) R_P(\lambda, \pi, P_0)\big), \qquad P \in \mathcal{P}^L(M).$$

That they are smooth in λ follows again from (7.6). Notice that

$$\int_{i\mathfrak{a}^*/\mathrm{Lat}(S)} \operatorname{tr}\big(I_{P_0}(\pi_\Lambda, f) R_P(\lambda, \pi_\Lambda, P_0)\big)\, d\Lambda$$

depends only on the restriction of f to $L(\mathbf{Q}_S)^1$ and the orbit of π under $i\mathfrak{a}^*$. We denote this last function of λ by

$$\phi_P(\lambda, f, \pi, P_0),$$

where now f is taken to be a function in $C_c^\infty(L(\mathbf{Q}_S)^1)$ and π is a representation in $\Pi(M(\mathbf{Q}_S)^1)$. Then $\{\phi_P(\lambda, f, \pi, P_0) \colon P \in \mathcal{P}^L(M)\}$ is an (L, M) family which satisfies properties analogous to those described above. In particular, we can associate a number

$$\phi_M^L(f, \pi) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}^L(M)} \phi_P(\lambda, f, \pi, P_0) \theta_P(\lambda)^{-1}$$

to every $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$ and $\pi \in \Pi(M(\mathbf{Q}_S)^1)$.

8. The distributions $J_{M,\gamma}$ and $J_{M,\pi}$

In this section, S continues to be a finite set of valuations on $\mathbf{Q}$, and $M \subset L$ are fixed groups in $\mathfrak{L}(M_0)$. We will use the examples of the last section to define two families of tempered distributions on $L(\mathbf{Q}_S)$. They are to be regarded as local approximations to the distributions J_o^L and J_χ^L.

Our first distributions are similar to those studied in [1(b)]. They are obtained by taking orbital integrals on $L(\mathbf{Q}_S)$, weighted by the function $v_M^L(x)$. We need a lemma to guarantee that they are tempered. Recall the functions Ξ^L. and σ^L defined in Section 5.

Lemma 8.1. *If $\gamma \in L(\mathbf{Q}_S)_{\text{reg}} \cap M$ and n is sufficiently large,*

$$\int_{L(\mathbf{Q}_S)_\gamma \backslash L(\mathbf{Q}_S)} \Xi^L(x^{-1}\gamma x)\left(1 + \sigma^L(x^{-1}\gamma x)\right)^{-n} v_M^L(x)\,dx$$

is finite. (Since $v_M^L(x)$ is left M-invariant, the integrand is $L(\mathbf{Q}_S)_\gamma$-invariant.)

Proof. Write the variable x as $\prod_{v \in S} x_v$ and let $\gamma = \prod_{v \in S}\gamma_v$. Since

$$H_P(x) = \sum_{v \in S} H_P(x_v), \quad \text{and}$$

$$v_M^L(x) = \frac{1}{q!}\sum_{P \in \mathcal{P}^L(M)} \left(-\lambda(H_P(x))\right)^q \theta_P(\lambda)^{-1},$$

$q = \dim(A_M/A_L)$, we can rewrite the above integral as a sum of products of integrals over $L(\mathbf{Q}_v)_{\gamma_v} \backslash L(\mathbf{Q}_v)$. If v is Archimedean, the convergence of the resulting integral can be proved as in [1(b), Lemma 7.2]. If v is discrete one knows that any finite dimensional $L(\mathbf{Q}_v)$ module over $\mathbf{Q}_v$ has a basis of eigenvectors for any given split torus in $L(\mathbf{Q}_v)$, such that the corresponding lattice is stabilized by K_v. This fact, together with [5(b), Lemma 13] (see also the proofs of Corollary 4.7.3 and 4.8.4 of [12]), allows us to transcribe the Archimedean proof. The argument is the same, so we need not present the details. □

If $\gamma \in L(\mathbf{Q}_S)_{\text{reg}} \cap M$ and $f \in \mathcal{C}(L(\mathbf{Q}_S))$ define

$$J_{M,\gamma}^L(f) = |D^L(\gamma)|^{1/2}\int_{L(\mathbf{Q}_S)_\gamma \backslash L(\mathbf{Q}_S)} f(x^{-1}\gamma x) v_M^L(x)\,dx.$$

By the last lemma, the integral converges absolutely, and each $J_{M,\gamma}^L$ is a tempered distribution. If $Q \in \mathcal{F}^L(M)$, we have

$$J_{M,\gamma}^{M_Q}(f_Q) = |D^L(\gamma)|^{1/2}\int_{L(\mathbf{Q}_S)_\gamma \backslash L(\mathbf{Q}_S)} f(x^{-1}\gamma x) v_M^Q(x)\,dx. \tag{8.1}$$

This follows from the change of variables formula

$$|D^L(\gamma)|^{1/2}\int_{N_Q(\mathbf{Q}_S)}\phi\Big((m^{-1}\gamma m)^{-1}n^{-1}(m^{-1}\gamma m)n\Big)\,dn$$
$$= |D^{M_Q}(\gamma)|^{1/2}\delta_Q^L(\gamma)^{1/2}\int_{N_Q(\mathbf{Q}_S)}\phi(n)\,dn, \qquad m\in M_Q(\mathbf{Q}_S).$$

Notice that if $L = M$, $J^L_{M,\gamma}$ is just the ordinary orbital integral I^M_γ, defined in Section 5.

Lemma 8.2. *For $\gamma \in L(\mathbf{Q}_S)_{\mathrm{reg}} \cap M$, $f \in \mathcal{C}(L(\mathbf{Q}_S))$, and $y \in L(\mathbf{Q}_S)$,*

$$J^L_{M,\gamma}(f^y) = \sum_{Q\in\mathcal{F}^L(M)} J^{M_Q}_{M,\gamma}(f_{Q,y}).$$

Proof. With a change of variables we see that $J^L_{M,\gamma}(f^y)$ equals

$$|D^L(\gamma)|^{1/2}\int_{L(\mathbf{Q}_S)_\gamma\backslash L(\mathbf{Q}_S)} f(x^{-1}\gamma x)v^L_M(xy)\,dx.$$

If $P\in\mathcal{P}^L(M)$,

$$v^L_P(\lambda, xy) = e^{-\lambda(H_P(xy))}$$
$$= e^{-\lambda(H_P(K_P(x)y))}e^{-\lambda(H_P(x))}$$

where $K_P(x)$ is any element in $K_S = \Pi_{v\in S}K_v$ such that $xK_P(x)^{-1}$ belongs to $P(\mathbf{Q}_S)$. This equals

$$u_P(\lambda, x, y)v^L_P(x),$$

where

$$u_P(\lambda, x, y) = e^{-\lambda(H_P(K_P(x)y))}, \qquad P\in\mathcal{P}^L(M),$$

is an (L, M) family. It follows from Lemma 6.3 that

$$v^L_M(xy) = \sum_{Q\in\mathcal{F}^L(M)} v^Q_M(x)u'_Q(x,y).$$

We see from the discussion of Section 6 that $u'_Q(x,y)$ is same as the function defined in Section 2. As a function of x, $u'_Q(x,y)$ is left $Q(\mathbf{Q}_S)$ invariant. We write

$$|D^L(\gamma)|^{1/2}\int_{L(\mathbf{Q}_S)_\gamma\backslash L(\mathbf{Q}_S)} f(x^{-1}\gamma x)v^Q_M(x)u'_Q(x,y)\,dx$$

as

$$|D^L(\gamma)|^{1/2}\int_{K\cap L(\mathbf{Q}_S)}\int_{N_Q(\mathbf{Q}_S)}\int_{M_Q(\mathbf{Q}_S)_\gamma\backslash M_Q(\mathbf{Q}_S)} f(k^{-1}n^{-1}m^{-1}\gamma mnk)$$
$$\cdot v^Q_M(m)u'_Q(k,y)\,dm\,dn\,dk.$$

This equals

$$|D^{M_Q}(\gamma)|^{1/2}\int_{M_Q(\mathbf{Q}_S)_\gamma\backslash M_Q(\mathbf{Q}_S)} f_{Q,y}(m^{-1}\gamma m)v_M^{M_Q}(m)\,dm = J_{M,\gamma}^{M_Q}(f_{Q,y}).$$

The lemma is proved. □

Our second family of distributions has actually already been defined. We introduce new notation only to point out the analogy with the distributions we have just defined. If $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$ and $f \in \mathcal{C}(L(\mathbf{Q}_S))$, define

$$J_{M,\pi}^L(f) = \phi_M^L(f,\pi) = \mathrm{tr}\big(I_{P_0}^L(\pi,f)R_M^L(\pi,P_0)\big),$$

for any $P_0 \in \mathcal{P}^L(M)$. If $Q \in \mathcal{F}^L(M)$ we have the analogue of (8.1),

$$J_{M,\pi}^{M_Q}(f_Q) = \mathrm{tr}\big(I_{P_0}^L(\pi,f)R_M^Q(\pi,P_0)\big), \tag{8.2}$$

for any $P_0 \in \mathcal{P}^L(M)$. This formula follows from Lemma 7.1 along with the fact, noted in Section 7, that the right hand side is independent of P_0.

Lemma 8.3. *For $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$, $f \in \mathcal{C}(L(\mathbf{Q}_S))$ and $y \in L(\mathbf{Q}_S)$,*

$$J_{M,\pi}^L(f^y) = \sum_{Q\in\mathcal{F}^L(M)} J_{M,\pi}^{M_Q}(f_{Q,y}).$$

Proof. Fix $P_0 \in \mathcal{P}^L(M)$. Then $J_{M,\pi}^L(f^y) = \phi_M^L(\pi,f^y)$ equals

$$\lim_{\lambda\to 0}\sum_{P\in\mathcal{P}^L(M)} \mathrm{tr}\big(I_{P_0}(\pi,f^y)R_P(\lambda,\pi,P_0)\big)\theta_P(\lambda)^{-1}.$$

Substituting for

$$I_{P_0}(\pi,f^y) = I_{P_0}(\pi,y)^{-1}I_{P_0}(\pi,f)I_{P_0}(\pi,y)$$

and

$$R_P(\lambda,\pi,P_0) = R_{P|P_0}(\pi)^{-1}R_{P|P_0}(\pi_\lambda)$$

yields the trace of

$$\lim_{\lambda\to 0}\sum_{P\in\mathcal{P}^L(M)} I_{P_0}(\pi,f)I_{P_0}(\pi,y)R_{P|P_0}(\pi)^{-1}R_{P|P_0}(\pi_\lambda)I_{P_0}(\pi,y)^{-1}\theta_P(\lambda)^{-1}.$$

In this expression we can replace $I_{P_0}(\pi,y)^{-1}$ by $I_{P_0}(\pi_\lambda,y)^{-1}$ without changing the final limit. We obtain the trace of the operator

$$I_{P_0}(\pi,f)\cdot\lim_{\lambda\to 0}\sum_{P\in\mathcal{P}^L(M)} R_{P|P_0}(\pi)^{-1}\cdot U_P(\lambda,\pi,y)\theta_P(\lambda)^{-1}\cdot R_{P|P_0}(\pi_\lambda), \tag{8.3}$$

where

$$U_P(\lambda,\pi,y)=I_P(\pi,y)I_P(\pi_\lambda,y)^{-1}.$$

One sees immediately from the definition of induced representation that for $\phi\in\mathcal{H}_P(\pi)=\mathcal{H}_P^L(\pi)$, and $k\in K\cap L(\mathbf{Q}_S)$,

$$(U_P(\lambda,\pi,y)\phi)(k)=e^{-\lambda(H_P(ky))}\phi(k)=u_P(\lambda,k,y)\phi(k). \tag{8.4}$$

In particular, any derivative in λ of $U_P(\lambda,\pi,y)$ is a bounded operator. By (6.2),

$$U_P(\lambda,\pi,y)\theta_P(\lambda)^{-1}=\sum_{\{Q\in\mathcal{F}^L(M):\,Q\supset P\}}U_Q'(\lambda,\pi,y)\theta_P^Q(\lambda)^{-1}.$$

Substitute this expression in (8.3). Take the sum over P inside the sum over Q. We shall show that the limit in λ can be taken inside the sum over Q. That is, that

$$I_{P_0}(\pi,f)\sum_{\{P\in\mathcal{P}^L(M):\,P\subset Q\}}R_{P|P_0}(\pi)^{-1}U_Q'(\lambda,\pi,y)R_{P|P_0}(\pi_\lambda)\cdot\theta_P^Q(\lambda)^{-1}$$

has a limit as λ approaches 0. Suppose that $P_0\subset Q$. If we can show that the limit exists in this case, then the limit will exist for an arbitrary P_0' in $\mathcal{P}^L(M)$. In fact, it will just be the conjugate by $R_{P_0'|P_0}(\pi)$ of the limit for P_0. When we evaluate the trace, the two limits will be equal. Therefore, we may assume $P_0\subset Q$. Now it is clear that if $m\in M_Q(\mathbf{A})$,

$$u_Q(\lambda,mx,y)=u_Q(\lambda,x,y).$$

It follows from this fact, (8.4), and (7.5) that if $P\in\mathcal{P}^L(M)$, $P\subset Q$,

$$R_{P|P_0}(\pi)^{-1}U_P(\lambda_Q,\pi,y)=U_{P_0}(\lambda_Q,\pi,y)R_{P|P_0}(\pi)^{-1}.$$

(Recall that λ_Q is the projection of λ onto $i\mathfrak{a}_Q^*$.) Therefore

$$R_{P|P_0}(\pi)^{-1}U_Q'(\lambda,\pi,y)=U_Q'(\lambda,\pi,y)R_{P|P_0}(\pi)^{-1}.$$

Our notation here is confusing. The operator $U_Q'(\lambda,\pi,y)$ on the left hand side has been obtained, via (6.3), from $U_P(\lambda_Q,\pi,y)$, while on the right hand side, $U_Q'(\lambda,\pi,y)$ is obtained from $U_{P_0}(\lambda_Q,\pi,y)$, and is independent of P.

Now the existence of the required limit follows from Lemma 6.2. It follows that $\phi_M^L(\pi,f^y)$ equals the sum over $Q\in\mathcal{F}^L(M)$ of the trace of the operator

$$I_{P_0}(\pi,f)U_Q'(\pi,y)R_M^Q(\pi,P_0) \tag{8.5}$$

where P_0 can be taken to be any group in $\mathcal{P}^L(M)$ with $P_0\subset Q$.

This operator acts on $\mathcal{H}_{P_0}(\pi)=\mathcal{H}_{P_0}^L(\pi)$. But $\mathcal{H}_{P_0}(\pi)$ can be interpreted as the space of square integrable, $(K\cap Q(\mathbf{Q}_S))$-equivariant functions from $K\cap L(\mathbf{Q}_S)$ to the Hilbert space $\mathcal{H}_{P_0\cap M_Q}^{M_Q}(\pi)$. Now (7.5) and (8.4) tell us how to interpret the operators $R^Q_M(\pi, P_0)$ and $U'_Q(\pi, y)$ in this picture. Then (8.5) becomes an integral operator with kernel $K(k_1, k_2)$ equal to

$$\int_{M_Q(\mathbf{Q}_S)} I^{M_Q}_{P_0 \cap M_Q}(\pi, m)\cdot\delta_Q(m)^{1/2}\int_{N_Q(\mathbf{Q}_S)} f(k_1^{-1}mnk_2)u'_Q(k_2, y)\,dn\,dm$$
$$\cdot R^{M_Q}_M(\pi, P_0 \cap M_Q).$$

Therefore

$$\int_{K\cap M_Q(\mathbf{Q}_S)\backslash K\cap L(\mathbf{Q}_S)} \operatorname{tr}(K(k,k))\,dk$$

equals

$$\operatorname{tr}\left(I^{M_Q}_{P_0\cap M_Q}(\pi, f_{Q,y})R^{M_Q}_M(\pi, P_0\cap M_Q)\right).$$

This is just $J^{M_Q}_{M,\pi}(f_{Q,y})$. The lemma is proved. □

9. The map ϕ^L_M

Given groups $M \subset L$ in $\mathcal{L}(M_0)$, a function f in $\mathcal{C}(L(\mathbf{Q}_S))$ and a class π in $\Pi_{\text{temp}}(M(\mathbf{Q}_S))$, we defined

$$\phi^L_M(f, \pi) = \operatorname{tr}\left(I_{P_0}(\pi, f)R^L_M(\pi, P_0)\right), \qquad P_0 \in \mathcal{P}^L(M),$$

in Section 7. Let $\phi^L_M(f)$ be the map that sends $\pi \in \Pi_{\text{temp}}(M(\mathbf{Q}_S))$ to $\phi^L_M(f, \pi)$. According to Lemma 8.3,

$$\phi^L_M(f^y) = \sum_{Q\in\mathcal{F}^L(M)} \phi^{M_Q}_M(f_{Q,y})$$

for any $y \in L(\mathbf{Q}_S)$. In this section we shall show that

$$\phi^L_M: f \to \phi^L_M(f)$$

is a continuous map from $\mathcal{C}(L(\mathbf{Q}_S))$ to $\mathcal{I}(M(\mathbf{Q}_S))$. This will establish all the hypotheses of Section 4 (modulo Assumption 5.1, of course) for the case that $U(L) = \mathcal{C}(L(\mathbf{Q}_S))$, $V(M) = \mathcal{I}(M(\mathbf{Q}_S))$ and ϕ^L_M is as just defined. The proof of continuity is essentially a result in local harmonic analysis. In order not to stray too far afield, we shall be brief. The reader familiar with Harish-Chandra's work on the harmonic analysis on the Schwartz space will have no trouble with the details. (See [5(a)] and [12].)

Fix $P_0 \in \mathcal{P}^L(M)$. Given $f \in \mathcal{C}(L(\mathbf{Q}_S))$ and $Q \in \mathcal{F}^L(M)$, let $\phi'_Q(f, P_0)$ be the map that sends $\pi \in \Pi_{\text{temp}}(M(\mathbf{Q}_S))$ to

$$\phi'_Q(f, \pi, P_0) = \operatorname{tr}\left(I_{P_0}(\pi, f)R'_Q(\pi, P_0)\right).$$

LEMMA 9.1. *$\phi'_Q(f, P_0)$ belongs to $\mathcal{I}(M(\mathbf{Q}_S))$. In fact,*

$$f \to \phi'_Q(f, P_0)$$

is a continuous map from $\mathcal{C}(L(\mathbf{Q}_S))$ to $\mathcal{I}(M(\mathbf{Q}_S))$.

Proof. Let $\mathfrak{M} = \prod_{v \in S} M_v$ be a Levi S-subgroup of M. We need to show that for any n and any $D \in \mathrm{Diff}(\oplus_v i\mathfrak{a}_v^*)$,

$$f \to \sup_{\sigma \in \Pi_{\mathrm{temp}}(\mathfrak{M}_S)} (1 + \|\sigma\|)^n |D_\sigma \phi'_Q(f, \sigma^M, P_0)|$$

is a continuous seminorm on $\mathcal{C}(L(\mathbf{Q}_S))$. Now $\phi'_Q(f, \sigma^M, P_0)$ is the value at $t = 0$ of a linear combination of functions

$$\left(\frac{d}{dt}\right)^q \phi_R(t\lambda, f, \sigma^M, P_0), \qquad R \supset Q,$$

with $q = \dim(A_Q/Z)$, and λ a fixed point in $i\mathfrak{a}_Q^*$. We can take

$$\sigma = \bigotimes_{v \in S} \sigma_v, \qquad \sigma_v \in \Pi_{\mathrm{temp}}(M_v(\mathbf{Q}_v))$$

and

$$f = \prod_{v \in S} f_v, \qquad f_v \in \mathcal{C}(M_v(\mathbf{Q}_v)).$$

Then $\phi_R(t\lambda, f, \sigma^M, P_0)$ equals the product over $v \in S$ of

$$\mathrm{tr}\Big(I_{P_v}(\sigma_v, f_v) R_{P'_v|P_v}(\sigma_v)^{-1} R_{P'_v|P_v}(\sigma_{v,t\lambda})\Big),$$

for groups P_v and P'_v in $\mathcal{P}^L(M_v)$. We must show that for $D \in \mathrm{Diff}(i\mathfrak{a}_v^*)$ and q and n nonnegative integers, the value at $t = 0$ of

$$\sup_{\sigma_v \in \Pi_{\mathrm{temp}}(M_v(\mathbf{Q}_v))} (I + \|\sigma_v\|)^n \left|\left(\frac{d}{dt}\right)^q D_{\sigma_v} \mathrm{tr}\Big(I_{P_v}(\sigma_v, f_v) R_{P'_v|P_v}(\sigma_v)^{-1} R_{P'_v|P_v}(\sigma_{v,t\lambda})\Big)\right|$$

is a continuous seminorm on $\mathcal{C}(L(\mathbf{Q}_v))$.

The case of v discrete poses no problem. For then, f_v is bi-invariant under an open compact subgroup of K_v. This means that the operator

$$I_{P_v}(\sigma_v, f_v) R_{P'_v|P_v}(\sigma_v)^{-1} R_{P'_v|P_v}(\sigma_{v,t\lambda})$$

is of finite rank. Moreover, the operator vanishes unless σ_v belongs to a subset of $\Pi_{\mathrm{temp}}(M_v(Q_v))$ which is compact (in the obvious sense). These facts are easy consequences of a result of Harish-Chandra [5(c), Lemma 3]. The continuity of our seminorms follows from the definition of $\mathcal{C}(L(\mathbf{Q}_v))$.

Now suppose that v is Archimedean. For simplicity assume that $L = G$. If T is an operator on $\mathcal{H}_{P_v}(\sigma_v)$ let $\|T\|_1$ be the trace of its positive semi-definite square root. For W in $\Pi(K_v)$ let P_W be the projection of $\mathcal{H}_{P_v}(\sigma_v)$ onto $\mathcal{H}_{P_v}(\sigma_v)_W$, and define

$$I_{P_v}(\sigma_v, f_v)_{W_1, W_2} = P_{W_1} I_{P_v}(\sigma_v, f_v) P_{W_2}.$$

For any n and any $D \in \mathrm{Diff}(i\mathfrak{a}_v^*)$, there is a continuous seminorm $\|\cdot\|$ on $\mathcal{C}(G(\mathbf{Q}_v))$ such that

$$\| D_{\sigma_v} I_{P_v}(\sigma_v, f_v)_{W_1, W_2} \|_1 \leq \| f \| (1 + \| W_1 \|)^{-n} (1 + \| W_2 \|)^{-n} (1 + \| \sigma_v \|)^{-n},$$

for all $f \in \mathcal{C}(G(\mathbf{Q}_v))$, $\sigma_v \in \Pi_{\mathrm{temp}}(M_v(\mathbf{Q}_v))$ and $W_1, W_2 \in \Pi(K_v)$. This can be established fairly readily from the definitions. It is essentially the easy half of the theorem stated in [1(a)]. The estimate we require then follows from this inequality, (7.6), and the fact that

$$\sum_{W_1, W_2 \in \Pi(K_v)} (1 + \| W_1 \|)^{-n} (1 + \| W_2 \|)^{-n} (1 + \| \sigma_v \|)^{-n}$$

is bounded independently of $\sigma_v \in \Pi_{\mathrm{temp}}(M_v(\mathbf{Q}_v))$ for n sufficiently large. □

COROLLARY 9.2. *ϕ_M^L maps $\mathcal{C}(L(\mathbf{Q}_S))$ continuously to $\mathcal{I}(M(\mathbf{Q}_S))$.*

Proof. By Lemma 6.3,

$$\phi_M^L(f) = \sum_{Q \in \mathcal{P}^L(M)} \phi_Q'(f, P_0), \qquad f \in \mathcal{C}(L(\mathbf{Q}_S)),$$

for any $P_0 \in \mathcal{P}^L(M)$. The corollary follows from the lemma. □

Now suppose $M_1 \in \mathcal{L}^L(M)$ and that P_1 is a group in $\mathcal{P}^L(M_1)$ which contains a given $P_0 \in \mathcal{P}^L(M)$. Suppose also that $Q \in \mathcal{F}^L(M_1)$. Then for any $f \in \mathcal{C}(L(\mathbf{Q}_S))$, $\phi_Q'(f, P_1)$ is a function in $\mathcal{I}(M_1(\mathbf{Q}_S))$. Then $\phi_Q'(f, P_1)_M$ is a function in $\mathcal{I}(M(\mathbf{Q}_S))$. Its value at $\pi \in \Pi_{\mathrm{temp}}(M(\mathbf{Q}_S))$ is $\phi_Q'(f, \pi^{M_1}, P_1)$, which by (7.8) equals $\phi_Q'(f, \pi, P_0)$. Thus

$$\phi_Q'(f, P_1)_M = \phi_Q'(f, P_0). \tag{9.1}$$

The map ϕ_M^L is particularly simple on spherical functions. If f is bi-invariant under the maximal compact subgroup $K \cap L(\mathbf{Q}_S)$ of $L(\mathbf{Q}_S)$, it follows from (7.3) that $\phi_M^L(f)$ vanishes if $M \neq L$. Finally, suppose that $\tilde{S}$ is a disjoint union of S and $\{v\}$. Let f be a function in $\mathcal{C}(L(\mathbf{Q}_S))$, and f_v a Schwartz function on $L_v(\mathbf{Q}_v)$ which is bi-invariant under $K_v \cap L_v(\mathbf{Q}_v)$. Then if $\tilde{f} = f \cdot f_v$,

$$\phi_M^L(\tilde{f}) = \phi_M^L(f) \cdot f_{v, M}. \tag{9.2}$$

10. The invariant distributions $I_{M,\gamma}$

We have just seen that the maps

$$\phi_M^L \colon \mathcal{C}(L(\mathbf{Q}_S)) \to \mathcal{I}(M(\mathbf{Q}_S)), \qquad M \subset L,$$

satisfy the assumptions of Section 4. Suppose that γ is an element in $L(\mathbf{Q}_S)_{\text{reg}} \cap M$. We defined the distribution $J^L_{M,\gamma}$ in Section 8. If $M_1 \in \mathcal{L}^L(M_0)$ and does not contain M, define $J^{M_1}_{M,\gamma}$ to be 0. Then by Lemma 8.2,

$$J^L_{M,\gamma}(f^y) = \sum_{Q \in \mathcal{F}^L(M_0)} J^{M_Q}_{M,\gamma}(f_{Q,y}),$$

for all y in $L(\mathbf{Q}_S)$. It follows from Proposition 4.1 that there are invariant tempered distributions

$$I^L_{M,\gamma}, \quad \gamma \in L(\mathbf{Q}_S)_{\text{reg}} \cap M,\ L, M \in \mathcal{L}(M_0),$$

on $L(\mathbf{Q}_S)$ such that

$$J^L_{M,\gamma}(f) = \sum_{M_1 \in \mathcal{L}^L(M_0)} \hat{I}^{M_1}_{M,\gamma}\big(\phi^L_{M_1}(f)\big),$$

for all L and M. Observe that $I^L_{M,\gamma}$ will vanish unless $M \subset L$. Although they are defined by a simple formula, these distributions are in some ways rather complicated. For example,

$$I^G_{M,\gamma}(f) = I_{M,\gamma}(f), \qquad f \in \mathcal{C}(G(\mathbf{Q}_S)),$$

depends only on the image of f in $\mathcal{I}(G(\mathbf{Q}_S))$. However, as a distribution on $\mathcal{I}(G(\mathbf{Q}_S))$, $I_{M,\gamma}$ has no simple formula.

Notice that we could have used the distributions $\{J^L_{M,\pi}\}$ instead of $\{J^L_{M,\gamma}\}$ to obtain invariant distributions $\{I^L_{M,\pi}\}$. However, as the distributions $J^L_{M,\pi}$ were used to define the maps ϕ^L_M, this leads to nothing new. In fact, $I^L_{M,\pi}$ vanishes if $M \neq L$, and

$$I^M_{M,\pi}(f) = J^M_{M,\pi}(f) = \phi(f,\pi) = \operatorname{tr}\pi(f).$$

The rest of this section will be devoted to proving a useful property of the distributions. Suppose that $M \subset M_1 \subset L_1$ are groups in $\mathcal{L}(M_0)$, and that γ belongs to $L_1(\mathbf{Q}_S)_{\text{reg}} \cap M$. We shall show that $\hat{I}^{L_1}_{M_1,\gamma}(\phi)$ can be expressed as a linear combination of

$$\big\{\hat{I}^L_{M,\gamma}(\phi_L) \colon M \subset L \subset L_1\big\}, \qquad \phi \in \mathcal{I}(L_1(\mathbf{Q}_S)).$$

First, we shall prove two lemmas. Suppose that $M \in \mathcal{L}(M_0)$ is fixed, and that $\{c_P(\lambda)\colon P \in \mathcal{P}(M)\}$ is a (G, M) family. We assume in addition that if $M \subset L \subset L_1$, and $Q \in \mathcal{P}(L_1)$, the number c^Q_L is independent of Q. We denote it by $c^{L_1}_L$.

LEMMA 10.1. *If* $L_1 \supset M$,

$$\sum_{L \in \mathcal{L}^{L_1}(M)} c^L_M \cdot \phi^{L_1}_L(f)_M = \sum_{Q \in \mathcal{F}^{L_1}(M)} c'_Q \cdot \phi^{M_Q}_M(f_Q),$$

for any $f \in \mathcal{C}(L_1(\mathbf{Q}_S))$.

Proof. Notice that each side of the required formula is an element in $\mathcal{I}(M(\mathbf{Q}_S))$. By (9.1), the value of $\phi_L^{L_1}(f)_M$ at $\pi \in \Pi_{\text{temp}}(M(\mathbf{Q}_S))$ is

$$\sum_{Q \in \mathcal{P}^{L_1}(L)} \phi'_Q(f, \pi, P_0),$$

for fixed $P_0 \in \mathcal{P}^{L_1}(M)$. Since $c_M^L = c_M^Q$, the value of the left hand side of the required formula at π is

$$\sum_{Q \in \mathcal{F}^{L_1}(M)} c_M^Q \cdot \phi'_Q(f, \pi, P_0) = \sum_{Q \in \mathcal{F}^{L_1}(M)} c'_Q \cdot \phi_M^Q(f, \pi, P_0)$$
$$= \sum_Q c'_Q \cdot \phi_M^{M_Q}(f_Q, \pi),$$

by Lemmas 6.3 and 7.1. This is just the value at π of the right hand side. ∎

LEMMA 10.2. *Suppose that $L_1 \supset M$ and that γ belongs to $L_1(\mathbf{Q}_S)_{\text{reg}} \cap M$. Then*

$$\sum_{L \in \mathcal{L}^{L_1}(M)} c_M^L \hat{I}_{L,\gamma}^{L_1}(\phi) = \sum_{L \in \mathcal{L}^{L_1}(M)} c_L^{L_1} \hat{I}_{M,\gamma}^{L}(\phi_L),$$

for all $\phi \in \mathcal{I}(L_1(\mathbf{Q}_S))$.

Proof. Let f be any function in $\mathcal{C}(L_1(\mathbf{Q}_S))$ such that $\phi(f) = \phi$. The left hand side of the required formula equals

$$\sum_{L \in \mathcal{L}^{L_1}(M)} c_M^L J_{L,\gamma}^{L_1}(f) \tag{10.1}$$

minus the expression

$$\sum_{\{L, M_1 : M \subset L \subset M_1 \subsetneq L_1\}} c_M^L \hat{I}_{L,\gamma}^{M_1}\left(\phi_{M_1}^{L_1}(f)\right). \tag{10.2}$$

We shall prove the lemma by induction on $\dim(L_1/M)$. Apply the induction hypothesis to the sum over L in (10.2), and then add the result to the right hand side of the required formula. We obtain

$$\sum_{\{L, M_1 : M \subset L \subset M_1 \subset L_1\}} c_L^{M_1} \cdot \hat{I}_{M,\gamma}^{L}\left(\phi_{M_1}^{L_1}(f)_L\right).$$

By Lemma 10.1, this equals

$$\sum_{Q \in \mathcal{F}^{L_1}(M)} c'_Q \sum_{L \in \mathcal{L}^{M_Q}(M)} \hat{I}_{M,\gamma}^{L}\left(\phi_L^{M_Q}(f_Q)\right). \tag{10.3}$$

On the other hand, (10.1) equals

$$|D^{L_1}(\gamma)|^{1/2} \int_{L_1(\mathbf{Q}_S)_\gamma \backslash L_1(\mathbf{Q}_S)} f(x^{-1}\gamma x) \left(\sum_{L \in \mathcal{L}^{L_1}(M)} c_M^L v_L^{L_1}(x) \right) dx.$$

The sum in the brackets equals

$$\sum_{Q\in\mathcal{F}^{L_1}(M)} c_M^Q v_Q'(x) = \sum_{Q\in\mathcal{F}^{L_1}(M)} c_Q' v_M^Q(x),$$

by Lemma 6.3 and Corollary 6.4. It follows from (8.1) that the expression (10.1) equals

$$\sum_{Q\in\mathcal{F}^{L_1}(M)} c_Q' J_{M,\gamma}^{M_Q}(f_Q).$$

Recalling the inductive definition of the distributions $I_{M,\gamma}^{M_Q}$, we see that this equals (10.3). □

We shall apply the last lemma with

$$c_P(\lambda) = e^{\lambda(X_P)}, \qquad P\in\mathcal{P}(M),$$

with

$$X_P = \sum_\beta r_\beta X_P^\beta,$$

as in Section 7. The constants r_β are still to be chosen. Fix $M_1 \in \mathcal{L}(M)$. Choose positive numbers l_β for each of the reduced roots of (M_1, A) such that for every $R\in\mathcal{P}^{M_1}(M)$, the sum over all reduced roots β of (R, A) of $l_\beta\beta^\vee$ belongs to the chamber in $\mathfrak{a}$ associated to R. Fix $t\in\mathbf{R}$. For any reduced root β of (G, A) let $r_\beta = tl_\beta$ if β vanishes on $\mathfrak{a}_{M_1}$, and let $r_\beta = 0$ otherwise. Now suppose that $L\in\mathcal{L}(M)$. We claim that c_M^L vanishes unless $L\subset M_1$. For as we saw in Section 7,

$$c_M^L = \lim_{\lambda\to 0}\sum_{R\in\mathcal{P}^L(M)} e^{\lambda(X_R)}\theta_R^L(\lambda)^{-1}.$$

It is the volume in $\mathfrak{a}_M^L$ of the convex hull of the points $\{X_R: R\in\mathcal{P}^L(M)\}$ defined in Section 7. Each X_R is orthogonal to $\mathfrak{a}_L$, and by our choice of $\{r_\beta\}$, it is also orthogonal to $\mathfrak{a}_{M_1}$. Therefore, if L is not contained in M_1, the points $\{X_R\}$ all lie in a proper subspace of $\mathfrak{a}_M^L$. The convex hull then has volume 0; c_M^L then does vanish. On the other hand, if $L = M_1$, each point X_R lies in the chamber of $\mathfrak{a}_M^L$ corresponding to R. The volume of the convex hull does not vanish by the results of Section 6. Therefore $c_M^{M_1}\neq 0$.

Now fix $L_1\in\mathcal{L}(M_1)$. For $\phi\in\mathcal{I}(L_1(\mathbf{Q}_S))$,

$$\sum_{L\in\mathcal{L}^{L_1}(M)} c_M^L \hat{I}_{L,\gamma}^{L_1}(\phi)$$

is a polynomial in t, whose highest term is $c_M^{M_1}\hat{I}_{M_1,\gamma}^{L_1}(\phi)$. The right hand side of the identity in Lemma 10.2 is also a polynomial in t. Equating the highest terms, we find that $\hat{I}^{L_1}_{M_1,\gamma}(\phi)$ equals the sum over those $L \in \mathcal{L}^{L_1}(M)$ such that $\dim(\mathfrak{a}^{L_1}_L) = \dim(\mathfrak{a}^{M_1}_M)$, of

$$\left(c^{M_1}_M\right)^{-1} c^{L_1}_L \hat{I}^L_{M,\gamma}(\phi_L).$$

For any L in this last sum, there are natural maps

$$\mathfrak{a}^{M_1}_M \to \mathfrak{a}^{L_1}_M \to \mathfrak{a}^{L_1}_L.$$

The composition gives us a map from $\mathfrak{a}^{M_1}_M$ to $\mathfrak{a}^{L_1}_L$. Suppose that it is not an isomorphism. Then by our choice of $\{r_\beta\}$, the images in $\mathfrak{a}^{L_1}_L$ of vectors $\{X_Q\colon Q \in \mathcal{P}^{L_1}(M)\}$ span a proper subspace of $\mathfrak{a}^{L_1}_L$. It follows that $c^{L_1}_L = 0$. Thus we may include only those L in the sum for which the map is an isomorphism, or what is the same thing, for which

$$\mathfrak{a}^{L_1}_M = \mathfrak{a}^L_M \oplus \mathfrak{a}^{M_1}_M.$$

We have established

LEMMA 10.3. *Suppose that $M \subset M_1 \subset L_1$ are groups in $\mathcal{L}(M_0)$. Then for every $L \in \mathcal{L}^{L_1}(M)$ there is a constant $d(L)$, which vanishes unless*

$$\mathfrak{a}^{L_1}_M = \mathfrak{a}^L_M \oplus \mathfrak{a}^{M_1}_M,$$

such that for all $\phi \in \mathcal{I}(L_1(\mathbf{Q}_S))$ and $\gamma \in L_1(\mathbf{Q}_S)_{\text{reg}} \cap M$,

$$\hat{I}^{L_1}_{M_1,\gamma}(\phi) = \sum_{L \in \mathcal{L}^{L_1}(M)} d(L) \hat{I}^L_{M,\gamma}(\phi_L). \qquad \square$$

11. A splitting theorem

It is important to be able to express the distributions $I^L_{M,\gamma}$ on $L(\mathbf{Q}_S)$ in terms of distributions on the local groups $L(\mathbf{Q}_v)$. As before, $M \subset L$ are groups in $\mathcal{L}(M_0)$ and S is a finite set of valuations on $\mathbf{Q}$.

THEOREM 11.1. *Let $\phi \in \mathcal{I}(L(\mathbf{Q}_S))$ and $\gamma \in L(\mathbf{Q}_S)_{\text{reg}} \cap M$. Suppose that S is a disjoint union of two subsets S' and S'' and that ϕ and γ decompose relative to the product $L(\mathbf{Q}_S) = L(\mathbf{Q}_{S'})L(\mathbf{Q}_{S''})$ as $\phi = \phi'\phi''$ and $\gamma = \gamma'\gamma''$ respectively. Then*

$$\hat{I}^L_{M,\gamma}(\phi) = \sum_{M_1 \in \mathcal{L}^L(M)} \hat{I}^{M_1}_{M,\gamma'}(\phi'_{M_1}) \hat{I}^L_{M_1,\gamma''}(\phi'').$$

Proof. We shall prove the theorem by induction on $\dim(L/M)$. Suppose that

$$f = f' \cdot f'', \qquad f' \in \mathcal{C}(L(\mathbf{Q}_{S'})), \qquad f'' \in \mathcal{C}(L(\mathbf{Q}_{S''})),$$

is any function such that $\phi(f') = \phi'$ and $\phi(f'') = \phi''$. Then $\hat{I}^L_{M,\gamma}(\phi)$ is the difference between $J^L_{M,\gamma}(f)$ and

$$(11.1) \qquad \sum_{\{M_2:\, M\subset M_2 \subsetneq L\}} \hat{I}^{M_2}_{M,\gamma}\left(\phi^L_{M_2}(f)\right).$$

Apply the induction hypothesis to each summand. Then

$$\hat{I}^{M_2}_{M,\gamma} = \sum_{\{M_1:\, M\subset M_1\subset M_2\}} \hat{I}^{M_2|M_1}_{M,\gamma'} \otimes \hat{I}^{M_2}_{M_1,\gamma''},$$

where $\hat{I}^{M_2|M_1}_{M,\gamma'} \otimes \hat{I}^{M_2}_{M_1,\gamma''}$ stands for the invariant distribution on $\mathcal{I}(M_2(\mathbf{Q}_{S'}) \otimes \mathcal{I}(M_2(\mathbf{Q}_{S''}))$ that maps a function $\psi' \otimes \psi''$ to $\hat{I}^{M_1}_{M,\gamma'}(\psi'_{M_1})\hat{I}^{M_2}_{M_1,\gamma''}(\psi'')$. Therefore, (11.1) can be written as the difference between

$$(11.2) \qquad \sum_{\{M_1, M_2:\, M\subset M_1\subset M_2\subset L\}} \left(\hat{I}^{M_2|M_1}_{M,\gamma'} \otimes \hat{I}^{M_2}_{M_1,\gamma''}\right)\left(\phi^L_{M_2}(f)\right)$$

and

$$\sum_{\{M_1:\, M\subset M_1\subset L\}} \hat{I}^{M_1}_{M,\gamma'}\left(\phi'_{M_1}\right) I^L_{M_1,\gamma''}(\phi'').$$

This last expression is just what we want. We will therefore be done if we can show that (11.2) equals $J^L_{M,\gamma}(f)$.

The value of $\phi^L_{M_2}(f)$ at $\pi \in \Pi_{\text{temp}}(M_2(\mathbf{Q}_S))$ is obtained from the (L, M_2) family

$$\phi_P(\lambda, f, \pi, P_2), \qquad P \in \mathcal{P}^{L_1}(M_2),$$

for a fixed group P_2 in $\mathcal{P}^L(M_2)$. If $\pi = \pi' \otimes \pi''$,

$$\phi_P(\lambda, f, \pi, P_2) = \phi_P(\lambda, f', \pi', P_2)\phi_P(\lambda, f'', \pi'', P_2).$$

We can apply Lemma 6.3 to this product of (L, M_2) families. It follows that

$$\phi^L_{M_2}(f) = \sum_{Q\in\mathcal{F}^L(M_2)} \phi'_Q(f', P_2)\phi^Q_{M_2}(f'', P_2).$$

Now $\phi'_Q(f', P_2)_{M_1}$ is an element in $\mathcal{I}(M_1(\mathbf{Q}_{S'}))$. As we observed in Section 9,

$$\phi'_Q(f', P_2)_{M_1} = \phi'_Q(f', P_1),$$

for any group $P_1 \in \mathcal{P}^L(M_1)$, contained in P_2. Also, Lemma 7.1 allows us to write $\phi^Q_{M_2}(f'', P_2)$ as $\phi^{M_Q}_{M_2}(f''_Q)$. Consequently, $(\hat{I}^{M_2|M_1}_{M,\gamma'} \otimes \hat{I}^{M_2}_{M_1,\gamma''})(\phi^L_{M_2}(f))$ equals

$$(11.3) \qquad \sum_{Q\in\mathcal{F}^L(M_2)} \hat{I}^{M_1}_{M,\gamma'}\left(\phi'_Q(f', P_1)\right)\hat{I}^{M_2}_{M_1,\gamma''}\left(\phi^{M_Q}_{M_2}(f''_Q)\right).$$

Substitute (11.3) for the summand in (11.2). Since $I^{M_1}_{M,\gamma'}(\phi'_Q(f', P_1))$ is independent of M_2, and

$$\sum_{\{M_2:\, M_1\subset M_2\subset M_Q\}} \hat{I}^{M_2}_{M_1,\gamma''}\left(\phi^{M_Q}_{M_2}(f''_Q)\right) = J^{M_Q}_{M_1,\gamma''}(f''_Q),$$

we obtain

$$\sum_{\{M_1:\, M\subset M_1\subset L\}} \Bigl(\sum_{Q\in\mathcal{F}^L(M_1)} \hat{I}^{M_1}_{M,\gamma'}(\phi'_Q(f',P_1)) J^{M_Q}_{M_1,\gamma''}(f''_Q) \Bigr).$$

According to (8.1), the expression in the brackets equals the integral over y in $L(\mathbf{Q}_{S''})_{\gamma''}\backslash L(\mathbf{Q}_S'')$ of the product of

$$|D^L(\gamma'')|^{1/2} f''(y^{-1}\gamma'' y)$$

with

$$(11.4) \qquad \sum_{Q\in\mathcal{F}^L(M_1)} \hat{I}^{M_1}_{M,\gamma'}(\phi'_Q(f',P_1)) v^Q_{M_1}(y).$$

The value at $\pi' \in \Pi_{\mathrm{temp}}(M_1(\mathbf{Q}_{S'}))$ of the function

$$\sum_{Q\in\mathcal{F}^L(M_1)} v^Q_{M_1}(y)\phi'_Q(f',P_1)$$

in $\mathcal{I}(M_1(\mathbf{Q}_{S'}))$ is

$$\sum_{Q\in\mathcal{F}^L(M_1)} v^Q_{M_1}(y)\phi'_Q(f',\pi',P_1) = \sum_Q v'_Q(y)\phi^Q_{M_1}(f',\pi',P_1)$$
$$= \sum_Q v'_Q(y)\phi^{M_Q}_{M_1}(f'_Q,\pi'),$$

by Lemmas 6.3 and 7.1. Therefore (11.4) equals

$$\sum_{Q\in\mathcal{F}^L(M_1)} v'_Q(y)\hat{I}^{M_1}_{M,\gamma'}(\phi^{M_Q}_{M_1}(f'_Q)).$$

Since $v'_Q(y)$ is independent of M_1, (11.2) equals the integral over y in $L(\mathbf{Q}_{S''})_{\gamma''}\backslash L(\mathbf{Q}_{S''})$ of the sum over all $Q\in\mathcal{F}^L(M)$ of the product of

$$|D^L(\gamma'')|^{1/2} f''(y^{-1}\gamma'' y)v'_Q(y)$$

with

$$\sum_{\{M_1:\, M\subset M_1\subset M_Q\}} \hat{I}^{M_1}_{M,\gamma'}(\phi^{M_Q}_{M_1}(f'_Q)).$$

This last expression is just $J^{M_Q}_{M,\gamma'}(f'_Q)$, which by (8.1) equals the integral over x in $L(\mathbf{Q}_{S'})_{\gamma'}\backslash L(\mathbf{Q}_{S'})$ of

$$|D^L(\gamma')|^{1/2} f'(x^{-1}\gamma' x)v^Q_M(x).$$

But

$$\sum_{Q\in\mathcal{F}^L(M)} v'_Q(y)v^Q_M(x) = v^L_M(xy).$$

Moreover,

$$|D^L(\gamma')|^{1/2} \cdot |D^L(\gamma'')|^{1/2} = |D^L(\gamma)|^{1/2},$$

and

$$f'(x^{-1}\gamma' x) f''(y^{-1}\gamma'' y) = f\big((xy)^{-1}\gamma(xy)\big).$$

It follows that (11.2) equals $J_{M,\gamma}^L(f)$. This is what we were required to prove. □

If we combine the theorem with Lemma 10.3 we obtain

COROLLARY 11.2. *For every group*

$$\mathfrak{L} = \prod_{v \in S} L_v, \qquad L_v \in \mathfrak{L}^L(M),$$

there is a constant $c(\mathfrak{L})$, *which equals* 0 *unless*

$$\mathfrak{a}_M^L = \bigoplus_{v \in S} \mathfrak{a}_M^{L_v},$$

such that for all

$$\phi = \prod_{v \in S} \phi_v, \qquad \phi_v \in \mathcal{I}(L(\mathbf{Q}_v)),$$

and

$$\gamma = \prod_{v \in S} \gamma_v, \qquad \gamma_v \in \mathfrak{L}(\mathbf{Q}_v)_{\text{reg}} \cap M,$$

we have

$$\hat{I}_{M,\gamma}^L(\phi) = \sum_{\mathfrak{L}} c(\mathfrak{L}) \prod_{v \in S} \hat{I}_{M,\gamma_v}^{L_v}(\phi_{v,L_v}).$$

Proof. Let v be a valuation in S, and let S′ be the complement of v in S. Then we can decompose $\phi = \phi'\phi_v$ and $\gamma = \gamma'\gamma_v$ relative to the product $L(\mathbf{Q}_S) = L(\mathbf{Q}_{S'}) \cdot L(\mathbf{Q}_v)$. By the theorem, $\hat{I}_{M,\gamma}^L(\phi)$ equals

$$\sum_{M_1 \in \mathfrak{L}^L(M)} \hat{I}_{M,\gamma'}^{M_1}(\phi'_{M_1}) \hat{I}_{M_1,\gamma_v}^L(\phi_v).$$

Applying Lemma 10.3, we obtain

$$\sum_{M_1 \in \mathfrak{L}^L(M)} \sum_{L_v \in \mathfrak{L}^L(M)} d(L_v) \hat{I}_{M,\gamma'}^{M_1}(\phi'_{M_1}) \hat{I}_{M,\gamma_v}^{L_v}(\phi_{v,L_v}),$$

where $d(L_v)$ vanishes unless

$$\mathfrak{a}_M^L = \mathfrak{a}_M^{M_1} \oplus \mathfrak{a}_M^{L_v}.$$

The corollary follows by induction on the number of elements in S. □

We will need a slight generalization of this corollary. Suppose that v is a valuation. We have been studying the distributions

$$I^L_{M,\gamma_v}, \qquad \gamma_v \in L(\mathbf{Q}_v)_{\mathrm{reg}} \cap M,$$

on $\mathcal{C}(L(\mathbf{Q}_v))$. It has always been understood that the Levi subgroups $M \subset L$ were defined over $\mathbf{Q}$. This is clearly not necessary. For any pair $M_v \subset L_v$ of Levi subgroups of G defined over $\mathbf{Q}_v$ for which K_v is admissible (a condition we will assume for the rest of the paper), we could just as easily have defined distributions

$$I^{L_v}_{M_v,\gamma_v}, \qquad \gamma_v \in L_v(\mathbf{Q}_v)_{\mathrm{reg}} \cap M_v,$$

on $L_v(\mathbf{Q}_v)$. Lemma 10.3 would certainly continue to be valid.

We will still, however, retain the fixed Levi subgroups $M \subset L$ defined over $\mathbf{Q}$. Then there is a surjective map

$$h_{ML}: \mathfrak{a}_M \to \mathfrak{a}_L$$

such that

$$h_{ML}(H_M(m)) = H_L(m),$$

for any m in $M(\mathbf{Q}_S)$. Suppose that $\mathfrak{L} = \prod_v L_v$ is a Levi S-subgroup of L. There is a natural map from $\mathfrak{a}_{L_v} = \mathrm{Hom}(X(L_v)_{\mathbf{Q}_v}, \mathbf{R})$ onto $\mathfrak{a}_L = \mathrm{Hom}(X(L)_{\mathbf{Q}}, \mathbf{R})$. We therefore have a surjective map

$$h_L: \bigoplus_{v \in S} \mathfrak{a}_{L_v} \to \mathfrak{a}_L.$$

Define a map $H_{\mathfrak{L}}$, with values in $\oplus_{v \in S} \mathfrak{a}_{L_v}$, by letting

$$H_{\mathfrak{L}}(x) = \bigoplus_{v \in S} (\log p_v) H_{L_v}(x_v),$$

for any $x = \prod_v x_v$ in $\mathfrak{L}_S$. If $\chi \in X(L)_{\mathbf{Q}}$,

$$\langle h_L(H_{\mathfrak{L}}(x)), \chi \rangle = \sum_v (\log p_v) \langle H_{L_v}(x_v), \chi \rangle.$$

The exponential of this number is

$$\prod_v p_v^{\langle H_{L_v}(x_v), \chi \rangle} = \prod_v |\chi(x_v)|_v = e^{\langle H_L(x), \chi \rangle}.$$

In other words,

$$h_L(H_{\mathfrak{L}}(x)) = H_L(x).$$

Now fix an S-subgroup $\mathfrak{M} = \prod_{v \in S} M_v$ of M. We have the maps

$$h_M: \bigoplus_v \mathfrak{a}_{M_v} \to \mathfrak{a}_M$$

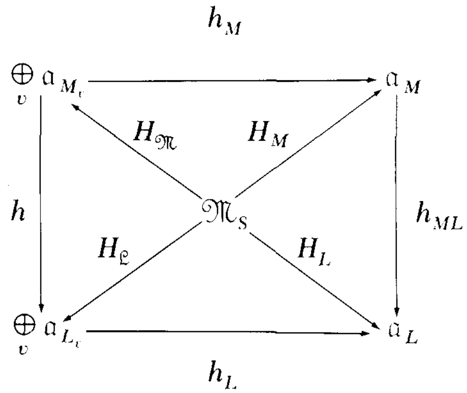
and

$$H_{\mathfrak{M}}: \mathfrak{M}_S \to \bigoplus_v \mathfrak{a}_{M_v},$$

defined above. Let us write $\mathcal{L}^L(\mathcal{M}, S)$ for the set of levi S-subgroups $\mathcal{L} = \prod_v L_v$ of L such that $M_v \subset L_v$ for each v. If $\mathcal{L}$ is any group in $\mathcal{L}^L(\mathcal{M}, S)$, we have a surjective map

$$h\colon \bigoplus_v \mathfrak{a}_{M_v} \to \bigoplus_v \mathfrak{a}_{L_v}.$$

This leads to a commutative diagram:



All the maps are surjective. The kernels of h and h_{ML} equal $\oplus_v \mathfrak{a}_{M_v}^{L_v}$ and $\mathfrak{a}_M^L$ respectively. So we have a map

$$\bigoplus_{v \in S} \mathfrak{a}_{M_v}^{L_v} \to \mathfrak{a}_M^L. \tag{11.5}$$

We shall be most concerned with those $\mathcal{L} \in \mathcal{L}^L(\mathcal{M}, S)$ for which this map is an isomorphism. When this happens, we also have $\ker h \cap \ker h_M = 0$, and the equivalent property that $\ker H_{\mathcal{M}} = \ker \mathcal{H}_{\mathcal{L}} \cap \ker H_M$.

COROLLARY 11.3. *For every* $\mathcal{L} = \prod_{v \in S} L_v$ *in* $\mathcal{L}^L(\mathcal{M}, S)$ *there is a constant* $d(\mathcal{L})$, *which equals* 0 *unless the map* (11.5) *is an isomorphism, such that for all*

$$\phi = \prod_v \phi_v, \qquad \phi_v \in \mathcal{I}(L(\mathbf{Q}_v)),$$

and

$$=$$

$$\gamma = \prod_v \gamma_v, \qquad \gamma_v \in L(\mathbf{Q}_v)_{\text{reg}} \cap M_v,$$

we have

$$\hat{I}^L_{M,\gamma}(\phi) = \sum_{\mathcal{L} \in \mathcal{L}^L(\mathcal{M}, S)} d(\mathcal{L}) \prod_{v \in S} \hat{I}^{L_v}_{M_v, \gamma_v}(\phi_{v, L_v}).$$

Proof. By the last corollary, $\hat{I}^L_{M,\gamma}(\phi)$ is the sum over all groups

$$\mathcal{M}' = \prod_{v \in S} M'_v, \qquad M'_v \in \mathcal{L}^L(M),$$

of

$$c(\mathfrak{M}')\prod_{v\in S}\hat{I}^{M'_v}_{M,\gamma_v}(\phi_{v,M'_v}).$$

Now, apply Lemma 10.3 to each of the distributions $\hat{I}^{M'_v}_{M,\gamma_v}(\phi_{v,M'_v})$. We obtain the sum over Levi subgroups L_v in $\mathcal{L}^{M'_v}(M_v)$, defined over $\mathbf{Q}_v$, of

$$d(L_v)\hat{I}^{L_v}_{M_v,\gamma_v}(\phi_{v,L_v}).$$

The constant $d(L_v)$ will be 0 unless

$$\mathfrak{a}^{M'_v}_{M_v}=\mathfrak{a}^{M}_{M_v}\oplus\mathfrak{a}^{L_v}_{M_v}. \tag{11.6}$$

This last condition means that the natural map

$$\mathfrak{a}^{L_v}_{M_v}\to\mathfrak{a}^{M'_v}_{M}$$

is an isomorphism. Let $\mathcal{L}=\prod_v L_v$. In view of (11.6) $\mathfrak{M}'$ is uniquely determined by $\mathcal{L}$. Define

$$d(\mathcal{L})=c(\mathfrak{M}')\prod_{v\in S}d(L_v).$$

The map (11.5) is the composition of

$$\bigoplus_{v\in S}\mathfrak{a}^{L_v}_{M_v}\to\bigoplus_{v\in S}\mathfrak{a}^{M'_v}_{M}\to\mathfrak{a}^{L}_{M}.$$

If the first map is not an isomorphism, $d(L_v)=0$ for some v. If the second is not an isomorphism, $c((\mathfrak{M}))=0$. Therefore $d(\mathcal{L})$ is 0 unless the map (11.5) is an isomorphism. □

If ϕ and γ are as in the corollary, and $\mathcal{L}$ belongs to $\mathcal{L}^L(\mathfrak{M},S)$, we shall write

$$\phi_{\mathcal{L}}=\prod_{v\in S}\phi_{v,L_v}$$

and

$$\hat{I}^{\mathcal{L}}_{\mathfrak{M},\gamma}(\phi_{\mathcal{L}})=\prod_{v\in S}\hat{I}^{L_v}_{M_v,\gamma_v}(\phi_{v,L_v})$$

in the next section. Corollary 11.3 then is the formula

$$\hat{I}^{L}_{M,\gamma}(\phi)=\sum_{\mathcal{L}\in\mathcal{L}^L(\mathfrak{M},S)}d(\mathcal{L})\hat{I}^{\mathcal{L}}_{\mathfrak{M},\gamma}(\phi_{\mathcal{L}}).$$

12. Compact support

We have studied the maps

$$\phi^L_M:\mathcal{C}(L(\mathbf{Q}_S))\to\mathcal{I}(M(\mathbf{Q}_S)),$$

for groups $M\subset L$ in $\mathcal{L}(M_0)$, and a finite set S of valuations on $\mathbf{Q}$. In Section 7, we also defined a number $\phi_M^L(f, \pi)$ for $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$ and $\pi \in \Pi(M(\mathbf{Q}_S)^1)$. We will take π to be a tempered representation. Then

$$\phi_M^L(f): \pi \to \phi_M^L(f, \pi)$$

is a complex valued function on $\Pi_{\text{temp}}(M(\mathbf{Q}_S)^1)$. Our goal is to show that it belongs to $\mathcal{I}_c(M(\mathbf{Q}_S)^1)$. From Corollary 9.2 and its very definition, we know that $\phi_M^L(f)$ is obtained from a function in $\mathcal{I}(M(\mathbf{Q}_S))$ by the projection (5.3). We have only to show that the orbital integrals of $\phi_M^L(f)$ are compactly supported.

THEOREM 12.1. *If $M \subset L$, ϕ_M^L maps $C_c^\infty(L(\mathbf{Q}_S)^1)$ continuously to $\mathcal{I}_c(M(\mathbf{Q}_S)^1)$.*

Proof. We shall prove the theorem by induction on $\dim(L/M)$. Suppose that $\mathcal{T} = \prod_{v \in S} T_v$, where for each v, T_v is a maximal torus of M defined over $\mathbf{Q}_v$. Set $\mathcal{T}_S = \prod_{v \in S} T_v(\mathbf{Q}_v)$, and $\mathcal{T}_S^1 = \mathcal{T}_S \cap M(\mathbf{Q}_S)^1$. We must show that for every compact subset C of $L(\mathbf{Q}_S)^1$ there is a bounded subset D of $\mathcal{T}_S^1 \cap M(\mathbf{Q}_S)_{\text{reg}}$ such that whenever f is a function in $C_c^\infty(L(\mathbf{Q}_S)^1)$ which is supported on C, the function

$$\gamma \to \hat{I}_\gamma^M(\phi_M^L(f)), \qquad \gamma \in \mathcal{T}_S^1 \cap M(\mathbf{Q}_S)_{\text{reg}},$$

is supported on D.

LEMMA 12.2. *For each $\phi \in \mathcal{I}_c(L(\mathbf{Q}_S)^1)$ there is a compact subset C of $\mathcal{T}_S^1 \cap L(\mathbf{Q}_S)_{\text{reg}}$, depending only on* supp ϕ *such that $\hat{I}_{M,\gamma}^L(\phi) = 0$ if γ does not belong to C.*

Proof. For v in S define A_v to be the split component of the torus T_v. Let M_v be the centralizer of A_v in M. It is a Levi subgroup of M defined over $\mathbf{Q}_v$, so $\mathcal{M} = \prod_{v \in S} M_v$ is a Levi S-subgroup of M, and $\mathcal{T}_S^1 \cap L(\mathbf{Q}_S)_{\text{reg}}$ is contained in $L(\mathbf{Q}_S)_{\text{reg}} \cap \mathcal{M}_S$. Therefore, Corollary 11.3 tells us that for any $\gamma \in \mathcal{T}_S^1 \cap L(\mathbf{Q}_S)_{\text{reg}}$ and $\phi \in \mathcal{I}(L(\mathbf{Q}_S))$,

$$\hat{I}_{M,\gamma}^L(\phi) = \sum_{\mathcal{L} \in \mathcal{L}^L(\mathcal{M}, S)} d(\mathcal{L}) \hat{I}_{\mathcal{M},\gamma}^{\mathcal{L}}(\phi_{\mathcal{L}}).$$

Now, $\mathcal{T}_S^1 \cap L(\mathbf{Q}_S)_{\text{reg}}$ is also contained in $L(\mathbf{Q}_S)^1$. This means that $\hat{I}_{M,\gamma}^L(\phi)$ depends only on the function

$$\pi \to \int_{i\mathfrak{a}_L^*/\text{Lat}(L,S)} \phi(\pi_\Lambda)\, d\Lambda$$

on $\Pi_{\text{temp}}(L(\mathbf{Q}_S)^1)$. We therefore may identify ϕ with a function on $L(\mathbf{Q}_S)^1$, which we assume belongs to $\mathcal{I}_c(L(\mathbf{Q}_S)^1)$.

Fix $\mathcal{L} \in \mathcal{L}^L(\mathcal{M}, S)$. Then $H_{\mathcal{L}}$ maps the center of $\mathcal{L}_S$ surjectively onto $\oplus_{v \in S} \mathfrak{a}_{L_v}$. With this fact it is easy to show that the function

$$\hat{I}_{\mathcal{M},\gamma}^{\mathcal{L}}(\phi_{\mathcal{L}}), \qquad \gamma \in \mathcal{T}_S^1 \cap L(\mathbf{Q}_S)_{\text{reg}},$$

vanishes unless $H_{\mathfrak{L}}(\gamma)$ belongs to a compact subset of $\oplus_{v\in S}\mathfrak{a}_{L_v}$ which depends only on $\operatorname{supp}\phi$. On the other hand, we can assume that $d(\mathfrak{L})\neq 0$, so by Corollary 11.3 the map (11.5) is an isomorphism. As we observed in the preamble to the corollary, this implies that

$$\ker H_{\mathfrak{M}} = \ker H_{\mathfrak{L}} \cap \ker H_M.$$

Now $\mathfrak{T}_S$ is a subgroup of $\mathfrak{M}_S$. The kernel of $H_{\mathfrak{M}}$ in $\mathfrak{T}_S$ is compact. $\mathfrak{T}_S^1$ is by definition just the kernel of H_M in $\mathfrak{T}_S$. Therefore, the restriction of $H_{\mathfrak{L}}$ to $\mathfrak{T}_S^1$ is a proper map. Thus the map

$$\gamma \to \hat{I}^{\mathfrak{L}}_{\mathfrak{M},\gamma}(\phi_{\mathfrak{L}}), \qquad \gamma\in\mathfrak{T}_S^1\cap L(\mathbf{Q}_S)_{\mathrm{reg}},$$

is supported on a bounded set, which depends only on $\operatorname{supp}\phi$. This proves the lemma. ∎

Lemma 12.3. *For every compact subset C of $L(\mathbf{Q}_S)^1$ there is a bounded subset D of $\mathfrak{T}_S^1\cap L(\mathbf{Q}_S)_{\mathrm{reg}}$ such that for any function f in $C_c^\infty(L(\mathbf{Q}_S)^1)$ supported on C, the function*

$$\gamma \to J^L_{M,\gamma}(f), \qquad \gamma\in\mathfrak{T}_S^1\cap L(\mathbf{Q}_S)_{\mathrm{reg}}$$

is supported on D.

If $L = M$, $J^L_{M,\gamma}(f)$ equals $I^M_\gamma(f)$, the invariant orbital integral of f. The lemma in this case is a well known result of Harish-Chandra. The proof for arbitrary L is no different. ∎

We can now finish the proof of the theorem. $\hat{I}^M_\gamma(\phi^L_M(f))$ equals

$$J^L_{M,\gamma}(f) - \sum_{\{M_1:\, M\subsetneq M_1\subset L\}} \hat{I}^{M_1}_{M,\gamma}\big(\phi^L_{M_1}(f)\big).$$

It follows from the last two lemmas and our induction hypothesis that if f is supported on C, $\hat{I}^M_\gamma(\phi^L_M(f))$ vanishes for γ outside a fixed bounded subset of $\mathfrak{T}_S^1\cap L(\mathbf{Q}_S)_{\mathrm{reg}}$. But $\mathfrak{T}_S^1\cap L(\mathbf{Q}_S)_{\mathrm{reg}}$ is dense in $\mathfrak{T}_S^1\cap M(\mathbf{Q}_S)_{\mathrm{reg}}$. The theorem follows. □

13. The invariant distributions $I_{\mathfrak{o}}$ and I_χ

With the completion of the proof of Theorem 12.1 we have reached our goal. We have shown that for every S the maps

$$\phi^L_M: C_c^\infty\big(L(\mathbf{Q}_S)^1\big) \to \mathcal{I}_c\big(M(\mathbf{Q}_S)^1\big)$$

are continuous and satisfy (4.2). Assume that S contains the Archimedean valuation. Then there are invariant distributions $\{I^L_{\mathfrak{o}}: \mathfrak{o}\in\mathfrak{O}\}$ and $\{I^L_\chi: \chi \in \mathfrak{X}\}$ on

$L(\mathbf{Q}_S)^1$, and by Theorem 4.2,

$$\sum_{\mathfrak{o}\in\mathcal{O}} I^L_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} I^L_{\chi}(f)$$

for any $f \in C_c^\infty(L(\mathbf{Q}_S)^1)$.

As we observed in Section 4, we can think of $I^L_{\mathfrak{o}}$ and I^L_{χ} as invariant distributions on $C_c^\infty(L(\mathbf{A})^1)$. However, we had better check that the final distributions are independent of S. Suppose that S' is a larger set of valuations, the disjoint union of S and S_1. There is a natural injection of $C_c^\infty(L(\mathbf{Q}_S)^1)$ into $C_c^\infty(L(\mathbf{Q}_{S'})^1)$. The image, f', of a function f in $C_c^\infty(L(\mathbf{Q}_S)^1)$ is the product of f with the characteristic function of $\Pi_{v\in S_1}K_v$. We must verify that $I^L_{\mathfrak{o}}(f)$ equals $I^L_{\mathfrak{o}}(f')$. By definition, $J^L_{\mathfrak{o}}(f) = J^L_{\mathfrak{o}}(f')$. On the other hand we can map any function ϕ in $\mathcal{I}_c(L(\mathbf{Q}_S)^1)$ to the function in $\mathcal{I}_c(L(\mathbf{Q}_{S'})^1)$ whose value at

$$\pi\otimes\pi_1,\ \pi\in\Pi_{\text{temp}}(L(\mathbf{Q}_S)),\ \pi_1\in\Pi_{\text{temp}}(L(\mathbf{Q}_{S_1}))$$

is $\phi(\pi)$ if π_1 is of class one and is 0 otherwise. It follows from (9.2) that for any $M \in \mathcal{L}^L(M_0)$ the image of $\phi^L_M(f)$ in $\mathcal{I}_c(M(\mathbf{Q}_{S'})^1)$ equals $\phi^L_M(f')$. Now

$$I^L_{\mathfrak{o}}(f) = J^L_{\mathfrak{o}}(f) - \sum_{\{M\in\mathcal{L}^L(M_0):\, M\neq L\}} \hat{I}^M_{\mathfrak{o}}(\phi^L_M(f)).$$

It follows by induction on $\dim L$ that $I^L_{\mathfrak{o}}(f) = I^L_{\mathfrak{o}}(f')$. Similarly $I^L_{\chi}(f) = I^L_{\chi}(f')$. Thus, the distributions $I^L_{\mathfrak{o}}$ and I^L_{χ} are independent of S.

We can therefore regard $\hat{I}^L_{\mathfrak{o}}$ and $\hat{I}^L_{\chi}$ as distributions on $\mathcal{I}_c(L(\mathbf{A})^1)$, the direct limit over all S of the spaces $\mathcal{I}_c(L(\mathbf{Q}_S)^1)$. Notice that ϕ^L_M extends to a continuous map from $C_c^\infty(L(\mathbf{A})^1)$ to $\mathcal{I}_c(M(\mathbf{A})^1)$. In fact, $\mathcal{H}_P(\pi)$, $I_{P_0}(\pi, f)$, $R_{P|P_0}(\pi)$ and $\phi^L_M(f)$ can all be defined directly for $f \in C_c^\infty(L(\mathbf{A}))$ and $\pi \in \Pi_{\text{temp}}(M(\mathbf{A}))$. If f^1 is the restriction of f to $L(\mathbf{A})^1$, the value of $\phi^L_M(f^1)$ at a class in $\Pi_{\text{temp}}(M(\mathbf{A})^1)$ is the integral over all π in the associated orbit in $\Pi_{\text{temp}}(M(\mathbf{A}))$ of

$$\lim_{\lambda\to 0} \sum_{P\in\mathcal{P}^L(M)} \phi_P(\lambda, f, \pi, P_0)\theta_P(\lambda)^{-1}$$

$$= \lim_{\lambda\to 0} \sum_{P\in\mathcal{P}^L(M)} \operatorname{tr}(I_{P_0}(\pi, f)R_{P|P_0}(\pi)^{-1}R_{P|P_0}(\pi_\lambda))\theta_P(\lambda)^{-1}.$$

As they are defined, $\hat{I}^L_{\mathfrak{o}}$ and $\hat{I}^L_{\chi}$ appear to depend on all the arbitrary choices we made in Section 1. We shall show that, as distributions on $\mathcal{I}_c(L(\mathbf{A})^1)$, they do not.

We shall first consider changing the maximal compact subgroup K. Fix $f \in C_c^\infty(L(\mathbf{A}))$. For the moment, write $\phi^L_M(f, K)$ for $\phi^L_M(f)$. We wish to study the dependence on K. Given $P \in \mathcal{P}^L(M)$ and $\pi \in \Pi_{\text{temp}}(M(\mathbf{A}))$, define

$$(\gamma_P(\pi)\psi)(nmk) = \delta_P(m)^{1/2}\pi(m)\psi(k),$$

for $\psi \in \mathcal{H}_P(\pi)$, $n \in N_P(\mathbf{A})$, $m \in M(\mathbf{A})$ and $k \in K \cap L(\mathbf{A})$. Then $\gamma_P(\pi)$ maps $\mathcal{H}_P(\pi)$ isomorphically onto a Hilbert space $\tilde{\mathcal{H}}_P(\pi)$ of functions on $L(\mathbf{A})$. $\tilde{\mathcal{H}}_P(\pi)$ is what is usually taken for the underlying space of the induced representation. It is independent of K. Now

$$\begin{aligned}\phi_P(\lambda, f, \pi, P_0) &= \operatorname{tr}\left(I_{P_0}(\pi, f)R_{P|P_0}(\pi)^{-1}R_{P|P_0}(\pi_\lambda)\right)\\ &= \operatorname{tr}\left(\gamma_{P_0}(\pi)I_{P_0}(\pi, f)R_{P|P_0}(\pi)^{-1}R_{P|P_0}(\pi_\lambda)\gamma_{P_0}(\pi)^{-1}\right)\\ &= \operatorname{tr}\left(\tilde{I}_{P_0}(\pi, f)\tilde{R}_{P|P_0}(\pi)^{-1}\cdot\gamma_P(\pi)\gamma_P(\pi_\lambda)^{-1}\cdot\tilde{R}_{P|P_0}(\pi_\lambda)\cdot\gamma_{P_0}(\pi_\lambda)\gamma_{P_0}(\pi)^{-1}\right),\end{aligned}$$

where

$$\tilde{I}_{P_0}(\pi, f) = \gamma_{P_0}(\pi)I_{P_0}(\pi, f)\gamma_{P_0}(\pi)^{-1}$$

and

$$\tilde{R}_{P|P_0}(\pi) = \gamma_P(\pi)R_{P|P_0}(\pi)\gamma_{P_0}(\pi)^{-1}.$$

If $\phi \in \tilde{\mathcal{H}}_{P_0}(\pi)$,

$$\left(\tilde{I}_{P_0}(\pi, y)\phi\right)(x) = \phi(xy), \qquad x, y \in L(\mathbf{A}),$$

so the operator $\tilde{I}_{P_0}(\pi, f)$ is independent of K. It does, however, depend on our choice of Haar measure on $L(\mathbf{A})$. For suitable λ and ϕ, $(\tilde{R}_{P|P_0}(\pi_\lambda)\phi)(x)$ is the product of convergent intertwining integrals

$$\int_{N_P(\mathbf{Q}_v)\cap N_{P_0}(\mathbf{Q}_v)\backslash N_P(\mathbf{Q}_v)} \phi(nx)\,dn$$

with some scalar normalizing factors. The integrals clearly do not depend on K. We will assume from now on that each scalar factor is also independent of K and that the normalized operators are independent of any choice of Haar measure. Then the only terms in our expression for $\phi_P(\lambda, f, \pi, P_0)$ that depend on K are the operators

$$\gamma_P(\pi)\gamma_P(\pi_\lambda)^{-1}: \tilde{\mathcal{H}}_P(\pi_\lambda) \to \tilde{\mathcal{H}}_P(\pi).$$

Notice that

$$\left(\gamma_P(\pi)\gamma_P(\pi_\lambda)^{-1}\phi\right)(x) = \phi(x)e^{-\lambda(H_P(x))},$$

for any $\phi \in \tilde{\mathcal{H}}_P(\pi_\lambda)$.

Now suppose that K is replaced by another admissible maximal compact subgroup K^*. The Haar measures fixed in Section 1 were tied, via (1.1), to our choice of K. We must therefore take a different set of Haar measures on all our groups, subject only to the restrictions of Section 1. If the new Haar measure on $L(\mathbf{A})$ differs from the old one by a factor $\mu(L)$, the operator $\tilde{I}_{P_0}(\pi, f)$ will have to be replaced by $\mu(L)\tilde{I}_{P_0}(\pi, f)$. For $P \in \mathcal{P}^L(M)$ and $x \bullet L(\mathbf{A})$ we have the vector $H_P^*(x)$ in $\mathfrak{a}_M$, associated to the decomposition

$$L(\mathbf{A}) = N_P(\mathbf{A})M(\mathbf{A})(K^* \cap L(\mathbf{A})).$$

We also have the Hilbert spaces $\mathcal{H}_P^*(\pi)$ and the operators

$$\gamma_P^*(\pi): \mathcal{H}_P^*(\pi) \to \tilde{\mathcal{H}}_P(\pi).$$

If ϕ belongs to $\tilde{\mathcal{H}}_P(\pi)$,

$$\left(\gamma_P^*(\pi)\gamma_P^*(\pi_\lambda)^{-1}\left(\gamma_P(\pi)\gamma_P(\pi_\lambda)^{-1}\right)^{-1}\phi\right)(x) = u_P(\lambda, x, K^*)\phi(x),$$

where

$$u_P(\lambda, x, K^*) = e^{-\lambda(H_P^*(x))}e^{\lambda(H_P(x))}.$$

It follows that

$$\gamma_P^*(\pi)\gamma_P^*(\pi_\lambda)^{-1} = \gamma_P(\pi)U_P(\lambda, \pi, K^*)\gamma_P(\pi_\lambda)^{-1}, \tag{13.1}$$

where $U_P(\lambda, \pi, K^*)$ is the operator on $\mathcal{H}_P(\pi)$ defined by

$$(U_P(\lambda, \pi, K^*)\phi)(k) = u_P(\lambda, k, K^*)\phi(k), \qquad k \bullet K \cap L(\mathbf{A}). \tag{13.2}$$

We can now calculate the analogue for K^* of $\phi_P(\lambda, f, \pi, P_0)$. We need only replace $\gamma_P(\pi)\gamma_P(\pi_\lambda)^{-1}$ by the right hand side of (13.1) in the formula above for $\phi_P(\lambda, f, \pi, P_0)$, and then multiply by $\mu(L)$. The result is

$$\mu(L)\cdot\operatorname{tr}\left(I_{P_0}(\pi, f)R_{P|P_0}(\pi)^{-1}U_P(\lambda, \pi, K^*)R_{P|P_0}(\pi_\lambda)U_{P_0}(\lambda, \pi, K^*)^{-1}\right).$$

To obtain $\phi_M^L(\pi, f, K^*)$, we multiply this expression by $\theta_P(\lambda)^{-1}$, sum over $P \bullet \mathcal{P}^L(M)$, and let λ approach 0. Now $U_{P_0}(\lambda, \pi, K^*)$ is independent of P and its value at $\lambda = 0$ is 1. Therefore $\phi_M^L(\pi, f, K^*)$ equals the trace of the operator

(13.3)

$$\mu(L)I_{P_0}(\pi, f)\cdot\lim_{\lambda\to 0}\sum_{P\in\mathcal{P}^L(M)} R_{P|P_0}(\pi)^{-1}\cdot U_P(\lambda, \pi, K^*)\theta_P(\lambda)^{-1}\cdot R_{P|P_0}(\pi_\lambda).$$

We can now argue exactly as in the proof of Lemma 8.3. Formulas (13.2) and (13.3) take the place of (8.4) and (8.3). Notice that $\{u_P(\lambda, x, K^*): P \bullet \mathcal{P}^L(M)\}$ is an (L, M) family. If $Q \in \mathcal{F}^L(M)$ and $m \in M_Q(\mathbf{A})$,

$$u_Q(\lambda, mx, K^*) = u_Q(\lambda, x, K^*).$$

It follows from this fact, (13.2) and (7.5) that if P_0 and P are groups in $\mathcal{P}^L(M)$ which are both contained in Q,

$$R_{P|P_0}(\pi)^{-1}U_Q'(\lambda, \pi, K^*) = U_Q'(\lambda, \pi, K^*)R_{P|P_0}(\pi)^{-1}.$$

As in Lemma 8.3, we see that $\phi_M^L(f, \pi, K^*)$ equals the sum over $Q \in \mathcal{F}^L(M)$ of the trace of the operator

$$I_{P_0}(\pi, f)U'_Q(\pi, K^*)R^Q_M(\pi, P_0),$$

where P_0 can be taken to be any group in $\mathcal{P}^L(M)$ with $P_0 \subset Q$. The rest of the proof of Lemma 8.3 carries over without further difficulty. We obtain

LEMMA 13.1.

$$\phi^L_M(f, \pi, K^*) = \mu(L) \sum_{Q \in \mathcal{F}^L(M)} \phi^{M_Q}_M(f_{Q,K^*}, \pi, K),$$

where f_{Q,K^*} is the function in $C^\infty_c(M(\mathbf{A}))$ whose value at $m \in M(\mathbf{A})$ is

$$\delta_Q(m)^{1/2} \int_{K \cap L(\mathbf{A})} \int_{N_Q(\mathbf{A})} f(k^{-1}mnk)u'_Q(k, K^*)\, dn\, dk. \qquad \square$$

Next, suppose that J^L is one of the distributions J^L_o or J^L_χ. Identify f with its restriction to $L(\mathbf{A})^1$, and write

$$J^L(f, K) = J^L(f)$$

to denote the dependence on K. This dependence can be studied by transcribing almost verbatim the proof of Theorem 3.2. The result is the formula

$$J^L(f, K^*) = \mu(L) \sum_{Q \in \mathcal{F}^L(M_0)} c(M_Q)c(L)^{-1} J^{M_Q}(f_{Q,K^*}, K).$$

Now let $I^L(\cdot, K)$ be the invariant distribution defined by Proposition 4.1. We claim that its Fourier transform, $\hat{I}(\cdot, K)$, is independent of K. Indeed, arguing as in the proof of Proposition 4.1, we find that

$$I^L(f, K^*) - \mu(L)I^L(f, K)$$

equals the sum over all $Q \in \mathcal{F}^L(M)$, with $Q \neq L$, of the product of $\mu(L)c(M_Q)c(L)^{-1}$ with

$$J^{M_Q}(f_{Q,K^*}, K) - \sum_{M \in \mathcal{L}^{M_Q}(M_0)} c(M)c(M_Q)^{-1} \hat{I}^M(\phi^{M_Q}_M(f_{Q,K^*}, K), K^*).$$

Now $\phi^{M_Q}_M(f_{Q,K^*}, K)$ is an element in $\mathcal{I}_c(M(\mathbf{A})^1)$. Therefore we may assume by induction on $\dim L$ that

$$\hat{I}^M(\phi^{M_Q}_M(f_{Q,K^*}, K), K^*) = \hat{I}^M(\phi^{M_Q}_M(f_{Q,K^*}, K), K).$$

It follows that the above expression vanishes. We have shown that $I^L(f, K^*) = \mu(L)I^L(f, K)$.

Now suppose that instead of fixing f we fixed an element ϕ in $\mathcal{I}_c(L(\mathbf{A})^1)$. Take f to be any function such that $\phi = \phi^L_L(f, K)$. The second choice of Haar measures will necessitate replacing f by $\mu(L)^{-1}f$. Therefore $\hat{I}^L(\phi, K)$ equals $\hat{I}^L(\phi, K^*)$. We have established

PROPOSITION 13.2. *As distributions on $\mathcal{I}_c(L(\mathbf{A})^1)$, $\hat{I}^L_{\mathfrak{o}}$ and $\hat{I}^L_{\chi}$ are independent of K. They are also independent of the Haar measures chosen in Section* 1. □

In Section 1 we also fixed M_0, a minimal Levi subgroup of G. The distributions $\hat{I}_{\mathfrak{o}}$ and $\hat{I}_{\chi}$ are independent of this choice as well. Any other minimal Levi subgroup equals $y^{-1}M_0y$, for some y in $G(\mathbf{Q})$. Then $y^{-1}Ky$ is a maximal compact subgroup of G which is admissible relative to $y^{-1}M_0y$. If L is in $\mathcal{L}(M_0)$, $y^{-1}Ly$ belongs to $\mathcal{L}(Y^{-1}M_0y)$. We can transfer all the Haar measures chosen in Section 1 by conjugating by y^{-1}. It is a simple exercise to check that $I^L_{\mathfrak{o}}(f) = I^{y^{-1}Ly}_{\mathfrak{o}}(f^y)$ and $I^L_{\chi}(f) = I^{y^{-1}Ly}_{\chi}(f^y)$ for $f \in C_c^\infty(L(\mathbf{A})^1)$. It follows that $\hat{I}_{\mathfrak{o}}$ and $\hat{I}_{\chi}$ are independent of M_0.

14. An example

We shall conclude our paper with a look at the example of inner twistings of GL_n. Special cases have recently been studied by Flath [3(a)] and by Deligne and Kazdan (unpublished). For simplicity of notation we will stick to $\mathrm{GL}_n(\mathbf{Q})$ although we could just as easily work, through restriction of scalars, with an arbitrary number field. Suppose that D is a division algebra of degree d^2 over $\mathbf{Q}$. Let G and G_1 be the general linear groups of ranks m and $n = md$ over D and $\mathbf{Q}$ respectively. The local Langlands conjecture states that for every valuation v on $\mathbf{Q}$ there is an injection

$$\Pi_{\mathrm{temp}}(G(\mathbf{Q}_v)) \to \Pi_{\mathrm{temp}}(G_1(\mathbf{Q}_v)) \tag{14.1}$$

with certain properties. This would yield an injection

$$\Pi_{\mathrm{temp}}\left(G(\mathbf{A})^1\right) \to \Pi_{\mathrm{temp}}\left(G_1(\mathbf{A})^1\right).$$

For every $f \in C_c^\infty(G(\mathbf{A})^1)$ we could then define a complex valued function ϕ_1 on $\Pi_{\mathrm{temp}}(G_1(\mathbf{A})^1)$ by letting $\phi_1(\pi_1)$ equal $\mathrm{tr}(\pi(f))$ if π_1 is the image of $\pi \in \Pi_{\mathrm{temp}}(G(\mathbf{A})^1)$, and letting it equal 0 otherwise. The conditions on (14.1) are such that ϕ_1 should belong to $\mathcal{I}_c(G_1(\mathbf{A})^1)$. The ultimate goal would be to establish a correspondence between the automorphic representations of G and G_1 by comparing the identity

$$\sum_{\mathfrak{o} \in \mathcal{O}_1} \hat{I}_{\mathfrak{o}}(\phi_1) = \sum_{\chi \in \mathfrak{X}_1} \hat{I}_{\chi}(\phi_1),$$

the invariant trace formula for G_1, with the trace formula for G.

We will attempt something more modest here. Now $\mathcal{O}$ and $\mathcal{O}_1$ can be identified with the semisimple conjugacy classes in $G(\mathbf{Q})$ and $G_1(\mathbf{Q})$ respectively.

The theory of division algebras gives an injection

$$\mathcal{O} \to \mathcal{O}_1. \tag{14.2}$$

It is easy to describe the image; it is also easy to say what the image of the map

$$C_c^\infty\big(G(\mathbf{A})^1\big) \to \mathcal{I}_c\big(G_1(\mathbf{A})^1\big) \tag{14.3}$$

should be. If ϕ_1 belongs to the expected image of (14.3) but $\mathfrak{o} \in \mathcal{O}_1$ does not belong to the image of (14.2), $\hat{I}_{\mathfrak{o}}(\phi_1)$ ought to vanish. We will prove this when the class $\mathfrak{o}$ is *unramified*, in the sense of [1(d)].

First of all let us recall some notions for the general linear group over a fixed field E. The characteristic polynomial identifies the semisimple conjugacy classes in $\mathrm{GL}_n(E)$ with the polynomials in $E[X]$ of degree n and nonzero constant term. Regular semisimple conjugacy classes correspond to polynomials with distinct roots. If $\mathfrak{o}$ is a regular semisimple class we define a partition

$$\mathfrak{p}(\mathfrak{o}) = (n_1, \dots, n_r), \qquad n_1 \geq n_2 \geq \cdots \geq n_r,$$

of n from the degrees of the irreducible factors of the characteristic polynomial. A partition can also be defined for any Levi subgroup, M, of GL_n defined over E. It is the unique partition

$$\mathfrak{p}(M) = (n_1, \dots, n_r), \qquad n_1 \geq \cdots \geq n_r,$$

of n such that M is isomorphic to $\prod_{i=1}^r \mathrm{GL}_{n_i}$. Note that r is the dimension of the space $\mathfrak{a}_M$. We can partially order the partitions of n by setting $\mathfrak{p}_1 \leq \mathfrak{p}_2$ whenever there are Levi subgroups $M_1 \subset M_2$ of GL_n such that $\mathfrak{p}_1 = \mathfrak{p}(M_1)$ and $\mathfrak{p}_2 = \mathfrak{p}(M_2)$. Then $\mathfrak{p}(\mathfrak{o}) \leq \mathfrak{p}(M)$, for a given $\mathfrak{o}$ and M, if and only if $\mathfrak{o}$ intersects $M(E)$. Notice that if

$$\mathfrak{p}(k) = (k, k, \dots, k)$$

for a given divisor k of n, $\mathfrak{p}(k) \leq (n_1, \dots, n_r)$ if and only if k divides each integer n_i.

Lemma 14.1. *Suppose that M is a Levi subgroup of GL_n and that $\mathfrak{a}_M^{M_1} \cap \mathfrak{a}_M^{M_2} = \{0\}$ for groups M_1 and M_2 in $\mathcal{L}(M)$ (defined, of course, over E). Suppose also that $\mathfrak{p}(k) \leq \mathfrak{p}(M_1)$ and $\mathfrak{p}(k) \leq \mathfrak{p}(M_2)$ for some divisor k of n. Then $\mathfrak{p}(k) \leq \mathfrak{p}(M)$.*

Proof. The condition on M_1 and M_2 is equivalent to $\mathfrak{a}_M = \mathfrak{a}_{M_1} + \mathfrak{a}_{M_2}$. Now for $i = 1, 2$, let

$$\mathfrak{p}(M_i) = (n_{i1}, \dots, n_{ir_i}).$$

Then k divides each n_{ij}, and $M_i = \prod_{j=1}^{r_i} M_{ij}$, where M_{ij} is isomorphic to $\mathrm{GL}_{n_{ij}}$. The intersection of $\mathfrak{a}_{M_1}$ and $\mathfrak{a}_{M_2}$ is a space of dimension at least one. Therefore

$$\dim \mathfrak{a}_M = \dim \mathfrak{a}_{M_1} + \dim \mathfrak{a}_{M_2} - \dim(\mathfrak{a}_{M_1} \cap \mathfrak{a}_{M_2}) \leq r_1 + r_2 - 1.$$

Now M is the subgroup of M_i defined by a subset of the simple roots of M_i with respect to some ordering on a maximal split torus. It follows that

$$M = \prod_{j=1}^{r_i} (M \cap M_{ij}).$$

We claim that one of the groups M_{ij} is contained in M. Assume the contrary, and suppose that $r_1 \leq r_2$. For each j, $M \cap M_{2j}$ will be a proper subgroup of M_{2j}. The length of $\mathfrak{p}(M)$ will be no less than $2r_2$. Since this must also equal $\dim \mathfrak{a}_M$, we obtain a contradiction. We have shown that $M_{2j} \subset M$ for some j. Therefore, $M_{2j} \subset M_{1h}$ for some h. Let $n' = n - n_{2j}$. Then k divides n' and we have a partition $\mathfrak{p}'(d)$ of n'. There is clearly a unique subgroup G' of GL_n, isomorphic to $\mathrm{GL}_{n'}$, such that M_2 is contained in $M_{2j}G'$. Then $M' = M \cap G'$, $M_1' = M_1 \cap G'$ and $M_2' = M_2 \cap G'$ are Levi subgroups of G' such that $\mathfrak{a}_{M'}^{M_1'} \cap \mathfrak{a}_{M'}^{M_2'} = \{0\}$. The partition $\mathfrak{p}(M_1')$ is obtained by replacing n_{1h} by $n_{1h} - n_{2j}$. Since k divides $n_{1h} - n_{2j}$, $\mathfrak{p}'(k) \leq \mathfrak{p}(M_1')$. Similarly $\mathfrak{p}'(k) \leq \mathfrak{p}(M_2')$. It follows by induction on n that $\mathfrak{p}'(k) \leq \mathfrak{p}(M')$. Since $\mathfrak{p}(M)$ is obtained from $\mathfrak{p}(M')$ by adjoining n_{2j}, $\mathfrak{p}(k) \leq \mathfrak{p}(M)$. □

For each valuation v on $\mathbf{Q}$ we have the invariant, $\mathrm{inv}_v(D)$, of D at v. It is an element in $\mathbf{Q}/\mathbf{Z}$, and

$$\sum_v \mathrm{inv}_v(D) = 0.$$

Let d_v be the order of $\mathrm{inv}_v(D)$. Then d is the least common multiple of the integers $\{d_v\}$. The image of the map (14.1) should be the set of induced cuspidal representations

$$\sigma^G, \qquad \sigma \in \Pi_{\mathrm{cusp}}(M_v(\mathbf{Q}_v)),$$

where M_v is a Levi subgroup of G_1 defined over $\mathbf{Q}_v$ such that $\mathfrak{p}(d_v) \leq \mathfrak{p}(M_v)$. Said another way, the image of the map (14.3) will be

$$\mathcal{I}_c^G(G_1(\mathbf{A})^1) = \lim_{\vec{S}} \mathcal{I}_c^G(G_1(\mathbf{Q}_S)^1),$$

where $\mathcal{I}_c^G(G_1(\mathbf{Q}_S)^1)$ is the space of functions ϕ_1 in $\mathcal{I}_c^G(G_1(\mathbf{Q}_S)^1)$ such that $\phi_{1,\mathfrak{M}} = 0$ for any Levi S-subgroup $\mathfrak{M} = \prod_{v \in S} M_v$ of G_1 for which the property

$$\mathfrak{p}(d_v) \leq \mathfrak{p}(M_v), \qquad v \in S,$$

fails to hold. Next, suppose that $\mathfrak{o}$ is an unramified class in $\mathcal{O}_1$; in the present situation this means a regular semisimple conjugacy class in $G_1(\mathbf{Q})$. For each v, $\mathfrak{o}$ generates a regular semisimple conjugacy class in $G_1(\mathbf{Q}_v) = \mathrm{GL}_n(\mathbf{Q}_v)$, so we obtain a partition $\mathfrak{p}_v(\mathfrak{o})$ of n. It follows from the theory of division algebras that $\mathfrak{o}$ is in the image of the map (14.2) if and only if $\mathfrak{p}(d_v) \leq \mathfrak{p}_v(\mathfrak{o})$ for all v.

THEOREM 14.2. *Suppose that* $\phi_1 \in \mathcal{I}_c^G(G_1(\mathbf{A})^1)$ *and that* $\hat{I}_\mathfrak{o}(\phi_1) \neq 0$ *for a given unramified class* $\mathfrak{o} \in \mathcal{O}_1$. *Then* $\mathfrak{o}$ *belongs to the image of the map* (14.2).

Proof. Let f_1 be any function in $C_c^\infty(G_1(\mathbf{A})^1)$ such that $\phi(f_1) = \phi_1$. The results of [1(d), §8] allow us to express $J_\mathfrak{o}(f_1)$ as a weighted orbital integral. There are parabolic subgroups $P_0 \subset P_1$ of G_1, with Levi components $M_0 \subset M_1$, all defined over $\mathbf{Q}$, such that P_0 is minimal and $\mathfrak{p}(M_1) = \mathfrak{p}(\mathfrak{o})$. Let γ_1 be a point in $M_1(\mathbf{Q}) \cap \mathfrak{o}$. If T is a suitably regular point in $\mathfrak{a}_{P_0}^+$, $J_\mathfrak{o}^T(f_1)$ equals

$$\operatorname{vol}\big(A_{M_1}(\mathbf{R})^0 \cdot G(\mathbf{Q})_{\gamma_1} \backslash G(\mathbf{A})_{\gamma_1}\big) \int_{G(\mathbf{A})_{\gamma_1}\backslash G(\mathbf{A})} f_1(x^{-1}\gamma_1 x) v(x, T)\, dx,$$

where $v(x,T)$ is the volume of the convex hull of the projection of

$$\Big\{ s^{-1}T - s^{-1}H_{P_2}(w_s x) : s \in \bigcup_{P_2 \supset P_0} \Omega(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) \Big\}$$

onto $\mathfrak{a}_{M_1}^G$ (see [1(d), (8.7)]). Suppose that $s \in \Omega$ and that $P_0' = w_s^{-1} P_0 w_s$. Then

$$\begin{aligned} s^{-1}H_{P_0}(w_s x) &= s^{-1}\big(H_{P_0}(w_s) + H_{P_0}(\tilde{w}_s x)\big) \\ &= -H_{P_0}\big(w_s^{-1}\big) + H_{P_0'}(x) \\ &= s^{-1}T_0 - T_0 + H_{P_0'}(x), \end{aligned}$$

by Lemma 1.1. It follows easily that $v(x, T_0)$ equals

$$v_{M_1}(x) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M_1)} e^{-\lambda(H_P(x))} \theta_P(\lambda)^{-1}.$$

Choose a finite set S of valuations on $\mathbf{Q}$, containing the Archimedean valuation and all the places at which D does not split, such that

(i) f_1 belongs to $C_c^\infty(G_1(\mathbf{Q}_S)^1)$.

(ii) $|D(\gamma_1)|_v = 1$ for all v not in S.

(iii) $\{x \in G_1(\mathbf{Q}_v) : x^{-1}\gamma_1 x \in K_v\} = G_1(\mathbf{Q}_v)_{\gamma_1} \cdot K_v$ for all v not in S.

Then

$$|D(\gamma_1)| = \prod_{v \in S} |D(\gamma_1)|_v = 1.$$

Moreover, the integral over $G(\mathbf{A})_{\gamma_1} \backslash G(\mathbf{A})$ above can be taken over $G(\mathbf{Q}_S)_{\gamma_1} \backslash G(\mathbf{Q}_S)$. If we identify γ_1 with its image in $M_1(\mathbf{Q}_S)$, we obtain

$$J_\mathfrak{o}(f_1) = \operatorname{vol}\big(A_{M_1}(\mathbf{R})^0 \cdot G(\mathbf{Q})_{\gamma_1} \backslash G(\mathbf{A})_{\gamma_1}\big) J_{M_1, \gamma_1}(f_1).$$

It follows from the definitions of Section 4 that

$$\hat{I}_\mathfrak{o}(\phi_1) = \operatorname{vol}\big(A_M(\mathbf{R})^0 \cdot G(\mathbf{Q})_{\gamma_1} \backslash G(\mathbf{A})_{\gamma_1}\big) \hat{I}_{M_1, \gamma_1}(\phi_1). \tag{14.4}$$

We have not used any special properties of GL_n in deriving this formula. It holds for any reductive group.

Our theorem will now be proved by combining the corollaries of Theorem 11.1 with Lemma 14.1. The first corollary states that

$$\hat{I}_{M_1,\gamma_1}(\phi_1) = \sum_{\mathcal{L}} c(\mathcal{L}) \hat{I}^{\mathcal{L}}_{M_1,\gamma_1}(\phi_{1,\mathcal{L}}),$$

the sum being taken over groups

$$\mathcal{L} = \prod_{v \in S} L_v, \qquad L_v \supset M_1.$$

By assumption, there is an $\mathcal{L}$ such that

$$c(\mathcal{L}) \hat{I}^{\mathcal{L}}_{M_1,\gamma_1}(\phi_{1,\mathcal{L}}) \neq 0,$$

so that

$$\mathfrak{a}^G_{M_1} = \bigoplus_{v \in S} \mathfrak{a}^{L_v}_{M_1},$$

and

$$\mathfrak{p}(d_v) \leq \mathfrak{p}(L_v), \qquad v \in S.$$

Fix $v \in S$. For any prime p, let p^r be the highest power of p which divides d_v. Since the invariants of D sum to 0, there must be a $w \in S$, distinct from v, such that p^r divides d_w. Therefore $\mathfrak{p}(p^r) \leq \mathfrak{p}(L_v)$ and $\mathfrak{p}(p^r) \leq \mathfrak{p}(L_w)$. Since $\mathfrak{a}^{L_v}_{M_1} \cap \mathfrak{a}^{L_w}_{M_1} = \{0\}$, we can apply Lemma 14.1. We see that $\mathfrak{p}(p^r) \leq \mathfrak{p}(M_1)$. It follows that $\mathfrak{p}(d_v) \leq \mathfrak{p}(M_1)$.

We have identified γ_1 with its image in $M_1(\mathbf{Q}_S)$, so we shall write

$$\gamma_1 = \prod_{v \in S} \gamma_v, \qquad \gamma_v \in M_1(\mathbf{Q}_v).$$

For each v, choose a Levi subgroup M_v of G_1, defined over $\mathbf{Q}_v$, with $\gamma_v \in M_v \subset M_1$, and $\mathfrak{p}(M_v) = \mathfrak{p}_v(\mathfrak{o})$. Let $\mathcal{M} = \prod_{v \in S} M_v$. By Corollary 11.3, $\hat{I}_{M_1,\gamma_1}(\phi_1)$ equals the sum over all $\mathcal{L}$ in $\mathcal{L}(\mathcal{M}, S)$ of

$$d(\mathcal{L}) \hat{I}^{\mathcal{L}}_{\mathcal{M},\gamma_1}(\phi_{1,\mathcal{L}}).$$

This summand will be nonzero for some $\mathcal{L}$. The conditions for the nonvanishing of $d(\mathcal{L})$ and $\phi_{1,\mathcal{L}}$ imply that

$$\mathfrak{a}^{M_1}_{M_v} \cap \mathfrak{a}^{L_v}_{M_v} = \{0\},$$

and

$$\mathfrak{p}(d_v) \leq \mathfrak{p}(L_v)$$

for all $v \in S$. Turning again to Lemma 14.1, we obtain

$$\mathfrak{p}(d_v) \leq \mathfrak{p}(M_v) = \mathfrak{p}_v(\mathfrak{o})$$

for all $v \in S$. Since S contains all valuations for which $d_v > 1$, $\mathfrak{o}$ belongs to the image of (14.2). Our theorem is proved. □

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